

FOCK'S THEORY OF HYDROGEN ATOM AND QUANTUM SPACE*

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It is shown that Fock's theory of hydrogen atom gives an example of the nonrelativistic Snyder-like Quantum Geometry (QG).

Показано, что теория Фока атома водорода является примером нерелятивистской квантовой геометрии снайдеровского типа.

1. INTRODUCTON

Fock's theory of hydrogen atom [1] is a very bright landmark in the history of quantum theory. Many physicists trust that the meaning of this theory is much more general and important than the explanation of the symmetry of one specific atomic system though very important one. They believe that its actual meaning still must be understood, that it contains some "signal from the future", some features of the true quantum theory of particle interactions. The search for this more general sense of [1] can only be based on attempts to develop Fock's idea in this or in that way, making "experiments" on it, for example, generalizing it to the case of the relativistic quantum (field) theory. And attempts to interpret [1] in a new sense take place indeed. Works based on the historical paper [1] have permanently appeared since the time it was published in 1935, certifying the faith of the authors in its more profound destination.

In the present paper an attempt has been made to look at Fock's approach from the point of view of Quantum Geometry and noncommutative differential calculus. This work is a development of the article [9].

There are no grounds to transfer the geometric notions derived from the macroscopic experience to small (microscopic) distances. From the philosophical point of view this question has been discussed since ancient times. The term

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”*αμερος*” has been introduced simultaneously with ”*ατομος*” and denotes the smallest, indivisible portion of the space.

The suggestions to consider geometry as a subject of quantization naturally appeared almost simultaneously with quantum theory itself. So, Quantum Geometry (QG) is as old as quantum theory itself. One of the first QG models was suggested by H.Snyder [2,3]. In this approach usual commuting position operators were changed for the noncommuting quantities of a concrete form. W.Pauli [3] stressed that new Snyder’s coordinate operators could be considered as boosts of the momentum space of constant curvature (De Sitter or Anti De Sitter momentum space). In other words, the Snyder quantization of space-time is based on the substitution of the pseudo-Euclidean geometry of the momentum space by the De Sitter geometry.

To be more exact, actually Snyder didn’t use the connection of his quantum coordinates with generators of the isometry group of the curved momentum space (see [3]). The idea to consider the momentum space with non-Euclidean geometry as a cornerstone of the theory with quantum space-time belongs to Yu.A.Gelfand [5] and was developed by I.E.Tamm [6] and others [7–12].

Such a change of the geometry of the momentum space leads to the modification of the procedure of extension of the S matrix off the mass shell [11], i.e., to the different dynamical description. In fact, the statement on the geometry of the momentum space off the mass shell is an additional axiom of quantum field theory (QFT). Actually in the standard QFT, this axiom is accepted without saying. In the nonrelativistic theory, the extension off the energy shell on the ground of Schrödinger equation in the momentum space and Lippmann–Schwinger equation must be considered.

We can think that some background interaction exists, which modifies the geometry of the momentum space. See in this connection Ref.8. As we have stressed above, all other axioms are fulfilled, including the standard translation invariance. The last means that there are relative coordinates (properly defined) which are the subjects of quantization [11]. In consequence of the change of the geometry of the p space, the space-time becomes quantum (noncommutative).

We stress that physical meaning of the geometry and topology of the momentum space has not obtained clear physical interpretation yet. The space-time groups considered in QFT as the covariance groups are the isometry groups of space-time.

It is worth mentioning a series of recent papers [14–16] where it has been shown that the curved momentum space and the corresponding Snyder-like quantum space naturally arise when considering the 2+1 model of gravity interacting with the scalar field. (The canonical momenta belong to the hyperboloid in four-dimensional projective hyperboloid.)

The explicit character of Snyder’s approach to space-time quantization has a remarkable consequence: we can define the spectrum of a commutative set

of operators constructed from \hat{x}_μ and other generators of the De Sitter group. As was shown in [11], the formulation of the generalized causality condition and QFT in terms of the points of this new numerical quantum space-time is as comprehensive procedure as it is in the usual QFT with the Minkovskian space-time. In this approach the structure of the singular field-theoretic functions is entirely reconstructed as compared to the standard QFT, and the corresponding perturbation theory is free of ultraviolet divergences.

In the present paper, we shall use Fock's theory of hydrogen atom symmetry to show that it is in fact the realization of the picture described above in the nonrelativistic case: The Coulomb field fulfills the role of the background interaction mentioned above, which provides the non-Euclidean geometry of momentum space. The modified shifts of the last (which are up to some similarity transformation the Runge–Lenz vector's components) can be considered as nonrelativistic analogs of Snyder's coordinates (2).

The paper is organized as follows. In Section 2, we consider Snyder's theory. In Section 3, we recall briefly the necessary moments of Fock's theory of hydrogen atom. Sections 4, 5 are devoted to the analysis of the spectrum and matrix elements of unitary irreducible representations of the isometry group of momentum space and the interpretation of the spectrum as quantum space (QS) is given. It is shown that the Schrödinger equation in QS is a differential-difference equation with the increment equal to Bohr's radius. In Section 6, an introduction to the noncommutative differential calculus with impact to apply it to the Schrödinger equation in QS is given. Section 7 contains the theory of the Schrödinger equation in QS as a noncommutative differential equation in QS. In the last 8th Section an example of integrable case (q-oscillator) of the generalized Schrödinger equation is given.

2. SNYDER APPROACH

In this approach the usual quantum mechanical coordinate operators

$$x_\mu = i\hbar \frac{\partial}{\partial p^\mu}, \quad \mu = 0, 1, 2, 3, \quad g_{\mu\nu} = \text{diag}(1, -1, -1, -1), \quad (1)$$

i.e., the generators of translations of the Minkovski momentum space, are substituted by Snyder quantum coordinates \hat{x}_μ , i.e., the generators of De Sitter boosts:

$$\hat{x}_\mu = il_0 \left(p^4 \frac{\partial}{\partial p^\mu} - p_\mu \frac{\partial}{\partial p^4} \right), \quad [\hat{x}_\mu, \hat{x}_\nu] = -il_0^2 \hat{M}_{\mu\nu}, \quad (2)$$

where l_0 , the "fundamental length", indicates the scale at which the effects of QG become appreciable. A plausible candidate for this role is Planck's length $l_{Planck} = \sqrt{\frac{c\hbar}{G}}$.

3. FOCK'S THEORY OF HYDROGEN ATOM

The Schrödinger equation for Hydrogen atom (we limit ourselves with the case of continuum part of spectrum)

$$H\psi(\vec{x}) = \left(E_p - \frac{e^2}{|\vec{x}|} \right) \psi(\vec{x}) = E_q \psi(\vec{x}) = \frac{q^2}{2\mu} \psi(\vec{x}), \quad (3)$$

where $E_p = \frac{p^2}{2\mu}$, $E_q = \frac{q^2}{2\mu}$, in the momentum representation takes the form

$$(p^2 - q^2) \psi(\vec{p}) = \frac{\hbar}{\pi^2 a} \int \frac{d^3 p'}{|\vec{p}' - \vec{p}|^2} \psi(\vec{p}'), \quad (4)$$

where a is the atomic unit of length (Bohr's radius)

$$a = \frac{\hbar^2}{\mu e^2}. \quad (5)$$

Recall that atomic units of energy, momentum and time are correspondingly

$$e_a = \frac{\mu e^4}{\hbar^2}, \quad \pi_a = \frac{\mu e^2}{\hbar}, \quad \tau_a = \frac{\hbar^3}{\mu e^4}. \quad (6)$$

Following V.A.Fock [1] we take into account the non-Euclidean geometry of momentum space, naturally arising here. For the continuous part of the energy spectrum it is pseudo-Euclidean 3-dimensional space of negative curvature (Lobachevsky space). Introducing 4-dimensional projective momenta P_μ ($\mu = 0, 1, 2, 3$) we have

$$\vec{P} = \frac{2q \vec{p}}{\vec{p}^2 - q^2}, \quad P_0 = \frac{\vec{p}^2 + q^2}{\vec{p}^2 - q^2}, \quad q = \sqrt{2\mu E}, \quad (7)$$

$$P_0^2 - \vec{P}^2 = 1, \quad (8)$$

or inverse relation

$$\vec{p} = \frac{q \vec{P}}{P_0 - 1}, \quad p^2 = q^2 \frac{P_0 + 1}{P_0 - 1}. \quad (9)$$

Equation (8) describes the two-sheet hyperboloid (the upper sheet corresponds to $1 \leq P_0 < +\infty$; the lower pole, to $-\infty < P_0 \leq -1$).

It is convenient to use the hyperspherical coordinates

$$\begin{aligned} \vec{P} &= \sinh \alpha \vec{n}, \quad P_0 = \pm \cosh \alpha, \\ \vec{n} &= \frac{\vec{P}}{|\vec{P}|} = (\sin \theta \cos \phi, \sin \theta \sin \phi, \cos \theta). \end{aligned} \quad (10)$$

The volume element in the Lobachevsky space is

$$d\Omega_P = \frac{d^3 P}{|P_0|} = \sinh^2 \alpha \sin \theta d\alpha d\theta d\phi. \quad (11)$$

The distance $s(P, P')$ of two points of the Lobachevsky space (8) is given by

$$\cosh s(P, P') = \left| 1 - \frac{(P - P')^2}{2} \right|. \quad (12)$$

In terms of P_μ equation (4) takes the form

$$\Phi_r(P) = \frac{r}{2\pi^2 a} \int \frac{d\Omega_{P'}}{(P - P')^2} \Phi_r(P'), \quad (13)$$

where

$$\Phi_r(P) = (p^2 - q^2)^2 \psi(\vec{p}), \quad r = \frac{\hbar}{q}. \quad (14)$$

This equation is manifestly invariant under the group of motions of the Lobachevsky momentum space which is Lorentz group. The generators of the Lorentz group boosts

$$\hat{x}_i = -i \left[P_0 \frac{\partial}{\partial P^i} - P_i \frac{\partial}{\partial P^0} \right] \quad (15)$$

up to some similarity transformation coincide with the additional integrals of motion of the Coulomb problem, i.e., Runge–Lenz invariants. From the other side their similarity to Snyder coordinates (2) is evident.

4. WAVE FUNCTIONS AND DIFFERENTIAL-DIFFERENCE SCHRÖDINGER EQUATION IN QUANTUM SPACE

As was shown by V.A.Fock [1] the solutions of the Schrödinger equation in momentum space (13) are the eigen-functions of the Laplace–Beltrami operator on the Lobachevsky space (8), or the Casimir operator of the Lorentz group:

$$\left(\vec{x}^2 - \frac{1}{\hbar^2} \vec{L}^2 \right) \Phi_r(P) = \left(1 + \frac{r^2}{a^2} \right) \Phi_r(P), \quad (16)$$

where \vec{L} is the vector of angular momentum operators. The solutions of (16) are the matrix elements of unitary irreducible (infinite dimensional) representations of the Lorentz group. For the principal series of unitary representations of the Lorentz group the parameter r runs over the interval

$$0 \leq r < \infty \quad (17)$$

which coincides with the physically admissible region of variation (see (14)). From the other side the interval (17) is the range of variation of the relative distance in the three-dimensional Euclidean space. Taking these properties of r into account we interpret the parameter r as the relative distance in the quantum space [8, 10].

Let us consider the quantities

$$\Phi_r(P) = \langle \vec{r} | \vec{P} \rangle = \left| P_0 - \vec{P} \vec{n}_r \right|^{-1 - i \frac{r}{a}}, \quad (18)$$

where

$$\vec{r} = r \vec{n}_r, \quad \vec{n}_r^2 = 1. \quad (19)$$

Expression (18) from the one hand is the solution of the equation (16), from the other hand it is the generating function for the radial solutions of the Schrödinger equation in the momentum space (13). The expression (18) plays the role of the plane wave in quantum r space. The radial solutions of the Schrödinger equation in the momentum space can be obtained from the expansion in spherical harmonics:

$$\langle \vec{r} | \vec{P} \rangle = 4\pi \sum_{l=0}^{\infty} \sum_{m=-l}^{m=l} i^l \langle \rho, l, | \alpha \rangle \langle \vec{n}_r | l, m \rangle \langle l, m | \vec{n} \rangle, \quad (20)$$

where ρ is the dimensionless parameter:

$$\rho = \frac{r}{a} \quad (21)$$

and

$$\langle l, m | \vec{n} \rangle = \langle l, m | \theta, \phi \rangle = Y_{lm}(\vec{n}), \quad \langle \vec{n} | l, m \rangle = Y_{lm}^*(\vec{n}). \quad (22)$$

The functions $\langle \rho, l | \alpha \rangle$ are the radial wave functions of the Coulomb problem in the momentum space. They can be obtained also as the Fourier transforms of the radial solutions in configurational space. We present the different representations of $\langle \rho, l | \alpha \rangle$ in terms of Legendre functions $P_{i\rho - \frac{1}{2}}^{-l - \frac{1}{2}}(\cosh \alpha)$:

$$\langle \rho, l | \alpha \rangle = (-i)^l \sqrt{\frac{\pi}{2 \sinh \alpha}} \frac{\Gamma(i\rho + l + 1)}{\Gamma(i\rho + 1)} P_{i\rho - \frac{1}{2}}^{-l - \frac{1}{2}}(\cosh \alpha), \quad (23)$$

the Gegenbauer functions $C_{i\rho-l-1}^{l+1}(\cosh \alpha)$:

$$\langle \rho, l | \alpha \rangle = (-i)^l \sqrt{\frac{\pi}{2 \sinh \alpha}} \left(\frac{\sinh \alpha}{2} \right)^{l+\frac{1}{2}} \frac{\Gamma(2l+2)\Gamma(i\rho-l)}{\Gamma(i\rho+1)\Gamma(l+\frac{3}{2})} C_{i\rho-l-1}^{l+1}(\cosh \alpha), \quad (24)$$

and hypergeometric function ${}_2F_1(\alpha, \beta; \gamma; z)$:

$$\langle \rho, l | \alpha \rangle = (-i)^l \sqrt{\frac{\pi}{2 \sinh \alpha}} \left(\frac{\sinh \alpha}{2} \right)^{l+\frac{1}{2}} \frac{\Gamma(i\rho+l+1)}{\Gamma(i\rho+1)\Gamma(l+\frac{3}{2})} \times \quad (25)$$

$$\times {}_2F_1\left(i\rho+l+1, -i\rho+l+1; l+\frac{3}{2}; -\sinh^2 \frac{\alpha}{2}\right) =$$

$$= (-i)^l \sqrt{\frac{\pi}{2 \sinh \alpha}} \left(\frac{\sinh \alpha}{2} \right)^{l+\frac{1}{2}} \frac{\Gamma(i\rho+l+1)}{\Gamma(i\rho+1)\Gamma(l+\frac{3}{2})} e^{\alpha(i\rho-l-1)} \times \quad (26)$$

$$\times {}_2F_1(-i\rho+l+1, l+1; 2l+2; 2e^{-\alpha} \sinh \alpha) =$$

$$= \sqrt{\frac{\pi}{2}} \left(-i \frac{\sinh \alpha}{2} \right)^l \frac{\Gamma(i\rho+l+1)}{\Gamma(i\rho+1)\Gamma(l+\frac{3}{2})} \times \quad (27)$$

$$\times {}_2F_1\left(\frac{i\rho+l+1}{2}, \frac{-i\rho+l+1}{2}; l+\frac{3}{2}; -\sinh^2 \alpha\right).$$

These different representations are convenient for performing the contraction limit when we consider in the next section the correspondence with usual (non-quantum) space limit. The following orthogonality and completeness conditions for the radial solutions are valid

$$\frac{2}{\pi} \int_0^\infty \sinh^2 \alpha \, d\alpha \langle \rho, l | \alpha \rangle \langle \alpha | \rho', l \rangle = \frac{\delta(\rho - \rho')}{\rho^2}, \quad (28)$$

$$\frac{2}{\pi} \int_0^\infty \rho^2 \, d\rho \langle \alpha | \rho, l \rangle \langle \rho, l | \alpha' \rangle = \frac{\delta(\alpha - \alpha')}{\sinh^2 \alpha}, \quad (29)$$

and corresponding conditions for the plane waves (18):

$$\frac{1}{(2\pi)^3} \int \langle \vec{r} | \vec{P} \rangle \langle \vec{P} | \vec{r}' \rangle d\Omega_P = \delta(\vec{r} - \vec{r}'), \quad (30)$$

$$\frac{1}{(2\pi)^3} \int \langle \vec{P} | \vec{r} \rangle \langle \vec{r} | \vec{P}' \rangle d^3r = \delta(\vec{P} - \vec{P}') P_0. \quad (31)$$

The plane wave in quantum space and its radial part obey the following equations off the energy shell, i.e., for $E_p \neq E_q$ or $p \neq q$:

$$\begin{aligned} e_a \left[\cosh\left(i\frac{\partial}{\partial\rho}\right) + \frac{i}{\rho} \sinh\left(i\frac{\partial}{\partial\rho}\right) - \frac{\Delta_{\theta,\phi}}{\rho^2} e^{i\frac{\partial}{\partial\rho}} - 1 \right] \langle \vec{r} | \vec{P} \rangle = \\ = \hat{H}_0 \langle \vec{r} | \vec{P} \rangle = E_P \langle \vec{r} | \vec{P} \rangle, \end{aligned} \quad (32)$$

$$\begin{aligned} e_a \left[\cosh\left(i\frac{\partial}{\partial\rho}\right) + \frac{i}{\rho} \sinh\left(i\frac{\partial}{\partial\rho}\right) - \frac{l(l+1)}{\rho^2} e^{i\frac{\partial}{\partial\rho}} - 1 \right] \langle \rho, l | \alpha \rangle = \\ = \left(\hat{H}_{0l} - E_P \right) \langle \rho, l | \alpha \rangle, \end{aligned} \quad (33)$$

where $E_P = e_a(|P_0| - 1) = 2e_a \sinh^2 \frac{\alpha}{2}$, $e_a = \frac{\mu e^4}{\hbar^2}$ is the atomic unit of energy. The strong argument for the idea that plane wave $\langle \vec{r} | \vec{P} \rangle$ describes the free motion in the quantum r space is the existence of three more differential-difference operators \hat{p}_i for which $\langle \vec{r} | \vec{P} \rangle$ is the eigenfunction with eigenvalues equal to the momentum components

$$\hat{p}_i \langle \vec{r} | \vec{P} \rangle = p_i \langle \vec{r} | \vec{P} \rangle, \quad (34)$$

where

$$\begin{aligned} \hat{p}_1 = \pi_a \left\{ \sin\theta \cos\phi \left(e^{i\frac{\partial}{\partial\rho}} - \hat{H}_0 \right) - i \left(\frac{\cos\theta \cos\phi}{\rho} \frac{\partial}{\partial\theta} - \frac{\sin\phi}{\rho \sin\theta} \frac{\partial}{\partial\phi} \right) e^{i\frac{\partial}{\partial\rho}} \right\} \\ \hat{p}_2 = \pi_a \left\{ \sin\theta \sin\phi \left(e^{i\frac{\partial}{\partial\rho}} - \hat{H}_0 \right) - i \left(\frac{\cos\theta \sin\phi}{\rho} \frac{\partial}{\partial\theta} + \frac{\cos\phi}{\rho \sin\theta} \frac{\partial}{\partial\phi} \right) e^{i\frac{\partial}{\partial\rho}} \right\} \\ \hat{p}_3 = \pi_a \left\{ -\cos\theta \left(e^{i\frac{\partial}{\partial\rho}} - \hat{H}_0 \right) + i \frac{\sin\theta}{\rho} \frac{\partial}{\partial\theta} e^{i\frac{\partial}{\partial\rho}} \right\} \end{aligned} \quad (35)$$

$\pi_a = \frac{\mu e^2}{\hbar}$ is the atomic unit of momentum.

It looks now quite natural to make the next step and introduce the interaction term $V(r)$ into the free differential-difference Schrödinger equation in quantum space (32). From the usual point of view this corresponds to some perturbations for the Coulomb potential. We stress that there are even integrable cases for such differential-difference equations (see [13] and references therein).

5. CONTRACTION

The important requirement to the theory with curved momentum space is its correspondence with the usual theory. In the physical regime, when we can neglect the effects of curvature, all relations must go over into the usual ones. Let us first analyse this problem in the momentum space (7). The vicinities of the tops of both poles

$$P_0 \approx \pm 1 \quad (36)$$

carry the flat geometry. For these regions we are in the regime when the Inönü–Wigner contraction [17] is an actual approach. For example Snyder's quantum coordinate operators (15) go over into usual coordinate operators (1) in these regions.

The tops of the hyperboloid (8) are:

$$P_0 = 1 \text{ or } q \cong 0, \text{ i.e. } \rho \gg \frac{\pi_a}{|\vec{p}|}, \quad (37)$$

$$P_0 = -1 \text{ or } q \rightarrow \infty, \text{ i.e. } |\vec{p}| \ll \frac{\pi_a}{\rho}. \quad (38)$$

In classical physics the small p -s correspond to great impact parameters. In this regime the scattered particle slightly feels the Coulomb field. In the case of the bound states we must consider the orbits corresponding to big values of principal quantum number n . In the contraction limit all finite-difference relations reduce to standard differential relations of Quantum Mechanics. For example the differential-difference operators of momentum (35) reduce to usual momentum operators

$$\hat{p}_i e^{\frac{i\vec{r}\vec{p}}{\hbar}} = -i\hbar \frac{\partial}{\partial x_i} e^{\frac{i\vec{r}\vec{p}}{\hbar}} = p_i e^{\frac{i\vec{r}\vec{p}}{\hbar}}, \quad (39)$$

and the plane wave (18) converts to usual exponential function:

$$\begin{aligned} & \left| P_0 - \vec{P}\vec{n}_r \right|^{-1-i\frac{r}{a}} \\ \langle \vec{r} | \vec{P} \rangle &= \exp \left\{ - \left(1 + i\frac{r}{a} \right) \ln \left(P_0 - \vec{P}\vec{n}_r \right) \right\} \approx \\ & \approx \exp \left\{ - \left(1 + i\frac{r}{a} \right) \ln \left(1 - \vec{P}\vec{n}_r + \dots \right) \right\} \approx \\ & \approx \exp \left\{ i\frac{r}{a} \vec{P}\vec{n}_r + \dots \right\} \approx e^{i\frac{\vec{P}\vec{x}}{\hbar}}. \end{aligned} \quad (40)$$

6. NONCOMMUTATIVE DIFFERENTIAL CALCULUS AND FINITE-DIFFERENCE DERIVATIVES

We start this section with an historical remark. Referring to the second Snyder's paper on quantum space (the second paper of [2]) we invite the reader to convince that his generalization of Maxwell's equations for the case of QG is based, in fact, on a version of the noncommutative differential calculus.

Let us show that the finite-difference Schrödinger equation (33) is naturally described in terms of noncommutative differential calculus [18–23]. This calculus can be naturally and most easily introduced on a ground of the theory of differential forms as its deformation. We shall limit ourselves with the differential calculus over the associative algebra A over \mathbb{R} or \mathbb{C} . In our case the necessity to consider an algebra over \mathbb{C} follows from the form of the finite-difference Schrödinger equation, containing shifts by the imaginary quantity ia . This is general property of the finite-difference Schrödinger equation (33) corresponding to the continuous part of the spectrum of hydrogen atom, requiring to consider the wave functions in the complex ρ plane. Finite linear combinations of elements of A and finite products are again elements of A . The multiplication is associative. A differential calculus on A is a \mathbb{Z} -graded associative algebra over \mathbb{C}

$$\Omega(A) = \sum_{r=0} \oplus \Omega^r(A), \quad (41)$$

$$\Omega^0(A) = A, \quad \Omega^r(A) = \{0\} \forall r < 0. \quad (42)$$

The elements of $\Omega^r(A)$ are called r forms. There exist an exterior derivative operator d which satisfies the following conditions

$$d^2 = 0 \quad (43)$$

and

$$d(\omega\omega') = (d\omega)\omega' + (-1)^r\omega d\omega', \quad (44)$$

where ω and ω' are r and r' forms, respectively. A is the commutative algebra generated by the coordinate functions x^i , $i = 1, \dots, n$. In the standard differential calculus on usual manifolds differentials commute with functions:

$$[x^i, dx^j] = 0, \quad i, j = 1, \dots, n \quad (45)$$

in terms of real coordinates x^i . For us it is essential that (45) can be generalized (deformed) in different ways with (41)–(44) still true. Let us consider in more detail the deformation of (45) of the form

$$[x^i, x^j] = 0, \quad (46)$$

$$[x^i, dx^j] = \sum_{k=1}^n dx^k C^{ij}_k, \quad (47)$$

where the C^{ij}_k are (complex) constants which are constrained by the requirement of a consistent differential calculus.

- Let us apply d to (46) and use (47), this gives

$$d[x^i, x^j] = -[x^j, dx^i] + [x^i, dx^j] = \quad (48)$$

$$= -\sum_{k=1}^n dx^k C^{ji}_k + \sum_{k=1}^n dx^k C^{ij}_k = 0,$$

or

$$C^{ij}_k = C^{ji}_k. \quad (49)$$

This means in particular

$$[x^i, dx^j] = [x^j, dx^i]. \quad (50)$$

The last relation can be proved directly:

$$\begin{aligned} [x^i, dx^j] &= (dx^i) x^j - x^j dx^i = \\ &= \underbrace{d(x^i x^j - x^j x^i)}_{=0} + x^j dx^i - x^i dx^j = [x^j, dx^i]. \end{aligned} \quad (51)$$

- Taking the commutator of dx^i with (46) we obtain

$$\begin{aligned} &[[x^i, dx^j], dx^k] = \\ &= x^i \left\{ dx^k x^j + \sum_{l=1}^n dx^l C^{jk}_l \right\} - x^j \left\{ dx^k x^i + \sum_{l=1}^n dx^l C^{ik}_l \right\} - \\ &\quad - dx^k (x^i x^j - x^j x^i) = \\ &\left\{ dx^k x^i + \sum_{m=1}^n dx^m C^{ik}_m \right\} x^j + \sum_{l=1}^n \left\{ dx^l x^i + \sum_{m=1}^n dx^m C^{il}_m \right\} C^{jk}_l - \quad (52) \\ &\left\{ dx^k x^j + \sum_{m=1}^n dx^m C^{jk}_m \right\} x^i - \sum_{l=1}^n \left\{ dx^l x^j + \sum_{m=1}^n dx^m C^{jl}_m \right\} C^{ik}_l - \end{aligned}$$

$$dx^k (x^i x^j - x^j x^i) = \sum_{l,m=1}^n dx^m (C^{il}_m C^{jk}_l - C^{jl}_m C^{ik}_m) = 0,$$

or

$$\sum_{l=1}^n C^{ik}_l C^{jl}_m = \sum_{l=1}^n C^{jk}_l C^{il}_m, \quad (53)$$

using (49) the last equation can also be written in a form

$$\sum_{l=1}^n C^{k[i}_l C^{j]l}_m = 0. \quad (54)$$

This means that n matrices C^i with entries C^{ik}_l mutually commute.

• Taking the commutator of x^k with (47) also yields (53) and therefore no additional conditions.

Acting with d on (47) and using the Leibniz rule (44) we obtain the classical commutation rule

$$dx^i dx^j = -dx^j dx^i \quad (55)$$

for differentials. The equations obtained by commuting x^k through these relations are identically satisfied.

The Hodge $*$ operator (or duality transformation) for the noncommutative differential forms is introduced by the standard formula

$$* (dx^{i_1} \cdots dx^{i_k}) = \frac{1}{(n-k)!} \sum \epsilon_{i_1 \dots i_k i_{k+1} \dots i_n} dx^{i_{k+1}} \cdots dx^{i_n}. \quad (56)$$

For convenience we shall make difference between right $\vec{*}$ and left $\overleftarrow{*}$ Hodge operators. By definition $\vec{*}$ acts on the forms of the type

$$\sum (dx^{i_1} \cdots dx^{i_k} f(x)), \quad (57)$$

$\overleftarrow{*}$ acts on the forms of the type

$$\sum (f(x) dx^{i_1} \cdots dx^{i_k}) \quad (58)$$

in both cases by the standard formula (56). Action of the operator $\vec{*}$ on the forms of the type (58) and action of the operator $\overleftarrow{*}$ on the forms of the type (57) gives 0. Correspondingly we introduce right and left δ operations

$$\vec{\delta} = \vec{*} d \vec{*}, \quad \overleftarrow{\delta} = \overleftarrow{*} d \overleftarrow{*}. \quad (59)$$

Let A be the algebra of all functions on \mathbb{C} . In what follows we consider the one-dimensional case. It is generated by canonical coordinate function of one

variable $\psi(\rho) = \rho$. One of the simplest deformations of the ordinary differential calculus on A is

$$[d\rho, \rho] = \frac{i}{2} d\rho, \quad (60)$$

where i (in dimensional units ia) is the step in the finite-difference Schrödinger equation. To establish the connection between the noncommutative differential calculus and finite-difference operations (in Schrödinger eq.) is our goal here. This is a special case of the commutation structure (47) considered above. Similar relations are encountered when considering the differential calculus on the lattice [18–23]. Equation (60) can be rewritten in a form

$$d\rho \rho = \left(\rho + \frac{i}{2} \right) d\rho, \quad (61)$$

which can be generalized to the total algebra A as

$$d\rho \psi(\rho) = \psi \left(\rho + \frac{i}{2} \right) d\rho. \quad (62)$$

Then we can introduce the generalized derivatives (left and right) corresponding to our deformed differential calculus. For the left derivative we write

$$d\psi(\rho) = \left(\vec{\partial} \psi(\rho) \right) d\rho. \quad (63)$$

From Leibniz rule (44) we have

$$\begin{aligned} d(\psi(\rho) \varphi(\rho)) &= d\rho \left(\vec{\partial} (\psi(\rho) \varphi(\rho)) \right) = (d\psi(\rho)) \varphi(\rho) + \psi(\rho) (d\varphi(\rho)) = \\ &= d\rho \left(\vec{\partial} \psi(\rho) \right) \varphi(\rho) + \psi(\rho) d\rho \left(\vec{\partial} \varphi(\rho) \right) \end{aligned} \quad (64)$$

after using (62)

$$d(\psi(\rho) \varphi(\rho)) = d\rho \left(\vec{\partial} \psi(\rho) \right) \varphi(\rho) + d\rho \psi \left(\rho + \frac{i}{2} \right) \left(\vec{\partial} \varphi(\rho) \right). \quad (65)$$

Now from the commutativity rule (46)

$$\psi(x) \varphi(x) = \varphi(x) \psi(x) \quad (66)$$

it follows also that equivalent Leibniz rule is valid

$$\begin{aligned} d(f(x) g(x)) &= dx \left(\vec{\partial} (f(x) g(x)) \right) = (dg(x)) f(x) + g(x) (df(x)) = \\ &= dx \left(\vec{\partial} g(x) \right) f(x) + dx g \left(x + \frac{i}{2} \right) \left(\vec{\partial} f(x) \right). \end{aligned} \quad (67)$$

Equalizing (65) and (67) we obtain

$$\left(\overrightarrow{\partial} \psi(\rho)\right) \left[\varphi(\rho + i\frac{\rho}{2}) - \varphi(\rho)\right] = \left(\overrightarrow{\partial} \varphi(\rho)\right) \left[\psi(\rho + \frac{I}{2}) - \psi(\rho)\right], \quad (68)$$

or

$$\frac{\overrightarrow{\partial} \psi(\rho)}{\psi(\rho + \frac{i}{2}) - \psi(\rho)} = \frac{\overrightarrow{\partial} \varphi(\rho)}{\varphi(\rho + \frac{i}{2}) - \varphi(\rho)} = \text{const}, \quad (69)$$

where *const* is the same for any function under differentiation. To calculate this *const* we choose

$$\psi(\rho) = \rho. \quad (70)$$

This gives

$$d\rho = d\rho \left(\overrightarrow{\partial} \rho\right) \implies \left(\overrightarrow{\partial} \rho\right) = 1 \quad (71)$$

and

$$\text{const} = \frac{\overrightarrow{\partial} \rho}{(\rho + \frac{i}{2}) - \rho} = \frac{2}{i}. \quad (72)$$

The ultimate expression for the left partial derivative is

$$\overrightarrow{\partial} \psi(\rho) = \frac{\psi(\rho + \frac{i}{2}) - \psi(\rho)}{\frac{i}{2}}. \quad (73)$$

The expression for the right derivative $\overleftarrow{\partial} \psi(\rho)$ is obtained in a similar way and has the form

$$\overleftarrow{\partial} \psi(\rho) = \frac{\psi(\rho) - \psi(\rho - \frac{i}{2})}{\frac{i}{2}}. \quad (74)$$

7. NONCOMMUTATIVE DIFFERENTIAL CALCULUS AND SCHRÖDINGER EQUATION IN QUANTUM SPACE

In this section we shall apply the noncommutative differential calculus of previous section to the Schrödinger equation (33). Let us exclude the "first finite-difference radial derivative" from this equation making a substitution

$$\psi_l(\rho) = \frac{\langle \rho, l | \alpha \rangle}{\rho}. \quad (75)$$

Recall that similar substitution excludes the first radial derivative in the usual Schrödinger (differential) equation. Taking into account the following rules for finite-difference operations $\sinh\left(i\frac{\partial}{\partial\rho}\right)$ and $\cosh\left(i\frac{\partial}{\partial\rho}\right)$:

$$\sinh i\frac{\partial}{\partial\rho} \psi(\rho) \varphi(\rho) =$$

$$= \sinh i \frac{\partial}{\partial \rho} \psi(\rho) \cosh i \frac{\partial}{\partial \rho} \varphi(\rho) + \cosh i \frac{\partial}{\partial \rho} \psi(\rho) \sinh i \frac{\partial}{\partial \rho} \varphi(\rho) \quad (76)$$

$$\begin{aligned} & \cosh i \frac{\partial}{\partial \rho} \psi(\rho) \varphi(\rho) = \\ & = \cosh i \frac{\partial}{\partial \rho} \psi(\rho) \cosh i \frac{\partial}{\partial \rho} \varphi(\rho) + \sinh i \frac{\partial}{\partial \rho} \psi(\rho) \sinh i \frac{\partial}{\partial \rho} \varphi(\rho) \end{aligned} \quad (77)$$

and relations

$$\cosh i \frac{\partial}{\partial \rho} \frac{1}{\rho} = \frac{\rho}{\rho^2 + 1}, \quad \sinh i \frac{\partial}{\partial \rho} \frac{1}{\rho} = \frac{-i}{\rho^2 + 1}, \quad (78)$$

we obtain

$$\begin{aligned} & e_a \left[\cosh \left(i \frac{\partial}{\partial \rho} \right) - \frac{l(l+1)}{2\rho} e^{i \frac{\partial}{\partial \rho}} \frac{1}{\rho} - 1 + V(\rho) - E_P \right] \psi_l(\rho) = \\ & = e_a \left[2 \sinh^2 \left(\frac{i}{2} \frac{\partial}{\partial \rho} \right) - \frac{l(l+1)}{2\rho} e^{i \frac{\partial}{\partial \rho}} \frac{1}{\rho} + V(\rho) - E_P \right] \psi_l(\rho) = \\ & = \left(\rho \hat{H}_{0l} \frac{1}{\rho} + V(\rho) - E_P \right) \psi_l(\rho) = 0. \end{aligned} \quad (79)$$

Let us consider the expression

$$\begin{aligned} & \frac{1}{2} \left(\overrightarrow{\delta} d + \overleftarrow{\delta} d \right) \psi(\rho) = \frac{1}{2} \left(\overrightarrow{\delta} + \overleftarrow{\delta} \right) d \psi(\rho) = \frac{1}{2} \left(\overrightarrow{*} d \overrightarrow{*} d + \overleftarrow{*} d \overleftarrow{*} d \right) \psi(\rho) = \\ & = \frac{1}{2} \left(\overrightarrow{\delta} + \overleftarrow{\delta} \right) \left(\left(\overrightarrow{\partial} \psi(\rho) \right) d\rho + d\rho \left(\overleftarrow{\partial} \psi(\rho) \right) \right) = \\ & = \frac{1}{2} \left(\overrightarrow{\delta} + \overleftarrow{\delta} \right) \left[\left(\frac{\psi(\rho + \frac{i}{2}) - \psi(\rho)}{\frac{i}{2}} \right) d\rho + d\rho \left(\frac{\psi(\rho) - \psi(\rho - \frac{i}{2})}{\frac{i}{2}} \right) \right] = \\ & = \frac{1}{2} \left(\overrightarrow{*} + \overleftarrow{*} \right) d \left[\frac{\psi(\rho + \frac{i}{2}) - \psi(\rho - \frac{i}{2})}{\frac{i}{2}} \right] = \quad (80) \\ & - \left(\overrightarrow{*} + \overleftarrow{*} \right) \left[\left(\psi(\rho + i) - \psi(\rho) - \psi(\rho + \frac{i}{2}) + \psi(\rho - \frac{i}{2}) \right) d\rho + \right. \\ & \left. + d\rho \left(\psi(\rho - i) - \psi(\rho) + \psi(\rho + \frac{i}{2}) - \psi(\rho - \frac{i}{2}) \right) \right] = \\ & = -2 \left(\cosh i \frac{\partial}{\partial \rho} - 1 \right) \psi(\rho) = -4 \sinh^2 \frac{i}{2} \frac{\partial}{\partial \rho} \psi(\rho). \end{aligned}$$

Using this result we can define left and right momentum operators in the form

$$\vec{p} = -i\hbar \vec{*} d, \quad \overleftarrow{p} = -i\hbar \overleftarrow{*} d \quad (81)$$

and free particle momentum operator

$$\hat{p} = \frac{1}{2} (\vec{p} + \overleftarrow{p}), \quad (82)$$

where \vec{p} is not a vector but the right momentum operator! So, the first term in (79) takes the form

$$\frac{\hat{p}^2}{2\mu} = \frac{1}{2\mu} \left(-2\pi_0 \sinh \frac{i}{2} \frac{\partial}{\partial \rho} \right)^2. \quad (83)$$

The centrifugal term in (79) can be written in the form

$$\frac{l(l+1)}{\rho(\rho+i)} e^{i\frac{\partial}{\partial \rho}} = \frac{1}{\rho} \frac{1}{\lambda} \frac{1}{\rho}, \quad (84)$$

where

$$\vec{\lambda} = \sqrt{l(l+1)} \left(1 - \frac{\vec{p}}{2\pi_0} \right). \quad (85)$$

Ultimately we obtain the Schrödinger equation in quantum space in terms of noncommutative differential calculus as

$$H\psi_l(\rho) = \left(\frac{\hat{p}^2}{2\mu} + \frac{1}{\rho} \frac{1}{\lambda} \frac{1}{\rho} + V(\rho) \right) \psi_l(\rho) = E_P \psi_l(\rho). \quad (86)$$

8. LINEAR OSCILLATOR = q -OSCILLATOR IN QUANTUM SPACE

From the usual point of view the interaction term $V(\rho)$ in the differential-difference Schrödinger equation (32) corresponds to the perturbed Coulomb potential. Let us consider an example of integrable case for the Schrödinger equation with interaction. We write the ladder operators

$$\begin{aligned} a^\pm &= \mp \frac{i}{\sqrt{2}\pi_a \cos \frac{r}{2l_0}} e^{\pm \frac{1}{2} \left(\frac{r}{\lambda_0} \right)^2} \left(\hat{p} \right) e^{\mp \frac{1}{2} \left(\frac{r}{\lambda_0} \right)^2} \\ &= \pm \frac{i\sqrt{2}}{\cos \frac{r}{2l_0}} e^{\pm \frac{1}{2} \left(\frac{r}{\lambda_0} \right)^2} \left(\sinh \frac{ia}{2} \frac{\partial}{\partial r} \right) e^{\mp \frac{1}{2} \left(\frac{r}{\lambda_0} \right)^2}, \end{aligned} \quad (87)$$

\hat{p} is the noncommutative differential operator of radial momentum introduced in previous section, ω is the frequency, λ_0 is a parameter of dimension of length:

$$\lambda_0 = \sqrt{\frac{\hbar}{\mu\omega}}. \quad (88)$$

The ladder operators (87) obey the deformed commutation relation

$$[a^-, a^+]_q = qa^-a^+ - q^{-1}a^+a^- = 2(q^{-1} - q), \quad (89)$$

which guarantees the exact solubility of this finite-difference problem, q is a dimensionless quantity, parameter of deformation, which is expressed in terms of physical parameters:

$$q = e^{-\frac{a^2}{4\lambda_0^2}} = e^{-\frac{\hbar\omega}{4e_a}} = e^{-\frac{\omega\hbar^3}{4\mu e^4}}. \quad (90)$$

We introduce the Hamiltonian

$$\hat{H} = \frac{1}{2} \{a^-, a^+\}_q = \frac{1}{2} \{qa^-a^+ + q^{-1}a^+a^-\} \quad (91)$$

obeying deformed commutation relations with ladder operators

$$\left[a^\pm, \hat{H} \right]_{q^{\mp 1}} = \pm (q^2 - q^{-2}) a^\pm, \quad (92)$$

which in fact guarantee the integrability, and obtain the energy spectrum

$$E_n = 2e_a \left(e^{\frac{\hbar\omega}{2e_a}(n+\frac{1}{2})} - \cosh \frac{\hbar\omega}{4e_a} \right). \quad (93)$$

This integrable case can be easily identified with the well-known q oscillator.

REFERENCES

1. **Fock V.A.** — Zs. Phys., 1935, v.98, p.145.
2. **Snyder H.** — Phys. Rev., 1947, v.71, p.38; *ibid* 1947, v.72, p.68.
3. **Pauli W.**, unpublished, see footnote in the first Snyder's paper.
4. **Yang C.N.** — Phys. Rev., 1947, v.72, p.874.
5. **Gelfand Yu.A.** — Sov. Phys. JETP, 1963, v.16, p.184; *ibid* 1964, v.17, p.848.
6. **Tamm I.E.** — Collection of Scientific Works, Nauka Publishers, Moscow, 1975.
7. **Kadyshevsky V.G.** — Sov. Phys. JETP, 1961, v.14, p.1340; Sov. Phys. Doklady, 1963, v.7, p.235; *ibid*, 1963, v.7, p.1031.

8. **Mir-Kasimov R.M.** — Sov. Phys. JETP, 1966, v.22, p.629; *ibid*, 1966, v.22, p.807.
9. **Mir-Kasimov R.M.** — Sov. Phys. JETP, 1967, v.25, p.348.
10. **Kadyshevsky V.G., Mir-Kasimov R.M., Skachkov N.B.** — Nuovo Cimento, 1968, v.55A, p.233.
11. **Donkov A., Kadyshevsky V.G., Mateev M., Mir-Kasimov R.M.** — In: Proceedings of V.A.Steklov Mathematical Institute, CXXXVI, 1975, p.85.
12. **Kadyshevsky V.G., Fursaev D.V.** — Theor. and Math. Phys, 1990, v.83, p.197.
13. **Mir-Kasimov R.M.** — Phys. Lett. B, 1991, v.259, p.79; J. Phys., 1991, v.A24, p.4283; Phys. Lett. B, 1996, v.378, p.181; Turkish Journal of Physics, 1997, v.21, p.472; Int. Journ. Mod. Phys., 1997, v.12, No.1, p.24; Yadernaya Fizika, (Physics of Atomic Nuclei), 1998, v.61, No.11, p.1951.
14. **Deser S., Jackiw R., 't Hooft G.** — Ann. Phys., 1988, v.117, p.685.
15. **Witten E.** — Nucl. Phys., 1988, v.B311, p.46.
16. **'t Hooft G.** — Class. Quant. Grav., 1996, v.13, p.1023.
17. **Inönü E., Wigner E.** — Nuovo Cimento, 1952, v.IX, p.705.
18. **Woronowich S.L.** — Commun. Math. Phys., 1989, v.122, p.125.
19. **Wess J., Zumino B.** — Nucl. Phys. (Proc.Suppl.), 1990, v.B18, p.302.
20. **Dubois-Violette M., Kerner R., Madore J.** — Journ. Math. Phys., 1990, v.31, p.323.
21. **Connes A.** — Noncommutative Geometry, Academic Press, 1994.
22. **Madore J.** — An Introduction to Noncommutative Geometry and Its Physical Applications, Cambridge University Press, 1995.
23. **Dimakis A., Müller-Hoissen F.** — Journ. Math. Phys., 1998, v.40, p.1518.