«ФИЗИКА ЭЛЕМЕНТАРНЫХ ЧАСТИЦ И АТОМНОГО ЯДРА»
1993, ТОМ 24, ВЫП. 1

УДК 539.12.01

EFFECTIVE CHIRAL LAGRANGIANS AND NAMBU – JONA-LASINIO MODEL

M.K. Volkov

Joint Institute for Nuclear Research, Dubna

The review is presented of papers devoted to the construction of effective meson Lagrangians on the basis of studying QCD at low energies. Two different approaches to the solution of this problem are given as an example. In the first case, meson fields are introduced as a chiral phase of quark fields and external sources directly in the QCD Lagrangian. In other papers, mesons are treated as composite quark-antiquark objects. More attention is paid to the Nambu — Jona-Lasinio model, from the second group of models, and to its various physical applications. The main advantage of the last model is its mechanism of spontaneous breaking of chiral symmetry.

Дается обзор работ, посвященных построению эффективных мезонных лагранжианов на основе изучения КХД в области низких энергий. Приводятся примеры двух различных подходов к решению этой проблемы. В первом случае мезонные поля вводятся в качестве киральной фазы кварковых полей и внешних источников непосредственно в лагранжиане КХД. В других работах мезоны рассматриваются как составные кварк-антикварковые объекты. Наибольшее внимание уделено модели Намбу — Иона-Лазинио, относящейся ко второй группе моделей, и ее различным физическим приложениям. Основное преимущество последней модели связано с заложенным в ее основу механизмом спонтанного нарушения киральной симметрии.

1. INTRODUCTION

Quantum chromodynamics is well known to be a successful theory in the description of high-energy interactions of hadrons. On the other hand, the low-energy physics of hadrons is also well described by effective chiral Lagrangians developed much earlier than QCD [1—3]. As at low energies the direct use of QCD is difficult owing to a large value of the coupling constant, attempts have repeatedly been undertaken to obtain such effective chiral Lagrangians immediately from QCD rather than from group-theoretical considerations, and to consider them as a low-energy representation of QCD. In recent years, a lot of papers has been devoted to that problem. Here we present a brief review of some of the most characteristic investigations.

All these attempts can be divided into two groups. In the first of them, the authors introduce pseudoscalar meson fields, considering them to be external fields, as a chiral phase of quark fields directly into the QCD Lagrangian [4—7]. In the second group, a consistent bosonization of QCD is carried out, with all types of mesons treated as quark-antiquark bound states [8—13]. We consider here the most typical examples of these approaches.

The main attention in this work is paid to the Nambu — Jona-Lasinio model (NJL model). This model belongs to the second group. The NJL model at quark level was first studied in 1976 in [14]. In the last decade interest in this model was continuously increasing [15—19] thus expanding the range of its application in recent years [20—32]. More thorough studies were made of chiral anomalies and Skyrme-like terms with fourth-order derivatives as well as of intrinsic properties of the model connected with nondiagonal transitions of mesons [16,18,21,22], mixing of flavours [13,20—23], dependence of the model parameters on temperature and density [24—26], gluon condensates [26,27], phase transitions [28], relations to QCD [9,11,13,29] and the description of diquarks [30—31], baryons and light nuclei [11,29,32]. This list, being far from complete, shows a sufficiently wide spectrum of possible applications on the NJL model. Here we describe the connection of the NJL model with QCD, the main points of this model and its some applications to the meson physics.

The paper is organized as follows. In section 2 we describe models of the first type [4—7]. We shall show how the effective chiral Lagrangian can be obtained from QCD in the limit of large N_c and low energies. The next section is devoted to the problem of scalar particles and conformal anomaly. In section 4 we show how the NJL model is connected with QCD, and in sections 5—7 we describe the main points of the NJL model. Finally, in section 8 we present different applications of the NJL model to the low-energy meson physics.

2. DERIVATION OF EFFECTIVE CHIRAL LAGRANGIAN FROM QCD IN THE LIMIT OF LARGE N_c AND LOW ENERGIES

In 1985—1986 two papers appeared [4,12] devoted to similar problems of derivation of an effective meson chiral Lagrangian from QCD at large N_c (the number of colours) and low energies. It has been shown that a nonlinear chiral Lagrangian together with the anomalous Wess — Zumino term can be obtained as a low-energy approximation of QCD at large N_c under the assumption that the chiral symmetry in the space of flowers is spontaneously broken. Then QCD is reduced to a pure pseudoscalar theory provided that more heavy scalar, vector, and axial mesons are neglected; in this case pions represent a chiral phase of quark fields or Goldstone modes of the dynamically broken chiral symmetry. These modes are conserved only in the parts of QCD action without complete local chiral invariance.

Integration over all the colour degrees of freedom can be performed under quite general assumption on the behaviour of the arising potential. As a result, we get a chiral Lagrangian for pion fields and a number of standard formulae for such quantities as F_π (the pion decay constant), m_q (the mass of a constituent quark), and $<\bar{q}q>_0$ (the quark condensate).

The model thus obtained leads to the Skyrme model in which we may study the problem of stability of the soliton.

Now we shall describe basic features of the model [4]. The effective chiral Lagrangian follows from QCD by using identical transformations within the $1/N_c$ expansion. The change of variables of quark fields is made in the functional integral, which separates the degrees of freedom responsible for the spontaneous breaking of chiral symmetry and corresponding to pseudoscalar mesons. Use is also made of the hypothesis of spontaneous breaking of chiral symmetry (SBCS).

The QCD Lagrangian (chiral symmetry $U(n)\times U(n)$ and colour symmetry $SU(N_c)$) is of the form

$$\mathcal{L}(\bar{q}, q, G) = -\frac{N_c}{4g^2} G^a_{\mu\nu}(x)\, G^{\mu\nu}_a(x) +$$

$$+\, i\bar{q}(x)\gamma^\mu(\partial_\mu - iG_\mu(x))q(x). \qquad (G_\mu = G^a_\mu T_a). \tag{1}$$

Meson fields are described by operators $\bar{q}\gamma^5 t^a q$, where t^a are generators of $U(n)$ and $\mathrm{tr}\,(t^a t^b) = 2\delta^{ab}$. Consider the generating functional

$$Z(\eta) = Z_0^{-1} \int d\mu(G) d\bar{q} dq \exp\{i\int dx[\mathcal{L}(x) + \bar{q}\gamma^5 t^a q\eta_a(x)]\}, \tag{2}$$

where $d\mu(G)$ is the measure that includes gauge fixing terms and ghost fields. To single out chiral phase of quark fields, one performs the transformation

$$q^\Omega = (\Omega^+ P_R + P_L)q, \quad \bar{q}^\Omega = \bar{q}\,(\Omega P_L + P_R), \tag{3}$$

where $\Omega(x) = \exp\{i\pi(x)\}$, $\pi = 2\pi^a t_a$, $P_{R/L} = \frac{1}{2}(1 \pm \gamma^5)$. With the help of the Faddeev — Popov procedure

$$1 \equiv \Delta(\bar{q}, q) \int d\mu(\Omega)\, \delta\,(\bar{q}^\Omega \gamma^5 t^a q^\Omega) \tag{4}$$

for the transformed fields q, Z is written in the form

$$Z(\eta) = Z_0^{-1} \int d\mu(G) d\bar{q}\, dq\, d\mu(\Omega) d\varphi\, J \Delta(\bar{q}, q) \exp \{ \mathrm{i} \int dx [\mathcal{L}(x) +$$

$$+ \mathrm{i}\bar{q}(x) \hat{L} P_R q(x) + \bar{q}^{\Omega^{-1}}(x) \gamma^5 t^a q^{\Omega^{-1}}(x) \eta^a(x) + \bar{q}(x) \gamma^5 t^a q(x) \varphi^a(x)] \}. \quad (5)$$

Here use was made of the δ-function in the exponential form with the Lagrange multiplier φ^a, $\hat{L} = \gamma^\mu L_\mu$, $L_\mu = \Omega^{-1} \partial_\mu \Omega$ and J is the Jacobian of the change of variables, $q \to q^{\Omega^{-1}}$. Owing to the δ-function in (4) the pseudoscalar meson fields cannot simultaneously appear as phase of quark fields (see eq. (3)) and as costituent operators $\bar{q}\gamma^5 t^a q$ in the functional $Z(\eta)$. In what follows, they appear only as chiral phases of quark fields. An analogous procedure is employed also in [12]. There the transformation (3) is related to the transition from current quarks to constituent quarks with a large dynamical mass.

The Jacobian J results in the Wess — Zumino terms and Abelian anomaly with a π^0-meson necessary for solving the $U_A(1)$ problems

$$\ln J = \mathrm{i}\mathcal{L}_{\mathrm{WZ}} + \mathrm{i} \frac{\sqrt{2\pi}}{8\pi^2} \operatorname{tr} \int dx\, G_{\mu\nu}(x)\, G_{\rho\sigma}(x)\, \varepsilon^{\mu\nu\rho\sigma} \pi^0(x). \quad (6)$$

Now let us consider the remaining part of the functional $Z(\eta)$. We should integrate over all the colour variables and obtain the effective action for $\pi^a(x)$. Integration over gluon fields yields the factor

$$\exp \{ \mathrm{i} S_0 (\bar{q} T^a \gamma^\mu q, \pi^0) \} \quad (7)$$

and the functional S_0 contains the bilocal combinations of the fields $\bar{q}^i_\kappa(x) q^j_\kappa(y)$ (κ is the colour index) singlet in colour (i and j are flavours).

Now the bilocal collective variable $\xi^{ij}(x, y)$ is introduced which is singlet in colour [4,10],

$$\begin{aligned} &\exp \{ \mathrm{i} S_0 \} = \int d\xi\, \delta(N_c \xi^{ij}(x, y) - \bar{q}^i_\kappa(x) q^j_\kappa(y)) \exp \{ -\mathrm{i} V(\xi, \pi^0) \}, \\ &\delta(N_c \xi - \bar{q} q) = \int d\lambda \exp \{ \mathrm{i} \int dx\, dy\, \lambda^{ij}(x, y)\, [N_c \xi^{ij} - \bar{q}^i_\kappa(x) q^j_\kappa(y)] \} \end{aligned} \quad (8)$$

and then one can integrate over $\bar{q}$ and q

$$Z(\eta) = Z^{-1} \int d\mu(\Omega) d\xi\, d\lambda\, d\varphi\, \Delta \exp \{ \mathrm{i} [-V(\xi, \pi^0) + \mathcal{L}_{WZ}(L) + N_c \operatorname{tr} \lambda \xi -$$

$$- \mathrm{i}\, N_c \, \mathrm{tr} \ln \left(\mathrm{i}\hat{\partial} + \mathrm{i}\, \hat{L} P_R - \lambda + \varphi^a t^a \gamma^5 + \ldots \right)]\}. \tag{9}$$

The functional $V(\xi, \pi^0)$ can be splitted into two parts

$$V(\xi, \pi^0) = V(\xi, 0) + V'(\xi, \pi^0), \tag{10}$$

where $V'(\xi, \pi^0)$ includes at least one source, π^0. The fermion lines are associated with the factor $\delta_{\kappa\kappa'}\, \xi$. At large N_c the behaviour of $V'(\xi, \pi^0)$ is determined by purely Yang — Mills diagrams and does not depend on ξ: $V'(\xi, \pi^0) = V'_0(\pi^0) + O(1/N_c)$.

The diagrams corresponding to $V(\xi, 0)$ contain at least one quark loop and lead to an expansion of the form

$$V(\xi, 0) = N_c [V_0(\xi) + \frac{1}{N_c} V_1(\xi) + \ldots]. \tag{11}$$

To integrate over the remaining variables in the leading order in N_c, one expands the action in the exponential (9) around the stationary point*

$$\xi^{ij}_{\mathrm{st}}(x - y) = -\mathrm{i} \left(\frac{1}{\mathrm{i}\, \hat{\partial} - \lambda_{\mathrm{st}}} \right)^{ij}, \quad \lambda^{ij}_{\mathrm{st}}(x - y) = \left(\frac{\delta V_0}{\delta \xi_{\mathrm{st}}} \right)^{ij}, \quad \varphi^a = \pi^a = 0. \tag{12}$$

From the assumption that the $U(n) \times U(n)$ symmetry is broken down to the diagonal $U(n)$ it follows that $\xi^{ij}_{\mathrm{st}} = \delta^{ij} \xi_{\mathrm{st}}$, $\lambda^{ij}_{\mathrm{st}} = \delta^{ij} \lambda_{\mathrm{st}}$. Though explicit solutions ξ and λ are not known since $V_0(\xi)$ is not known, it is sufficient to assume that $\tilde{\lambda}_{\mathrm{st}}(p)$ at small momenta tend to a nonzero constant, $\tilde{\lambda}_{\mathrm{st}}(p)\,|_{p=0} = c \neq 0$. This implies that quarks acquire a dynamical mass, i.e. chiral symmetry gets spontaneously broken (the quark propagator in the leading order in N_c coincides with $\tilde{\xi}_{\mathrm{st}}(p)$). This point is crucial in both papers [4,12] for constructing effective chiral Lagrangians. Computation of the effective action for fields π^a is only a technical problem slightly different in both the papers.

---

* An analogous procedure is also employed in paper [12].

Let us turn back to paper [4]. At low energies leading are the terms quadratic in currents L_μ. In the vicinity of the stationary point the exponent of the exponential (9) is of the form

$$\mathfrak{L}_{\mathrm{WZ}}(L) - V_0'(\pi^0) + N_c \,\mathrm{tr}\, \{-\frac{1}{2}\hat{L}P_R\phi\hat{L}P_R - \frac{\mathrm{i}}{2}\hat{L}P_R\phi\lambda + \frac{1}{2}\lambda\phi\lambda + \\ + \frac{1}{2}\phi\gamma^5\phi\gamma^5\varphi + \lambda\xi + \frac{1}{2}\xi K\xi + \ldots\}, \tag{13}$$

where ϕ and K are coefficient functions in the expansion of tr ln $(\mathrm{i}\hat{D})$ and $V_0(\xi)$.

Then integrating over ξ, λ and φ successively we get

$$\mathfrak{L}_{\mathrm{eff}} = -\frac{F_\pi^2}{4}\,\mathrm{tr}\, L_\mu L_\mu - V_0'(\pi^0) - \\ - \frac{N_c}{48\pi^2}\int_0^1 d\tau\, \varepsilon^{\mu\nu\rho\sigma}\,\mathrm{tr}\,[L_5(\tau)\, L_\mu(\tau)\, L_\nu(\tau)\, L_\rho(\tau)\, L_\sigma(\tau)], \tag{14}$$

where $F_\pi = 93$ MeV is the pion-decay constant. Upon expanding the functional $V_0'(\pi^0)$ in powers of π^0 one obtains an extra mass term for the η' meson.

Note that a most important point in all models considered here is the hypothesis about the spontaneous breaking of chiral symmetry. This phenomena is a natural result of the assumption on the form of the potential $V_0(\xi)$, i.e. of the condition $[\tilde{\lambda}_{\mathrm{st}}(p) = \frac{\delta V_0}{\delta \xi_{\mathrm{st}}}]_{p=0} = c \neq 0$.

A rather different formulation of that hypothesis is given in [6]. There one considers the eigenvalues K of the total Dirac operators $\mathrm{i}D = \mathrm{i}\partial + gG + V + \gamma_5 A - S + \mathrm{i}\gamma_5 P$ with external vector, axial, scalar, and pseudoscalar fields, V_μ, A_μ, S and P, and gluon fields G_μ. In the Euclidean space, $\mathrm{i}\hat{D}q_{k\alpha} = Kq_{k\alpha}$. These eigenvalues define the determinant of the Dirac operator and thus the generating functional. Then the low-energy region is separated in an asymmetric way:

$$-\Lambda + M \leq K \leq \Lambda + M \quad (0 \leq M \leq \Lambda), \tag{15}$$

where Λ is a spectral parameter, and M is the mass of a constituent quark. For $M \neq 0$ the quark condensate is nonvanishing

$$\langle \bar{q}q \rangle_0 = -\frac{N_c}{2\pi^2}\left(\Lambda^2 M + \frac{M^3}{3}\right). \tag{16}$$

This expression can be easily derived from the definition $\langle \bar{q}q \rangle_0 = i N_c \, \mathrm{tr} \int d^4p \, G_q(p)$, where $G_q(p) = \frac{1}{(2\pi)^4}\frac{1}{m - \hat{p}}$ is the momentum representation for the quark propagator integrated in the limits (15). In this way the spontaneous breaking of chiral symmetry is phenomenologically introduced into the model.

As we see, SBCS in all models cannot be shown to follow directly from the construction of the models, but is rather an extra condition imposed either on an unknown potential [4,12], or on a spectrum of eigenvalues of the Dirac operator [6]. In this regard, the Nambu — Jona-Lasinio (NJL) model to be considered in the next sections is a more self-consistent model directly reproducing the mechanism of SBCS at low energies.

To conclude this section, we briefly indicate how the parameters Λ and M in the model [6] are expressed through the quark and gluon condensates. The quark condensate is described by equation (16); to deduce an equation for the gluon condensate we consider «radial» fluctuations of the quark field $q(x) \to \exp(-\sigma(x)) q(x)$ with the scalar field $\sigma(x) = \sigma_0(x) + t^a \sigma_a(x)$. Singlet fluctuations change the magnitude of the condensate $\langle \bar{q}q \rangle_0$ therefore they should be suppressed for stability of the region L. In order to investigate the possibility of the suppression of these fluctuations these authors construct the effective action $W_{\mathrm{eff}}(\sigma)$ for the field $\sigma(x)$ generated by the conformal triangular anomaly; here gluons play a leading role. Therefore, calculating $W_{\mathrm{eff}}(\sigma)$ one may put $m_q = 0$ and $A_\mu = V_\mu = S = P = 0$ so that $i\hat{D} = i(\hat{\partial} + g\hat{G})$. Along with $\hat{D}$ there is considered the conformally transformed operator $\hat{D}_\sigma = e^\sigma \hat{D} e^\sigma$. The corresponding generating functional is of the form

$$Z_q(G, \sigma) = \int d\bar{q}\, dq \, \exp\left(-\int d^4x \, \bar{q} \hat{D}_\sigma q\right).$$

The conformal-invariant part $Z_{\mathrm{inv}}(G)$ follows upon integration over $\sigma(x)$

$$Z_{\mathrm{inv}}^{-1} = \int d\sigma \, Z_1^{-1}(G, \sigma).$$

Then the conformal-non-invariant part $Z_{\mathrm{conf}}(G)$ and effective action $W_{\mathrm{eff}}(\sigma)$ are given by the formulae

$$Z_{\mathrm{conf}}(G) = \int d\sigma \, Z_q(G, 1)\, Z_q^{-1}(G, \sigma) \equiv \int d\sigma \, e^{-W_{\mathrm{eff}}(\sigma, G)},$$

$$W_{\mathrm{eff}}(\sigma, G) = \int_0^1 ds \int d^4x \, 2\, \mathrm{tr}\left(\sigma(x) \langle x | \theta(\Lambda^2 - (i\hat{D}_{s\sigma} - M)^2 | x \rangle\right). \tag{17}$$

By using the methods proposed in [6] for the effective potential one obtains the expression

$$\frac{N_f}{4\pi^2}\left\{\frac{N_f}{8}(e^{-8\sigma_c}-1)(6\Lambda^2M^2-\Lambda^4-M^4)+\frac{\sigma_c}{6}g^2\sum(G^a_{\mu\nu})^2\right\}. \tag{18}$$

Stability of the low-energy region (15) implies that the effective potential should possess a minimum at $\sigma_c = 0$, i.e. at the value of the quark condensate chosen above (see (16)). As a result, we arrive at the condition

$$\begin{gathered} 6N_c(6\Lambda^2M^2-\Lambda^4-M^4) = \langle g^2\sum(G^a_{\mu\nu})^2\rangle_0, \\ \langle g^2\sum(G^a_{\mu\nu})^2\rangle_0 > 0. \end{gathered} \tag{19}$$

Thus, the quark and gluon condensates become directly connected with the parameters Λ and M. In [6] the following estimates were found: for $\langle\bar{q}q\rangle_0 = -((200\div250)\ \text{MeV})^3$ and $\langle\frac{g^2}{4\pi}\sum(G^a_{\mu\nu})^2\rangle_0 = ((350\div400)\ \text{MeV})^4$ we have $\Lambda = (475\div610)$ MeV and $M = (250\div300)$ MeV. In derivation of (19) these authors made use of the low-energy approximation

$$g^2\sum(G^a_{\mu\nu})^2 \approx \langle g^2\sum(G^a_{\mu\nu})^2\rangle_0. \tag{20}$$

As a result, the quantum fluctuations of the gluon fields are neglected. However, note that these quantum gluon fields play an important role in calculating effective four-quark interaction, as has been shown in [8—13, 16,18]. They may produce additional terms in equation (19) of the form (const $\frac{M^2}{G}$), where G is the four-quark coupling constant (see, for instance the gap equation in the NJL model). This, in turn, may somewhat change the values of the parameters M and Λ*.

The nonlinear Lagrangian for the fields V_μ, A_μ, S, and P can be obtained by the method suggested by Andrianov, Bonora [6].

---

*Just now we get acquainted with the recent paper [33] in which four-quark interactions were taken into account, which led to an increasing value of Λ, Λ = 750 MeV. This value is close to the value used in the NJL model.

Here we have for the first time met with the notion of conformal symmetry. As it is of much significance for the construction of effective Lagrangians, we will dwell upon its physical consequences in the next section.

3. SCALE SYMMETRY, CONFORMAL ANOMALY, AND EFFECTIVE LAGRANGIAN OF A GLUONIUM (DILATON)

Scale symmetry plays an important role in the field theory and is employed in constructing effective Lagrangians. Here we shall only review a part of the voluminous literature devoted to that problem.

Let us recall basic ideas related to scale transformations of fields and Lagrangians [34]. Under a scale transformation of coordinates

$$x \to \lambda x$$

scalar and spinor fields are transformed as follows:

$$\varphi(\lambda x) \to \lambda^{-1}\varphi(x), \quad \psi(\lambda x) \to \lambda^{-3/2}\psi(x).$$

As a result, a simplest Lagrangian of the form

$$\mathcal{L}(x) = -\frac{1}{2}(\partial_\mu \varphi)^2 + i\bar{\psi}\hat{\partial}\psi + g\bar{\psi}\varphi\psi + h\varphi^4$$

is transformed in the following way: $\mathcal{L}(\lambda x) \to \lambda^{-4}\mathcal{L}(x)$ and gives rise to a scale-invariant action. It is easy to notice that mass terms break scale invariance.

If masses of light current quarks are set to be zero ($m^0_{u,d,s} = 0$), a QCD Lagrangian should be scale-invariant. There, however, arises an internal scale with mass dimensionality at the quantum level [35,36]

$$\mu = M_0 \exp\left\{-\frac{8\pi^2}{bg_0^2}\right\}, \tag{21}$$

where M_0 is the mass of an ultraviolet regulator, g_0 is the bare coupling constant, $g_0 = g(M_0)$ and $b = (11N_c - 3N_f)/3$. This effect breaks scale invariance. The naive trace of the energy-momentum tensor should equal zero, however, owing to the gluon anomaly

$$\sigma(x) = \theta_{\mu\mu}(x) = \frac{\beta(\alpha_s)}{4\alpha_s}(G^a_{\mu\nu}(x))^2, \tag{22}$$

where $\beta(\alpha_s) = -b\alpha_s^2/2\pi + O(\alpha_s^2)$ is the Gell-Mann — Low function.

In quantum field theory, the classical scale invariance gives rise to the relation [37]*

$$\lim_{q\to 0} i \int d^4x\, e^{iqx} <T\{Q(x), \sigma(0)\}> = -d_n <Q>, \qquad (23)$$

where $Q(x)$ is an arbitrary local operator constructed of gluons and quarks, and d_n is its normal dimension. In what follows we will utilize a very important particular case of the relation (23) when $Q(x) = \sigma(x) = \frac{\beta(\alpha_s)}{4\alpha_s}(G^a_{\mu\nu})^2$, i.e., the Ward identity

$$i \int d^4x <T\{\sigma(x), \sigma(0)\}> = -4<\sigma>. \qquad (24)$$

Now let us determine the low-energy (tree) interaction Lagrangian for $\sigma(x)$ obeying the Ward scale identity (24). Note that the solution is entirely determined by the sign of vacuum energy, it is stable only when $\varepsilon_{vac} < 0$.

The normal dimension of $\sigma(x)$ equals four. Therefore, if one takes the kinetic term in the form

$$\mathcal{L}_{kin} = \text{const}\, (\partial_\mu \sigma(x))^2\, (\sigma(x))^{-3/2}, \qquad (25)$$

the corresponding part of the action will be scale-invariant and $\mathcal{L}_{kin}$ will not contribute to $\theta_{\mu\mu}$.

Let us now construct the potential part $V(\sigma)$ so as to satisfy the condition $\theta_{\mu\mu} = \sigma$. Under an infinitesimal scale transformation, when $\lambda = 1 + \varepsilon$, the field σ and potential part of the action transform as follows

$$\sigma \to (1 - 4\varepsilon)\sigma, \quad \Delta S_{pot} = -\Delta \int d^4x\, V(\sigma) = -\int d^4x \left(4V - 4\sigma \frac{\delta V}{\delta\sigma}\right). \qquad (26)$$

Equating this change of S to the quantity $\int d^4x\, \theta_{\mu\mu}(x)$ we arrive at the equation that ensures the identity (24) to be valid [38]

$$4V - 4\sigma \frac{\delta V}{\delta \sigma} = \sigma. \qquad (27)$$

---

*A simple derivation of the relation (23) can be found in the paper by Shifman [36].

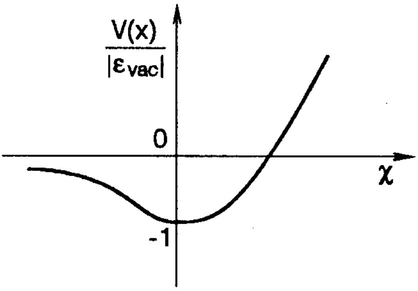

Fig. 1. The potential part of the effective Lagrangian of a gluonium (dilaton)

It has a simple solution*

$$V = -\frac{\sigma}{4}(\ln\sigma + \text{const}'). \tag{28}$$

The constants in (25) and (28) can be expressed through the mass of σ, m_σ, and the vacuum energy, $\varepsilon_{\text{vac}} \equiv \langle 0|\theta_{00}|0\rangle \equiv \frac{1}{4}\langle 0|\theta_{\mu\mu}|0\rangle$. Recall that $\varepsilon_{\text{vac}} < 0$. The field σ can be written in the form $\sigma = 4\varepsilon_{\text{vac}}\exp\chi$, then the gluonium (dilaton) effective Lagrangian acquires the form

$$\mathcal{L}_{\text{eff}} = -\frac{\varepsilon_{\text{vac}}}{2m_\sigma^2}(\partial_\mu\chi)^2\exp\left(\frac{\chi}{2}\right) + \varepsilon_{\text{vac}}(\chi - 1)\exp(\chi). \tag{29}$$

Let us demonstrate how the effective Lagrangian describing both the gluonium and quarkonium fields may be constructed. We will follow [41,42]. Besides the dilaton field $d(x)$ (gluonium) we consider the pion fields $\pi_i(x)$ ($i = 1,2,3$) and singlet scalar field $S(x)$.The quarkonium fields, π_i and S are described by the chiral $SU(2)\times SU(2)$ sigma-model. The gluonium field d is invariant under chiral transformations. The total Lagrangian has the form

$$\mathcal{L} = \frac{1}{2}\{(\partial_\mu d)^2 + (\partial_\mu S)^2 + (\partial_\mu\pi_i)^2\} - V(d, S, \pi_i), \tag{30}$$

where the potential V obeys an equation of the type (27)

$$\sigma = -H_0\left(\frac{d}{f_d}\right)^4 = 4V - d\frac{\delta V}{\delta d} - S\frac{\delta V}{\delta S} - \pi_i\frac{\delta V}{\delta\pi_i}. \tag{31}$$

Here $H_0 = -\langle\sigma\rangle$ and f_d is an analog of the pion constant f_π

$$-\langle 0|\sigma|d\rangle = m_d^2 f_d, \quad f_d = \langle 0|d|0\rangle. \tag{32}$$

---

*Note that an expression like (28) was obtained by Leutwyler [39] and Voloshin, Ter-Martirosyan [40] on the basis of the one-loop (rather than tree) approximation of gluon fluctuations in a condensate (background) field.

Equation (31) again provides the validity of the Ward identity (24). The potential V does not contain fields with derivatives and is a function of two $SU(2)\times SU(2)$ invariants, d and $(s^2+\pi_i^2)^{1/2}$.

Then from equation (31) one can determine the following form of the potential

$$V(d, S, \pi_i) = H_0\left(\frac{d}{f_d}\right)^4 \ln\frac{d}{C} + d^4 f\left(\frac{\sqrt{S^2+\pi_i^2}}{d}\right), \tag{33}$$

where C is an arbitrary constant of mass dimensionality and f is an arbitrary dimensionless function of the argument $(\sqrt{S^2}+\pi_i^2/d)$.

Following [42], the function f can be taken in the form

$$f\left(\frac{\sqrt{S^2+\pi_i^2}}{d}\right) = -\frac{1}{6}\frac{G_0}{f_d^4}\ln\left(\frac{C}{f_\pi}\frac{\sqrt{S^2+\pi_i^2}}{d}\right) + \frac{G_0}{24 f_\pi^4}\frac{(S^2+\pi_i^2)^2}{d^4}, \tag{34}$$

where $G_0 = \frac{8}{b}H_0$. Then the potential V acquires the form

$$V(d,S,\pi_i) = \frac{G_0}{24}\left(\frac{d}{f_d}\right)^4\left[11N_c \ln\frac{d}{C} - 4\ln\frac{\sqrt{S^2+\pi_2}}{f_\pi}\right] + \frac{1}{24}\frac{G_0}{f_\pi^4}(S^2+\pi_i^2)^2. \tag{35}$$

If fluctuations of the gluon field are neglected, i.e., if $d = f_d$ in (35), one arrives at the following expression for the potential of the sigma-model

$$V_{SM}(S, \pi_i) = -\frac{G_0}{6}\ln\frac{\sqrt{S^2+\pi_i^2}}{f_\pi} + \frac{1}{24}\frac{G_0}{f_\pi^4}(S^2+\pi_i^2)^2.$$

An analogous potential was obtained in [43] directly from QCD with the use of the procedure of bosonization described in the previous section. The obtained Lagrangians allow us to estimate the masses of the gluonium and mesons of the scalar nonet, parameters of mixing of the gluonium with neutral scalar mesons and to describe basic decays of these scalar states [41,42].

4. QCD AND NAMBU — JONA-LASINIO MODEL [9]

Recall that the generating functional of QCD (in Euclidean space) takes the form

$$e^{-W} = \int d\bar{q}dqDG \exp\left(-\int dx\, \{\bar{q}\,(\hat{\partial} + m^0)q + \bar{q}\hat{G}q + \frac{1}{8g^2}\operatorname{tr}(G_{\mu\nu})^2\}\right). \tag{36}$$

Here $\hat{G} = \gamma^{\mu} G^{a}_{\mu} T^{a}$, m^0 is the current quark mass. Gauge fixing terms and the Faddeev — Popov factor for the $SU(N_c)$ colour gauge symmetry are included in the gluon measure. When $m^0 = 0$, the Lagrangian has $U(N_f) \times U(N_f)$ global flavour symmetry.

Integrating again over the gluon fields yields

$$e^{-W} = \int d\bar{q}dq \exp\Big(-\Big\{\int dx\, \bar{q}\,(\hat{\partial} + m^0)q + \sum_{n=2}^{\infty} \frac{1}{n!} \int dx_1 \ldots dx_n\, \Gamma^{a_1\ldots a_n}_{\mu_1\ldots\mu_n} j^{a_1}_{\mu_1}(x_1) \ldots j^{a_n}_{\mu_n}(x_n)\Big\}\Big), \tag{37}$$

where $j^{a}_{\mu}(x) = \bar{q}(x)\gamma_{\mu} T^{a} q(x)$ is a chiral singlet local current coupling to G^{a}_{μ}; $\Gamma^{a_1\ldots a_n}_{\mu_1\ldots\mu_n}$ are chiral singlets related to one-gluon irreducible Green functions that contain the whole gluon dynamics. Fortunately, the principal features of the low-energy dynamics can be understood without detailed information on the functions Γ.

We introduce a bilocal field $\eta(x, y)$ describing mesons at low energies (see Sec.2) [4,9,10]

$$1 \equiv \int D\zeta D\eta \exp\Big(\int dxdy\ \operatorname{tr} \eta(x, y)\, (\zeta(y, x) - q(y)\, \bar{q}(x)\Big).$$

Then

$$e^{-W} = \int d\bar{q}dqD\zeta D\eta \exp\Big(-\Big\{\int dx\bar{q}(\hat{\partial} + m^0)q - \int dxdy\, \eta(\zeta - q\bar{q})\Big\} - \Gamma(\zeta)\Big),$$

where

$$\Gamma(\zeta) = \sum_{n=2}^{\infty} \frac{1}{n!} \int dx_1 \ldots dx_n\, \Gamma^{a_1\ldots a_n}_{\mu_1\ldots\mu_n}(x_1 \ldots x_n) \times$$
$$\times \operatorname{tr}\, [\gamma^{\mu_1} T_{a_1} \zeta(x_1, x_2) \gamma^{\mu_2} T_{a_2} \zeta(x_2, x_3) \ldots T_{a_n} \zeta(x_n, x_1)\,].$$

Integrating over ζ and defining

$$e^{-G(\eta)} \equiv \int D\zeta \exp\{-(\Gamma(\zeta) - \operatorname{tr} \eta\zeta)\},$$

we get

$$e^{-W} = \int d\bar{q}dqD\eta \exp\{-(\int dx\, \bar{q}(\hat{\partial} + m^0)q + \int dxdy\, \bar{q}(x)\, \eta(x, y)\, q(y) + G(\eta))\}, \tag{38}$$

where $G(\eta)$ is a nonlocal functional containing any powers of meson fields.

Further, use is to be made of a scale parameter Λ when considering low-energy physics.

The following assumptions are then made:

1) Gluon confinement

$\Gamma^{a_1 \dots a_n}_{\mu_1 \dots \mu_n}(x_1, \dots x_n)$ should be a colour singlet for $|x_n - x_m| > \Lambda^{-1}$, $n \neq m$; glueballs should have masses larger than Λ. Then $\Gamma^{(n)}$ is completely defined by the gluon condensate and has no poles. From Lorentz invariance it follows then that

$$\Gamma^{a_1 \dots a_n}_{\mu_1 \dots \mu_n}(x_1, \dots , x_n) \approx \delta^{a_1 \dots a_n} \delta_{\mu_1 \dots \mu_n} C_n \Lambda^n ,$$

where C_n are constants on the scale $|x_n - x_m| > \Lambda^{-1}$.

2) Quark confinement

$\eta(x, y)$ and $\zeta(x, y)$ are strongly localized on the scale Λ^{-1} so that no free quarks would exist at energies lower than Λ. Therefore the following expansions are possible

$$\Lambda^{-4}\eta(x, y) = \eta(z)f(t) + \eta_\mu(z)t_\mu f'(t) + \frac{1}{2}\eta_{\mu\nu}(z)t_\mu t_\nu f''(t) + \dots ,$$

$$\zeta(x, y) = \zeta(z)g(t) + \dots ,$$

where $z = \frac{1}{2}(x + y)$ and $t = \frac{\Lambda}{2}(x - y)$; f and g rapidly vanish when $t^2 \to \infty$; $q(x)$ can also be expanded around z.

$$q(x) = q(z) + \Lambda^{-1} t_\mu \partial^t_\mu q(z + \frac{t}{\Lambda})|_{t=0} + \dots .$$

Then every term of (38) can be written as follows

$$\int\int dx\, dy\, \bar{q}(x)\, \eta(x, y)\, q(y) = \int dz\, \bar{q}(z)\, \eta(z)\, q(z) \int dt\, f(t) + \dots ,$$

$$\int\int dx\, dy\, \zeta(x, y)\, \eta(x, y) = \int dz\, \zeta(z)\, \eta(z) \int dt\, f(t)\, g(t) + \dots ,$$

$$\Lambda^4 \int\int dx\, dy\, \gamma_\mu \zeta(x, y)\, \gamma_\mu \zeta(y,x) = \int dz (\gamma_\mu \zeta(z))^2 \int dt\, g^2(t) + \dots \text{ etc.}$$

Note that only g and the meson wave function f are integrated over momenta larger than Λ.

Apart from the localization of $\eta(x, y)$, one must further require the suppression of disintegration of the meson into a quark-antiquark pair above the threshold ($\approx$0.6 GeV) (quark confinement).

In the spinor space the local fields η are expanded as follows

$$\eta = \eta^S + \mathrm{i}\gamma_5\eta^P + \gamma_\mu\eta_\mu^V + \gamma_5\gamma_\mu\eta_\mu^A = \bar{\sigma} + \mathrm{i}\gamma^5\bar{\varphi} + \hat{V} + \gamma^5\hat{A} = M + \gamma_\mu N_\mu \,,$$

$$(\bar{a} = a_a\lambda^a, \lambda^a \text{ are the Gell-Mann matrices}).$$

Here the different terms correspond to the composite operators $(\bar{q}\lambda^a q)$, $(\bar{q}\gamma_5\lambda^a q)$, $(\bar{q}\gamma_\mu\lambda^a q)$, $(\bar{q}\gamma_5\gamma_\mu\lambda^a q)$ with quantum numbers (0^{++}, 0^{-+}, 1^{--}, 1^{++}), respectively. The gluon potential $G(\eta)$ takes then the following form

$$G(\eta) = \int dx \left(-E_0 + \left\{ \frac{\mu^2}{4} \operatorname{tr} \left(\tfrac{1}{2}\hat{M}M - N_\mu N_\mu\right) + \ldots \right\} + \right.$$
$$\left. + \left\{ \frac{\varepsilon}{4} \operatorname{tr}((\hat{M}M)^2 - N_\mu N_\mu \hat{M}M + \tfrac{1}{2}\hat{M}N_\mu M N_\mu + 8(N_\mu N_\mu)^2) + \ldots \right\} + O\left(\frac{\eta^6}{\Lambda^2}\right) \right), \quad (39)$$

where E_0 is the vacuum-energy density. The function $G(\eta)$ is locally chiral-invariant, does not contain derivatives, and is even in η. Terms in $G(\eta)$ are of the order of N_c^2, 1, $1/N_c$, ..., respectively.

To the first approximation, where the ε terms are omitted, $G(\eta)$ contains only one parameter μ^2, and this construction may generate four meson multiplets. As will be seen, we may achieve a quantitative description of meson physics without a detailed analysis of the gluon dynamics. Integrating over η, from (38) and (39) we get

$$\mathrm{e}^{-W} = \int [d\bar{q}\, dq]_\Lambda \exp \left(-\int dx \left\{ \bar{q}(\hat{\partial} + m^0)q - \frac{G_1}{2}[(\bar{q}\lambda^a q)^2 + (\bar{q}\mathrm{i}\gamma_5\lambda^5 q)^2] + \right.\right.$$
$$\left.\left. + \frac{G_2}{2}[(\bar{q}\gamma_\mu\lambda^a q)^2 + (\bar{q}\gamma_5\gamma_\mu\lambda^a q)^2] \right\} \right), \quad (40)$$

where $G_1 = 4/\mu^2$, $G_2 = 2/\mu^2$. This is just the NJL model. The obtained four-quark interaction is the Fierz transformation of the interaction $(\bar{q}\gamma_\mu\lambda^a q)^2$, i.e., a consequence of the vector-like structure of QCD. However, the relation $G_1 = 2G_2$ for coupling constants of scalar (pseudoscalar) and vector (axialvector) channels may be changed if either the functions f or ε-terms are taken into consideration (different for π and ρ) (see (39)).

For a special treatment of the $U(1)$ problem it is necessary to include the $U(1)$ anomaly into $G(\eta)$, for instance, as a term with π_S^2 (see Sec.2)*.

5. NJL MODEL AND LINEAR σ-MODEL

The basic independent quantities of the NJL model are the masses of constituent quarks connected with the quark condensate, the cut-off parameter Λ determining the boundary of the region of spontaneous breaking of chiral symmetry (SBCS) and the four-quark interaction constants G_1 and G_2. The purpose of the present part is to show how the standard linear σ model describing the scalar and pseudo-scalar mesons can be obtained from the effective four-quark interaction of the NJL type.

Let us consider the following effective quark Lagrangian of strong interactions which is invariant under $SU(3)_{\rm color}\otimes U(3)_L\otimes U(3)_R$ symmetry (for the case $m^0 = 0$)

$$\mathcal{L}(\bar q q) = \bar q(\mathrm{i}\hat\partial - m^0)q + 2G_1\sum_{a=0}^{n^2-1}\left\{(\bar q\frac{\lambda_a}{2}q)^2 + (\bar q\,\mathrm{i}\gamma^5\frac{\lambda_a}{2}q)^2\right\} -$$

$$- 2G_1\sum_{a=0}^{n^2-1}\left\{(\bar q\gamma^\mu\frac{\lambda_a}{2}q)^2 + (\bar q\gamma^\mu\gamma^5\frac{\lambda_a}{2}q)^2\right\}. \qquad (41)$$

Here summation over the color indices is implicit; λ_a are generators of the flavour $U(n)$ group; $m^0 = \mathrm{diag}\,(m_1^0, m_2^0, ..., m_n^0)$ is the bare quark mass matrix which explicitly breaks chiral and diagonal $U(n)$ flavour symmetry; G_1 and G_2 are universal quark coupling constants with the dimension of length squared. The meson fields can be introduced and phenomenological meson Lagrangian can be derived by a standard procedure on the basis of generating functionals $Z(\eta, \bar\eta)$ [14—18]

$$Z(\eta,\bar\eta) = \frac{1}{N}\int d\bar q\, dq \exp\left\{\mathrm{i}\int d^4x[\mathcal{L}(\bar q, q) + \eta\bar q + \bar\eta q]\right\} =$$

$$= \frac{1}{N'}\int d\bar q dq \prod_{a=0}^{8} d\tilde S_a dP_a dV_a dA_a \exp\left\{\mathrm{i}\int d^4x[\mathcal{L}'(\bar q, q, \tilde S, P, V, A) + \eta\bar q + \bar\eta q]\right\} =$$

---

*Recently, the NJL model has also been derived from a relativistic version of the potential model based on QCD [13].

$$= \frac{1}{N''}\int \prod_{a=0}^{8} dS_a dP_a dV_a dA_a \times$$

$$\times \exp\left\{ i\int d^4x\left[\mathcal{L}''(S, P, V, A) + i\int d^4y\, \bar{\eta}(x) D^{-1}(x, y)\, \eta(y)\right]\right\}, \quad (42)$$

where N, N', N'' are normalization constants, and η, $\bar{\eta}$ are external sources,

$$\mathrm{i}\, D(x, y) = [\mathrm{i}\hat{\partial} - m + S + \mathrm{i}\,\gamma_5 P + \hat{V} + \hat{A}\gamma_5]\delta^{(4)}(x - y) =$$

$$= [\mathrm{i}\hat{\partial} + \hat{A}_R + M]P_R + [\mathrm{i}\hat{\partial} + \hat{A}_L + M^+]P_L - m. \quad (43)$$

Here $m = \mathrm{diag}\,(m_1, m_2, \ldots, m_n)$ are masses of constituent quarks, $P_{R/L} = \frac{1}{2}(1 \pm \gamma_5)$ are projection operators, V and A are vector and axial-vector fields, respectively,

$$\hat{V} = \sum_{a=0}^{n^2-1} V^a_\mu \lambda_a \gamma^\mu, \quad \hat{A} = \sum_{a=0}^{n^2-1} A^a_\mu \lambda_a \gamma^\mu. \quad (44)$$

$(A_{R/L})_\mu = V_\mu \pm A_\mu$. S and P are scalar and pseudoscalar fields,

$$S = \sum_{a=0}^{n^2-1} S^a \lambda_a, \quad P = \sum_{a=0}^{n^2-1} P^a \lambda_a, \quad M = S + \mathrm{i}P. \quad (45)$$

The Lagrangians $\mathcal{L}'$ and $\mathcal{L}''$ are given by

$$\mathcal{L}'(\bar{q}, q, \tilde{S}, P, V, A) = \bar{q}[\mathrm{i}\hat{\partial} - m^0 + \tilde{S} + \mathrm{i}\gamma_5 P + \hat{V} + \hat{A}\gamma_5]q -$$

$$- \frac{\mathrm{tr}\,(\tilde{S}^2 + P^2)}{4G_1} + \frac{\mathrm{tr}\,(V_\mu^2 + A_\mu^2)}{4G_2}, \quad (46)$$

$$\mathcal{L}''(S, P, V, A) =$$

$$= - \frac{\mathrm{tr}\,(\tilde{S}^2 + P^2)}{4G_1} + \frac{\mathrm{tr}\,(V_\mu^2 + A_\mu^2)}{4G_2} - \mathrm{i}\,\mathrm{tr}\ln\,[\mathrm{i}\hat{D}(x - y)]\big|_{x=y}. \quad (47)$$

In what follows we consider only the case of three flavours ($n = 3$) and in this section our consideration will be restricted to scalar and pseudoscalar fields, S and P.

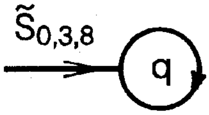

Fig.2. A loop, tadpole.

Only identical transformations were employed for introducing boson fields into the functional $Z(\eta, \bar{\eta})$. However, when one passes from the Lagrangian $\mathcal{L}'$ to $\mathcal{L}''$, there occurs an important phenomenon caused by rearrangement of the vacuum due to spontaneous breaking of chiral symmetry. As a result, the light masses of current quarks, m^0, change to heavier masses of constituent quarks, m. This is a result of vacuum expectation values of the originally introduced fields $\tilde{S}$ ($\tilde{S}_0$, $\tilde{S}_3$, $\tilde{S}_8$) in (46) being nonzero. In terms of diagrams, these expectation values are described by loop diagrams, tadpoles (see Fig.2). Redefining the quark mass matrix m one can go over to the physical scalar fields S with zero vacuum expectation values,

$$-m^0 + \tilde{S} = -m + S \to \langle S_0, S_3, S_8 \rangle_0 = 0. \tag{48}$$

In this way there appear masses of the constituent quarks.

Equation (48) represents an analog of the gap equation in a superconductor. Upon calculating the vacuum expectation value of both the parts of that equation, the latter acquires the form [16]

$$m_i^0 = m_i\,[1 - 8G_1 I_1(m_i)] = m_i + 2G_1 \langle \bar{q}_i q_i \rangle_0 \,. \tag{49}$$

Here $\langle \bar{q}_i q_i \rangle_0$ is the vacuum condensate of quarks and is a quadratically divergent quark loop (Fig.2). We will use an invariant cut-off Λ in the Euclidean region of momenta determining the boundary of the region of SBCS

$$I_1(m_i) = \frac{N_c}{(2\pi)^4} \int \frac{d_e^4 k\, \theta(\Lambda^2 - k^2)}{m_i^2 + k^2} = \frac{3}{(4\pi)^2} \left[\Lambda^2 - m_i^2 \ln \left(\frac{\Lambda^2}{m_i^2} + 1 \right) \right],$$

$$I_2(m_i, m_j) = \frac{N_c}{(2\pi)^4} \int \frac{d_e^4 k\, \theta(\Lambda^2 - k^2)}{(m_i^2 + k^2)(m_j^2 + k^2)} = \tag{50}$$

$$= \frac{3}{(4\pi)^2} \frac{1}{(m_i^2 - m_j^2)} \left[m_i^2 \ln \left(\frac{\Lambda^2}{m_i^2} + 1 \right) - m_j^2 \ln \left(\frac{\Lambda^2}{m_j^2} + 1 \right) \right].$$

Let us now show how one can derive a standard sigma model that describes masses and interactions of scalar and pseudoscalar mesons from the Lagrangian (47). The functional (42) may be written (without external sources),

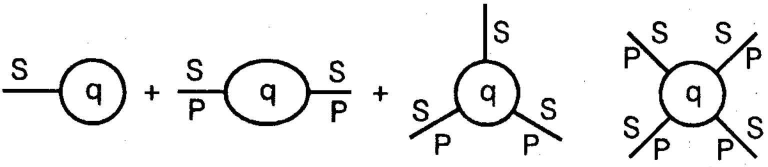

Fig. 3. Expansion of the fermion determinant. The mesons interact through quark loops

$$Z = (\det \mathrm{i}D)^N \int \prod_0^8 dS_a dP_a \exp\left\{-\mathrm{i}\int d^4x \frac{\mathrm{tr}\,[(S - m + m^0)^2 + P^2]}{4G_1}\right\}, \quad (51)$$

where $\det \mathrm{i}D = \det(\mathrm{i}D_0 + \widetilde{M}) = \exp \mathrm{tr} \ln (\mathrm{i}D_0 + \widetilde{M})$ is the quark determinant arising upon integration over quark fields, $\mathrm{i}D_0 = \mathrm{i}\hat{\partial} - m$, and $\widetilde{M} = MP_R + M^+P_L = S + \mathrm{i}\gamma_5 P$. There exist various methods for computing this fermion determinant (see, for instance, [18]). In this section we will use the simplest expansion in powers of external meson lines corresponding to the one-loop quark approximation,

$$-\mathrm{i}\,\mathrm{tr} \ln [\mathrm{i}D_0 + \widetilde{M}] = \mathrm{i}\,\mathrm{tr} \sum_{n=1}^{\infty} \frac{1}{n} [\mathrm{i}D_0^{-1}\widetilde{M}]^n. \quad (52)$$

This method was employed in [14-17]. To obtain the standard sigma model, it suffices to consider divergent quark loops of four types drawn in Fig.3. A total set of these diagrams together with quadratic meson terms entering into (47) is described by the Lagrangian

$$\mathfrak{L}^{(4)} = -\frac{\mathrm{tr}\,[(S-m+m^0)^2+P^2]}{4G_1} + \mathrm{tr}\,\{[p^2I_2 + 2(I_1+m^2I_2)][(S-m)^2+P^2] -$$

$$-I_2([(S-m)^2+P^2]^2 - [(S-m), P]_-^2\}. \quad (53)$$

The first term is the kinetic term (p is the meson momentum). Let us renormalize the fields so as to supply the kinetic term with a correct coefficient

$$S_a = g_a S_a^R, \quad P_a = g_a P_a^R, \quad g_a = [4I_2(m_i, m_j)]^{-1/2}, \quad (a \equiv (i, j)). \quad (54)$$

Then the Lagrangian (53) acquires the form corresponding to the standard sigma-model

$$\mathfrak{L}(S,P) = \frac{1}{4}\,\mathrm{tr}\left\{(\partial_\mu S)^2 + (\partial_\mu P)^2 + g_a^2\left[\frac{2}{g_a}\left(\frac{m-m^0}{G_1} - 8mI_1(m)\right)S - \right.\right.$$

$$\left.\left. - \left(\frac{1}{G_1} - 8I_1(m)\right)(S^2+P^2) - \left(S^2 - \frac{1}{g_a}\{m,S\}_+ + P^2\right)^2 + \left[\left(S - \frac{m}{g_a}\right), P\right]_-^2\right]\right\} -$$

$$- \mathrm{i}\,\mathrm{tr}\,\ln\left\{1 + \frac{g_a}{\mathrm{i}\hat{\partial} - m}(S + \mathrm{i}\gamma_5 P)\right\}' =$$

$$= \frac{1}{4}\,\mathrm{tr}\left\{(\partial_\mu S)^2 + (\partial_\mu P)^2 - (M_{S_a} S_a \lambda^a)^2 - (M_{P_a} P_a \lambda^a)^2 + 4mgS(S^2 + P^2) - \right.$$

$$\left. -g^2\left[(S^2+P^2)^2 - \left[\left(S - \frac{m}{g}\right), P\right]_-^2\right]\right\} - \mathrm{i}\,\mathrm{tr}\,\ln\left\{1 + \frac{g}{\mathrm{i}\hat{\partial} - m}[S + \mathrm{i}\gamma_5 P]\right\}'. \quad (55)$$

The index R is omitted here and in what follows; M_{S_a} and M_{P_a} are masses of scalar and pseudoscalar mesons. Using the condition of minimum of $\mathfrak{L}$ with respect to the variable S,

$$\left.\frac{\delta \mathfrak{L}}{\delta S}\right|_{S,P=0} = 0,$$

or the absence of terms linear in S in the Lagrangian $\mathfrak{L}(S,P)$ we again arrive at the gap equation (49). We shall discuss the masses of scalar and pseudoscalar mesons somewhat later, upon consideration of nondiagonal $P \to A$ transitions. The prime of the last term in (55) means that this term contains converging parts of the quark loop diagrams (including loop diagrams with an arbitrary number of external meson lines).

Now, let us fix the parameters m and Λ. To this end we construct the axial current on the basis of the Lagrangian (55)

$$J_{5\mu}^{(\pi^-)} = S_0 \partial_\mu \pi^- - \pi^- \partial_\mu S_0 + \frac{m_u}{g_u}\partial_\mu \pi^- + \dots . \quad (56)$$

If we then apply it to the decay $\pi^- \to \mu\bar{\nu}$, we arrive at the Goldberger — Treiman identity

$$\frac{m_u}{g_u} = F_\pi = 93\ \mathrm{MeV}\ \text{(pion decay constant)}. \quad (57)$$

In the next section, we will show that upon renormalization of the vector fields there appears the following connection between the constants g_u and g_ρ (g_ρ is the decay constant for the ρ-meson, $g_\rho^2/4\pi \approx 3$)

$$g_\rho = \sqrt{6}\, g_u\,. \tag{58}$$

Using this connection, the experimental value of g_ρ, taken from the decay $\rho \to 2\pi$, and the Goldberger — Treiman identity (57) we obtain the following estimates for the parameters of our model:

$$\begin{gathered} m_u = 234\ \text{MeV},\ \Lambda = 1.05\ \text{GeV}, \\ <\bar{q}_u q_u>_0 = -4m_u I_1(m_u) = -(255\ \text{MeV})^3 \end{gathered} \tag{59}$$

and for masses of the pion and σ-particle:

$$\begin{gathered} m_\pi^2 = \frac{g_u^2}{G_1}[1 - 8G_1 I_1(m_u)] = \frac{1}{G_1}\frac{m_u^0}{m_u}\frac{m_u^2}{F_\pi^2} \approx -2\frac{m_u^0}{F_\pi^2}<\bar{q}_u q_u>_0\,, \\ m_\sigma^2 = m_\pi^2 + 4m_u^2\,. \end{gathered} \tag{60}$$

These formulae represent the known Gell-Mann — Oakes — Renner formula for the pion mass [44] and standard model mass of the σ-particle, $m_\sigma \approx 500$ MeV. The constant of four-quark interaction G_1 influences only the mass of mesons but not their interaction constants. From (60) we get $G_1 \approx 7\ \text{GeV}^{-2}$, and from (49) we derive the estimate for the current mass, $m_u^0 = 5$ MeV. Masses of the remaining members of the pseudoscalar nonet are also well described within model when the mass of the strange quark is introduced and gluon anomaly and $P \to A$ transitions are taken into account [16,18].

6. VECTOR AND AXIAL-VECTOR MESONS AS COMPOSITE $q\bar{q}$-STATES

In the previous section, it was shown how the NJL model leads to the linear sigma-model describing interactions of scalar and pseudoscalar mesons. To this end, vector and axial-vector fields were neglected in the Fermion determinant, and the one-loop approximation with divergent quark loops was considered.

For a common description of scalar, pseudoscalar, vector, and axial-vector fields it is convenient to employ the so-called «heat kernel» technique used and expounded in detail in [18]. The merits of this technique consist in that it, on the one hand, allows one to preserve the explicit dependence of the theory on the physical important parameter Λ related to the scale of SBCS*, and, on the other hand, provides gauge invariance of the theory. The latter plays an important role for ensuring vector dominance of our theory upon introducing electroweak interactions and external electromagnetic fields, *W* and *Z* bosons, and also the inclusion of gluon background fields producing gluon condensates.

For simplicity, we will here consider a more trivial regularization of Pauli — Villars with two subtractions. The parameter Λ will be the mass of the subtracted field**. The main purposes of this section are as follows: derivation of the Yang — Mills structure of the Lagrangian for the composite quark-antiquark vector and axial-vector fields from the NJL model, renormalization of these fields leading to the universal physical coupling constant g_ρ that, in particular, describes the decay $\rho \to 2\pi$, and obtaining the important relation

$$g_\rho = \sqrt{6}g \qquad (g \equiv g_u) \tag{61}$$

used in the previous section for determining the parameters m_u and Λ. We will follow the papers by Kikkawa [14] and Volkov [16]. The corresponding part of the fermion determinant det (i*D*) will be determined within the one-loop approximation with the above regulatization of divergent integrals.

Summing up divergent quark loops with two, three, and four external vector mesons (like it was done in the previous section), we arrive at the expression:

$$-\frac{1}{3}\,\mathrm{tr}\,\{I_2(m)(V_{\mu\nu} - \mathrm{i}[V_\mu, V_\nu]_-)^2\}, \tag{62}$$

---

*If one uses regularizations without the explicit dependence on the parameter Λ (for instance, dimensional regularization), then an important equation of the model, like the gap equation, gets distorted and incorrect relations are established between the masses of current and constituent quarks.

**Two fields with equal masses, $M_1 = M_2 = \Lambda$, are subtracted so as to annihilate the quadratic divergence in the loop diagram with two-meson lines; in this case a constant gauge-non-invariant term that corresponds to the quadratic divergence drops out.

Note that all the three regularizations mentioned here (cut-off on the upper limit Λ, Pauli — Villars, and heat kernel technique) give similar expressions for divergent parts of loop diagrams up to terms of the order $O(m_i^2/\Lambda^2)$ (see [16,18]).

where $V_{\mu\nu} = \lambda_a[\partial_\mu V_\nu^a - \partial_\nu V_\mu^a]$ and $[V_\mu, V_\nu]_-$ is the commutator of the operators V_μ. Upon renormalization

$$V_\mu^{ij} = \sqrt{\frac{3}{8I_2(m_i, m_j)}}\, V_\mu^{Rij} = \frac{g_{V_{ij}}}{2} V_\mu^{Rij} \tag{63}$$

for getting a right coefficient of the kinetic term, the vector part of the Lagrangian $\mathcal{L}''$ is reduced to the form

$$\mathcal{L}(V) = \frac{1}{4}\,\mathrm{tr}\left\{M_{V_{ij}}^2 V_\mu^2 - \frac{1}{2}\left(V_{\mu\nu} - \mathrm{i}\frac{g_{V_{ij}}}{2}[V_\mu, V_\nu]_-\right)^2\right\} -$$
$$- \mathrm{i}\,\mathrm{tr}\ln\left\{1 + \frac{1}{\mathrm{i}\hat{\partial} - m}\frac{g_{V_{ij}}}{2}\hat{V}\right\}', \tag{64}$$

where $g_{V_{ij}} = (\frac{2}{3} I_2(m_i, m_j))^{-1/2}$ and $M_{V_{ij}}^2 = (g_{V_{ij}})^2/4G_2$*. Hence, it is seen that if for scalar (pseudoscalar) and vector mesons one uses regularization with the same cut-off parameter Λ (SBCS scale), the constants g_ρ and g are connected by (61) (see formulae (63) and (54) derived in the previous section).

From (64) and (47) for the masses of vector mesons we get

$$M_\rho^2 = M_\omega^2 = \frac{3}{8G_2 I_2(m_u)}, \quad M_\varphi^2 = M_\rho^2 \frac{I_2(m_u)}{I_2(m_s)},$$
$$M_{K^*}^2 = M_\rho^2 \frac{I_2(m_u)}{I_2(m_u, m_s)} + \frac{3}{2}(m_s - m_u)^2. \tag{65}$$

Numerical estimations for the meson masses will be given in the next section, upon consideration of nondiagonal transitions $P \to A$ and calculation of the final values for parameters $m_u = m_d$, m_s, G_2 and Λ.

For the axial-vector mesons we will in the same way arrive at the Lagrangian

---

*Recall once more that as a result of the employed Pauli — Villars regularization, the expressions for I_1 and I_2 up to terms of the order of $O(m_i^2/\Lambda^2)$ coincide with I_1 and I_2 derived in the previous section [16].

$$\mathfrak{L}(A) = \frac{1}{4}\,\mathrm{tr}\,\Big\{(M^2_{V_{ij}} + 6m^2)A^2_\mu - \frac{1}{2}A^2_{\mu\nu} + \frac{g^2_{V_{ij}}}{8}\,[A_\nu, A_\mu]^2_-\Big\} -$$

$$- \mathrm{i}\,\mathrm{tr}\,\ln\Big\{1 + \frac{1}{\mathrm{i}\hat{\partial} - m}\frac{g_{V_{ij}}}{2}\hat{A}\gamma_5\Big\} \qquad (66)$$

from which for their masses we obtain

$$M^2_{a_1} = M^2_{A_u} = M^2_\rho + 6m^2_\rho,\ M^2_{A_{1/2}} = M^2_{K^*} + 6m_u \cdot m_s\,,\ M^2_{A_s} = M^2_\varphi + 6m^2_s\,. \quad (67)$$

Selecting all the diverging quark loops with scalar, pseudoscalar, vector, and axial-vector meson lines in the fermion determinant det (iD) we get the Lagrangian for interaction of these fields,

$$\mathfrak{L}(S', P, V, A) = -\frac{1}{2G_1}\,\mathrm{tr}\,(gm^0 S') - \frac{1}{4}\,\mathrm{tr}\,\{\mu^2(S'^2 + P^2) - M^2_V(V^2_\mu + A^2_\mu) +$$

$$+ g^2[(S'^2 + P^2)^2 - [S', P]^2_-] - (D_\mu S')^2 - (D_\mu P)^2 + \frac{1}{2}G^{\mu\nu}_V G_{V\mu\nu} + \frac{1}{2}G^{\mu\nu}_A G_{A\mu\nu}\} -$$

$$- \mathrm{i}\,\mathrm{tr}\,\ln\Big\{1 + \frac{1}{\mathrm{i}\hat{\partial} - m}\,[g(S + \mathrm{i}\gamma_5 P) + \frac{g_V}{2}(\hat{V} + \hat{A}\gamma_5)]\Big\}'\,, \qquad (68)$$

where

$$S' = S - \frac{m}{g}\,,\quad \mu^2 = g^2[\frac{1}{G_1} - 8(I_1(m) + m^2 I_2(m))],$$

$$D_\mu S' = \partial_\mu S' - \mathrm{i}\frac{g_V}{2}[V, S']_- - \frac{g_V}{2}\{A_\mu, P\}_+\,,$$

$$D_\mu P = \partial_\mu P - \mathrm{i}\frac{g_V}{2}[V, P]_- + \frac{g_V}{2}\{A_\mu, P\}_+\,, \qquad (69)$$

$$G^{\mu\nu}_V = \partial^\mu V^\nu - \partial^\nu V^\mu - \mathrm{i}\frac{g_V}{2}([V^\mu, V^\nu]_- + [A^\mu, A^\nu]_-),$$

$$G^{\mu\nu}_A = \partial^\mu A^\nu - \partial^\nu A^\mu + \mathrm{i}\frac{g_V}{2}([A^\mu, V^\nu]_- + [V^\mu, A^\nu]_-).$$

The last term of the Lagrangian (68) written in the form −i tr ln {—"—}′ describes a finite part of the determinant det (iD) connected with anomalous quark loops resulting in the Wess — Zumino terms, with loops having five and more external meson lines, with finite parts of diverging diagrams, etc.

The Lagrangian (68) contains nondiagonal terms describing $S \to V$ and $P \to A$ transitions

$$\begin{aligned} \Delta \mathcal{L}_{SV} &= i\sqrt{\frac{3}{8}}\,\mathrm{tr}\,\{\partial_\mu S\,[V_\mu, m]_-\}, \\ \Delta \mathcal{L}_{PA} &= -\sqrt{\frac{3}{8}}\,\mathrm{tr}\,\{\partial_\mu P\,[A_\mu, m]_+\}. \end{aligned} \tag{70}$$

The terms of the first type are proportional to the mass difference of constituent quarks $(m_i - m_j)$ and arise for $n_f = 3$ only in the description of strange mesons [18]. A much more important role in chiral meson Lagrangians is attributed to the second-type transitions (70), to be thoroughly analysed in the next section.

7. NONDIAGONAL TRANSITIONS IN THE NJL MODEL

The nondiagonal terms that describe $S \to V$ and $P \to A$ mixing appear in the NJL model due to the SBCS and dynamical masses acquired by quarks, m_i. As is known, these transitions play an important role in chiral phenomenological Lagrangians (see a review by Gasiorowicz, Geffen [45]). Here we will show how these terms affect the values of basic parameters of our model (m_i, Λ, G_1, G_2).

The first-type transitions $(S \to V)$ when $n_f = 3$ appear only for strange particles and result in an insignificant additional renormalization of scalar fields of the form [18]*

$$S'_{ij} = Y_{ij}^{-1/2} S_{ij}, \quad Y_{ij} = \left[1 - \frac{3}{2}\frac{(m_i - m_j)^2}{M^2_{V_{ij}}}\right]^{-1}. \tag{71}$$

These renormalizations amount to about 10% which does not go beyond the accuracy of the model, and they do not lead to essential physical consequences. Therefore we shall neglect them in what follows.

---

*It is assumed that $m_u \approx m_d \neq m_s$. The renormalization (71) results from diagonalization of the Lagrangian via introduction of the physical fields V'_μ, S'

$$V^{ij}_\mu = V'^{ij}_\mu + \sqrt{\frac{3}{2}}\,\frac{(m_i - m_j)}{M^2_{V_{ij}}}\,Y^{1/2}_{ij}\,\partial_\mu S'_{ij}.$$

More important are transitions of the second type ($P \rightarrow A$) and they deserve a more accurate consideration.

Let us present one more diagonalization of the Lagrangian through introducing physical fields A'_μ [16,18,46]

$$A^{ij}_\mu = A'^{ij}_\mu + \sqrt{\frac{3}{2}}\,\frac{(m_i + m_j)}{M^2_{A_{ij}}}\,\partial_\mu P^{ij}. \tag{72}$$

Then pseudoscalar fields acquire an extra renormalization

$$P'_{ij} = Z^{-1/2}_{ij} P_{ij}, \quad Z_{ij} = \left[1 - \frac{3}{2}\frac{(m_i + m_j)^2}{M^2_{A_{ij}}}\right]^{-1} \tag{73}$$

and the constant g_P will now differ from the constant g_S by the factor $Z^{1/2}$

$$g^{ij}_P = Z^{1/2}_{ij} g^{ij}_S, \quad (g^{ij}_S = g^{ij}). \tag{74}$$

Using the Goldberger — Treiman identity, $g^{ij}_P = \frac{m_i + m_j}{2F_{ij}}$, and the relation $g_V = \sqrt{6}\, g_S$, we arrive at the following equation for the masses of constituent quarks

$$\left(\frac{m_i + m_j}{2F_{ij}}\right)^2 \left(1 - \frac{3}{2}\frac{(m_i + m_j)^2}{M^2_{A_{ij}}}\right) = \frac{g^2_{V_{ij}}}{6} \tag{75}$$

from which we get for the mass of the u quark

$$m^2_u = \frac{M^2_{a_1}}{12}\left[1 - \sqrt{1 - \left(\frac{2g_\rho F_\pi}{M_{a_1}}\right)^2}\,\right], \quad Z^{-1} = \frac{1}{2}\left[1 + \sqrt{1 - \left(\frac{2g_\rho F_\pi}{M_{a_1}}\right)^2}\,\right]. \tag{76}$$

Here the parameters m_u and Z are expressed in terms of the physical observables, g_ρ, F_π and M_{a_1} (M_{a_1} is the mass of the axial-vector meson a_1).

Note that of all the above observables, the mass M_{a_1} is by now measured with the least accuracy. In fact, there are large discrepancies between experiments on detection of the mass and width of the a_1 meson made in 1981 on hadron reactions $\pi N \rightarrow 3\pi N$ and experiments performed in 1986 on the study of τ-lepton decays. In the first experiments it was established that the mass and width of the a_1 meson equal [47]

$$M_{a_1} = 1275\pm28\ \text{MeV}\,, \quad \Gamma_{a_1} = 316\pm45\ \text{MeV}. \tag{77}$$

At the same time, the analysis of decay $\tau \to \nu_\tau 3\pi$ gave values in wider and completely different limits [48].

$$1046\ \text{MeV} \le M_{a_1} \le 1194\ \text{MeV}, \qquad 400\ \text{MeV} \le \Gamma_{a_1} \le 520\ \text{MeV}. \tag{78}$$

From formulae (76) it follows that the masses m_u and M_{a_1} are tightly connected with each other and they are subjected to be constraints

$$1 - \left(\frac{2g_\rho F_\pi}{M_{a_1}}\right)^2 \ge 0, M_{a_1} \ge 2g_\rho F_\pi = 1140\ \text{MeV}, m_u \le 330\ \text{MeV}. \tag{79}$$

The constraints on the mass of the constituent u quark are consistent with the standard ideas. The constraints on the mass of the a_1 meson indicate that in the experiments of Ruckstuhl et al. [48] and Albrecht et al. [48] in which the values $M_{a_1} = 1056$ MeV and $M_{a_1} = 1046$ MeV were obtained, probably, the data were not analysed quite accurately. Indeed, a subsequent analysis performed by [49] in which new ideas of the form of the vertex $a_1\rho\pi$ were used demonstrated the validity of this assertion and gave results for experimental data of 1986 more close to the original values (77)

$$M_{a_1} \approx 1250\ \text{MeV}. \tag{80}$$

Inequality (79) resulting from (76) allows us to understand under which conditions the Weinberg sum rule, $M_{a_1}^2 = 2M_\rho^2$ [50], and KSFR relation, $M_\rho^2 = 2g_\rho^2F_\pi^2$ [51], hold valid. They can be derived with the minimal mass of the a_1 meson, $M_{a_1} = 2g_\rho F_\pi$ (see (79)) and by using the formula $M_{a_1}^2 = M_\rho^2 + 6m_u^2$, for the a_1 meson mass obtained in the previous section. Then, inserting the value $m_u^2 = M_{a_1}^2/12$ obtained from (76) into the formula for the a_1 meson mass we immediately arrive at the above relations*.

---

*It is interesting to notice that formula $M_{a_1}^2 = M_\rho^2 + 6m_u^2$ (76) is consistent with the formula only when $M_{a_1} = 2g_\rho F_\pi$.

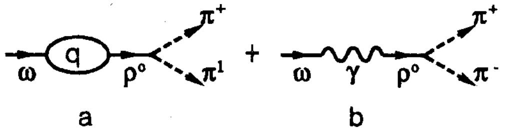

Fig. 4. Diagrams, describing the decay $\omega \to 2\pi$

Now we have all the formulae necessary for fixing the model parameters, $m_u \approx m_d$, m_s, Λ, G_1 and G_2 upon which we may proceed to calculate the meson masses and other physical quantities (like F_K, F_s, m_u^0, m_s^0, etc.).

From formula (76) we get $m_u = 280$ MeV that corresponds to $M_{a_1} = 1260$ MeV [52]. Using the formula obtained in the previous section $g_\rho^2 I_2(m_u) = 3/2$ we estimate the cut-off parameter Λ

$$\Lambda = 1280\,\text{MeV}. \tag{81}$$

From formula (65) of the previous section for the ρ meson mass we evaluate the constant G_2

$$G_2 = \left(\frac{g_\rho}{2M_\rho}\right)^2 \approx 16\,\text{GeV}^{-2}. \tag{82}$$

If the strange-quark mass is taken to equal 460 MeV, for masses of vector mesons φ and K^* we get from the same formulae

$$M_\varphi = 1025\,\text{MeV and } M_{K^*} = 916\,\text{MeV},$$

which is in satisfactory agreement with experiment,

$$M_\varphi = 1020\,\text{MeV and } M_{K^*} = 892\,\text{MeV}.$$

Now let us determine the mass difference for the u and d quarks by using, for instance, the decay $\omega \to 2\pi$ [53].

The amplitude of the $\omega \to 2\pi$ decay is described by the two diagrams shown in Fig.4 and has the form

$$T_{\omega \to 2\pi} = C\,(p^+ - p^- \gamma^\mu \omega_\mu \pi^+ \pi^- . \tag{83}$$

Here, p^+ and p^- are the π^+ and π^- momenta, and the constant $C = C_1 + C_2$ consists of two parts; C_1 describes the process of the strong transition $\omega \to \rho^0$ (Fig.4a) which takes place on account of the mass difference of the u and d quarks

$$C_1 = \frac{8(\pi\alpha_\rho)^{3/2} M_\omega^2}{3(M_\rho^2 - M_\omega^2 + iM_\rho\Gamma_\rho)} [I_2(m_u) - I_2(m_d) \approx \frac{6}{(4\pi)^2}\ln\frac{m_d}{m_u}],$$

C_2 describes the process of the electromagnetic transition $\omega \to \rho^0$ (Fig.4b). It has the opposite sign to C_1

$$C_2 = -\sqrt{\frac{\pi}{\alpha_\rho}}\,\frac{2\alpha M_\rho^2}{3(M_\rho^2 - M_\omega^2 + iM_\rho\Gamma_\rho)} \quad (\alpha = \frac{1}{137}).$$

Using the experimental $\omega \to 2\pi$ decay width, which is 286 KeV, we obtain for the mass difference of the u and d quarks the value

$$\Delta = m_d - m_u = 4.5\,\text{MeV}. \tag{84}$$

Now we shall proceed to describe the masses of pseudoscalar mesons. From formula (55) for the Lagrangian $\mathcal{L}(S, P)$ we arrive at the following expressions for the masses of pseudoscalar mesons [16]

$$M_{\pi^0}^2 = \frac{1}{2}(C_{uu} + C_{dd}), \quad M_{\pi^\pm}^2 = C_{ud} + (m_d - m_u)^2,$$

$$M_{K^\pm}^2 = C_{us} + (m_s - m_u)^2, \quad M_{K^0}^2 = C_{ds} + (m_s - m_d)^2, \tag{85}$$

$$M^2_{\substack{\eta \\ \eta'}} = \frac{1}{2}\left[C_{uu} + C_{dd} + d \mp \sqrt{(d - \frac{C_{ss} - C_{uu}}{3})^2 + \frac{8}{9}(C_{ss} - C_{uu})^2}\right],$$

where

$$C_{ij} = \frac{Z_{ij}}{4I_2(m_i, m_j)}[\frac{1}{G_1} - 4(I_1(m_i) + I_1(m_j)].$$

The term $d = 0.8\ \text{GeV}^2$ is due to the gluon anomalies taken into account [16]. It causes mixing of singlet-octet components of pseudoscalar mesons, η and η'. With the value of $d = 0.8\ \text{GeV}^2$ we get the mixing angle $\theta = -18°$*.

---

*The $U_A(1)$ anomaly could be taken into accout more accurately by introducing the t'Hooft determinant that breaks chiral symmetry and leads to mixing of flavours [20, 54].

With the use of the π^0 meson mass we may fix the last free parameter, $G_1 = 4.7\ \mathrm{GeV}^{-2}$.Then for the masses of constituent quarks $m_u = 280$ MeV, $m_d = 284.5$ MeV, $m_s = 460$ MeV we get

$$M_{K^\pm} = 493\ \mathrm{MeV},\ M_{K^0} = 497\ \mathrm{MeV},$$

$$M_\eta = 520\ \mathrm{MeV},\ M_{\eta'} = 1027\ \mathrm{MeV}. \tag{86}$$

The corresponding experimental values are as follows [52]:

$$M_{K^\pm} = 493.6\ \mathrm{MeV},\ M_{K^0} = 497.7\ \mathrm{MeV},$$

$$M_\eta = 549\ \mathrm{MeV},\ M_{\eta'} = 958\ \mathrm{MeV}. \tag{87}$$

Agreement is quite satisfactory.

The gap equation (formula (49) of Sec.5) provides the following values for the masses of current quarks:

$$m_u^0 = 3\ \mathrm{MeV},\ m_d^0 = 4\ \mathrm{MeV},\ m_s^0 = 90\ \mathrm{MeV}. \tag{88}$$

Diagonalization of the $P - A$ terms leads not only to renormalization of a number of fields and constants and to the change of basic model parameters, but also to appearance of extra diagrams that are important for obtaining self-consistent results. Recall that under the redefinition of the field A_μ

$$A_\mu^{ij} = A_\mu^{'ij} + \frac{g_\rho F_{ij} Z_{ij}}{M^2_{A_{ij}}} \partial_\mu P^{'ij} \tag{72'}$$

the diagrams with external lines A_μ^{ij} may transform into the diagrams with gradients of pseudoscalar mesons $\partial_\mu P^{'ij}$.

As an illustration, we present the calculation of two typical strong vertices with corrections due to $P - A$ transitions [55]. These are the vertices $\varepsilon\pi\pi$ and π^4 that play an important role in the description of $\pi\pi$ scattering in the sigma model (ε is a scalar isoscalar meson consisting of u and d quarks). If the $\pi \to a_1$ transitions are taken into account, the above vertices acquire the form

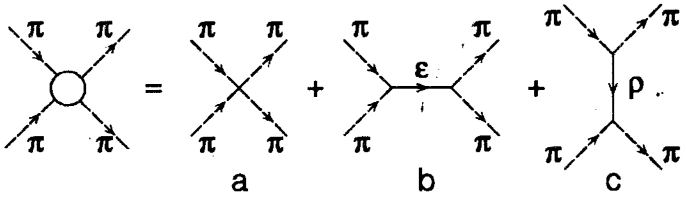

Fig. 5. Diagrams of $\pi\pi$ scattering in the tree approximation

$$\mathcal{L}_{\varepsilon\pi\pi} = 2m_u g_\pi Z^{1/2}\varepsilon\left\{\boldsymbol{\pi}^2 + \frac{1}{2m_u^2}\left(\frac{Z-1}{Z}\right)\left[\boldsymbol{\pi}\partial_\pi^2\boldsymbol{\pi} + \left(\frac{Z+1}{Z}\right)(\partial_\mu\boldsymbol{\pi})^2\right]\right\},$$
$$\mathcal{L}_{\pi^4} = -\frac{g_\pi^2}{2}Z\left\{\boldsymbol{\pi}^4 - \left(\frac{Z-1}{Z}\right)^2\frac{(\boldsymbol{\pi}\partial_\mu\boldsymbol{\pi})^2}{m_u^2} + \left(\frac{Z-1}{Z}\right)^4\frac{(\partial_\mu\boldsymbol{\pi}\partial_\mu\boldsymbol{\pi})^2-(\partial_\mu\boldsymbol{\pi}\partial_\nu\boldsymbol{\pi})^2}{12m_u^4}\right\}. \tag{89}$$

Let us show that the formula obtained describes $\pi\pi$ scattering in the complete agreement with low-energy theorems.

The diagrams describing $\pi\pi$ scattering in the tree approximation are drawn in Fig.5.

Using the formula $M_\varepsilon^2 = M_\pi^2 + 4m_u^2$ and retaining only terms with the lowest derivatives we get for the diagrams 5a and 5b from (89)

$$\mathcal{L}'_{\pi^4} = \frac{g_\pi^2}{2m_u^2 Z}\left[(\boldsymbol{\pi}\partial_\mu\boldsymbol{\pi})^2 + (Z-1)\boldsymbol{\pi}^2(\partial_\mu\boldsymbol{\pi})^2\right].$$

Using the Lagrangian $\mathcal{L}_{\rho\pi\pi} = g_\rho(\boldsymbol{\pi}\times\boldsymbol{\pi})\boldsymbol{\rho}$, we get for the diagram 5c

$$\mathcal{L}''_{\pi^4} = \frac{g_\rho^2}{2M_\rho^2}\left[(\boldsymbol{\pi}\partial_\mu\boldsymbol{\pi})^2 - \boldsymbol{\pi}^2(\partial_\mu\boldsymbol{\pi})^2\right] = -\frac{g_\rho^2}{2M_\rho^2}(\boldsymbol{\pi}\times\partial_\mu\boldsymbol{\pi})^2.$$

As we saw earlier, in the region where the low-energy sum rules hold valid, it is to be assumed that $M_{a_1} = 2g_\rho F_\pi$, $M_\rho^2 = 6m_u^2$, $Z = 2$. Then, using these relations and formula (75) we obtain for the total scattering amplitude the following expression

$$\mathcal{L}_{\pi^4} = \mathcal{L}'_{\pi^4} + \mathcal{L}''_{\pi^4} =$$
$$= \frac{g_\pi^2}{2m_u^2 Z}\left[2(\boldsymbol{\pi}\partial_\mu\boldsymbol{\pi})^2 + (Z-2)\boldsymbol{\pi}^2(\partial_\mu\boldsymbol{\pi})^2\right] = \frac{g_\pi^2}{2m_u^2}(\boldsymbol{\pi}\partial_\mu\boldsymbol{\pi})^2 \tag{90}$$

satisfying all the requirements of the low-energy theorems for pion-pion scattering.

A thorough analysis of the role of $P \to A$ transitions in the NJL model was performed by M.Wakamatsu [56].

8. APPLICATIONS

In the previous sections it has been shown that all the known phenomenological chiral Lagrangians describing low-energy physics of scalar, pseudoscalar, vector, and axial-vector mesons can easily be constructed within the NJL model. Besides, this model allows one to describe the deviations from the exact chiral group $U(n)\times U(n)$, which can be seen in calculations of the mass spectrum of mesons and constants F_π, F_K, and F_s. This is possible because the NJL model describes meson vertices in the one-loop quark approximation, and the difference of masses of constituent quarks (u, d) and s, that breaks the chiral group, may be taken into account. In what follows, we will show a number of cases where the difference of masses m_u and m_s plays an important role in the description of meson interactions. The NJL model allows us to describe all basic decays of pseudoscalar, vector, and axial-vector mesons (for $n_f = 3$) and their intrinsic properties, in particular, electromagnetic and weak radii, polarizabilities, scattering lengths. Since a detailed account of applications of the model is given in review papers [16,57], we shall here only demonstrate the most typical and interesting examples.

1) Strong Decays of Vector Mesons. Basic decays of vector mesons occur in a strong channel of the VPP type and are specified by constants g_ρ, g_{K^*} and g_φ defined by formulae (63) and (65) of Sec.6. Formula (65) can be written in the form

$$\frac{g_\varphi^2}{g_\rho^2} = \frac{\alpha_\varphi}{\alpha_\rho} = \frac{M_\varphi^2}{M_\rho^2}, \quad \frac{g_{K^*}^2}{g_\rho^2} = \frac{\alpha_{K^*}}{\alpha_\rho^2} = \frac{M_{K^*}^2}{M_\rho^2} - \frac{3}{2}\frac{(m_s - m_u)^2}{M_\rho^2},$$

from which we obtain for the above constants the following ratios

$$g_{K^*}^2 = 1.26 g_\rho^2, \quad g_\varphi^2 = 1.75 g_\rho^2. \tag{91}$$

Then one can calculate the partial widths of decays $V \to P_1 + P_2$

$$\Gamma(\rho \to \pi\pi) = \frac{\alpha_\rho M_\rho}{12}\left[1 - \left(\frac{2M_\pi}{M_\rho}\right)^2\right]^{3/2} = 156 \text{ MeV},$$

$$\Gamma(K^* \to K\pi) = \frac{\alpha_{K^*} M_{K^*}}{16}\left\{\left[1 - \left(\frac{M_K - M_\pi}{M_{K^*}}\right)^2\right]\left[1 - \left(\frac{M_K + M_\pi}{M_{K^*}}\right)^2\right]\right\}^{3/2} = 54 \text{ MeV},$$

$$\Gamma(\varphi \to \bar{K}K) = \frac{\alpha_\varphi M_\varphi}{12}\left[1 - \left(\frac{2M_K}{M_\varphi}\right)^2\right]^{3/2} = 3.4 \text{ MeV}. \tag{92}$$

The theoretical values of these partial widths are in good agreement with experiment [52]*,

$$\Gamma_{\rho \to \pi\pi} = (149 \pm 3) \text{ MeV}, \Gamma_{K^* \to K\pi} = (50 \pm 1) \text{ MeV},$$

$$\Gamma_{\varphi \to \bar{K}K} = (3.7 \pm 0.2) \text{ MeV}.$$

2) Radiative Decays of Pseudoscalar and Vector Mesons. The most typical radiative decays of pseudoscalar and vector mesons are two-particle decays of the type $P \to \gamma\gamma$, $P \to V\gamma$ and $V \to P\gamma$. All these processes are well described by anomalous triangular quark diagrams [58]. These diagrams give the Wess — Zumino terms connected with the imaginary part of the determinant det $(\mathrm{i}\hat{D})$ or with the last term of the Lagrangian (68) from Sec.6. These processes are thoroughly described in review papers [16,57]; here we only cite Table 1 of theoretical and experimental values for those decays. A particular note concerns the decays $K^* \to K\gamma$ for which our model provides a better agreement with experiment due to the difference of $m_s - m_u$ (the deviation from the group $U(3) \times U(3)$).

The partial width of the decay $V \to P\gamma$ is of the form

$$\Gamma(V \to P\gamma) = \frac{\alpha\alpha_V}{6}\frac{C_{VP}^2}{F_P^2}\left(\frac{M_V^2 - M_P^2}{4\pi M_V}\right)^3. \tag{93}$$

---

*The value $\alpha_\rho \approx 3$ is taken according to the old experimental data $\Gamma_{\rho \to \pi\pi} = 153$ MeV (Particle Data Group 1988).

Table 1. Partial widths of radiative decays $P \to \gamma\gamma$, $P \to V\gamma$ and $V \to P\gamma$ of low-lying mesons

Decay	Numerical values of partial widths (KeV)	
	Theory ($\theta = -18°$)	Experiment [52]
$\pi^0 \to \gamma\gamma$	$7.6 \cdot 10^{-3}$	$(7.5 \pm 0.3) \cdot 10^{-3}$
$\eta \to \gamma\gamma$	0.63	0.46 ± 0.01
$\eta' \to \gamma\gamma$	4.5	4.5 ± 0.35
$\eta' \to \rho^0\gamma$	67	62 ± 3
$\eta' \to \omega\gamma$	7.5	6.2 ± 0.6
$\omega \to \pi^0\gamma$	830	720 ± 40
$\rho \to \pi\gamma$	87	118 ± 30
$\rho \to \eta\gamma$	65	57 ± 10
$\omega \to \eta\gamma$	8.5	$3.9^{+1.9}_{-1.5}$
$\varphi \to \pi^0\gamma$	5.3	5.8 ± 0.6
$\varphi \to \eta\gamma$	69	56 ± 3
$\varphi \to \eta'\gamma$	0.56	< 1.8
$K^{*+} \to K^+\gamma$	52	50 ± 4
$K^{*0} \to K^0\gamma$	130	120 ± 10

The coefficients C_{VP} are respectively

$$C_{\rho\pi} = 1, \quad C_{\omega\pi} = 3, \quad C_{\omega\eta} = \sin\bar{\theta}, \quad C_{\rho\eta} = 3\sin\bar{\theta},$$
$$C_{\varphi\eta} = 2\cos\bar{\theta}, \quad C_{\varphi\eta'} = 2\cos\bar{\theta}, \quad C_{\varphi\pi^0} = 3\sin\beta.$$

Here $\bar{\theta} = \theta_0 - \theta$, θ is the singlet-octet mixing angle ($\theta = -18°$); θ_0 is the ideal mixing angle ($\sin\theta_0 = 1/\sqrt{3}$), $\beta = 3°$ is the ω–φ-mixing angle. For the decays $K^* \to K\gamma$ the coefficients C_{K^*K} depend on the quark mass difference

$$C_{K^{*+}K^+} = \frac{\lambda^2 - 6\lambda - 1}{2(\lambda^2 - 1)} + \frac{\lambda(2\lambda^2 + 1)}{(\lambda^2 - 1)} \ln\lambda^2 = 1.22,$$

$$C_{K^{*0}K^0} = 1 + \frac{\lambda \ln\lambda^2}{\lambda^2 - 1} = 1.97,$$

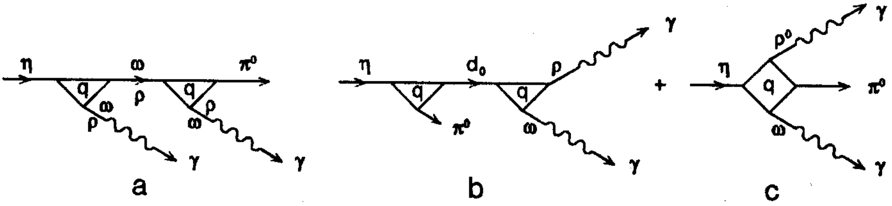

Fig. 6. Diagrams describing the $\eta \rightarrow \pi^0\gamma\gamma$ decay

where $\lambda = m_s/m_u = 1.64$ ($m_u = 280$ MeV, $m_s = 460$ MeV). As can be seen from Table 1, quite good agreement is achieved with experiment. In particular, for the ratio of these decays we get

$$\frac{\Gamma(K^{*0} \rightarrow K^0\gamma)}{\Gamma(K^{*+} \rightarrow K^+\gamma)} = 2.6,$$

whereas the experimental value equals 2.4. If one neglects the quark mass difference, i.e., if one puts $\lambda = 1$, then $C_{K^{*+}K^+} = 1$, $C_{K^{*0}K^0} = 2$, and that ratio will significantly differ from the experimental value,

$$\frac{\Gamma(K^{*0} \rightarrow K^0\gamma)}{\Gamma(K^{*+} \rightarrow K^+\gamma)} = 4 \quad (\lambda = 1).$$

3) Decay $\eta \rightarrow \pi^0\gamma\gamma$. The decay $\eta \rightarrow \pi^0\gamma\gamma$ is of interest since it clearly demonstrates merits of the linear sigma-model as compared to nonlinear chiral Lagrangians in describing some processes. Actually, in the nonlinear model it is difficult to get a correct result for the width of this decay [59]. Let us see what is the reason for this.

The decay $\eta \rightarrow \pi^0\gamma\gamma$ is described both by anomalous vertices of the type of Wess — Zumino terms (vertices $\eta\omega\omega$, $\eta\rho^0\rho^0$, $\pi^0\rho^0\omega$, Fig.6*a*) linked by intermediate vector mesons and by strong vertices corresponding to divergent quark diagrams and entering into the main Lagrangian of the sigma-model (see Lagrangian (68) of Sec.6) (vertices $\eta\pi^0\rho^0\omega$, $a_0(980)\eta\pi^0$ and $a_0(980)\omega\rho^0$ Fig.6*b*,*c*). The latter two vertices are connected by intermediate scalar mesons $a_0(980)$.

If one passes to the nonlinear chiral model, the contribution of the diagrams of Fig.6*b*,*c* disappears and only the diagrams with intermediate vec-

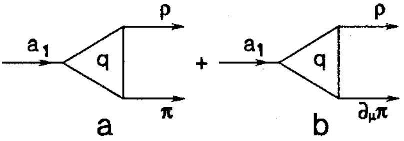

Fig. 7. Diagrams describing the $a_1 \to \rho\pi$ vertex. Diagrams *b* results from $\pi \to a_1$ diagonalization

tor mesons survive*. In terms of the current algebra this means that the commutator of neutral currents vanishes. However, the contribution from diagrams 6a amounts only to 30% of the experimental value. If one takes for the mass $a_0(980)$ not a theoretical but experimental value, the diagrams 6*b* and 6*c* do not cancel each other, which means the violation of chiral symmetry that occurs in reality. Then contributions of these diagrams are comparable with those of the diagrams 6a, and together with interference terms they will produce quite satisfactory result for the width of decay $\eta \to \pi^0\gamma\gamma$ [60]

$$\Gamma^{(a)}_{\eta\to\pi^0\gamma\gamma} = 0.4\ \text{eV},\ \Gamma^{(b+c)}_{\eta\to\pi^0\gamma\gamma} = 0.3\ \text{eV},\ \Gamma^{(\text{int})}_{\eta\to\pi^0\gamma\gamma} = 0.4\ \text{eV},\ \Gamma_{\eta\to\pi^0\gamma\gamma} = 1.1\ \text{eV},$$

whereas the experimental value equals

$$\Gamma^{(\text{exp})}_{\eta\to\pi^0\gamma\gamma} = (0.85\pm0.26)\ \text{eV}.$$

4) Axial-Vector Mesons. The Vertex $a_1\rho\pi$ and Decays $a_1 \to \rho\pi$, $a_1 \to \pi\gamma$, $\pi^- \to e\bar{\nu}\gamma$, $\tau \to \nu3\pi$. The most typical decay of axial mesons is a decay of the type $A \to VP$. The appropriate vertex is contained in the Lagrangian (68) of Sec.6 and is drawn in Fig.7*a*. Upon *P–A diagonalization*, a divergent part of the diagram 7*a* cancels out with a divergent part of the diagram 7*b* (the vertex *PA* ($A \to \partial_\mu\pi$)) upon which we are left with the remaining convergent part that enters into the last term of the Lagrangian (68). Therefore we should make use of the momentum expansion of quark loops and consider, at least, q^2-terms of those vertices. Strictly speaking, this step is beyond the scope of the approximation employed earlier and requires special substantiation. Nevertheless, we will try to consider some physical consequences of this approximation that, as will be seen below, gives reasonable results**.

---

*This can easily be verified using the formula $M^2_{a_0} = M^2_\pi + 4m^2_u$, expressing the vertex $a^0_0\eta\pi^0$ through the mass m_u and F_π and taking the limit $M_{a_0} = \infty$, tending the mass m_u to infinity.

**Note that the heat kernel technique described in [18] is also related to the momentum expansion of quark loops. The q^2-approximation was considered by many authors [3,10,59,62—64].

The diagram 7*a* along with the divergent part and q^2-terms result in the amplitude [64].

$$T^{(a)}_{a_1\to\rho\pi} = ig_\rho^2 F_\pi\{Zg^{\mu\nu} + \kappa\,[p^\mu q^\nu - g^{\mu\nu}pq + g^{\mu\nu}(q^2+p^2)]\}\varepsilon_\mu(Q)\,\varepsilon_\nu(q). \quad (94)$$

Here $\kappa = (8\pi^2F_\pi^2)^{-1}$, $Q = p + q$, p, q are the momenta of the pion and the ρ mesons, $\varepsilon_\mu(Q)$ and $\varepsilon_\nu(q)$ are the polarization vectors of the a_1 and ρ mesons. The diagram 7*b* gives the additional contribution*

$$T^{(b)}_{a_1\to\rho\pi} = ig^2F_\pi Z\frac{Q^2-q^2}{M^2_{a_1}}g^{\mu\nu}\varepsilon_\mu(Q)\,\varepsilon_\nu(q). \quad (95)$$

Summing up (94) and (95) one may easily see that on the mass shell the contribution from the divergent part of the diagram 7*a* (the first term of (94)) cancels out with the corresponding contribution from the diagram 7*b*, which results in (we put $p^2 = M_\pi^2$ and neglect the last small term in (94))

$$T_{a_1\to\rho\pi} = -ig_\rho^2F_\pi\left\{Z\frac{M_\rho^2}{M^2_{a_1}}g^{\mu\nu} + \kappa[M_\rho^2g^{\mu\nu} + (q^\mu p^\nu - g^{\mu\nu}qp)]\right\}\varepsilon_\mu(Q)\varepsilon_\nu(q) =$$

$$= -ig_\rho^2F_\pi\kappa\{q^\mu p^\nu - g^{\mu\nu}qp + cg^{\mu\nu}q^2\}\varepsilon_\mu(Q)\varepsilon_\nu(q).\ \left(C = 1 + \frac{Z}{\kappa M^2_{a_1}},\ q^2 = M_\rho^2\right). \quad (96)$$

In the low-energy limit ($Z = 2$, $M^2_{a_1} = 2M_\rho^2$) the factor of the first term is $ZM_\rho^2/M^2_{a_1} = 1$; thus, we obtain the known chiral-symmetric result for the local part of the given meson vertex. The expression in parentheses coincides with the κ term proposed in [61] on the basis of the chiral symmetry as well. All this indicates a close relation between our approach and the phenomenology based on chiral Lagrangian which has shown itself to be good.

As in the method of chiral Lagrangians, we have not yet got rigorous theoretical proofs for the q^2-approximation used here. So we shall dwell upon indirect reasons. The $a_1\rho\pi$ vertex has some properties similar to those of the anomalous $\omega\rho\pi$ vertex (see 2) of this section). Calculated through the quark loop, the latter vertex has no diverging part and begins directly with q^2 terms.

---

*One must take two q^2 expansion steps in diagram 7*a* and only one step in diagram 7*b* in order to stay within the approximation used.

In this case the q^2-approximation describes a wide class of decays in the energy interval from 100 MeV to 1 GeV ($\pi^0 \to \gamma\gamma$, $\rho \to \pi\gamma$, $\omega \to \pi\gamma$, $(\eta, \eta') \to \gamma\gamma$, etc., see Table 1). When the $a_1\rho\pi$ vertex is calculated after taking into account two diagrams (see Fig.7), the expansion also begins with q^2 terms. The amplitudes of the radiative decays $\rho \to \pi\gamma$ and $a_1 \to \pi\gamma$ associated with these meson vertices, look similar even outwardly:

$$L = \frac{e}{4} g_\rho \kappa F_\pi (\varepsilon^{\mu\nu\alpha\beta} \rho^-_{\alpha\beta} - 2\mathrm{i} a^{-\mu\nu}) F_{\mu\nu} \pi^+ + \text{h.c.} =$$

$$= \frac{e}{4} g_\rho \kappa F_\pi \varepsilon^{\mu\nu\alpha\beta} (\rho^-_{\alpha\beta} + \mathrm{i} a^{*-}_{\alpha\beta}) F_{\mu\nu} \pi^+ + \text{h.c.}, \qquad (97)$$

where $a^*_{\alpha\beta} = \frac{1}{2} \varepsilon_{\alpha\beta\mu\nu} a^{\mu\nu}$ is the dual $a^{\mu\nu} = \partial^\mu a^\nu - \partial^\nu a^\mu$ tensor. ($a^- \equiv a_1^-$).

In the region of energies equal to a_1 and ρ meson masses they describe the decay $a_1 \to \pi\gamma$

$$\Gamma_{a_1 \to \pi\gamma} = 400 \text{ keV}, \; \Gamma^{\text{exp}}_{a_1 \to \pi\gamma} = 640 \pm 240 \text{ keV} \; [65]$$

and the decay $\rho \to \pi\gamma$ (see Table 1) well.

In the low-energy region the Lagrangian (97) allows a good description of the axial and vector form factors of the radiative decay $\pi^- \to e\bar{\nu}\gamma$ [64]. The structure part of the $\pi^- \to e\bar{\nu}\gamma$ amplitude has the form

$$T_{\pi^- \to e\bar{\nu}\gamma} = eG_F \cos\theta_c \, [h_V \varepsilon^{\mu\nu\alpha\beta} p_\alpha q_\beta + \mathrm{i} h_A (g^{\mu\nu} pq - p^\mu q^\nu)] l_\nu^{(+)} \varepsilon_\mu(q). \qquad (98)$$

Here G_F is the Fermi constant, θ_c is the Cabibbo angle, p, q are the momenta of the pion and the photon, $l_\nu^{(+)}$ is the lepton current and h_V, h_A are the vector and axial-vector form factors. Then for the h_V we have the standard value

$$h_V = \frac{1}{8\pi^2 F_\pi}. \qquad (99)$$

For the h_A using (97) and the axial-vector dominance of the weak interaction [66]

$$L_W = \frac{G_F}{g_\rho} \cos\theta_c \, [M_\rho^2 \rho^+_\mu + Z^{-1} M^2_{a_1} a^+_{1\mu} - g_\rho F_\pi \partial_\mu \pi^+] l^\mu + \text{h.c.} \qquad (100)$$

we get

$$h_A = \frac{1}{8\pi^2 F_\pi Z}\,.$$

Then, the ratio of these form factors is equal to

$$\gamma = \frac{h_A}{h_V} = Z^{-1} = \frac{1}{2}\left[1 + \sqrt{1 - \left(\frac{2g_\rho F_\pi}{M_{a_1}}\right)^2}\,\right]. \qquad (101)$$

For the $M_{a_1} = 1.2$ GeV we get $\gamma = 0.65$. The experimental data are equal to

$$\gamma = 0.52 \pm 0.06 \ [67], \gamma = 0.67 \pm 0.04 \pm 0.16 \ [68].$$

Analogously we can describe the form factors of the decay $K^- \to e\bar{\nu}\gamma$ $(\mu\bar{\nu}\gamma)$ and $\pi^- \to e\bar{\nu}e^+e^-$ [64,69, 70].

Now let us calculate the $a_1 \to \rho\pi$ decay width using the amplitude (96)

$$\Gamma_{a_1 \to \pi\rho} = \frac{2}{3\pi M_{a_1}}\left(\frac{M_\rho^2 g_\rho^2}{32\pi^2 F_\pi}\right)^2\left(1 - \frac{M_\rho^2}{M_{a_1}^2}\right)\left\{\frac{M_{a_1}^2}{M_\rho^2}\left(1 + \frac{M^4}{2M_\rho^2 M_{a_1}^2}\right) + \right.$$

$$\left. + (1 + C)^2\left(2 + \frac{M^4}{4M_\rho^2 M_{a_1}^2}\right) - 3(1 + C)\,\frac{M^2}{M_\rho^2}\right\} \quad (M^2 = M_\rho^2 + M_{a_1}^2). \qquad (102)$$

Then for the $M_{a_1} = 1.2$ GeV we get

$$\Gamma_{a_1 \to \pi\rho} = 300 \text{ MeV}.$$

The interesting property of amplitude (96) is that in calculaton of the $a_1 \to \rho\pi$ decay width there appears a large negative interference term which reduces the sum of independent contributions of the gradient $(p^\mu q^\nu - g^{\mu\nu} pq)$ and the q^2 (with $g^{\mu\nu}$) parts of the amplitude almost by an order. It also turns out that in the interval 1.1 GeV $\leq M_{a_1} \leq$ 1.4 GeV the $a_1 \to \rho\pi$ decay width decreases with increasing a_1 meson mass.

If these properties of the amplitude $a_1 \to \rho\pi$ are included into analysis of experimental data on the decay $\tau \to \nu 3\pi$, the results of different groups [48] giving much different masses and widths of the a_1 meson become comparable with each other and also with data obtained earlier in analysing the processes $\pi N \to 3\pi N$ [47] (see [49] and Table 2).

Table 2. The a_1 parameters obtained by different collaborations with the $a_1\rho\pi$ amplitude taken as a constant and the results obtained with our representation of the amplitude*

Source [47, 48]	$T_{(a_1 \to \pi\rho)} = \text{const}$		$T_{(a_1 \to \pi\rho)}$ equals (96) [49]	
	a_1 Mass (MeV)	a_1 Width (MeV)	a_1 Mass (MeV)	a_1 Width (MeV)
DELCO (Ruckstuhl 1986)	$1056\pm20\pm15$	$476^{+132}_{-120}\pm54$	1242 ± 37	465^{+228}_{-143}
MARK II (Schmidke 1986)	$1194\pm14\pm10$	$462\pm56\pm30$	1260 ± 14	298^{+40}_{-34}
ARGUS (Albrecht 1986)	1046 ± 11	521 ± 27	1250 ± 9	488 ± 32
$\pi^- p \to p\pi^+\pi^-\pi^-$ (Daum 1981)	1280 ± 30	300 ± 50		
$\pi^- p \to p\pi^+\pi^-\pi^0$ (Dankowych 1981)	1240 ± 80	380 ± 100		

* The results [49] have been included into the Particle Data Group 1992.

5) Strong, Weak and Electromagnetic Kaon Decays and the $\Delta I = 1/2$ Rule. Here we shall consider kaon decays of the types $K \to \gamma\gamma$, $K_L \to \pi^0\gamma\gamma$, $K \to 2\pi$, $K \to 3\pi$ and present a theoretical explanation of the phenomenological $\Delta I = 1/2$ rule.

The above-mentioned decays are described by weak interactions with W-boson exchange, strong high-energy corrections with gluon exchange and strong low-energy interactions. The result of weak interactions with W-boson exchange and strong high-energy corrections with gluon exchange can be described by a local low-energy Lagrangian $\mathcal{L}_{\text{eff}}^{|\Delta S|=1}$ [71—73]

$$\mathcal{L}_{\text{eff}}^{|\Delta S|=1} = \frac{G_D}{\sqrt{2}} s_1 c_1 c_3 \sum_{i=1,2,3,5,6} C_i(\mu) Q_i(\mu) = \frac{G_F}{\sqrt{2}} s_1 c_1 c_3 Q. \qquad (103)$$

Here $G_F s_1 c_1 c_3 = 2.5\ \text{GeV}^{-2}$. s_1, c_1, c_3 are elements of the Kobayashi — Maskawa matrix [74]. $C_i(\mu)$ are numerical coefficients. $Q_i(\mu)$ are four-quark operators by Gilman, Wise [72], while $Q_6(\mu)$ is the «penguin»-operator [71],

μ is a normalization point. The appearance of the normalization point μ is due to the inclusion of the high-energy QCD interaction. The dependence on μ is contained in terms of the current quark-gluon coupling constant $\alpha_3(\mu) = \frac{2\pi}{9} \ln^{-1}(\frac{\mu}{\Lambda_3})$, calculated for three quark flavours where Λ_3 is the QCD-parameter ($\Lambda_3 \approx 0.12$ GeV). Numerical values of the coefficients C_i/μ are functions of μ and masses of heavy quarks (m_c, m_b and m_t). For standard normalization $\alpha_3(\mu) = 1$ ($\mu = 0.24$ GeV) one has [73]

$$C_1 = 1, C_2 = -1.6, C_3 = 0.033, C_5 = -0.02, C_6 = 0.1.$$

The effective Lagrangian (103) satisfies the selection rules: $\Delta S = 1$, $\Delta I = 1/2$ and $3/2$.

Amplitudes of considered kaon decays contain matrix elements of $Q_i(\mu)$ operators: $<x|Q_i(\mu)|K>$, where $|x>$ is any low-energy state. Values of matrix elements $<x|Q_i(\mu)|K>$ are determined by strong low-energy interactions. Thus the calculation of $<x|Q_i(\mu)|K>$ is an extremely involved problem which admits at present only model solutions.

Using results of [75] one can present matrix elements $<x|Q_i(\mu)|K>$ as the sum of two terms

$$<x|Q_i(\mu)|K> \; = \; <x|Q_i(\mu)|K>_{\mathrm{NPC}} + <x|Q_i(\mu)|K>_{\mathrm{PC}},$$

where $<x|Q_i(\mu)|K>_{\mathrm{NPC}}$ is a nonperturbative contribution (NPC) which is independent of μ and defined by the SBCS.

The term $<x|Q_i(\mu)|K>_{\mathrm{PC}}$ corresponds to a perturbative contribution (PC); the quantity $<x|Q_i(\mu)|K>_{\mathrm{PC}}$ is defined by quantum fluctuations with an energy less than μ. For sufficiently small μ ($\mu \sim 0.24$ GeV) nonperturbative contributions become essential. Thus all perturbative contributions for $\mu \sim 0.24$ GeV can be neglected.

In our model $<x|Q_i(\mu)|K>$ is naturally approximated by quark loops with virtual constituent quarks and the cut-off parameter $\Lambda = 1.28$ GeV that characterizes the scale of the chiral symmetry breaking.

Decays $K \to \gamma\gamma$

The $K \to \gamma\gamma$ amplitudes are defined by contact and pole diagrams in Fig.8. The pole diagrams are due to exchange of the pseudoscalar mesons π^0, η, η'

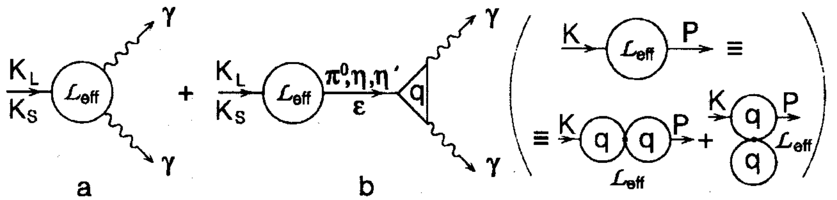

Fig. 8. Contact and pole diagrams of decays $K_{L,S} \to \gamma\gamma$ ($\varepsilon \equiv f_0$ (700))

and the scalar isoscalar meson $\varepsilon(f_0)$. Within our model accuracy (20-25%) the contribution of contact diagrams can be neglected [76].

In the pole approximation the decay $K \to \gamma\gamma$ amplitudes contain matrix elements of the $Q_i(\mu)$ operators calculated between $|K^0>$ and $|x>$ states, where $X = \pi^0$, η, η' or $\varepsilon(f_0)$. Since the value of the matrix elements of the above-mentioned transitions depends mainly on the operator Q_6, we write down these elements in the explicit form only for the «penguin» operator Q_6

$$Q_6 = [\bar{s}_a \gamma^\nu (1-\gamma^5) d_b \sum_{q=u,d,s} [\bar{q}_b \gamma_\nu (1-\gamma^5) q_a].$$

Here $a,b = 1,2,3$ are colour indices. Then

$$\begin{aligned} <\pi^0|Q_6|K^0> &= \rho X, \\ <\eta|Q_6|K^0> &= [-(\tfrac{2}{3}+\rho)\sin\bar{\theta} + \sqrt{2}\,\frac{F_s}{F_\pi}(\tfrac{1}{3}+\rho')\cos\bar{\theta}]X, \\ <\eta'|Q_6|K^0> &= [-(\tfrac{2}{3}+\rho)\cos\bar{\theta} - \sqrt{2}\,\frac{F_s}{F_\pi}(\tfrac{1}{3}+\rho')\sin\bar{\theta}]X. \end{aligned} \tag{104}$$

Here $X = <\pi^0|Q_1|K^0> = 3.5\cdot 10^{-3}$ (GeV)^4. The parameters ρ and ρ' are equal to

$$\begin{aligned} \rho &= 64(1+\lambda)\left(\frac{Zm_u F_\pi}{M_K F_K}\right)^2 \left[1 - \frac{\lambda F_K^2}{2(1+\lambda)F_\pi^2}\right] \approx 50, \\ \rho' &= 64\lambda(1+\lambda)\left(\frac{Zm_u F_\pi^2}{M_K F_K F_s}\right)^2 \left[1 - \frac{F_K^2}{2(1+\lambda)F_\pi^2}\right] \approx 60. \end{aligned} \tag{105}$$

Here $\lambda = m_s/m_u$ and M_K is the K-meson mass.

It is seen from the above formulae that in the case of exact $SU(3)$ symmetry ($m_u = m_s$, $F_\pi = F_K = F_s$) (104) and (105) result in a usual $SU(3)$ symmetric relation between the matrix elements $\langle K^0 | Q_6 | \pi^0, \eta, \eta' \rangle$ used in some papers [77,78].

Using formulae, given in [76,79] one can obtain the following values for the matrix elements of the transitions $K^0 \to \pi^0, \eta, \eta'$ for two different values of the angle θ

$$\theta = -18°: \langle \pi^0 | Q | K^0 \rangle = 4.9X; \ \langle \eta | Q | K^0 \rangle = 3X; \ \langle \eta' | Q | K^0 \rangle = -10.6X.$$

$$\theta = -20°: \langle \pi^0 | Q | K^0 \rangle = 4.9X; \ \langle \eta | Q | K^0 \rangle = 2.6X; \ \langle \eta' | Q | K^0 \rangle = -10.7X.$$

For the decay amplitude $K_L \to \gamma\gamma$ we have

$$T_{K_L \to \gamma\gamma} = \frac{\alpha G_F s_1 c_1 c_3}{3\pi F_\pi} \left\{ \frac{3 \langle \pi^0 | Q | K^0 \rangle}{M_K^2 - M_\pi^2} + \left(5 \sin\bar{\theta} - \sqrt{2}\,\frac{F_\pi}{F_s} \cos\bar{\theta}\right) \frac{\langle \eta | Q | K^0 \rangle}{M_K^2 - M_\eta^2} + \right.$$
$$\left. + \left(5 \cos\bar{\theta} + \sqrt{2}\,\frac{F_\pi}{F_s} \sin\bar{\theta}\right) \frac{\langle \eta' | Q | K \rangle}{M_K^2 - M_{\eta'}^2} \right\} = \begin{cases} 4.5 \cdot 10^{-9}\ \text{GeV}^{-1} (\theta = -18°) \\ 3.4 \cdot 10^{-9}\ \text{GeV}^{-1} (\theta = -20°) \end{cases}. \quad (106)$$

The experimental values are equal to

$$\Gamma_{K_L \to \gamma\gamma} = (7.24 \pm 0.35) \cdot 10^{-12}\text{eV}, \ T_{K_L \to \gamma\gamma} = 3.4 \cdot 10^{-9}\ \text{GeV}^{-1}.$$

It is seen that at $\theta = -20°$ one can obtain a good agreement between theoretical and experimental data.

Now let us consider the decay $K_S \to \gamma\gamma$. Here the main diagram is the pole diagram 8b with the ε meson ($f_0(700—800)$). The expressions of corresponding matrix elements are [79]

$$\langle \varepsilon | Q_1 | K^0 \rangle = \langle \varepsilon | Q_2 | K^0 \rangle = \langle \varepsilon | Q_3 | K^0 \rangle = 0,$$
$$\langle \varepsilon | Q_5 | K^0 \rangle = \frac{1}{3} \langle \varepsilon | Q_6 | K^0 \rangle = \frac{i}{3} \rho'' X.$$

The parameter ρ'' is defined by the expression

$$\rho'' = 64(1 + \lambda) Z^{3/2} \left(\frac{m_u F_\pi}{F_K M_K} \right)^2 \approx 70.$$

The decay $K_S \to \gamma\gamma$ amplitude is

$$T_{K_S \to \gamma\gamma} = -\frac{10}{9}\frac{\alpha}{\pi F_\pi Z^{1/2}}(G_F s_1 c_1 c_3)\frac{i\langle\varepsilon|Q|K^0\rangle}{M_K^2 - M_\varepsilon^2}\cos\delta_\varepsilon(M_K)\exp i\delta_\varepsilon(M_K) = \tag{107}$$

$$= \begin{cases} 2.7\cdot 10^{-9}\exp(\mathrm{i}\ 60^\circ)\ \mathrm{GeV}^{-1} & (m_\varepsilon = 0.7\ \mathrm{GeV}) \\ 2.3\cdot 10^{-9}\exp(\mathrm{i}\ 45^\circ)\ \mathrm{GeV}^{-1} & (m_\varepsilon = 0.8\ \mathrm{GeV}), \end{cases}$$

where

$$\delta_\varepsilon(M_K) = \mathrm{arctg}\left(\frac{M_K\Gamma_\varepsilon(M_K)}{M_\varepsilon^2 - M_K^2}\right), \quad \Gamma_\varepsilon(M_K) = \frac{3g^2_{\varepsilon\pi\pi}}{32\pi M_K}\left[1 - \frac{4M_\pi^2}{M_K^2}\right]^{1/2},$$

$$g_{\varepsilon\pi\pi}(M_K) = \frac{4m_u^2 Z^{1/2}}{F_\pi}.$$

Here $\Gamma_\varepsilon(M_K)$ and $g_{\varepsilon\pi\pi}(M_K)$ are the partial width and effective coupling constant of the decay $\varepsilon \to \pi\pi$ of the virtual meson ε (700—800) with the energy M_K.

The experimental value is equal to [52]

$$T_{K_S \to \gamma\gamma} = (5.0 \pm 1.3)\cdot 10^{-9}\ \mathrm{GeV}^{-1}.$$

Decay $K_L \to \pi^0\gamma\gamma$

If matrix elements of the transitions $K_L \to \pi^0, \eta, \eta'$ are known, one can describe the decay $K_L \to \pi^0\gamma\gamma$ by analogy with the decay $\eta \to \pi^0\gamma\gamma$ (see subsec.3 of this section). This decay is described by diagrams with intermediate vector ω and ρ mesons, with scalar f_0 (700—800), f_0 (975) and f_0 (1400) mesons and contact diagrams (see Fig.9). Besides, as shown in [77,78], a noticeable contribution to this process comes from diagrams with pion loops (see Fig.9*d*).

The decay $K_L \to \pi^0\gamma\gamma$ is also of interest since it is essential to take account of the difference of masses of s and (u,d) quarks when the transitions $(K_L \to \pi^0, \eta, \eta')$ are calculated. If this difference is neglected and use is made of the relation for matrix elements $\langle K_L|\pi^0, \eta, \eta'\rangle$ following from the group $SU(3)$, the relative contributions of diagrams with the intermediate η-meson are much reduced at physical values of (η_0, η_8) mixing angle

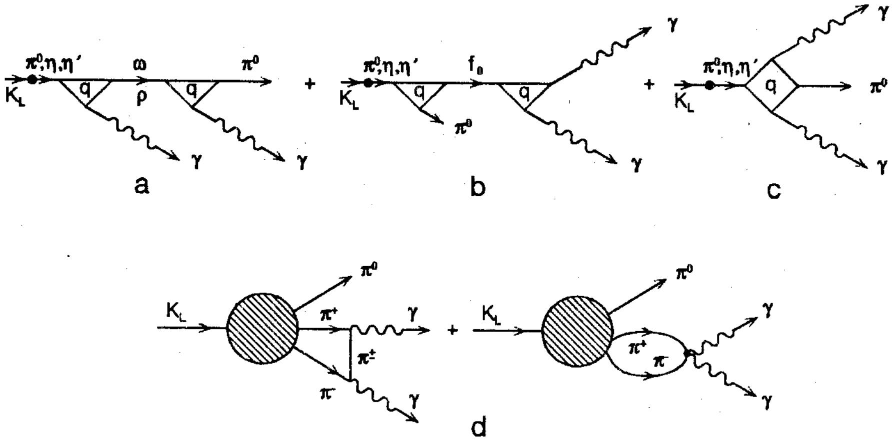

Fig. 9. Pole, contact and loop diagrams that describe the decay $K_L \to \pi^0\gamma\gamma$

$(-20° \le \theta \le -18°)$. Therefore, there remain the diagrams with intermediate π^0 and η'-mesons that have the same sign and reinforce each other [77,78]. Normalization to the matrix element with the intermediate π^0 meson gives rise to the following value for the amplitude with vector mesons (Fig.9*a*)

$$T_{K_L \to \pi^0\gamma\gamma} = T^{(\pi^0)}_{K_L \to \pi^0\gamma\gamma}\,[\overset{(\pi^0)}{1} - \overset{(\eta)}{0.09} + \overset{(\eta')}{0.21} = 1.1]\ (\theta = -18°,\ m_u = m_s).$$

However, when the mass difference of constituent s and (u, d) quarks is taken into account (see formulae (104) and (105)), the contribution of the diagram with the intermediate η meson highly increases; and since it is opposite in sign to the contributions from π^0 and η' mesons, the total value of the amplitude decreases drastically. Therefore, when the width of the decay $K_L \to \pi^0\gamma\gamma$ is computed, one may neglect the influence of the diagrams with intermediate vector mesons [80]

$$T_{K_L \to \pi^0\gamma\gamma} = T^{(\pi^0)}_{K_L \to \pi^0\gamma\gamma}\,[\overset{(\pi^0)}{1} - \overset{(\eta)}{1.27} + \overset{(\eta')}{0.266} = -0.04]\ (\theta = -18°,\ m_u \ne m_s).$$

A similar situation holds for contact diagrams (Fig.9*c*).

As a result, appreciable contributions to the amplitude of the decay $K_L \to \pi^0\gamma\gamma$ come only from the diagrams with the intermediate scalar mesons and pion loops* (Figs.9*b* and *d*). Theoretical estimates are as follows [80]

$$\Gamma^{(M_\varepsilon = 0.7\ \mathrm{GeV})}_{K_L \to \pi^0\gamma\gamma} = \begin{cases} 3.9\cdot 10^{-14}\ \mathrm{eV}\ \ (\theta = -18^\circ) \\ 3.3\cdot 10^{-14}\ \mathrm{eV}\ \ (\theta = -20^\circ) \end{cases};$$

$$\Gamma^{(M_\varepsilon = 0.8\ \mathrm{GeV})}_{K_L \to \pi^0\gamma\gamma} = \begin{cases} 3.0\cdot 10^{-14}\ \mathrm{eV}\ \ (\theta = -18^\circ) \\ 2.5\cdot 10^{-14}\ \mathrm{eV}\ \ (\theta = -20^\circ) \end{cases}.$$

The experimental value is equal to

$$\Gamma_{K_L \to \pi^0\gamma\gamma} = (2.7\pm 0.8)\cdot 10^{-14}\ \mathrm{eV}.$$

Decay $K \to 2\pi$

a) $\Delta I = 3/2$ *transitions.* The effective Lagrangian, obeying the selection rules $\Delta S = 1$ and $\Delta I = 3/2$, takes the form [79]

$$\mathcal{L}^{|\Delta I| = 3/2}_{\mathrm{eff}} = \frac{G_F}{\sqrt{2}} s_1 c_1 c_1 \cdot 0.2 \cdot Q_{|\Delta I| = 3/2}, \tag{108}$$

where

$$Q_{|\Delta I| = 3/2} = (\bar{s}_a L^\mu d_a)(\bar{u}_b L_\mu u_b) - (\bar{s}_a L^\mu d_a)(\bar{d}_b L_\mu d_b) + (\bar{s}_a L^\mu u_a)(\bar{u}_b L_\mu d_b),$$

$L^\mu = \gamma^\mu(1-\gamma^5)$ and a, b are colour indices. The decay amplitude $T^{3/2}(K \to 2\pi)$ is

$$T^{3/2}(K \to 2\pi) = 3.5\cdot 10^{-7}\, i \langle 2\pi | Q_{|\Delta I| = 3/2} | K\rangle\ \mathrm{GeV}^{-2}.$$

In our model the matrix elements $\langle 2\pi | Q_{|\Delta I| = 3/2} | K\rangle$ are defined by the contact quark diagrams drawn in Fig.10*a,b*.

Let us write down the results of the calculation

$$\langle \pi^+\pi^- | Q_{|\Delta I| = 3/2} | K^0\rangle = -\frac{1}{2}\langle \pi^0\pi^0 | Q_{3/2} | K^0\rangle = \frac{\sqrt{2}}{3}\langle \pi^+\pi^0 | Q_{3/2} | K^+\rangle,$$

---

*For the decay $\eta \to \pi^0\gamma\gamma$, the contribution of diagrams with pion loops to the decay amplitude is very small and can thus be neglected.

$$i\langle\pi^+\pi^0|Q_{3/2}|K^+\rangle =$$

$$= \frac{8M_K^2F_K}{1+\lambda}\left[1-\left(\frac{1+\lambda}{2}\right)^2\left(\frac{M_\pi F_\pi}{M_K F_K}\right)^2-(Z-1)\left(\frac{1+\lambda}{2}\right)\left(1+\frac{F_\pi^2}{F_K^2}\right)\right]=6.6\cdot 10^{-2}\,\mathrm{GeV}^3.$$

The numerical values of decay $K \to 2\pi$ ampltudes and partial widths due to the $\Delta I = 3/2$ transitions are presented in Table 3.

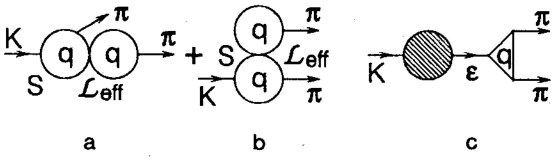

Fig. 10. Contact and pole diagrams of the decay $K \to 2\pi$ ($\varepsilon \equiv f_0$ (700))

Table 3. Numerical values of $K \to 2\pi$ amplitudes and partial widths

Decay	Theory				Experiment	
	$\Delta I = 3/2$		$\Delta I = 1/2$			
	T	Γ	T	Γ	T	Γ
$K^+\to\pi^+\pi^0$	2.3	1.8	0	0	1.84	1.13
$K^0\to\pi^+\pi^-$	1.1	0.4	30	300	27.7	253
$K^0\to\pi^0\pi^0$	2.2	0.8	30	150	26.3	116

Here T is an absolute value of $K \to 2\pi$ amplitude in units 10^{-8} GeV, and Γ is a partial width of a decay in units 10^{-17} GeV.

b) $\Delta I = 1/2$ transitions. The $\Delta I = 1/2$ transitions take place in the decays $K^0\to\pi^+\pi^-$ and $K^0\to\pi^0\pi^0$. The effective Lagrangian satisfying the selection rules $\Delta I = 1/2$ and $\Delta S = 1$ can be obtained by subtracting (108) from (103)

$$\mathcal{L}_{\rm eff}^{|\Delta I|=1/2}=\mathcal{L}_{\rm eff}^{|\Delta S|=1}-\mathcal{L}_{\rm eff}^{|\Delta I|=3/2}=\frac{G_F}{\sqrt{2}}s_1c_1c_3Q_{|\Delta I|=1/2}\,. \qquad (109)$$

The decay $K^0\to 2\pi$ amplitudes are defined both by contact and pole diagrams. The main contribution comes from the pole diagram with the ε

meson exchange (f_0 (700—800)). Within the model accuracy, the contribution of contact diagrams and pole diagrams with the exchange of other resonances can be neglected as compared to the ε meson contribution.

Then the decay $K^0 \to 2\pi$ amplitudes due to the $\Delta I = 1/2$ transitions are

$$A^{1/2}(K^0 \to 2\pi) = \frac{G_F}{\sqrt{2}} s_1 c_1 c_3 \mathrm{i} < \varepsilon | Q_{1/2} | K^0 > \frac{g_{\varepsilon\pi\pi}(M_K)}{M_\varepsilon^2 - M_K^2} \cos \delta_\varepsilon(M_K) \exp(\mathrm{i}\delta_\varepsilon(M_K)) =$$

$$= 3 \cdot 10^{-7} \exp(\mathrm{i}\delta_\varepsilon) \text{ GeV}. \qquad (110)$$

Here $\delta_\varepsilon = 60°$ is the phase of the amplitude $A^{1/2}(K^0 \to 2\pi)$ (see formula (107)). The numerical values of amplitudes and partial widths of decays $K^0 \to 2\pi$ that are due to the $\Delta I = 1/2$ transitions are presented on Table 3.

The theoretical values of decay $K \to 2\pi$ amplitudes are in satisfactory agreement with the experimental data and conform the phenomenological $\Delta I = 1/2$ rule. Strengthening of the $\Delta I = 1/2$ transitions in the linear σ model takes place by the exchange of the scalar meson f_0 (700—800). This scalar meson is a broad resonance which can be identified with the S_1 (910) state of Au, Morgan, Penington [81] (see also the new data in [81]).

Decay $K \to 3\pi$

Nonleptonic decays $K \to 3\pi$ proceed with a small energy release (~25 MeV per one particle of the decay), thus a soft-meson technique (a low-energy limit) is a good approach for their description. The $K \to 3\pi$ amplitudes can be connected with the $K^0 \to 2\pi$ amplitudes [82]. For instance, in case of the decay $K^+ \to \pi^+\pi^+\pi^-$ one obtains

$$T(K^+ \to \pi^+\pi^-\pi^+(q_+))|_{q_+=0} = \frac{\mathrm{i}}{\sqrt{2}F_\pi} T^{1/2}(K^0 \to \pi^+\pi^-).$$

The variant of the NJL model proposed here well describes the decays $K_L \to \pi^+\pi^-\gamma$, $\eta \to \pi^+\pi^-\gamma$, $\omega \to 3\pi$, $\varphi \to 3\pi$ [83], form factors of the decay K_{l4} [84] and so on.

6) Internal Structure of Mesons. In addition to various decays of mesons, the NJL model can describe their intrinsic properties, i.e., electromagnetic radii, polarizabilities, $\pi\pi$ and πK scattering lengths, etc. [16]. For illustration, we will here calculate the pion electromagnetic radius.

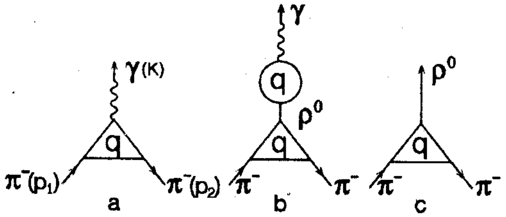

Fig. 11. Diagrams describing the pion electromagnetic radius

We demonstrate two versions of this calculation, one with the use of vector dominance and the second as a direct calculation. We start with the latter. The pion electromagnetic radius is described by two diagrams drawn in Figs. 11*a* and *b*. Transition $\rho^0\gamma$ through a quark loop leads to the expression [16]

$$\frac{1}{2}\frac{e}{g_\rho}F_{\mu\nu}\rho^0_{\mu\nu}\,.$$

Using then the local approximation for all diverging quark loops we get the following formula

$$F_{\pi^-}(k) = -ek^\nu\Big(1+\frac{k^2}{M_\rho^2-k^2}\Big)A_\nu \approx -ek^\nu\Big(1+\frac{k^2}{M_\rho^2}\Big)A_\nu \quad (k=p_1-p_2)\,,$$

which provides the known result:

$$<r^2>_{\pi^-} = \frac{6}{M_\rho^2}\,, \quad \sqrt{<r^2>_{\pi^-}} = 0.63\ \text{fm}$$

in good agreement with experiment,

$$\sqrt{<r^2>_{\pi^-}} = (0.663\pm0.023)\ \text{fm}.$$

An analogous expression for the pion radius follows from our model when use is made of the vector dominance. In this case, apart from diagrams 11*a* and *b* one should consider the diagram 11c that leads to the expression $\frac{g_\rho}{2}k^\nu\rho^0_\nu$. Upon making change vector fields, $\rho_\nu = \tilde{\rho}_\nu + \frac{e}{g_\rho}A_\nu$, the contributions of diagrams 11*a* and 11*c* cancel out and the remaining diagram 11*b* again gives the standard result corresponding to the vector dominance model

$$F_{\pi^-}(k) = -ek^{\nu}(1 - 1 + \frac{M_\rho^2}{M_\rho^2 - k^2})A_\nu = -ek^{\nu}(1 + \frac{k^2}{M_\rho^2})A_\nu \, .$$

The radii of K^0 and $K^{\pm}$ mesons are described in the same manner.

The model can also satisfactorily describe polarizabilities of pions and kaons [85,86] and $\pi\pi$ and πK scattering lengths [87].

The validity of q^2-expansions used in the model should be proved more rigorously, which, in particular, requires the solution of a highly complicated problem, the problem of quark confinement. Successful attempts along this line were undertaken in [11].

9. THE QUARK AND GLUON CONDENSATES IN NJL MODEL

The purpose of this section is to investigate a QCD-motivated NJL model containing a nonperturbative gluon condensate. We will then show how the basic parameters and model equations of the resulting chiral σ-model will change with this quantity taken into account [27].

Let us start with QCD and decompose the gluon field $\mathbf{G}_\mu^a$ into a condensate field G_μ^a and the quantum fluctuations g_μ^a around it

$$\mathbf{G}_\mu^a(x) = G_\mu^a(x) + g_\mu^a(x). \tag{111}$$

By assumption the first part of the field yields a nonvanishing gluon condensate

$$<\mathrm{vac}|\frac{g^2}{4\pi}:\mathbf{G}_{\mu\nu}^a(0)\mathbf{G}_a^{\mu\nu}(0):|\mathrm{vac}> = <\mathrm{vac}|\frac{g^2}{4\pi}G_{\mu\nu}^a(0)G_a^{\mu\nu}(0)|\mathrm{vac}>, \tag{112}$$

where $G_{\mu\nu}^a$ is the field strength tensor. Integrating in the generating functional of QCD over the quantum field $g_\mu^a(x)$ and approximating the (unknown) nonperturbative gluon propagator by a δ-function we get an effective chiral four-quark interaction of the NJL type. In this case, the condensate field $G_\mu^a(x)$ enters into the standard Lagragian of the NJL model through the covariant derivative of the quark field

$$D_\mu q = (\partial_\mu + ig\frac{\lambda_a}{2}G_\mu^a)q, \tag{113}$$

where g is the QCD coupling constant and $\lambda_a/2$ are the generators of the color group $SU(N_c)$.

The effective chiral quark Lagrangian describing interactions of composite scalar and pseudoscalar mesons in the presence of condensate gluon can be written as [16,18]

$$\mathcal{L}(q, G) = \bar{q}(\mathrm{i}\hat{D} - m^0)q + \frac{\kappa}{2}\,[(\bar{q}\tau^\alpha q)^2 + (\bar{q}\,\mathrm{i}\gamma_5\tau^\alpha q)^2], \qquad (114)$$

where $\hat{D} = \gamma^\mu D_\mu$, D_μ is the covariant derivative (113), τ^α are the Pauli matrices of $SU(2)_F$; ($\tau^0 \equiv \mathbb{1}$; summation over α is understood), and q are fields of current quarks with mass m^0*. Upon introducing meson fields, the Lagrangian (114) turns into the equivalent form

$$\mathcal{L}'(q, G, \tilde{\sigma}, \varphi) = -\frac{(\tilde{\sigma}_\alpha^2 + \varphi_\alpha^2)}{2\kappa} + \bar{q}(\mathrm{i}\hat{D} - m^0 + \tilde{\sigma} + \mathrm{i}\gamma_5\varphi)q \qquad (115)$$

with $\tilde{\sigma} = \sigma_\alpha\tau^\alpha$, $\varphi = \varphi_\alpha\tau^\alpha$. The vacuum expectation value of the isoscalar-scalar field $\tilde{\sigma}_0$ turns out to be nonzero ($<\tilde{\sigma}_0> \neq 0$). To pass to a physical field σ_0 with $<\sigma_0> = 0$ one usually performs a field shift leading to a new quark mass m to be identified with the mass of constituent quarks (see (48))

$$-m^0 + \tilde{\sigma}_0 = -m + \sigma_0; \quad \tilde{\sigma}_\alpha = \sigma_\alpha \quad (\alpha = 1,2,3), \qquad (116)$$

where m is determined from the gap equation (see below).

Let us for a moment neglect the gluon condensate in (115) ($\hat{D} \to \hat{\partial}$). Integrating in the generating functional associated with the Lagrangian (115) over the quark fields, evaluating the resulting quark determinant by a loop expansion and including thereby only second-order field derivatives gives then an expression corresponding to the linear σ-model (see (53))

$$\mathcal{L}'' = -\frac{\tilde{\sigma}_\alpha^2 + \varphi_\alpha^2}{2\kappa} +$$

---

*This type of interaction results from the (current × current) interaction of quarks due to gluon exchange after applying a Fierz transformation to color and Dirac indices. For simplicity, we shall omit here vector and axial-vector channels and consider an unbroken flavour group $SU(2)_F$ with equal quark masses $m_u^0 = m_d^0$.

$$+ \mathrm{Tr}\,\{[p^2 I_2 + 2(I_1 + m^2 I_2)][(\sigma - m)^2 + \varphi^2] - I_2[(\sigma - m)^2 + \varphi^2]^2\}. \quad (53')$$

Here I_1 and I_2 are divergent integrals regularized with a cut-off parameter Λ (see (50)).

Now let us see how the Lagrangian (53') changes when condensate gluon fields are taken into account. Here the effect of gluon condensate corrections will be calculated with an accuracy up to squared terms in the field strength $G^a_{\mu\nu}(x)$. For evaluating the quark determinant with external fields we shall use the «heat kernel» technique which has been proposed in papers [18,88]. Then, instead of (53') we get*

$$\mathcal{L}^G = \mathrm{Tr}\,\Big\{-\frac{\varphi^2+(\sigma-m+m^0)^2}{4\kappa} + [p^2(I_2+\frac{1}{96}\frac{G^2}{m^4}) + $$
$$+ 2[I_1+m^2 I_2+\frac{1}{48}\frac{G^2}{m^2}(1+\frac{1}{2})]][(\sigma-m)^2+\varphi^2] -$$
$$- (I_2+\frac{1}{96}\frac{G^2}{m^4})[(\sigma-m)^2+\varphi^2]^2\Big\}, \quad (117)$$

where

$$G^2 = \frac{\alpha}{\pi} <(G^a_{\mu\nu})^2>, \quad \alpha = \frac{g^2}{4\pi}.$$

Now let us determine physical quantities and model parameters. In order that the Lagrangian (117) takes its usual form (with the standard coefficient of the kinetic term) one should perform the following field renormalizations:

$$\sigma_a = g_\sigma \sigma^R_a, \quad \varphi_a = g_\varphi \varphi^R_a, \quad g_\sigma = g_\varphi = \frac{1}{2}(I_2 + \frac{G^2}{96 m^4})^{-1/2}. \quad (118)$$

Then, from the requirement for terms linear in σ to vanish we get a modified «gap» equation

$$m = m^0 + 8\kappa m I_1 + \kappa \frac{G^2}{6m}. \quad (119)$$

After renormalization of the meson fields we get for the square part of the Lagrangian (117) determining the mass terms (omitting the field index R),

---

*Higher order terms of the form $g^3 f_{abc} \frac{<G^a_{\mu\nu} G^b_{\nu\sigma} G^c_{\sigma\mu}>}{m^6} + \ldots$ give less essential contributions of about several percent [89], and will therefore, be neglected. Note that proper-time regularized integrals are replaced here by momentum cut-off regularized integrals I_1, I_2.

$$\mathcal{L}^G_m = -\frac{g_\sigma^2}{4}\,\mathrm{Tr}\,\Big\{\Big(\frac{1}{\kappa} - 8I_1 - \frac{G^2}{6m^2}\Big)(\sigma^2+\varphi^2) + (4m)^2\Big(I_2+\frac{G^2}{96m^4}\Big)\sigma^2\Big\} =$$

$$= -\frac{g_\sigma^2 m^0}{4\kappa m}\,\mathrm{Tr}\,(\sigma^2+\varphi^2) - m^2\,\mathrm{Tr}\,\sigma^2, \qquad (120)$$

where in the last line of eq. (120) use of the gap equation (119) has been made. We then get for the meson masses the known equations

$$m_\pi^2 = \frac{g_\varphi^2 m^0}{\kappa m} = \frac{m^0 m}{\kappa F_\pi^2} \approx$$

$$\approx -\frac{2m^0<\bar{q}q>}{F_\pi^2}\ \text{(Gell–Mann — Oakes — Renner formula)}, \qquad (121)$$

$$m_\sigma^2 = m_\pi^2 + 4m^2.$$

Here we have used the Goldberger — Treiman identity $g_\varphi = m/F_\pi$ and the expression for the total quark condensate $<\bar{q}q>$ which will be derived below (see eq. (127)). The Goldberger — Treiman identity leads to the following expression for the pion decay constant F_π

$$F_\pi^2 = \frac{m^2}{g_\varphi^2} = m^2\Big(4I_2 + \frac{G^2}{24m^2}\Big). \qquad (122)$$

Finally, the effective coupling constants of meson interactions can be read off from the interaction terms

$$\mathcal{L}_{\sigma\varphi^2} = g_\sigma m\,\mathrm{Tr}\,(\sigma\varphi^2),$$

$$\mathcal{L}_{(\sigma^2+\varphi^2)^2} = -\frac{g_\sigma^2}{4}\,\mathrm{Tr}\,(\sigma^2+\varphi^2)^2. \qquad (123)$$

Thus, the form of meson interactions remains unchanged after the inclusion of gluon condensate fields (see (55)). Only expression of the coupling constant $g_\sigma (g_\varphi)$ changes.

Until now we have not considered vector and axial-vector mesons. However, one should bare in mind that since axial-vector mesons do exist, nondiagonal transitions of the type $\pi \to a_1$ play an essential role in the NJL model (see Sec.7).

Then, (see (76)) the constituent quark mass is expressed

$$m_u^2 = \frac{M_{a_1}^2}{12}\left[1 - \sqrt{1 - \left(\frac{2g_\rho F_\pi}{M_{a_1}}\right)^2}\,\right].$$

It is seen from (76) that $1 - \left(\frac{2g_\rho F_\pi}{M_{a_1}}\right)^2 \geq 0$ and hence the minimal mass of the a_1 meson equals $M_{a_1} = 2g_\rho F_\pi = \sqrt{2}M_\rho = 1.1$ GeV, where in the last step the KSFR relation has been used. This is just Weinberg's relation. From this the estimates $m_u = 330$ MeV and $Z = 2$ follow.

The value of the gluon condensate is taken from the data on the hadron process $e^+e^- \rightarrow$ hadrons (see ref. [90])

$$G^2 = \frac{\alpha}{\pi} <G_{\mu\nu}^a G_a^{\mu\nu}> \; = \; [(410 \pm 80) \;\; \text{MeV}]^4. \tag{124}$$

Taking into account the additional renormalization constant Z due to $\pi - a_1$ mixing, eq. (122) together with eq. (50) gives for F_π

$$F_\pi^2 = \frac{N_c m^2}{(2\pi)^2 Z}\,[\ln(\frac{\Lambda^2}{m^2} + 1) - (1 + \frac{m^2}{\Lambda^2})^{-1} + \frac{\pi^2}{6N_c m^4}G^2]. \tag{125}$$

Hence, using the values $F_\pi = 93$ MeV, $N_c = 3$ and eq. (124) we find for the parameter Λ the estimate

$$\Lambda = 700 \text{ MeV}.$$

Let us first calculate that part of the quark condensate $<\bar{q}q>'$ which does not explicitly contain the gluon condensate G^2

$$<\bar{q}q>' = \text{Tr}\,(\frac{1}{i\hat{\partial} - m}) = -4mI_1 = -(200 \text{ MeV})^3.$$

The gap equation (119) can be rewritten in terms of quark and gluon condensates as

$$m = m^0 - 2\kappa <\bar{q}q>' + \frac{\kappa}{6m}\frac{\alpha}{\pi} <G_{\mu\nu}^a G_a^{\mu\nu}> \equiv m^0 - 2\kappa <\bar{q}q>, \tag{126}$$

where we have introduced the notion of the total quark condensate $<\bar{q}q>$ which includes also gluon condensate corrections,

$$<\bar{q}q> = <\bar{q}q>' - \frac{1}{12}\frac{\alpha}{\pi}\frac{<G^{a}_{\mu\nu}G^{\mu\nu}_{a}>}{m} = -(245\ \mathrm{MeV})^{3}. \qquad (127)$$

We see that this number is close to the standard value of the quark condensate.

With eqs. (121) and (126) we find the constant κ

$$\kappa^{-1} = \left(\frac{m_{\pi}F_{\pi}}{m}\right)^{2} - \frac{2<\bar{q}q>}{m} = 0.091\ \mathrm{GeV}^{2},\ \kappa = 11\ \mathrm{GeV}^{-2}.$$

Finally, we can determine the current quark mass m^{0},

$$m^{0} = \frac{m^{2}_{\pi}F^{2}_{\pi}\kappa}{m} = m + 2\kappa <\bar{q}q> \approx 5\ \mathrm{MeV}.$$

This is also a standard value. Thus, our model gives a reasonable self-consistent description of the most important model parameters and physical quantities.

The above investigation shows that corrections due to the gluon condensate which naturally arises in our QCD-motivated NJL model provide quite reasonable results. In particular, the gluon condensate $\frac{\alpha}{\pi} <G^{a}_{\mu\nu}G^{\mu\nu}_{a}>$ turns out to contribute to various quantities, like for instance g_{φ}, F_{π}, m and $<\bar{q}q>$, without changing thereby the form of meson mass formulae and interaction terms in the effective meson Lagrangian. Thus, in the considered approximation the main effects of the gluon condensate are the decrease in the value of the low-energy cut-off scale Λ from 1.25 GeV to 700 MeV and the increase in the coupling constant of the effective four-quark ineraction κ from 5 to 11 GeV^{-2}.

It has been argued in the literature that in models with a gluon condensate a dynamical gluon mass could appear. For example, the authors of ref. [91] presented a description of the gluon condensate based upon an analogy to the Landau — Ginzburg theory of superconductivity. As a result of that analysis they predicted a gluon mass of the order of 600 MeV. In our approach, in the case of a massive gluon, we would have the relation

$\kappa = \frac{g^2}{4\pi}\frac{4\pi}{2N_cM_G^2}$. At low energies one has $\frac{g^2}{4\pi} \approx 1$, which leads to an estimate $M_G \approx \sqrt{\frac{2\pi}{N_c\kappa}} \approx 440$ MeV which is not too different from the above value.

The effect of the gluon condensate in nonlinear chiral Lagrangians was studied in ref. [92] as well, particularly for estimating its influence on the low-energy coefficients. The coefficients of the G^2 terms obtained in their expressions for F_π and in the quark condensate coincide with the coefficients of the present paper. However, a definite advantage of the present approach is the existence of an inherent mechanism for spontaneous breaking of chiral symmetry and the appearance of constituent quarks and meson masses on the basis of a simple effective four-quark interaction arising from gluon exchange. In the above-mentioned paper the mechanism of spontaneous breaking of chiral symmetry is brought in from outside by using additional assumptions. Another essential difference between both approaches is that we cannot neglect the quark condensate $<\bar{q}q>'$ since it is an important contribution of the four-quark interaction caused by quantum fluctuations of the gluon field. Finally, let us mention that our bosonization approach is in some sense complementary to the more phenomenological approach of QCD sum rules [89]. Indeed, in our case composite hadrons arise as a result of two combined nonperturbative effects: i) by the ladder summation of (soft) gluon-mediated four-quark interactions and ii) the nonperturbative contributions of the quark and gluon condensates.

Here we have shown how it is possible to take into account the G^2 gluon corrections in NJL model. Let us remind that using the scale symmetry (see sec.3) we can introduce gluon condensate into our model in a more common method.

ACKNOWLEDGEMENTS

In conclusion I would like to express my gratitude to all my co-authors, especially to Prof. D.Ebert, for the intensive and fruitful collaboration.

REFERENCES

1. Weinberg S.— Phys.Rev.Lett., 1967, v.18, p.188—195; Phys.Rev., 1968, v.166, p.1568—1575;
Coleman S., Wess J., Zumino B.— Phys.Rev., 1969, v.177, p.2239—2250;
Callan C., Coleman S., Wess J., Zumino B.—Phys.Rev., 1969, v.177, p.2247—2258.

2. Pervushin V.N., Volkov M.K.— Phys.Lett., 1974, v.51B, p.356—358; ibid. 1974, v.51B, p.499—500; ibid. 1975, v.55B, p.405—408; ibid 1975, v.58B, p.177—179; ibid.1975, v.58B, p.74—78; Nuovo Cim, 1975, v.27A, p.277—293.
3. Volkov M.K., Pervushin V.N.— Essentially Nonlinear Field Theory, Dynamical Symmetry and Pion Physics, Moscow, Atomizdat, 1978, p.240.
4. Karchev N.I., Slavnov A.A.— Theor.Mat.Fiz., 1985, v.65, p.192—200.
5. Gasser J., Leutwyler H.— Ann.Phys., 1984, v.158, p.142—210.
6. Andrianov A.A. et al.— Theor.Mat.Fiz., 1987, v.70, p.63—74; Phys.Lett., 1987, v.186B, p.401—406;
   Andrianov A.A., Bonora L.— Nucl.Phys., 1984, v.B233, p.232—251.
7. Balog I.— Phys.Lett., 1984, v.149B, p.197—200.
8. Bel'kov A.A., Ebert D., Pervushin V.N.— Sov.J.Part.Nucl., 1991, v.22, p.5—62.
9. Ball R.— Proceedings of the Workshop on «Skyrmions and Anomalies», Mogilany, Poland, 1987, p.17;
   Meissner Ulf-G.— Phys.Rep., 1988, v.161, p.213—246.
10. Ebert D., Pervushin V.N., Reinhardt H.— Sov.J.Part.Nucl., 1979, v.10, p.1114—1156.
11. Praschifka J., Cahill R.T., Roberts C.D.— Int.Journ.Mod.Phys., 1989, v.4A, p.4929—4945; Phys.Rev., 1987, v.36D, p.209—220;
    Cahill R.T., Praschifka J., Burden C.J.— Aust.J.Phys., 1989, v.42, p.129—146; p.147—160; p.161—170; p.171—180.
12. Simic P.— Phys.Rev., 1986, v.34D, p.1903—1910; Phys.Rev.Lett. 1985, v.55, p.40—46.
13. Kalinovsky Yu.L., Kaschluhn L., Pervushin V.N.— Phys.Lett., 1989, v.231B, p.288—293;
    Kalinovsky Yu.L.et al.— Few-Body Systems, 1991, v.10, p.87—104.
14. Eguchi T.— Phys.Rev., 1976, v.D14, p.2755—2763;
    Kikkawa K.— Progr.Theor.Phys., 1976, v.56, p.947—955;
    Kleinert H.— Lectures at the Erice Summer School, ed. by Zichichi A., Plenum Press, New York, 1978, p.122.
15. Ebert D., Volkov M.K.— Yad.Fiz., 1982, v.36, p.1265—1277; Z.Phys., 1983, v.C16, p.205—210.
16. Volkov M.K.— Ann.Phys., 1984, v.157, p.282—303; Sov.J.Part. and Nuclei, 1986, v.17, p.186—203.
17. Ebert D.— Z.Phys., 1985, v.C28, p.433—440.
18. Ebert D., Reinhardt H.— Nucl.Phys., 1986, v.B271, p.188—226; Phys.Lett., 1986, v.B173, p.453—459.
19. Dhar A., Wadia S.R.— Phys.Rev.Lett., 1984, v.52, p.959—962;
    Dhar A., Shankar R., Wadia S.R.— Phys.Rev., 1985, v.D31, p.3256—3265.
20. Bernard V., Jaffe R.L., Meissner U.G.— Nucl.Phys., 1988, v.B308, p.753—790.
21. Wakamatsu M., Weise W.— Z.Phys., 1988, v.A331, p.173—186.
22. Wakamatsu M.— Ann.Phys., 1989, v.193, p.287—325.
23. Hatsuda T.— Phys.Lett., 1988, v.B206, p.385—389; Phys.Lett., 1988, v.B213, p.361—364.
24. Hatsuda T., Kunihiro T.— Phys.Lett., 1987, v.B185, p.304—310; Prog. Theor. Phys. Suppl., 1987, v.91, p.284—300.
    Kunihiro T. — Nucl.Phys., 1991, v.B351, p.593—622.
25. Bernard V., Meissner U.G.— Ann.Phys., 1991, v.206, p.50—62;
    Bernard V., Meissner U.G., Zahed I.— Phys.Rev., 1987, v.D36, p.819—827.
26. Ebert D., Kalinovsky Yu.L., Münchov L., Volkov M.K. — JINR Preprint E2-92-134, Dubna, 1992; Int. J. Mod. Phys. A (1993);
    Ebert D., Kalinovsky Yu.L., Volkov M.K. — JINR Preprint E2-92-268, Dubna, 1992; Phys. Lett. B (1993).
27. Ebert D., Volkov M.K.— Phys.Lett., 1991, v.B272, p.86—90.

28. Yamawaki K.— Preprint DPNU-90-42, 1990, p.12.
Kondo K., Shuto S., Yamawaki K.— Preprint DPNU-91-14, 1991, p.14.
29. Reinhardt H.— a) Phys.Lett., 1990, v.248, p.365—371; b) Phys.Lett., 1990, v.B244, p.316—326; Phys.Lett., 1991, v.B257, p.375—380.
30. Ebert D., Kaschluhn L., Kastelewicz G.— Phys.Lett., 1991, v.B264, p.420—426.
31. Vogl U.— Z.Phys., 1990, v.A337, p.191—196.
32. Ball R.D.— Int.J.Mod.Phys., 1990, v.A5, p.4391—4426.
33. Andrianov A.A., Andrianov V.A.— Preprint ITFA-90-30, Amsterdam, 1990, p.12; Preprint SI-91-11, Siegen, 1991, p.11.
34. Zumino B.— Brandeis Summer Inst.Theor.Phys., eds. Deser S. et al., MIT Press, Cambridge, 1970, p.83.
35. Migdal A.A., Shifman M.A.— Phys.Lett., 1982, v.B119, p.445—451;
36. Shifman M.A.— XXI Winter School LINF, Leningrad, 1986, p.3—45.
37. Vainstein A.I., et al. — Particles and Nucleus, 1982, v.13, p.542—583.
38. Ellis J., Lanik J.— Phys.Lett., 1985, v.B150, p.289—294; Phys.Lett., 1986, v.B175, p.93—99.
39. Leutwyler H.— Nucl.Phys., 1981, v.B179, p.129—141.
40. Voloshin M.B., Ter-Martirosyan K.A.— Theory of Gauge Int. of Elementary Particles, Moscow, Energoatomizdat (in Russian), 1984, p.296.
41. Gomm H., Schechter J.— Phys.Lett., 1985, v.B158, p.449—454;
Gomm H. et al.— Phys.Rev., 1986, v.D33, p.801—819.
42. Lanik J.— Acta Phys.Slov., 1985, v.35, p.343—355; JINR Rapid Comm. N20-86, 1986, p.10—18.
43. Andrianov A.A. et al.— Pis'ma ZhETF, 1986, v.43, p.557—561.
44. Gell—Mann M., Oakes R.J., Renner B.— Phys.Rev., 1968, v.175, p.2195—2201.
45. Gasiorowicz S., Geffen D.A. — Rev.Mod.Phys., 1969, v.41, p.531—573.
46. Volkov M.K., Osipov A.A.— Preprint JINR P2-85-390, Dubna, 1985, p.7.
47. Daum C. et al. — Nucl.Phys., 1981, v.B182, p.269—278;
Dankowych I. et al.— Phys.Rev.Lett., 1981, v.46, p.580—584.
48. Ruckstuhl W.B. et al. — Phys.Rev.Lett., 1986, v.56, p.2132—2139;
Albrecht H. et al.— Z.Phys., 1986, v.C33, p.7—14;
Schmidke W.B. et al.— Phys.Rev.Lett., 1986, v.57, p.527—531;
Band H.R. et al.— Phys.Lett., 1987, v.B198, p.297—302.
49. Ivanov Yu.P., Osipov A.A., Volkov M.K.— Z.Phys., 1991, v.C49, p.563—567.
50. Weinberg S.— Phys.Rev.Lett., 1967, v.18, p.507—511.
51. Kawarabayashi K., Suzuki M., — Phys.Rev.Lett., 1966, v.16, p.255—261;
Riazuddin, Fayyazuddin — Phys.Rev., 1966, v.147, p.1071—1086.
52. Rev. of Particle Properties — Phys.Rev., 1992, D45, part II.
53. Volkov M.K., Ivanov A.N.— Teor.Mat.Fiz., 1986, v.69, p.157—161.
54. Klimt S. et al. — Nucl.Phys., 1990, v.A516, p.429—450.
55. Volkov M.K., Osipov A.A.— Teor.Mat.Fiz., 1991, v.86, p.385—390.
56. Wakamatsu M.— Ann. Phys., 1989, v.193, p.287—302.
57. Ebert D., Ivanov A.N., Volkov M.K.— Fortschr.Phys., 1989, v.37, p.487—540.
58. Adler S.L.— Phys.Rev., 1969, v.177, p.2426—2440;
Bell J.S., Jackiw R.— Nuovo Cim., 1969, v.A60, p.47—58;
Bardeen W.A.— Phys.Rev., 1969, v.184, p.1849—1855.
59. Volkov M.K.— Part. and Nuclei, 1979, v. 10, p.693—726.
60. Volkov M.K., Kreopalov D.E.— Yad.Fiz., 1983, v.37, p.1297—1302.
61. Wess J., Zumino B.— Phys.Rev., 1967, v.163, p.1727—1738.
62. Golterman M.F.L., Hari Dass N.D. — Nucl.Phys., 1986, v.B227, p.739—750.
63. Bando M., Kugo T., Yamawaki K.— Phys.Rep., 1988, v.164, p.217—314.
64. Ivanov A.N., Nagy M., Volkov M.K.— Phys.Lett., 1988, v.B200, p.171—176.

65. Zielensky M. et al.— Phys.Rev.Lett., 1984, v.52, p.1195—1199.
66. Volkov M.K., Nagy M., Osipov A.A.— Theor.Mat.Fiz., 1991, v.87, p.57—61.
67. Bay A. et al. — Phys.Lett., 1986, v.B174, p.445—450.
68. Dominguez C.A., Sola I.— Preprint DESY 88—010, Hamburg, 1988, p.12.
69. Volkov M.K.— Phys.Lett., 1989, v.B222, p.298—300.
70. Volkov M.K. et al. — Czech.J.Phys., 1989, v.B39, p.1256—1262.
71. Vainstein A.I., Zakharov V.I., Shifman M.A.— JETP, 1977, v.72, p.1275—1290.
72. Gilman F.J., Wise M.B.— Phys.Rev., 1979, v.D20, p.1979—1986, p.2392—2403; D27, p.1128—1135.
73. Buras A.J., Slominsky W.— Nucl.Phys., 1985, v.B253, p.231—250.
74. Kobayashi M., Maskawa K.— Progr.Theor.Phys., 1973, v.49, p.652—664.
75. Vainstein A.I. et al.— Yad.Fiz., 1985, v.41, p.1063—1075.
76. Ivanov A.N., Troitskaya N.I., Volkov M.K.— Phys.Lett., 1986, v.B175, p.467—470.
77. Sehgal L.M.— Phys.Rev., 1988, D38, p.808—816.
78. Ecker G., Pich A., de Rafael E.— Nucl.Phys., 1988, v.B303, p.665—678.
79. Volkov M.K., Ivanov A.N., Troitskaya N.I.— Yad.Fiz., 1988, v.47, p.1157—1164.
80. Volkov M.K.— Yad.Fiz., 1991, v.53, p.764—770.
81. Au K.L., Morgan D., Penington M.R.— Phys.Lett., 1988, v.B167, p.229—233; Phys.Rev., 1988, v.D35, p.1633—1642;
    Svec M. et al. — Phys. Rev., 1992, v.D46, p.949—960.
82. De Alfaro V. et al.— Currents in Hadron Physics, Amsterdam — London, 1973, p.670.
83. Volkov M.K., Osipov A.A.— JINR Rapid Communications, No. 3[49]-91, Dubna,1991, p.67-70; Yad.Fiz., 1992, v.55, p.191—196.
84. Volkov M.K. et al.— Phys.Lett., 1989, v.B227, p.275—278; v.B232, p.568 E.
85. Volkov M.K., Ebert D.— Yad.Fiz., 1981, v.34, p.182—185.
86. Volkov M.K., Osipov A.A. — Yad.Fiz., 1985, v.41, p.1027—1034.
87. Volkov M.K., Osipov A.A.— Yad.Fiz., 1984, v.39, p.694—698.
88. Ebert D. et al.— Berlin — Zeuthen Preprint PHE-5-1991, p.23.
89. Shifman M.A., Vainshtein A.I., Zakharov V.I.— Nucl.Phys., 1979, v.B147, p.385—447, p.448—462.
    Novikov V.A. et al.— Fortschr.Phys., 1984, v.32, p.585—622.
    Nikolaev S.N., Radyushkin A.V.— Nucl.Phys., 1983, v.B213, p.285—297.
90. Bertlmann et al.— Z.Phys., 1988, v.C39, p.231—237.
91. Celenza L.S., Shakin C.M.— Phys.Rev., 1986, v.D36, p.1591—1602; ibid 1985, v.D35, p.2843—2850.
92. Espriu D., de Rafael E., J.Jaron— Nucl.Phys., 1990, v.B345, p.22-56.