

## COOPERATIVE ELECTROMAGNETIC EFFECTS

*V.I. Yukalov, E.P. Yukalova*

Joint Institute for Nuclear Research, Dubna  
and University of Western Ontario, London, Canada

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*V.I. Yukalov, E.P. Yukalova*

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Collective phenomena in strongly nonequilibrium systems interacting with electromagnetic field are considered. Such systems are described by complicated nonlinear differential or integro-differential equations. The aim of this review is to show that many nonlinear collective phenomena can be successfully treated by a recently developed method called the Scale Separation Approach whose name is due to the idea of separating different characteristic space-time scales existing in nonequilibrium statistical systems. This approach is rather general and can be applied to various nonequilibrium physical problems, several of which are discussed here. The problems considered not only serve as illustrations of the method but are quite important by themselves presenting interesting physical effects, such as Collective Liberation of Light, Turbulent Photon Filamentation, Superradiant Spin Relaxation, Negative Electric Current, and Magnetic Semiconfinement of Atoms.

Исследуются коллективные явления в сильнонеравновесных системах, взаимодействующих с электромагнитным полем. Такие системы описываются сложными нелинейными дифференциальными или интегродифференциальными уравнениями. Цель данного обзора — показать, что многие нелинейные коллективные явления могут быть успешно описаны с помощью развитого недавно метода, названного методом разделения масштабов, основанного на идее выделения различных характерных пространственно-временных масштабов, существующих в неравновесных статистических системах. Этот подход обладает достаточной общностью и может быть применен к всевозможным неравновесным физическим проблемам, некоторые из которых рассматриваются в данном обзоре. Рассмотренные проблемы не только играют роль иллюстрации метода, но важны и сами по себе, представляя интересные физические эффекты, такие, как коллективное освобождение света, турбулентное фотонное нитеобразование, сверхизлучательная спиновая релаксация, отрицательный электрический ток и магнитный полуконфайнмент атомов.

### 1. INTRODUCTION

Strongly nonequilibrium cooperative processes that occur in statistical systems interacting with electromagnetic field are described by complicated nonlinear differential and integro-differential equations. For treating such difficult problems, a general approach has been recently developed called the *Scale Separation Approach* whose basic idea is to present the evolution equations in such a form where it could be possible to separate several characteristic space-time scales. In many cases, different scales appear rather naturally being directly related to the physical properties of the considered system.

Since the scale separation approach makes the mathematical foundation for the following applications, we start the review with presenting the basic techniques of this approach. Then we demonstrate it by applying the method to

different physical problems related to strongly nonequilibrium processes occurring under the interaction of electromagnetic field with matter. The considered examples not only serve as illustrations of the method but are of importance as such since they concern interesting and rather nontrivial physical effects. For consideration, those effects are chosen that have been first correctly described or predicted by the authors. Among these effects, we would like to emphasize, as the most interesting, the following: *Collective Liberation of Light*, *Turbulent Photon Filamentation*, *Superradiant Spin Relaxation*, *Negative Electric Current*, and *Magnetic Semiconfinement of Atoms*.

The content of the report is as follows. In Section 2 the *Scale Separation Approach* is described. This method makes it possible to solve, or to strongly simplify, many complicated systems of nonlinear differential equations, including stochastic and partial-derivative equations. The mathematical procedure of solving nonlinear differential equations in the following applications is based on this approach. The examples we consider have mainly to do with the evolution equations describing strongly nonequilibrium statistical systems interacting with electromagnetic fields. We concentrate our attention on collective phenomena whose existence as such, as well as their properties, are due to nonlinear effects. This is why we constantly have to deal with nonlinear equations.

Resonant interactions of electromagnetic field with radiating systems are usually described by the Maxwell–Bloch equations, in which one often passes to the momentum representation by means of Fourier transform. But we prefer to work in the *Real-Space representation*, outlined in Section 3, which seems to be more convenient for employing the Scale Separation Approach. Another convenient trick we employ is the elimination of electromagnetic field from evolution equations. For this purpose, the operator Maxwell equations, supplemented by the Coulomb calibration, can be rewritten in the integral form connecting the vector potential with the retarded current formed by the radiating system. Substituting this vector potential into evolution equations eliminates from them electromagnetic field. In this way, we come to the system of equations not containing explicitly electromagnetic field, instead of which there appears an effective dipole interaction of radiating atoms. After eliminating electromagnetic field, we have less equations, although the price for this is that these equations become integro-differential. Nevertheless, the obtained equations are more convenient for applying to them our method of solution. Another important advantage of the derived equations is the possibility of taking into account quantum effects. Such effects are often principal, while the standard semiclassical Maxwell–Bloch equations cannot take account of them. To simplify evolution equations, not losing quantum effects, is the idea of the *Stochastic Mean-Field Approximation* of Section 4. Since cooperative electromagnetic phenomena are directly related to arising coherence, Section 5 gives the definitions for *Dynamical Characteristics of Coherence*.

The equations derived in the previous sections and the method of solution developed above are applied to several concrete systems exhibiting interesting physical properties. In Section 6, we suggest the theory of *Collective Liberation of Light*, which can occur in materials with polariton band gap. In Section 7, we consider the influence of external fields on radiation properties of resonant atoms, checking whether it is feasible to get *Amplification by Nonresonant Fields*. Section 8 discusses the so-called *Mössbauer Magnetic Anomaly* observed in some magnetic materials. In Section 9 the *Problem of Pattern Selection* is analyzed. This problem arises, for instance, when one needs to describe resonant media with spatially nonuniform electromagnetic structures. For treating the problem, we have suggested an original approach based on probabilistic analysis of possible spatiotemporal patterns. This method is applied, in Section 10, to describing *Turbulent Photon Filamentation* in resonant media.

Scale Separation Approach, being a general method, can be employed for treating strongly nonequilibrium systems of different physical nature. In Section 11, it is used for giving a thorough picture of *Superradiant Spin Relaxation* occurring in nonequilibrium nuclear magnets. This method also makes it possible to analyse nonlinear differential equations in partial derivatives. Such an analysis helps in finding conditions under which unusual nonlinear effects can happen. This is illustrated in Section 12 by describing a transient effect of *Negative Electric Current* in nonuniform semiconductors. Another novel effect of *Magnetic Semiconfinement of Atoms* is described in Section 13. Both these effects have been predicted by the authors. In Section 14, we discuss conditions when *Nuclear Matter Lasing* could be possible.

Throughout the review, we consider several physical systems of rather different nature. Because of this, it is more appropriate to give all details and to discuss the related literature in the corresponding sections, limiting the Introduction by a brief enumeration of the considered problems. Section 15 contains *Conclusion* summarizing main results.

## 2. SCALE SEPARATION APPROACH

Because of the pivotal role of this approach for treating physical problems in the following sections, we need to start by presenting its general scheme. It is possible to separate five main steps, or parts, of the approach: (i) stochastic quantization of short-range correlations; (ii) separation of variables onto fast and slow; (iii) averaging method for multifrequency systems; (iv) generalized expansion about guiding centers; and (v) selection of scales for space structures. Below, these steps are explicitly explained.

**2.1. Short-Range Stochastic Quantization.** When considering nonequilibrium processes in statistical systems, one needs to write evolution equations for

some averages  $\langle A_i \rangle$  of operators  $A_i(t)$ , where  $t$  is time and  $i = 1, 2, \dots, N$  enumerates particles composing the considered system. For simplicity, a discrete index  $i$  is employed, although everywhere in what follows one could mean an operator  $A(\vec{r}_i, t)$  depending on a continuous space variable  $\vec{r}_i$ .

There exists the well-known problem in statistical mechanics consisting in the fact that writing an evolution equation for  $\langle A_i \rangle$  one does not get a closed set of equations but a hierarchical chain of equations connecting correlation functions of higher orders. Thus, an equation for  $\langle A_i \rangle$  involves the terms as  $\sum_j \langle A_i B_j \rangle$  with double correlators  $\langle A_i B_j \rangle$ , and the evolution equations for the latter acquire the terms with triple correlators, and so on. The simplest way for making the system of equations closed is by resorting to the mean-field type decoupling  $\langle A_i B_j \rangle \rightarrow \langle A_i \rangle \langle B_j \rangle$ . When considering radiation processes, this decoupling is called the semiclassical approximation. Then the term  $\sum_j \langle A_i B_j \rangle$  reduces to  $\langle A_i \rangle \sum_j \langle B_j \rangle$ , so that one can say that  $\langle A_i \rangle$  is subject to the action of the mean field  $\sum_j \langle B_j \rangle$ . The semiclassical approximation describes well coherent processes, when long-range correlations between particles govern the evolution of the system, while short-range correlations, due to quantum fluctuations, are not important. However, the latter may become of great importance if there are periods of time when the long-range correlations are absent. For example, this may happen at the beginning of a nonequilibrium process when long-range correlations have had yet no time to develop. Then neglecting short-range correlations can lead to principally wrong results for the whole dynamics.

To include the influence of short-range correlations, the semiclassical approximation can be modified as follows:

$$\sum_j \langle A_i B_j \rangle = \langle A_i \rangle \left( \sum_j \langle B_j \rangle + \xi \right), \quad (1)$$

where  $\xi$  is a random variable describing local short-range correlations. It is natural to treat  $\xi$  as a Gaussian stochastic variable defined by its first,  $\langle \xi \rangle$ , and second,  $\langle |\xi|^2 \rangle$ , moments. According to the short-range character of local fields, we should set

$$\langle \xi \rangle = 0. \quad (2)$$

The second moment, aiming at taking into account incoherent local fluctuations, can be defined by means of the following reasoning. Consider the equality

$$\left\langle \left| \sum_j \langle A_i B_j \rangle \right|^2 \right\rangle = |\langle A_i \rangle|^2 \left( \left| \sum_j \langle B_j \rangle \right|^2 + \langle |\xi|^2 \rangle \right)$$

resulting from definitions (1) and (2). On the other hand, wishing to take into account both long-range coherent and short-range incoherent terms, one should write

$$\left| \sum_j \langle A_i B_j \rangle \right|^2 = |\langle A_i \rangle|^2 \left( \left| \sum_j \langle B_j \rangle \right|^2 + \sum_j |\langle B_j \rangle|^2 \right),$$

where the first term in the brackets corresponds to the coherent while the second term, to incoherent parts. Comparing the latter two equalities, we come to the conclusion that

$$\ll |\xi|^2 \gg = \sum_j |\langle B_j \rangle|^2. \quad (3)$$

As far as short-range correlations and fluctuations are often due to quantum effects, the manner of taking them into account by introducing a stochastic variable  $\xi$  can be named the stochastic quantization. Then the decoupling (1) may be termed the *stochastic mean-field approximation*. A similar kind of approximation has been used for taking account of quantum spontaneous emission of atoms in the problem of atomic superradiance [1]. Somewhat related ideas have also been used in the stochastic quantization of quantum field theory [2].

**2.2. Classification of Function Variations.** Employing the stochastic mean-field approximation makes it possible to write down a closed set of stochastic differential equations. The next step is to find such a change of variables which results in the possibility of separating the functional variables onto fast and slow. Let us consider, first, the variation of functions in time. Assume that we come to the set of equations of the form

$$\frac{du}{dt} = f, \quad \frac{ds}{dt} = \varepsilon g, \quad (4)$$

in which  $f = f(\varepsilon, u, s, \xi, t)$ ,  $g = g(\varepsilon, u, s, \xi, t)$ , and  $\varepsilon \ll 1$  is a small parameter. Equations (4) are complimented by initial conditions

$$u = u_0, \quad s = s_0 \quad (t = 0). \quad (5)$$

Here, for simplicity, we deal with only two functions,  $u$  and  $s$ , and one small parameter  $\varepsilon$ . The whole procedure is straightforwardly applicable to the case of many functions and several parameters.

Let the functions  $f$  and  $g$  be such that

$$\lim_{\varepsilon \rightarrow 0} f \neq 0, \quad \lim_{\varepsilon \rightarrow 0} \varepsilon g = 0. \quad (6)$$

Then from Eqs. (4) it follows that

$$\lim_{\varepsilon \rightarrow 0} \frac{du}{dt} \neq 0, \quad \lim_{\varepsilon \rightarrow 0} \frac{ds}{dt} = 0. \quad (7)$$

This permits us to classify the solution  $u$  as fast, compared to the slow solution  $s$ . In turn, the slow solution  $s$  is a *quasi-invariant* with respect to the fast solution  $u$ . Thus, we may classify the functions representing the sought solutions onto fastly and slowly varying in time.

In the case of partial differential equations, one has, in addition to time, a space variable  $\vec{r}$ . Then the notion of fast and slow functions can be generalized as follows [3]. Let  $\vec{r} \in \mathbf{V}$ , with  $V$  being the measure of the volume  $\mathbf{V}$ , and let  $t \in [0, T]$ , where  $T$  can be infinite. If one has

$$\lim_{\varepsilon \rightarrow 0} \ll \frac{1}{V} \int_{\mathbf{V}} \frac{\partial u}{\partial t} d\vec{r} \gg \neq 0, \quad \lim_{\varepsilon \rightarrow 0} \ll \frac{1}{T} \int_0^T \vec{\nabla} u dt \gg \neq 0, \quad (8)$$

while

$$\lim_{\varepsilon \rightarrow 0} \ll \frac{1}{V} \int_{\mathbf{V}} \frac{\partial s}{\partial t} d\vec{r} \gg = 0, \quad \lim_{\varepsilon \rightarrow 0} \ll \frac{1}{T} \int_0^T \vec{\nabla} s dt \gg = 0, \quad (9)$$

then the solution  $u$  can be called *fast on average* with respect to both space and time, as compared to  $s$  that is *slow on average*. In such a case,  $s$  is again a quasi-invariant with respect to  $u$ . In general, it may, of course, happen that one of the solutions is fast in time but slow in space, or vice versa, as compared to another solution. Note that in the Hamiltonian mechanics quasi-invariants with respect to time are called adiabatic invariants [4]. A generalization of this notion to the case of both space and time variables [3] is given by definition (9).

**2.3. Multifrequency Averaging Technique.** Let us continue considering the ordinary differential equations (4). The generalization to the case of partial differential equations can be done similarly to the way discussed at the end of the previous section. After classifying the function  $u$  as fast and  $s$  as slow, we may resort to the Krylov–Bogolubov averaging technique [5] extended to multifrequency systems.

Since the slow solution  $s$  is a quasi-invariant for the fast variable  $u$ , one considers the equation for the fast function, with the slow one kept fixed,

$$\frac{\partial X}{\partial t} = f(\varepsilon, X, z, \xi, t), \quad (10)$$

here  $s = z$  being treated as a constant parameter. The initial conditions for Eq. (10) is

$$X = u_0 \quad (t = 0). \quad (11)$$

The pair of solutions

$$X = X(\varepsilon, z, \xi, t), \quad z = const \quad (12)$$

are called the *generating solutions*. Substituting the solution  $X$  into the right-hand side of the equation for the slow function  $s$ , one defines the average

$$\bar{g}(\varepsilon, z) \equiv \ll \frac{1}{\tau} \int_0^\tau g(\varepsilon, X(\varepsilon, z, \xi, t), z, \xi, t) dt \gg, \quad (13)$$

in which  $\tau$  is the characteristic time of fast oscillations. In many cases, it is sufficient to set  $\tau \rightarrow \infty$ . In this way, we come to the equation

$$\frac{dz}{dt} = \varepsilon \bar{g}(\varepsilon, z), \quad (14)$$

with the initial condition

$$z = s_0 \quad (t = 0). \quad (15)$$

The solution to Eq. (14),

$$z = z(\varepsilon, t), \quad (16)$$

is to be substituted into  $X$  yielding

$$y(\varepsilon, \xi, t) = X(\varepsilon, z(\varepsilon, t), \xi, t). \quad (17)$$

Generating solutions (12) are the first crude approximations one starts with. More elaborate solutions (16) and (17) are termed *guiding centers*.

Notice two points that difference the considered way of obtaining the guiding centers (16) and (17) from the standard averaging method [5]. The first point is in retaining in Eq. (10) the small parameter  $\varepsilon$ , which makes it possible to correctly take into account important physical effects, such as attenuation. The standard manner of defining the generating solutions with setting  $\varepsilon = 0$  would result in essentially more rough approximations. The second difference is in the occurrence of the stochastic average in definition (13), since here we are dealing with stochastic differential equations.

**2.4. Generalized Asymptotic Expansion.** The generating solutions (12) play the role of the trial zero-order approximation, while the guiding centers (16) and (17) essentially improve the trial approximations. Higher-order corrections may be obtained by presenting the general solutions as asymptotic expansions about the guiding centers. Then,  $k$ -order approximations are written as

$$u_k = y(\varepsilon, \xi, t) + \sum_{n=1}^k y_n(\varepsilon, \xi, t) \varepsilon^n, \quad s_k = z(\varepsilon, t) + \sum_{n=1}^k z_n(\varepsilon, \xi, t) \varepsilon^n. \quad (18)$$

Such series are named *generalized asymptotic expansions* [6], since the expansion coefficients depend themselves on parameter  $\varepsilon$ . The right-hand sides of Eqs. (4) are also to be expanded about the guiding centers yielding

$$f(\varepsilon, u_k, s_k, \xi, t) \simeq f(\varepsilon, y, z, \xi, t) + \sum_{n=1}^k f_n(\varepsilon, \xi, t) \varepsilon^n,$$



$$g(\varepsilon, u_k, s_k, \xi, t) \simeq g(\varepsilon, y, z, \xi, t) + \sum_{n=1}^k g_n(\varepsilon, \xi, t) \varepsilon^n. \quad (19)$$

Then, expansions (18) and (19) are to be substituted in Eqs. (4) with equating the like terms with respect to the explicit powers of  $\varepsilon$ . Thus, in the first order, this gives

$$\frac{dy_1}{dt} = f_1(\varepsilon, \xi, t) - \bar{g}(\varepsilon, z) X_1(\varepsilon, \xi, t), \quad \frac{dz_1}{dt} = g(\varepsilon, y, z, \xi, t) - \bar{g}(\varepsilon, z), \quad (20)$$

where

$$X_1(\varepsilon, \xi, t) \equiv \frac{\partial}{\partial z} X(\varepsilon, z, \xi, t), \quad z = z(\varepsilon, t).$$

For the approximations of order  $n \geq 2$ , we get

$$\frac{dy_n}{dt} = f_n(\varepsilon, \xi, t), \quad \frac{dz_n}{dt} = g_n(\varepsilon, \xi, t). \quad (21)$$

The initial conditions for all  $n = 1, 2, \dots$  are

$$y_n = z_n = 0 \quad (t = 0). \quad (22)$$

The functions  $f_n$  and  $g_n$  depend on  $y_1, y_2, \dots, y_n$ , and on  $z_1, z_2, \dots, z_n$ , but it is important that the dependence on  $y_n$  and  $z_n$  is linear. The latter follows from the fact that expanding a function

$$f\left(y + \sum_{n=1}^k y_n \varepsilon^n\right) = \sum_{n=1}^k f_n \varepsilon^n$$

in powers of  $\varepsilon$ , one has

$$f_1 = f'(y)y_1, \quad f_2 = \frac{1}{2!} [f''(y)y_1 + f'(y)y_2],$$

$$f_3 = \frac{1}{3!} [f'''(y)y_1 + 2f''(y)y_2 + f'(y)y_3],$$

and so on. In this way, Eqs. (20) directly define  $y_1$  and  $z_1$ , and Eqs. (21) are linear equations, thus, being easily integrated.

Usually, one does not need the higher-order approximations since the main physics, in the majority of cases, is already well described by the guiding centers (16) and (17). The latter are good approximations to the exact solutions [7] in the time interval  $0 \leq t \leq T_s/\varepsilon$ , where  $T_s$  is a characteristic time of the slow-solution variation. In those cases when the higher-order approximations are important, each  $k$ -order approximant can also be improved by invoking some sort of summation [8] of asymptotic series (18), for instance, the self-similar summation [9–12].

**2.5. Selection of Space Structures.** The solutions to differential or integro-differential nonlinear equations in partial derivatives are generally nonuniform in space exhibiting the formation of different spatial structures. And it often happens that a given set of equations possesses several solutions corresponding to different spatial patterns or to different scales of such patterns [13]. When there is a family of solutions describing several possible patterns, the question arises which of these solutions, and respectively patterns, is preferable and in what sense could it be preferable. This problem of pattern selection is a general and very important problem constantly arising in considering spatial structures. In this subsection we delineate a simple way that in many cases helps to solve the problem of pattern selection. A more refined theory will be presented in Sections 9 and 10.

Assume that the obtained solutions describe spatial structures that can be parametrized by a multiparameter  $\beta$ , so that the  $k$ -order approximations  $u_k(\beta, \vec{r}, t)$  and  $s_k(\beta, \vec{r}, t)$  include the dependence on  $\beta$  whose value is, however, yet undefined. To define  $\beta$ , and respectively the related pattern, one may proceed in the spirit of the self-similar approximation theory [14–23], by treating  $\beta$  as a control function. According to the theory [14–23], control functions are to be defined from fixed-point conditions for an approximation cascade constructed for an observable quantity. For the latter, one may take the average energy defined as follows. The internal energy, which is a statistical average of the system Hamiltonian, is a functional  $E[u, s]$  of the solutions. Taking the  $k$ -order approximations for the latter and averaging the internal energy over the period of fast oscillations and over stochastic variables, one gets the average energy

$$E_k(\beta) \equiv \ll \frac{1}{\tau} \int_0^\tau E[u_k(\beta, \vec{r}, t), s_k(\beta, \vec{r}, t)] dt \gg . \quad (23)$$

For the sequence of approximations,  $\{E_k(\beta)\}$ , it is possible to construct an approximation cascade whose fixed point can be given by the condition

$$\frac{\partial}{\partial \beta} E_k(\beta) = 0 , \quad (24)$$

from which one gets the control function  $\beta = \beta_k$  defining the corresponding pattern. According to optimal control theory, control functions are defined so that to minimize a cost functional. The latter, in our case, is naturally represented by the average energy (23). Hence, when the fixed-point equation (24) has several solutions, one may select of them that one which minimizes the cost functional (23), so that

$$E_k(\beta_k) = \text{abs min}_{\beta} E_k(\beta) . \quad (25)$$

Equations (24) and (25) have a simple physical interpretation as the minimum conditions for the average energy (23). However, one should keep in mind that

there is no, in general, such a principle of minimal energy for nonequilibrium systems [13]. Therefore the usage of the ideas from the self-similar approximation theory [14–23] provides a justification for employing conditions (24) and (25) for nonequilibrium processes.

The scale separation approach presented in this section makes it possible to solve rather complicated sets of nonlinear differential equations describing various nonequilibrium phenomena in statistical systems. More details on this approach can be found in Refs. 24–28.

### 3. REAL SPACE REPRESENTATION

When considering the interaction of atoms with electromagnetic fields, one usually employs the so-called mode representation, expanding field operators over mode wave functions [29,30]. These can be either free-mode functions, that is plane waves, or resonator-mode functions depending on the resonator geometry. We prefer to deal with the real-space representation because of the following reasons: First, the evolution equations in this representation are written in a form more convenient for analysing temporal nonstationary behaviour of solutions. Second, it is more suitable for describing nonuniform solutions corresponding to self-organized space structures. And third, this representation is more appropriate for using the scale separation approach. Since the real space representation is rarely considered in literature, it is worth recalling in brief the derivation of the main equations in this representation [31]. To understand the basis of the main evolution equations is very important, for these equations will be constantly used in what follows. One more peculiarity of the consideration below, differencing it from the standard texts, is the comparison of the formulas for the cases of electro-dipole and magnetodipole transitions.

Let us have a system of radiators that can be atoms, molecules, nuclei, etc. Assume that the size of a radiator,  $a_0$ , is small as compared to the mean distance between them,  $a$ , as well as to the characteristic radiation wavelength  $\lambda$ ,

$$\frac{a_0}{a} \ll 1, \quad \frac{a_0}{\lambda} \ll 1, \quad (26)$$

while the relation between  $a$  and  $\lambda$  can be arbitrary. Canonical variables related to the electromagnetic field are the electric field  $\vec{E}$  and the vector potential  $\vec{A}$ , whose commutation relations are

$$[E^\alpha(\vec{r}, t), A^\beta(\vec{r}', t)] = 4\pi i c \delta_{\alpha\beta} \delta(\vec{r} - \vec{r}'),$$

$$[E^\alpha(\vec{r}, t), E^\beta(\vec{r}', t)] = [A^\alpha(\vec{r}, t), A^\beta(\vec{r}', t)] = 0,$$

where  $c$  is the light velocity and the indices  $\alpha, \beta = 1, 2, 3$ , or  $x, y, z$ , enumerate the Cartesian coordinates. The magnetic field is

$$\vec{H}(\vec{r}, t) = \vec{\nabla} \times \vec{A}(\vec{r}, t).$$

To uniquely define the latter, we invoke the Coulomb gauge condition

$$\vec{\nabla} \cdot \vec{A}(\vec{r}, t) = 0.$$

Here and in what follows the system of units is used where  $\hbar \equiv 1$ .

The radiator charges are described by the annihilation,  $\psi$ , and creation,  $\psi^\dagger$ , field operators with the commutation relations

$$[\psi(\vec{r}, t), \psi^\dagger(\vec{r}', t)]_{\mp} = \delta(\vec{r} - \vec{r}'), \quad [\psi(\vec{r}, t), \psi(\vec{r}', t)]_{\mp} = 0,$$

$$[\psi(\vec{r}, t), \vec{E}(\vec{r}', t)] = [\psi(\vec{r}, t), \vec{A}(\vec{r}', t)] = 0,$$

in which the indices minus or plus mean the commutators or anticommutators, respectively, depending on the Bose or Fermi statistics of the charges.

Assume that in addition to the quantum radiation fields  $\vec{E}$  and  $\vec{H}$  there are classical fields  $\vec{E}_0$  and  $\vec{H}_0$  for which we have

$$\vec{E}_0(\vec{r}, t) = -\vec{\nabla}\varphi_0(\vec{r}, t), \quad \vec{H}_0(\vec{r}, t) = \vec{\nabla} \times \vec{A}_0(\vec{r}, t), \quad \vec{\nabla} \cdot \vec{A}_0(\vec{r}, t) = 0.$$

These additional fields can be due to external sources or can be created by the matter which the radiators are inserted in.

Each radiator is also subject to the action of a scalar potential  $\varphi_i(\vec{r})$  representing all stationary Coulomb interactions. Thus, we may introduce the total scalar and vector potentials

$$\varphi_{tot}(\vec{r}, t) = \varphi_0(\vec{r}, t) + \sum_{i=1}^N \varphi_i(\vec{r}), \quad \vec{A}_{tot}(\vec{r}, t) = \vec{A}_0(\vec{r}, t) + \vec{A}(\vec{r}, t), \quad (27)$$

where  $N$  is the number of radiators. Then the local energy operator is defined as

$$\hat{H}(\vec{r}, t) = \frac{1}{2m_0} \left[ i \vec{\nabla} + \frac{e}{c} \vec{A}_{tot}(\vec{r}, t) \right]^2 + e \varphi_{tot}(\vec{r}, t), \quad (28)$$

where  $m_0$  is mass and  $e$ , charge of a particle. Omitting here the relativistic term  $e^2 \vec{A}_{tot}^2/c^2$  and using the Coulomb calibration, we have

$$\hat{H}(\vec{r}, t) = -\frac{\nabla^2}{2m_0} + \frac{ie}{m_0 c} \vec{A}_{tot}(\vec{r}, t) \cdot \vec{\nabla} + e \varphi_{tot}(\vec{r}, t).$$

The Hamiltonian of the system of radiators interacting with electromagnetic field and with matter is written as the sum

$$\hat{H} = \hat{H}_r + \hat{H}_f + \hat{H}_{rf} + \hat{H}_m + \hat{H}_{mf}, \quad (29)$$

in which the terms represent, respectively, the Hamiltonians of radiators, field, radiator–field interaction, matter, and matter–field interaction. The Hamiltonian of the system of radiators is

$$\hat{H}_r(t) = \int \psi^\dagger(\vec{r}, t) \left[ -\frac{\nabla^2}{2m_0} + e \sum_{i=1}^N \varphi_i(\vec{r}) \right] \psi(\vec{r}, t) d\vec{r}. \quad (30)$$

This includes also the direct interaction of radiators with matter by means of the effective scalar potentials  $\varphi_i(\vec{r})$ . The field Hamiltonian writes

$$\hat{H}_f(t) = \frac{1}{8\pi} \int \left[ \vec{E}^2(\vec{r}, t) + \vec{H}^2(\vec{r}, t) \right] d\vec{r}. \quad (31)$$

The radiator–field interaction is described by

$$\hat{H}_{rf}(t) = \int \psi^\dagger(\vec{r}, t) \left[ \frac{ie}{m_0 c} \vec{A}_{tot}(\vec{r}, t) \cdot \vec{\nabla} + e \varphi_0(\vec{r}, t) \right] \psi(\vec{r}, t) d\vec{r}. \quad (32)$$

The Hamiltonians of matter and of matter–field interaction are to be specified according to particular cases under consideration.

The size of a radiator, according to inequalities (26), is the smallest characteristic length. If  $\vec{r}_i$  is the center-of-mass of a radiator, we shall use the notation

$$\vec{E}_i(t) \equiv \vec{E}(\vec{r}_i, t), \quad \vec{H}_i(t) = \vec{H}(\vec{r}_i, t),$$

$$\vec{A}_i(t) \equiv \vec{A}(\vec{r}_i, t), \quad \vec{E}_{0i}(t) \equiv \vec{E}_0(\vec{r}_i, t), \quad \vec{H}_{0i}(t) = \vec{H}_0(\vec{r}_i, t).$$

For  $\vec{r}$  in the vicinity of  $\vec{r}_i$ , we may write

$$\varphi_0(\vec{r}, t) \simeq -\vec{r} \cdot \vec{E}_{0i}(t) \quad (\vec{r} \approx \vec{r}_i),$$

$$\vec{A}_0(\vec{r}, t) \simeq -\frac{1}{2} \vec{r} \times \vec{H}_{0i}(t), \quad \vec{A}(\vec{r}, t) \simeq \vec{A}_i(t) - \frac{1}{2} (\vec{r} - \vec{r}_i) \times \vec{H}_i(t).$$

The energy levels of each radiator are defined by the Schrödinger equation

$$\left[ -\frac{\nabla^2}{2m_0} + e \varphi_i(\vec{r}) \right] \psi_n(\vec{r} - \vec{r}_i) = E_n \psi_n(\vec{r} - \vec{r}_i),$$

where it is assumed that all radiators are identical and  $\varphi_i(\vec{r}) = \varphi(\vec{r} - \vec{r}_i)$ . The eigenfunctions  $\psi_n(\vec{r} - \vec{r}_i)$  form a complete orthonormal set enumerated by the indices  $n$  and  $i$ , so that

$$\int \psi_m^*(\vec{r} - \vec{r}_i) \psi_n(\vec{r} - \vec{r}_j) d\vec{r} = \delta_{mn} \delta_{ij}, \quad \sum_{in} \psi_n^*(\vec{r} - \vec{r}_i) \psi_n(\vec{r}' - \vec{r}_i) = \delta(\vec{r} - \vec{r}').$$

With these functions, we may define the density of transition current

$$\vec{j}_{mn}(\vec{r}) = -\frac{ie}{2m_0} \left[ \psi_m^*(\vec{r}) \vec{\nabla} \psi_n(\vec{r}) - \psi_n(\vec{r}) \vec{\nabla} \psi_m^*(\vec{r}) \right] \quad (33)$$

and the transition current

$$\vec{j}_{mn} = \int \vec{j}_{mn}(\vec{r}) d\vec{r}. \quad (34)$$

We also introduce the electric transition dipole

$$\vec{d}_{mn} = e \int \psi_m^*(\vec{r}) \vec{r} \psi_n(\vec{r}) d\vec{r} \quad (35)$$

and the magnetic transition dipole

$$\vec{\mu}_{mn} = \frac{1}{2c} \int \vec{r} \times \vec{j}_{mn}(\vec{r}) d\vec{r}. \quad (36)$$

Using the equality

$$\vec{\nabla} = m_0 \left[ \vec{r}, -\frac{\nabla^2}{2m_0} + e \varphi_i(\vec{r}) \right],$$

one can connect the electric transition current (34) and transition dipole (35) as

$$\vec{j}_{mn} = i \omega_{mn} \vec{d}_{mn}, \quad \omega_{mn} \equiv E_m - E_n. \quad (37)$$

The field operators can be expanded over the basis of the wave functions as

$$\psi(\vec{r}, t) = \sum_n \sum_{i=1}^N c_{ni}(t) \psi_n(\vec{r} - \vec{r}_i).$$

From the commutation relations for the field operators one has

$$\left[ c_{mi}(t), c_{ni}^\dagger(t) \right]_{\mp} = \delta_{mn} \delta_{ij}, \quad [c_{mi}(t), c_{nj}(t)]_{\mp} = 0.$$

The fact that each radiator is certainly in one of the states labelled by the index  $n$  is expressed by the unipolarity condition

$$\sum_n c_{ni}^\dagger(t) c_{ni}(t) = 1. \quad (38)$$

The wave functions  $\psi_n(\vec{r} - \vec{r}_i)$ , in agreement with inequalities (26), are localized in a small region of the size of a radiator. Such functions are called the localized orbitals. The localization condition can be represented by the equality

$$\int \psi_m^*(\vec{r} - \vec{r}_i) f(\vec{r}) \psi_n(\vec{r} - \vec{r}_j) d\vec{r} = 0 \quad (i \neq j),$$

in which  $f(\vec{r})$  is a finite function.

Using the notations and conditions introduced above, we transform the radiator Hamiltonian (30) to the form

$$\hat{H}_r(t) = \sum_n \sum_{i=1}^N E_n c_{ni}^\dagger(t) c_{ni}(t). \quad (39)$$

The radiator-field Hamiltonian (32) becomes

$$\hat{H}_{rf}(t) = - \sum_{mn} \sum_{i=1}^N c_{mi}^\dagger(t) c_{ni}(t) \left[ \vec{d}_{mn} \cdot \vec{E}_{0i}(t) + \frac{1}{c} \vec{j}_{mn} \cdot \vec{A}_i(t) + \vec{\mu}_{mn} \cdot \vec{B}_i(t) \right], \quad (40)$$

where

$$\vec{B}_i(t) = \vec{H}_{0i}(t) + \vec{H}_i(t) \quad (41)$$

is the total magnetic field.

From definitions (34) to (36), we have

$$\vec{d}_{mn}^* = \vec{d}_{nm}, \quad \vec{j}_{mn}^* = \vec{j}_{nm}, \quad \vec{\mu}_{mn}^* = \vec{\mu}_{nm}.$$

Because the wave functions are usually either symmetric or antisymmetric with respect to the spatial inversion, so that

$$|\psi_n(-\vec{r})| = |\psi_n(\vec{r})|, \quad (42)$$

then we see that  $\vec{d}_{nn} = \vec{j}_{nn} = 0$  but, in general,  $\vec{\mu}_{nn} \neq 0$ .

The next approximation that is usually involved is related to the situation when only a couple of radiator levels takes part in the considered process. This happens when the transition frequency

$$\omega_0 \equiv \omega_{21} = E_2 - E_1 > 0 \quad (43)$$

for these two levels is selected by means of an external alternating field whose frequency is close to the transition frequency (43). In this way, considering only two levels is equivalent to the quiresonance approximation. Then, it is convenient to introduce the transition operators

$$\sigma_i^-(t) = c_{1i}^\dagger(t) c_{2i}(t), \quad \sigma_i^+(t) = c_{2i}^\dagger(t) c_{1i}(t)$$

and the population-difference operator

$$\sigma_i^z(t) = c_{2i}^\dagger(t) c_{2i}(t) - c_{1i}^\dagger(t) c_{1i}(t) ,$$

so that

$$2c_{1i}^\dagger(t) c_{1i}(t) = 1 - \sigma_i^z(t) , \quad 2c_{2i}^\dagger(t) c_{2i}(t) = 1 + \sigma_i^z(t) .$$

The commutation relations for the introduced operators are

$$\begin{aligned} [\sigma_i^-, \sigma_j^+] &= -\delta_{ij} \sigma_i^z, & [\sigma_i^-, \sigma_j^-] &= [\sigma_i^+, \sigma_j^+] = 0, & [\sigma_i^-, \sigma_j^z] &= 2 \delta_{ij} \sigma_i^-, \\ [\sigma_i^+, \sigma_j^z] &= -2 \delta_{ij} \sigma_i^+, & [\sigma_i^-, \vec{A}_j] &= [\sigma_i^-, \vec{E}_j] = [\sigma_i^z, \vec{A}_j] = [\sigma_i^z, \vec{E}_j] = 0, \end{aligned}$$

where all operators are taken at coinciding times.

With the notation

$$\vec{d}_{21} \equiv \vec{d}, \quad \vec{\mu}_{21} \equiv \vec{\mu}, \quad (44)$$

we have  $\vec{d}_{12} = \vec{d}^*$ ,  $\vec{\mu}_{12} = \vec{\mu}^*$ , and consequently

$$\vec{j}_{12} = -i \omega_0 \vec{d}^*, \quad \vec{j}_{21} = i \omega_0 \vec{d}. \quad (45)$$

Since only the difference between level energies is measurable, one can set  $E_1 = 0$ . Then the radiator Hamiltonian (39) reduces to

$$\hat{H}_r(t) = \frac{1}{2} \sum_{i=1}^N \omega_0 [1 + \sigma_i^z(t)]. \quad (46)$$

Everywhere in what follows we assume that electromagnetic fields acting on a radiator do not change the classification of its energy levels. In the other case it would be impossible to talk about quiresonance. This implies that the interaction energies of a radiator with fields are assumed to be much smaller than  $\omega_0$ . Because of the latter, the term

$$\frac{1}{2} \sum_{i=1}^N [(\vec{\mu}_{11} + \vec{\mu}_{22}) + (\vec{\mu}_{22} - \vec{\mu}_{11}) \sigma_i^z(t)] \cdot \vec{B}_i(t) ,$$

entering the radiator-field Hamiltonian (40), can be neglected as compared to Eq. (46). As a result, we obtain

$$\hat{H}_{rf}(t) = - \sum_{i=1}^N \left[ \frac{1}{c} \vec{j}_i(t) \cdot \vec{A}_i(t) + \vec{d}_i(t) \cdot \vec{E}_{0i}(t) + \vec{\mu}_i(t) \cdot \vec{B}_i(t) \right], \quad (47)$$

where the notation

$$\vec{j}_i(t) = i \omega_0 \left[ \vec{d} \sigma_i^+(t) - \vec{d}^* \sigma_i^-(t) \right], \quad \vec{d}_i(t) = \vec{d} \sigma_i^+(t) + \vec{d}^* \sigma_i^-(t),$$



$$\vec{\mu}_i(t) = \vec{\mu} \sigma_i^+(t) + \vec{\mu}^* \sigma_i^-(t) \quad (48)$$

is used. The Hamiltonian of the matter–field interaction can be written analogously to the first term in Eq. (47) as

$$\hat{H}_{mf}(t) = -\frac{1}{c} \sum_{j=1}^{N_0} \vec{J}_{mj}(t) \cdot \vec{A}_j(t), \quad (49)$$

where  $N_0$  is the number of particles forming the matter and  $\vec{J}_{mj}$  is a local matter current having the structure of the operator  $\vec{J}_{mj} = (e/m)\vec{p}_j$ , with  $\vec{p}_j$  being the momentum of a  $j$ -particle.

The transition between the quantum states  $\psi_1$  and  $\psi_2$  can be either accompanied by the change of parity or not. Then from definitions (35) and (36) it follows that one has one of two possibilities:

$$\begin{aligned} \vec{d} \neq 0, \quad \vec{\mu} = 0 & \quad (\text{changed parity}); \\ \vec{d} = 0, \quad \vec{\mu} \neq 0 & \quad (\text{conserved parity}). \end{aligned} \quad (50)$$

Thus, we actually have to deal with only one of the dipole transitions, either with electric or with magnetic. Here we consider them in parallel in order to compare these two cases.

#### 4. STOCHASTIC MEAN-FIELD APPROXIMATION

Now it is necessary to write down the evolution equations for the operators entering the total Hamiltonian (29) whose terms are given by Eqs. (46), (31), (47), and (49). The Heisenberg equations yield

$$\frac{1}{c} \frac{\partial}{\partial t} \vec{E}(\vec{r}, t) = \vec{\nabla} \times \vec{H}(\vec{r}, t) - \frac{4\pi}{c} \vec{J}(\vec{r}, t), \quad \frac{1}{c} \frac{\partial}{\partial t} \vec{A}(\vec{r}, t) = -\vec{E}(\vec{r}, t), \quad (51)$$

which are, actually, the operator Maxwell equations, where the operator of current is

$$\vec{J}(\vec{r}, t) = \sum_{i=1}^N \left[ \vec{j}_i(t) - c \vec{\mu}_i(t) \times \vec{\nabla} \right] \delta(\vec{r} - \vec{r}_i) + \sum_{j=1}^{N_0} \vec{J}_{mj}(t) \delta(\vec{r} - \vec{r}_j). \quad (52)$$

For the transition operators we have

$$\frac{d\sigma_i^-}{dt} = -i\omega_0 \sigma_i^- + \left( k_0 \vec{d} \cdot \vec{A}_i - i \vec{d} \cdot \vec{E}_{0i} - i \vec{\mu} \cdot \vec{B}_i \right) \sigma_i^z \quad (53)$$

for the lowering operator, where  $k_0 \equiv \omega_0/c$ , and the Hermitian conjugate equation for the rising operator  $\sigma_i^+$ . For the population-difference operator we get

$$\begin{aligned} \frac{d\sigma_i^z}{dt} = & -2k_0 \left( \vec{d} \sigma_i^+ + \vec{d}^* \sigma_i^- \right) \cdot \vec{A}_i + \\ & + 2i \left( \vec{d} \sigma_i^+ - \vec{d}^* \sigma_i^- \right) \cdot \vec{E}_{0i} + 2i \left( \vec{\mu} \sigma_i^+ - \vec{\mu}^* \sigma_i^- \right) \cdot \vec{B}_i . \end{aligned} \quad (54)$$

From Eqs. (51), using the Coulomb calibration, we find the wave equation

$$\left( \nabla^2 - \frac{1}{c^2} \frac{\partial^2}{\partial t^2} \right) \vec{A}(\vec{r}, t) = -\frac{4\pi}{c} \vec{J}(\vec{r}, t) . \quad (55)$$

The solution of the latter has the form

$$\vec{A}(\vec{r}, t) = \vec{A}_{vac}(\vec{r}, t) + \frac{1}{c} \int \vec{J} \left( \vec{r}', t - \frac{|\vec{r} - \vec{r}'|}{c} \right) \frac{d\vec{r}'}{|\vec{r} - \vec{r}'|} , \quad (56)$$

in which  $\vec{A}_{vac}$  is the vacuum vector potential being a solution of the uniform wave equation. With the operator of current (52), the vector potential (56) can be written as the sum

$$\vec{A} = \vec{A}_{vac} + \vec{A}_{rad} + \vec{A}_{mat} \quad (57)$$

of the vacuum potential  $\vec{A}_{vac}$ , the radiator potential

$$\vec{A}_{rad}(\vec{r}_i, t) = \sum_j \frac{1}{c r_{ij}} \vec{j}_j \left( t - \frac{r_{ij}}{c} \right) + \sum_j \frac{\vec{r}_{ij}}{r_{ij}^3} \times \left( r_{ij} \frac{\partial}{\partial r_{ij}} - 1 \right) \vec{\mu}_j \left( t - \frac{r_{ij}}{c} \right) , \quad (58)$$

and of the matter potential

$$\vec{A}_{mat}(\vec{r}_i, t) = \sum_j \frac{1}{c r_{ij}} \vec{J}_{mj} \left( t - \frac{r_{ij}}{c} \right) , \quad (59)$$

where  $\vec{r}_{ij} \equiv \vec{r}_i - \vec{r}_j$ ,  $r_{ij} \equiv |\vec{r}_{ij}|$ , and the summation  $\sum_j$  does not include the term with  $j = i$ .

Our aim is to derive the evolution equations for the variables

$$u_i(t) \equiv \langle \sigma_i^-(t) \rangle , \quad s_i(t) \equiv \langle \sigma_i^z(t) \rangle , \quad (60)$$

in which the angle brackets mean the statistical averaging over the radiator degrees of freedom. For the double correlators, we shall employ the *mean-field-type decoupling*

$$\langle \sigma_i^\alpha \sigma_j^\beta \rangle = \langle \sigma_i^\alpha \rangle \langle \sigma_j^\beta \rangle \quad (i \neq j) . \quad (61)$$

The quantum effects due to self-action [29] can be taken into account by including into the evolution equations the attenuation terms defined by

$$\gamma \equiv \frac{4}{3} k_0^3 (d_0^2 + \mu_0^2) , \quad (62)$$

where  $d_0 \equiv |\vec{d}|$  and  $\mu_0 \equiv |\vec{\mu}|$ . More generally, one includes the phenomenological longitudinal and transverse attenuation parameters  $\gamma_1$  and  $\gamma_2$ .

To take into account the retardation, we may remember that the action of electromagnetic fields is characterized by the energies that are much smaller than  $\omega_0$ . That is, in the zero order one has  $\sigma_i^- \sim \exp(-i\omega_0 t)$ , as follows from Eq. (53). This suggests to treat the retardation by means of the formula

$$\langle \sigma_j^- \left( t - \frac{r_{ij}}{c} \right) \rangle = u_j(t) \exp(i k_0 r_{ij}) , \quad (63)$$

which can be called the *quasirelativistic approximation* since in the relativistic limit,  $c \rightarrow \infty$ , Eq. (63) becomes an identity.

Comparing the terms of the vector potential (58), induced by either electro-dipole or magnetodipole transitions, we notice their essential difference. Really, averaging over angles gives

$$\sum_j f(r_{ij}) \vec{r}_{ij} = 0 , \quad (64)$$

unless there is a special arrangement of radiators in space. Hence, the vector potential induced by magnetodipole transitions, in usual conditions, is negligibly small. Then for the averaged potential (58), we have

$$\langle \vec{A}_{rad}(\vec{r}_i, t) \rangle = i k_0^2 \sum_j \left( \vec{d} \varphi_{ij}^* u_j^* - \vec{d}^* \varphi_{ij} u_j \right) , \quad (65)$$

where

$$\varphi_{ij} \equiv \frac{\exp(i k_0 r_{ij})}{k_0 r_{ij}} . \quad (66)$$

The influence of vacuum fluctuations and of matter is characterized by the term

$$\xi_i(t) \equiv k_0 \vec{d} \cdot \left[ \vec{A}_{vac}(\vec{r}_i, t) + \vec{A}_{mat}(\vec{r}_i, t) \right] , \quad (67)$$

which we consider as a stochastic variable, whose properties are to be defined by additional conditions.

In this way, we come to the evolution equations for the transverse variable,

$$\frac{du_i}{dt} = -(i \omega_0 + \gamma_2) u_i - i s_i \left( \vec{d} \cdot \vec{E}_{0i} + \vec{\mu} \cdot \vec{H}_{0i} \right) +$$

$$+i k_0^3 s_i \vec{d} \cdot \sum_j \left( \vec{d} \varphi_{ij}^* u_j^* - \vec{d}^* \varphi_{ij} u_j \right) + s_i \xi_i, \quad (68)$$

and for the longitudinal variable,

$$\begin{aligned} \frac{ds_i}{dt} = & 2 i u_i^* \left( \vec{d} \cdot \vec{E}_{0i} + \vec{\mu} \cdot \vec{H}_{0i} \right) - 2 i u_i \left( \vec{d}^* \cdot \vec{E}_{0i} + \vec{\mu}^* \cdot \vec{H}_{0i} \right) - \\ & - 2 i k_0^3 \left( \vec{d} u_i^* + \vec{d}^* u_i \right) \cdot \sum_j \left( \vec{d} \varphi_{ij}^* u_j^* - \vec{d}^* \varphi_{ij} u_j \right) - \gamma_1 (s_i - \zeta) - 2(u_i^* \xi_i + u_i \xi_i^*), \end{aligned} \quad (69)$$

where  $\zeta \in [-1, 1]$  is a pumping parameter. An equation for  $u_i^*$  can be obtained by the complex conjugation of Eq. (68). Another useful equation is

$$\begin{aligned} \frac{d|u_i|^2}{dt} = & -2 \gamma_2 |u_i|^2 + s_i (u_i^* \xi_i + u_i \xi_i^*) - i s_i u_i^* \left( \vec{d} \cdot \vec{E}_{0i} + \vec{\mu} \cdot \vec{H}_{0i} \right) + \\ & + i s_i u_i \left( \vec{d}^* \cdot \vec{E}_{0i} + \vec{\mu}^* \cdot \vec{H}_{0i} \right) + \\ & + i k_0^3 s_i \left( u_i^* \vec{d} + u_i \vec{d}^* \right) \cdot \sum_j \left( \vec{d} \varphi_{ij}^* u_j^* - \vec{d}^* \varphi_{ij} u_j \right). \end{aligned} \quad (70)$$

Equations (68) to (70) are basic for describing nonequilibrium collective phenomena in radiating systems. The set of assumptions employed for deriving these equations can be briefly named the *stochastic mean-field approximation* since the mean-field-type decoupling (61) was used for the radiator correlators, but quantum effects are taken into account through the stochastic variable (67).

## 5. DYNAMICAL CHARACTERISTICS OF COHERENCE

One of the most important results of the cooperative behaviour of radiators is the appearance of coherent radiation. The level of coherence of electromagnetic fields can be described by the corresponding correlation functions [32]. Here we introduce another characteristic of coherence, which is convenient for considering the radiation from ensembles of radiators [33].

The energy density of the radiated electromagnetic field is

$$W \equiv \frac{1}{8\pi} \left( \vec{E}^2 + \vec{H}^2 \right), \quad (71)$$

where  $\vec{E} = \vec{E}(\vec{r}, t)$  and  $\vec{H} = \vec{H}(\vec{r}, t)$ . Differentiating Eq. (71) with respect to time, using the Maxwell equations (51), and defining the intensity of scattering

$$\frac{\partial W_s}{\partial t} \equiv \frac{1}{2} \left( \vec{J} \cdot \vec{E} + \vec{E} \cdot \vec{J} \right) \quad (72)$$

and the Poynting vector

$$\vec{S} \equiv \frac{c}{8\pi} \left( \vec{E} \times \vec{H} - \vec{H} \times \vec{E} \right), \quad (73)$$

we obtain the continuity equation

$$\frac{\partial}{\partial t} (W + W_s) + \text{div} \vec{S} = 0. \quad (74)$$

The intensity of radiation into the unit solid angle is

$$I(\vec{n}, t) \equiv \langle: \vec{n} \cdot \vec{S}(\vec{r}, t) :> r^2, \quad (75)$$

where  $\vec{n} \equiv \vec{r}/r$ ,  $r \equiv |\vec{r}|$ , and the colons imply the normal ordering of operators. To accomplish the latter, one separates the Hermitian operators into their conjugate parts, which, for instance, for the vector potential (58) reads as

$$\vec{A}_{rad}(\vec{r}, t) = \vec{A}^+(\vec{r}, t) + \vec{A}^-(\vec{r}, t), \quad (76)$$

where

$$\vec{A}^+(\vec{r}, t) = \sum_j \left[ \frac{i k_0 \vec{d}}{|\vec{r} - \vec{r}_j|} + \frac{1 + i k_0 |\vec{r} - \vec{r}_j|}{|\vec{r} - \vec{r}_j|^3} \vec{\mu} \times (\vec{r} - \vec{r}_j) \right] \sigma_j^+ \left( t - \frac{1}{c} |\vec{r} - \vec{r}_j| \right).$$

Respectively, the electromagnetic positive and negative fields related to Eq. (76) are

$$\vec{E}_{rad} \equiv -\frac{1}{c} \frac{\partial \vec{A}_{rad}}{\partial t} = \vec{E}^+ + \vec{E}^-, \quad \vec{H}_{rad} \equiv \vec{\nabla} \times \vec{A}_{rad} = \vec{H}^+ + \vec{H}^-.$$

In the time and space derivatives, we may employ, for differentiating  $\sigma_j^\pm$ , the relations

$$\left( \frac{1}{c} \frac{\partial}{\partial t} + \frac{\partial}{\partial r_{ij}} \right) \sigma_j^\pm \left( t - \frac{r_{ij}}{c} \right) = 0, \quad \left( \frac{\partial}{\partial r_{ij}} \pm i k_0 \right) \sigma_j^\pm \left( t - \frac{r_{ij}}{c} \right) = 0.$$

In the wave zone, where  $r \gg |\vec{r}_i|$  and  $|\vec{r} - \vec{r}_j| \simeq r - \vec{n} \cdot \vec{r}_j$ , ( $r \gg |\vec{r}_j|$ ), we have

$$\vec{A}^+(\vec{r}, t) \simeq i \frac{k_0}{r} \left( \vec{d} + \vec{\mu} \times \vec{n} \right) \sum_j \sigma_j^+ \left( t - \frac{r - \vec{n} \cdot \vec{r}_j}{c} \right), \quad (77)$$

from where

$$\vec{E}^+ = -i k_0 \vec{A}^+, \quad \vec{H}^+ = \vec{n} \times \vec{E}^+. \quad (78)$$

Then in the part of the Poynting vector (73), describing the radiation from the ensemble of radiators, one has

$$\vec{S}_{rad} = \frac{c}{4\pi} \vec{E}_{rad} \times \vec{H}_{rad}, \quad \vec{H}_{rad} = \vec{n} \times \vec{E}_{rad}.$$

For the corresponding part of the radiation intensity (75), we get

$$I_{rad}(\vec{n}, t) = \frac{cr^2}{4\pi} \langle : \vec{E}_{rad}^2 - (\vec{n} \cdot \vec{E}_{rad})^2 : \rangle . \quad (79)$$

Averaging the latter over stochastic variables and over fast oscillations yields

$$\bar{I}(\vec{n}, t) \equiv \frac{\omega_0}{2\pi} \int_0^{2\pi/\omega_0} \ll I_{rad}(\vec{n}, t) \gg dt , \quad (80)$$

the slow variables in the process of integration being kept fixed. For the radiation intensity (79), this results in

$$\bar{I}(\vec{n}, t) = \omega_0 \gamma \sum_{ij}^N f_{ij}(\vec{n}) \overline{\langle \sigma_i^+(t) \sigma_j^-(t) \rangle} , \quad (81)$$

where

$$f_{ij}(\vec{n}) \equiv \frac{3}{8\pi} |\vec{n} \times \vec{e}|^2 \exp(i k_0 \vec{n} \cdot \vec{r}_{ij}) \quad (82)$$

and  $\vec{e} = \vec{d}/d_0$  or  $\vec{\mu}/\mu_0$  depending on the type of radiation.

In the radiation intensity (81), we may separate the terms with the coinciding and with different indices, so that  $\sum_{ij} = \sum_{i=j} + \sum_{i \neq j}$ . This makes it possible to separate the radiation intensity into the incoherent and coherent parts,

$$\bar{I}(\vec{n}, t) = I_{inc}(\vec{n}, t) + I_{coh}(\vec{n}, t) , \quad (83)$$

so that the incoherent radiation intensity is

$$I_{inc}(\vec{n}, t) = \frac{1}{2} \omega_0 \gamma \sum_{i=1}^N f_{ii}(\vec{n}) [1 + s_i(t)] \quad (84)$$

while the coherent radiation intensity is

$$I_{coh}(\vec{n}, t) = \omega_0 \gamma \sum_{i \neq j}^N f_{ij}(\vec{n}) \overline{u_i^*(t) u_j(t)} . \quad (85)$$

Here the equality  $2\sigma_i^+ \sigma_i^- = 1 + \sigma_i^z$  was used. The total radiation intensity is given by the integral

$$I(t) \equiv \int \bar{I}(\vec{n}, t) d\Omega(\vec{n}) = I_{inc}(t) + I_{coh}(t) \quad (86)$$

over solid angles. Here the incoherent part is

$$I_{inc}(t) = \frac{1}{2} \omega_0 \gamma \sum_{i=1}^N [1 + s_i(t)] , \quad (87)$$

and the coherent part is

$$I_{\text{coh}}(t) = \omega_0 \gamma \sum_{i \neq j}^N f_{ij} \overline{u_i^*(t) u_j(t)}, \quad (88)$$

where

$$f_{ij} \equiv \int f_{ij}(\vec{n}) d\Omega(\vec{n}), \quad f_{ii} = 1. \quad (89)$$

Finally, the level of coherence can be defined [33] by means of the *coherence coefficients*

$$C_{\text{coh}}(\vec{n}, t) \equiv \frac{I_{\text{coh}}(\vec{n}, t)}{I_{\text{inc}}(\vec{n}, t)}, \quad C_{\text{coh}}(t) \equiv \frac{I_{\text{coh}}(t)}{I_{\text{inc}}(t)}. \quad (90)$$

The radiation is mainly incoherent when  $C_{\text{coh}} \ll 1$  and it is almost purely coherent if  $C_{\text{coh}} \gg 1$ .

## 6. COLLECTIVE LIBERATION OF LIGHT

A system of initially inverted atoms can, due to photon exchange, become strongly correlated, as a result emitting a coherent pulse. This effect of self-organization, accompanied by a coherent burst, is called the Dicke superradiance [34]. This phenomenon is well studied for atoms in vacuum [1,29,30], including different particular cases, such as superradiance in two-component systems [35–37], superradiance from ensembles of three-level molecules [1,38], two-photon superradiance [39,40], and so on (see citations in Refs. [41]). When radiating atoms or molecules are placed in a solid, they interact with phonons [42,43], which can lead to such interesting phenomena as the laser cooling of solids [44, 45].

When an atom is placed in a periodic dielectric structure, in which, due to periodicity, a photonic band gap develops, then spontaneous emission with a frequency inside the band gap can be rigorously forbidden [46,47]. This kind of matter, where photon band gap appears because of the structure periodicity in real space, has been called photonic band-gap materials. The photon band gap also appears in natural dense media due to photon interactions with optical collective excitations, such as phonons, magnons, or excitons [48,49]. One calls this type of the gap the polariton band gap since photons coupled with collective excitations of a medium are termed polaritons.

If a single resonance atom is placed in a medium with a photon band gap, and the atomic transition frequency lies inside this gap, then the spontaneous emission is suppressed, which is named the localization of light [46,47]. This

effect is caused by the formation of a photon–atom bound state [50–52]. When a collection of identical resonance atoms is doped into a medium with a photon band gap, so that the atomic transition frequency is inside this gap, then the atoms, in principle, can radiate because of the formation of a photonic impurity band within the photon band gap [50,53–55]. A model case of a concentrated sample, whose linear size  $L$  is much smaller than the radiation wavelength  $\lambda$ , has been considered for studying superradiance near a photonic band gap [56,57], when the transition frequency almost coincides with the frequency of the upper band edge. Here, following Ref. 58, we study the realistic case of a sample with  $\lambda \ll L$ .

Assume that the localization of light occurs for a single atom with an electric dipole transition, so that its population difference is always  $s_0 = s(0)$ . Considering an ensemble of resonance atoms, we resort to Eqs. (68), (69), and (70). For simplicity, we write  $u_i = u$  and  $s_i = s$ . Introduce the effective coupling parameters

$$g \equiv \frac{3\gamma}{4\gamma_2} \sum_j \frac{\sin(k_0 r_{ij})}{k_0 r_{ij}}, \quad g' \equiv \frac{3\gamma}{4\gamma_2} \sum_j \frac{\cos(k_0 r_{ij})}{k_0 r_{ij}}, \quad (91)$$

where  $\gamma \equiv 4k_0^3 d_0^2/3$ . In the absence of resonator imposing a selected mode,

$$g \approx g' \approx \frac{3\gamma}{4\gamma_2} \rho \lambda^3, \quad (92)$$

where  $\rho$  is the density of resonance atoms. It is convenient to introduce the effective frequency and effective attenuation defined, respectively, as

$$\Omega \equiv \omega_0 + \gamma_2 g' s, \quad \Gamma \equiv \gamma_2 (1 - g s). \quad (93)$$

These expressions include the influence of local fields [59] through the coupling parameters (91). Since the latter take into account the existence of an ensemble of atoms, we may call  $\Omega$  and  $\Gamma$  the *collective frequency* and *collective width*, respectively.

With these notations, Eq. (68) reduces to

$$\frac{du}{dt} = -(i\Omega + \Gamma) u + s \xi + \gamma_2 \vec{e}_d^2 (g + i g') s u^*, \quad (94)$$

where  $\xi = \xi_i$  and  $\vec{e}_d \equiv \vec{d}/d_0$ . Equation (69) becomes

$$\begin{aligned} \frac{ds}{dt} = & -4 \gamma_2 g |u|^2 - \gamma_1 (s - s_0) - 2 (u^* \xi + u \xi^*) - \\ & - 2 \gamma_2 \left[ (g + i g') (u^* \vec{e}_d)^2 + (g - i g') (u \vec{e}_d^*)^2 \right], \end{aligned} \quad (95)$$



where  $\zeta = s_0$  takes into account that for a single atom the localization of light occurs. And for Eq. (70), we have

$$\begin{aligned} \frac{d|u|^2}{dt} = & -2 \Gamma |u|^2 + s (u^* \xi + \xi^* u) + \\ & + \gamma_2 s \left[ (g + i g') (u^* \vec{e}_d)^2 + (g - i g') (u \vec{e}_d^*)^2 \right]. \end{aligned} \quad (96)$$

Let us accept the natural inequalities

$$\frac{\gamma_1}{\Omega} \ll 1, \quad \frac{\gamma_2}{\Omega} \ll 1, \quad \left| \frac{\Gamma}{\Omega} \right| \ll 1. \quad (97)$$

And, as always, we keep in mind that the interaction term (67) is small as compared to the frequency  $\Omega$ , or that  $\ll \xi \gg = 0$ , which tells that this term is small on average. Then, according to Sec. 2, we may classify the solution  $u$  as fast while  $s$  and  $|u|^2$  as slow. Solving Eq. (94), with  $s$  being a quasi-invariant, we get

$$u(t) = \left[ u_0 + s \int_0^t e^{(i\Omega + \Gamma)t'} \xi(t') dt' \right] e^{-(i\Omega + \Gamma)t}. \quad (98)$$

Introduce the notation

$$\alpha \equiv \lim_{\tau \rightarrow \infty} \frac{\text{Re}}{\tau \Gamma s} \int_0^\tau \ll \xi^*(t) u(t) \gg dt, \quad (99)$$

where Re means the real part and which, if  $\ll \xi \gg = 0$ , takes the form

$$\alpha = \lim_{\tau \rightarrow \infty} \frac{\text{Re}}{\tau \Gamma} \int_0^\tau dt \int_0^t e^{-(i\Omega + \Gamma)(t-t')} \ll \xi^*(t) \xi(t') \gg dt'.$$

When  $\xi(t)$  is a stochastic variable corresponding to a stationary random process, so that

$$\ll \xi^*(t) \xi(t') \gg = \ll \xi^*(t - t') \xi(0) \gg,$$

then the notation (99) becomes

$$\alpha = \lim_{\tau \rightarrow \infty} \frac{\text{Re}}{\tau \Gamma} \int_0^\tau dt \int_0^t e^{-(i\Omega + \Gamma)t'} \ll \xi^*(t') \xi(0) \gg dt'.$$

Defining a new function

$$w \equiv |u|^2 - \alpha s^2, \quad (100)$$

and averaging the right-hand sides of Eqs. (95) and (96) over time and over stochastic variables we get

$$\frac{ds}{dt} = -4 g \gamma_2 w - \gamma_1^* (s - \zeta^*), \quad \frac{d|u|^2}{dt} = -2 \Gamma w,$$

where

$$\gamma_1^* \equiv \gamma_1 + 4 \gamma_2 \alpha, \quad \zeta^* \equiv \frac{\gamma_1}{\gamma_1^*} s_0.$$

In what follows, we assume that the quantity (99), describing the intensity of interaction between atoms and matter, is small,

$$|\alpha| \ll 1. \quad (101)$$

To understand the structure of the atom–matter coupling  $\alpha$ , we may model the random variable  $\xi$  by the interaction of an atom with an ensemble of oscillators as

$$\xi(t) = \sum_{\omega} \gamma_{\omega} (b_{\omega} e^{-i\omega t} + b_{\omega}^{\dagger} e^{i\omega t}),$$

where  $b_{\omega}$  and  $b_{\omega}^{\dagger}$  are Bose operators. Then the atom-matter coupling is

$$\alpha = \sum_{\omega} \gamma_{\omega}^2 \left[ \frac{n_{\omega}}{(\omega - \Omega)^2 + \Gamma^2} + \frac{1 + n_{\omega}}{(\omega + \Omega)^2 + \Gamma^2} \right],$$

with  $n_{\omega} \equiv \ll b_{\omega}^{\dagger} b_{\omega} \gg$ . If the coupling  $\alpha$  is small, then  $\gamma_1^* \approx \gamma_1$ ,  $\zeta^* \approx s_0$ , and  $d|u|^2/dt \approx dw/dt$ . Therefore, we obtain the equations

$$\frac{ds}{dt} = -4 g \gamma_2 w - \gamma_1 (s - s_0), \quad \frac{dw}{dt} = -2 \gamma_2 (1 - g s) w. \quad (102)$$

For transient times, when  $t \ll \gamma_1^{-1}$ , Eqs. (102) can be solved explicitly, giving

$$s = -\frac{\gamma_0}{g\gamma_2} \tanh\left(\frac{t - t_0}{\tau_0}\right) + \frac{1}{g}, \quad w = \frac{\gamma_0^2}{4g^2\gamma_2^2} \operatorname{sech}^2\left(\frac{t - t_0}{\tau_0}\right), \quad (103)$$

where the integration constants  $\gamma_0 = \tau_0^{-1}$  and  $t_0$  are defined by the initial conditions  $u(0) = u_0$  and  $s(0) = s_0$ . For the radiation width  $\gamma_0$ , we get the equation

$$\gamma_0^2 = \Gamma_0^2 + 4g^2\gamma_2^2 (|u_0|^2 - \alpha_0 s_0^2), \quad (104)$$

where

$$\Gamma_0 \equiv \gamma_2(1 - gs_0), \quad \gamma_0 \equiv \frac{1}{\tau_0}, \quad \alpha_0 \equiv \alpha(0).$$

For the delay time, we find

$$t_0 = \frac{\tau_0}{2} \ln \left| \frac{\gamma_0 - \Gamma_0}{\gamma_0 + \Gamma_0} \right|. \quad (105)$$

Introducing the critical coupling

$$\alpha_c \equiv \frac{(1 - gs_0)^2}{4g^2 s_0^2} + \frac{|u_0|^2}{s_0^2}, \quad (106)$$

we may rewrite the radiation width as

$$\gamma_0 = 2g |s_0| \gamma_2 \sqrt{\alpha_c - \alpha_0}. \quad (107)$$

In the case of only one atom, we have to set  $g = 0$ . Then Eqs. (102) give

$$s = s_0, \quad w = (|u_0|^2 - \alpha_0 s_0^2) e^{-2\gamma_2 t} \quad (g = 0),$$

which means that the light is localized. But for an ensemble of atoms the radiation becomes possible.

To find out what happens at large times, when  $t \rightarrow \infty$ , we need to analyse the stationary solutions of Eqs. (102). There are two pairs of such solutions:

$$s_1^* = s_0, \quad w_1^* = 0 \quad (108)$$

and

$$s_2^* = \frac{1}{g}, \quad w_2^* = \frac{\gamma_1(gs_0 - 1)}{4g^2\gamma_2}. \quad (109)$$

The stability analysis [58] shows that the fixed point (108) is stable for  $gs_0 < 1$  and unstable for  $gs_0 > 1$ , when the point (109) becomes stable. When  $gs_0 < 1$ , the stationary point (108) is a stable node, while that (109) is a saddle point. In the interval  $1 < gs_0 \leq 1 + \gamma_1/8\gamma_2$ , the fixed point (108) is a saddle point, and that (109) is a stable node. For  $gs_0 > 1 + \gamma_1/8\gamma_2$ , the stationary solutions (108) correspond again to a saddle point, while the fixed point (109) becomes a stable focus. In the latter case, the pulsing regime of radiation is realized, with the asymptotic period between pulses

$$T_p = \frac{4\pi}{|\gamma_1^2 + 8(1 - gs_0)\gamma_1\gamma_2|^{1/2}}. \quad (110)$$

However, at finite times the radiation pulses are not periodic, so that the characteristic time (110) is an approximate period only for  $t \rightarrow \infty$ .

In this way, when a single atom cannot radiate because of the localization of light, an ensemble of atoms can emit coherent radiation, provided that the interaction between atoms is sufficiently strong, so that  $gs_0 > 1$ . This is why such an effect can be called the collective liberation of light. However, this liberation is not complete but only partial since  $s_2^* > 0$ .

## 7. AMPLIFICATION BY NONRESONANT FIELDS

An essential enhancement of radiation can occur due to correlations between radiators, which results in the emission of a coherent pulse. In order that these correlations could be sufficiently strong, it is usually required that the radiation wavelength would be much larger than the mean distance between radiators. If the latter is not the case, it is hardly probable that the self-organized coherence can develop. How would it be possible to amplify the radiation intensity for a system of radiators whose wavelength is smaller than or comparable with the mean distance between them? This question is of high importance for short-wave emission such as  $x$ -ray and  $\gamma$ -ray radiation. Coherent transient effects due to phase modulation of recoilless  $\gamma$  radiation have been considered both theoretically and experimentally [60–63]. A regenerated signal of gamma echo has been observed [64], which is similar to photon echo in optics [65]. In the present section we explore the conditions when *stationary* enhancement of short-wave radiation is feasible, being due to external nonresonant fields. Some preliminary results on the problem have been reported [66–68], based on simplified models. Here the problem is considered more accurately, using the main Eqs. (68) to (70). The latter, in the case of short-wave radiation, when the interaction of radiators can be neglected, take the form

$$\frac{du_i}{dt} = -(i\omega_0 + \gamma_2)u_i - is_i \vec{d} \cdot \vec{E}_{0i}, \quad (111)$$

$$\frac{ds_i}{dt} = 2i(u_i^* \vec{d} - u_i \vec{d}^*) \cdot \vec{E}_{0i} - \gamma_1(s_i - \zeta), \quad (112)$$

$$\frac{d|u_i|^2}{dt} = -2\gamma_2|u_i|^2 - is_i(u_i^* \vec{d} - u_i \vec{d}^*) \cdot \vec{E}_{0i}. \quad (113)$$

The initial conditions are  $u_i(0) = u_0$  and  $s_i(0) = s_0$ .

Assuming, as usual, the existence of small parameters

$$\frac{\gamma_1}{\omega_0} \ll 1, \quad \frac{\gamma_2}{\omega_0} \ll 1, \quad \frac{|\vec{d} \cdot \vec{E}_{0i}|}{\omega_0} \ll 1, \quad (114)$$

we see that  $u_i$  has to be classified as a fast solution while  $s_i$  and  $|u_i|^2$ , as slow ones. With  $s_i$  being a quasi-invariant, Eq. (111) gives

$$u_i(t) = e^{-(i\omega_0 + \gamma_2)t} \left[ u_0 - i s_i \vec{d} \cdot \int_0^t \vec{E}_{0i}(\tau) e^{(i\omega_0 + \gamma_2)\tau} d\tau \right].$$

Let the external field  $\vec{E}_{0i} = \vec{E}_{0i}(t)$  consist of two parts,

$$\vec{E}_{0i} = \vec{E}_0 + \vec{E}_1 e^{i(\vec{k} \cdot \vec{r}_i - \omega t)} + \vec{E}_1^* e^{-i(\vec{k} \cdot \vec{r}_i - \omega t)}, \quad (115)$$

one being a stationary nonresonant field  $\vec{E}_0$ , and another part is a pair of plane waves, which are in quasiresonance with the transition frequency,

$$\frac{|\Delta|}{\omega_0} \ll 1, \quad \Delta \equiv \omega - \omega_0. \quad (116)$$

Then the solution of Eq. (111) writes

$$u_i(t) = -\frac{s_i \vec{d} \cdot \vec{E}_0}{\omega_0 - i\gamma_2} + \frac{s_i \vec{d} \cdot \vec{E}_1}{\Delta + i\gamma_2} e^{i(\vec{k} \cdot \vec{r}_i - \omega t)} + \left( u_0 + \frac{s_i \vec{d} \cdot \vec{E}_0}{\omega_0 - i\gamma_2} - \frac{s_i \vec{d} \cdot \vec{E}_1}{\Delta + i\gamma_2} e^{i \vec{k} \cdot \vec{r}_i} \right) e^{-(i\omega_0 + \gamma_2) t}. \quad (117)$$

Substituting this into the right-hand side of Eq. (112) and averaging over time as

$$\lim_{\tau \rightarrow \infty} \frac{1}{\tau} \int_0^\tau f(s, t) dt,$$

we come to the equation

$$\frac{ds_i}{dt} = -\gamma_1^* (s_i - \zeta^*), \quad (118)$$

with

$$\gamma_1^* \equiv \gamma_1 + 4\gamma_2 \left( \frac{|\vec{d} \cdot \vec{E}_0|^2}{\omega_0^2 + \gamma_0^2} + \frac{|\vec{d} \cdot \vec{E}_1|^2}{\Delta^2 + \gamma_2^2} \right), \quad \zeta^* \equiv \frac{\gamma_1}{\gamma_1^*} \zeta.$$

The solution to Eq. (118) is

$$s_i(t) = s_0 e^{-\gamma_1^* t} + \zeta^* (1 - e^{-\gamma_1^* t}). \quad (119)$$

Calculating the correlation function

$$\overline{u_i^*(t) u_j(t)} = s^2(t) \left( \frac{|\vec{d} \cdot \vec{E}_0|^2}{\omega_0^2 + \gamma_2^2} + \frac{|\vec{d} \cdot \vec{E}_1|^2}{\Delta^2 + \gamma_2^2} e^{-i \vec{k} \cdot \vec{r}_{ij}} \right),$$

where, for simplicity, we set  $s_i = s$ , we find the incoherent and coherent radiation intensities (84) and (85), respectively, as

$$I_{inc}(\vec{n}, t) = \frac{3N}{16\pi} \omega_0 \gamma |\vec{n} \times \vec{e}_d|^2 [1 + s(t)],$$

$$I_{coh}(\vec{n}, t) = \frac{3N^2}{8\pi} \omega_0 \gamma |\vec{n} \times \vec{e}_d|^2 s^2(t) \times$$

$$\times \left[ F(k_0 \vec{n}) \frac{|\vec{d} \cdot \vec{E}_0|^2}{\omega_0^2 + \gamma_2^2} + F(k_0 \vec{n} - \vec{k}) \frac{|\vec{d} \cdot \vec{E}_1|^2}{\Delta^2 + \gamma_2^2} \right], \quad (120)$$

where  $\vec{n} \equiv \vec{r}/r$  and the form factor is

$$F(\vec{k}) \equiv \frac{1}{N^2} \sum_{i \neq j}^N e^{i\vec{k} \cdot \vec{r}_{ij}} = \left| \frac{1}{N} \sum_{i=1}^N e^{i\vec{k} \cdot \vec{r}_i} \right|^2. \quad (121)$$

As is seen from expressions (120) and (121), the maxima of coherent radiation occur in the directions satisfying the condition

$$(k_0 \vec{n} - \vec{k}) \cdot \vec{r}_i = 2\pi n_i \quad (n_i = 0, 1, 2, \dots). \quad (122)$$

This corresponds either to forward scattering, when all  $n_i = 0$ , and the periodicity of matter is not required, or to the scattering in the Bragg directions, for which the strict space periodicity of radiators is needed. The enhancement of coherent radiation in the directions defined by condition (122) is called the Borrmann effect [69,70], which for the case of  $\gamma$ -rays is sometimes termed the Kagan–Afanasiev effect [71,72].

The total radiation intensities (87) and (88) are

$$I_{inc}(t) = \frac{1}{2} N \omega_0 \gamma [1 + s(t)],$$

$$I_{coh}(t) = N^2 \varphi \omega_0 \gamma s^2(t) \left( \frac{|\vec{d} \cdot \vec{E}_0|^2}{\omega_0^2 + \gamma_2^2} + \frac{|\vec{d} \cdot \vec{E}_1|^2}{\Delta^2 + \gamma_2^2} \right), \quad (123)$$

where the shape factor is

$$\varphi \equiv \frac{3}{8\pi} \int |\vec{n} \times \vec{e}_d|^2 F(k_0 \vec{n} - \vec{k}) d\Omega(\vec{n}). \quad (124)$$

The value of the latter strongly depends on the shape of the considered sample. Thus, for pencil-like or disk-like shapes [29], one has

$$\varphi = \begin{cases} \frac{3\lambda}{8L}, & \frac{\lambda}{2\pi L} \ll 1, \quad \frac{R}{L} \ll 1 \\ \frac{3}{8} \left( \frac{\lambda}{\pi R} \right)^2, & \frac{\lambda}{2\pi R} \ll 1, \quad \frac{L}{R} \ll 1, \end{cases}$$

where  $R$  and  $L$  are the radius and length of a cylindrical sample, and  $\lambda \equiv \equiv 2\pi/k$ ,  $k \equiv |\vec{k}| = \omega/c$ .

Consider the stationary limit  $t \rightarrow \infty$ , keeping in mind the situation typical of Mössbauer experiments, when the alternating field is weak,

$$\frac{|\vec{d} \cdot \vec{E}_1|^2}{\gamma_1 \gamma_2} \ll 1, \quad (125)$$

and let us set, for simplicity,

$$\zeta = -1 \quad (126)$$

which means that there is no additional pumping except through the given field (115). Then Eq. (119) reduces to

$$\lim_{t \rightarrow \infty} s_i(t) = -1 + \frac{4\gamma_2}{\gamma_1} \left( \frac{|\vec{d} \cdot \vec{E}_0|^2}{\omega_0^2} + \frac{|\vec{d} \cdot \vec{E}_1|^2}{\Delta^2 + \gamma_2^2} \right).$$

For the coherence coefficient, defined in Eq. (90), we get

$$\lim_{t \rightarrow \infty} C_{\text{coh}}(t) = N \frac{\varphi\gamma_1}{2\gamma_2}. \quad (127)$$

The role of the nonresonant field  $\vec{E}_0$  can be characterized by the *switching factor* [24]

$$S(E_0, t) \equiv \frac{I(t)}{\lim_{E_0 \rightarrow 0} I(t)} \quad (128)$$

and its stationary limit

$$S(E_0) \equiv \lim_{t \rightarrow \infty} S(E_0, t). \quad (129)$$

For our case, we obtain

$$S(E_0) = 1 + \frac{\Delta^2 + \gamma_2^2}{\omega_0^2} \frac{|\vec{d} \cdot \vec{E}_0|^2}{|\vec{d} \cdot \vec{E}_1|^2}. \quad (130)$$

The switching factors (128) and (129) show how the radiation intensity is amplified when a nonresonant field  $\vec{E}_0$  is switched on, as compared to the situation when  $\vec{E}_0 = 0$ . As is seen from expression (130), the amplification can be quite noticeable only if  $|\vec{d} \cdot \vec{E}_0| \gg |\vec{d} \cdot \vec{E}_1|$ , so that to compensate the smallness of the parameters  $|\Delta|/\omega_0$  and  $\gamma_2/\omega_0$ .

## 8. MÖSSBAUER MAGNETIC ANOMALY

Stationary fields, electric or magnetic, can be due not to external sources but can arise in a sample as a result of phase transitions [73,74]. If an ensemble of radiators is incorporated into matter exhibiting a phase transition accompanied by the appearance of a constant field, the latter may influence some radiation characteristics. An interesting example of this kind is given by the gamma radiation of Mössbauer nuclei placed into magnetic materials. This example is especially intriguing because of long-standing controversy related to its interpretation.

There exists a number of experiments demonstrating the so-called magnetic anomaly of the Mössbauer effect in materials undergoing magnetic phase transition. This anomaly consists in an essential increase, up to 50%, of the area under the Mössbauer spectrum below the temperature of magnetic transition, as compared to the spectrum area in paramagnetic state above the transition temperature. A detailed discussion of these experiments can be found in the book [75] and review [76]. The controversy related to this anomaly concerns the explanation of the cause of the latter.

The area of the Mössbauer spectrum, for Mössbauer nuclei in a solid sample, is given by the integral

$$A_{abs} = f_M \int_{-\infty}^{+\infty} \sigma_{abs}(\omega) d\omega, \quad (131)$$

in which

$$f_M = \exp(-k_0^2 r_0^2) \quad (132)$$

is the Mössbauer factor,  $k_0 = \omega_0/c$ ,  $r_0$  is the mean-square deviation of the nucleus from a lattice site,

$$\sigma_{abs}(\omega) = \frac{\sigma_0 \Gamma_{abs}^2}{(\omega - \omega_0)^2 + \Gamma_{abs}^2} \quad (133)$$

is the absorption cross-section,  $\Gamma_{abs}$  is the absorption half-width,

$$\sigma_0 = \frac{2\pi(1 + 2I_1)}{k_0^2(1 + 2I_0)(1 + \alpha_e)} \quad (134)$$

is the cross-section of resonant absorption,  $I_0$  and  $I_1$  are the nuclear spins of the ground-state and excited levels, and  $\alpha_e$  is the electron conversion coefficient. After integrating Eq. (131), we have the spectrum area

$$A_{abs} = \pi f_M \sigma_0 \Gamma_{abs}. \quad (135)$$

It is important to emphasize that the Mössbauer anomaly, we consider here, has been observed only in the so-called absorption geometry, when absorbing Mössbauer nuclei are placed inside magnetic matter which is irradiated by an external source. Contrary to this, in the experiments with the so-called source geometry, when a radioactive source is incorporated into the magnetic matter, but absorbing Mössbauer nuclei are outside this matter, no magnetic anomaly has been observed [77–79]. Therefore it is clear that the considered Mössbauer anomaly is directly related to the action on Mössbauer nuclei of an effective magnetic field appearing below the critical point. But what is the origin of this anomaly?



Historically, the first suggestion was to ascribe the anomaly in the temperature behaviour of the spectrum area (135) to the influence of the appearing magnetic order on the Mössbauer factor (132). A number of citations having to do with this suggestion are listed in Refs. 75,76. This assumption implies that the mean-square deviation  $r_0$  defining the Mössbauer factor (132) is essentially influenced by arising magnetic order. The course of reasoning is as follows. Mössbauer nuclei doped into a solid are characterized by the same mean-square deviation as the particles forming the solid sample. The latter can be described by the Hamiltonian

$$\hat{H}_m = \sum_i \frac{\vec{p}_i^2}{2m} + \frac{1}{2} \sum_{i \neq j} \Phi(R_{ij}) - \sum_{i \neq j} I(R_{ij}) \vec{S}_i \cdot \vec{S}_j, \quad (136)$$

in which  $\Phi(R_{ij})$  is a potential of direct pair interactions while  $I(R_{ij})$  is that of exchange interactions,  $\vec{S}_i$  is a spin operator, and  $R_{ij} \equiv |\vec{R}_{ij}|$ , with  $\vec{R}_{ij} = \vec{R}_i - \vec{R}_j$ . The indices of summation in Eq. (136) run as  $i = 1, 2, \dots, N_0$ , with  $N_0$  being the number of lattice sites. Introduce the deviation from a lattice site,

$$\vec{u}_i \equiv \vec{R}_i - \vec{a}_i, \quad (137)$$

defined so that

$$\vec{a}_i = \langle \vec{R}_i \rangle, \quad \langle \vec{u}_i \rangle = 0. \quad (138)$$

Taking into account that  $|\vec{u}_i|$  is small as compared to the interparticle distance, one expands the interaction potential in powers of  $u_i^\alpha$  up to the second order, which results in the Hamiltonian

$$\hat{H}_m = U_0 + \hat{H}_p + \hat{H}_s + \hat{H}_{sp} + \hat{H}', \quad (139)$$

whose terms are explained below: the constant part of the lattice energy

$$U_0 = \frac{1}{2} \sum_{i \neq j} \Phi(a_{ij}); \quad a_{ij} \equiv |\vec{a}_{ij}|, \quad \vec{a}_{ij} \equiv \vec{a}_i - \vec{a}_j, \quad (140)$$

the phonon term

$$\hat{H}_p = \sum_i \frac{\vec{p}_i^2}{2m} + \frac{1}{2} \sum_{i \neq j} \sum_{\alpha\beta} \Phi_{ij}^{\alpha\beta} u_i^\alpha u_j^\beta, \quad (141)$$

in which  $\Phi_{ij}^{\alpha\beta} \equiv \partial^2 \Phi(a_{ij}) / \partial a_i^\alpha \partial a_j^\beta$ , the spin Hamiltonian

$$\hat{H}_s = - \sum_{i \neq j} I(a_{ij}) S_{ij}; \quad S_{ij} \equiv \vec{S}_i \cdot \vec{S}_j, \quad (142)$$

the term responsible for spin–phonon interactions,

$$\hat{H}_{sp} = - \sum_{i \neq j} \sum_{\alpha\beta} I_{ij}^{\alpha\beta} u_i^\alpha u_j^\beta S_{ij}, \quad (143)$$

where  $I_{ij}^{\alpha\beta} = \partial^2 I(a_{ij}) / \partial a_i^\alpha \partial a_j^\beta$ , and the term

$$\hat{H}' = - \sum_i \sum_\alpha u_i^\alpha \left( 1 + \frac{1}{2} \sum_\beta u_i^\beta \frac{\partial}{\partial a_i^\beta} \right) F_i^\alpha \quad (144)$$

related to the striction energy, where the striction force acting on the site  $i$  is given by the components

$$F_i^\alpha \equiv - \frac{\partial}{\partial a_i^\alpha} \sum_j [\Phi(a_{ij}) - 2I(a_{ij})S_{ij}].$$

The correct definition of the lattice sites in Eq. (138) presupposes that they serve as equilibrium positions for particles. This implies that the striction energy is to be zero on average,

$$\langle \hat{H}' \rangle = 0. \quad (145)$$

Then one invokes a kind of the semiclassical approximation

$$\langle u_i^\alpha S_{ij} \rangle = \langle u_i^\alpha \rangle \langle S_{ij} \rangle = 0, \quad \langle u_i^\alpha u_j^\beta S_{ij} \rangle = \langle u_i^\alpha u_j^\beta \rangle \langle S_{ij} \rangle,$$

decoupling the phonon and spin degrees of freedom, which suggests to present the operator term in the spin–phonon interaction (143) as

$$u_i^\alpha u_j^\beta S_{ij} = \langle u_i^\alpha u_j^\beta \rangle S_{ij} + u_i^\alpha u_j^\beta \langle S_{ij} \rangle - \langle u_i^\alpha u_j^\beta \rangle \langle S_{ij} \rangle. \quad (146)$$

Thus, the matter Hamiltonian (139) can be reduced to

$$\hat{H}_m = \bar{U}_0 + \hat{H}_p + \hat{H}_s, \quad (147)$$

with the renormalized terms

$$\bar{U}_0 = U_0 + \sum_{i \neq j} \sum_{\alpha\beta} I_{ij}^{\alpha\beta} \langle u_i^\alpha u_j^\beta \rangle \langle S_{ij} \rangle,$$

$$\hat{H}_p = \sum_i \frac{\vec{p}_i^2}{2m} + \frac{1}{2} \sum_{i \neq j} \sum_{\alpha\beta} D_{ij}^{\alpha\beta} u_i^\alpha u_j^\beta, \quad \hat{H}_s = - \sum_{i \neq j} J_{ij} S_{ij},$$

in which the striction energy, because of condition (145), is omitted and the renormalized interactions are

$$D_{ij}^{\alpha\beta} \equiv \Phi_{ij}^{\alpha\beta} - 2I_{ij}^{\alpha\beta} \langle S_{ij} \rangle, \quad J_{ij} \equiv I(a_{ij}) + \sum_{\alpha\beta} I_{ij}^{\alpha\beta} \langle u_i^\alpha u_j^\beta \rangle .$$

The renormalized dynamical matrix  $D_{ij}^{\alpha\beta}$  defines the effective phonon spectrum  $\omega_{ks}$  through the eigenvalue problem

$$\frac{1}{m} \sum_j \sum_\beta D_{ij}^{\alpha\beta} e^{-\vec{k} \cdot \vec{a}_{ij}} e_{ks}^\beta = \omega_{ks}^2 e_{ks}^\alpha ,$$

where  $\vec{e}_{ks}$  is a polarization vector; the index  $s$ , labelling polarizations. The spectrum and polarization vectors are assumed to be even functions of the wave vector, so that  $\omega_{ks} = \omega_{-ks}$  and  $\vec{e}_{ks} = \vec{e}_{-ks}$ . Polarization vectors form a complete orthonormal basis with the properties

$$\vec{e}_{ks} \cdot \vec{e}_{ks'} = \delta_{ss'} , \quad \sum_s e_{ks}^\alpha e_{ks}^\beta = \delta_{\alpha\beta} .$$

Expanding the deviation and momentum as

$$\vec{u}_i = \sum_{ks} \frac{\vec{e}_{ks}}{\sqrt{2mN_0\omega_{ks}}} (b_{ks} + b_{-ks}^\dagger) e^{i\vec{k} \cdot \vec{a}_i} ,$$

$$\vec{p}_i = -i \sum_{ks} \sqrt{\frac{m\omega_{ks}}{2N_0}} \vec{e}_{ks} (b_{ks} - b_{-ks}^\dagger) e^{i\vec{k} \cdot \vec{a}_i} ,$$

one transforms the renormalized phonon Hamiltonian to the standard form

$$\hat{H}_p = \sum_{ks} \left( b_{ks}^\dagger b_{ks} + \frac{1}{2} \right) .$$

After this, it is straightforward to calculate the correlators

$$\langle u_i^\alpha u_j^\beta \rangle = \frac{\delta_{ij}}{2N_0} \sum_{ks} \frac{e_{ks}^\alpha e_{ks}^\beta}{m \omega_{ks}} \coth \frac{\omega_{ks}}{2T} ,$$

in which  $T$  is temperature. Thus, one gets the mean-square deviation from the equation

$$r_0^2 \equiv \frac{1}{3} \sum_\alpha \langle u_i^\alpha u_i^\alpha \rangle = \frac{1}{6mN_0} \sum_{ks} \frac{1}{\omega_{ks}} \coth \frac{\omega_{ks}}{2T} . \quad (148)$$

In this way, the influence of magnetic order on the mean-square deviation comes from its influence on the phonon spectrum.

However, the magnitude of the spin-phonon interaction, renormalizing the dynamical matrix, is rather small, as compared to the magnitude of direct interactions [76], so that  $|I_{ij}^{\alpha\beta}/\Phi_{ij}^{\alpha\beta}| \sim 10^{-3}$ . Hence, magnetic order cannot influence much phonon frequencies, as well as the sound velocity

$$c_s \equiv \lim_{k \rightarrow 0} \frac{\omega_{ks}}{k} = - \lim_{k \rightarrow 0} \sum_j \sum_{\alpha\beta} D_{ij}^{\alpha\beta} \frac{(\vec{k} \cdot \vec{a}_{ij})^2}{2mk^2} e_{ks}^\alpha e_{ks}^\beta. \quad (149)$$

This conclusion is in agreement with all known experiments where phonon characteristics have been examined by means of neutron scattering, sound-velocity measurements, elastic and thermal investigations. The onset of magnetic order can change the Mössbauer factor not more than by 1%, which cannot explain the observed Mössbauer anomaly of the spectrum area (135).

Another explanation was advanced by Babikova et al. [78], supposing that magnetic order can influence the electron conversion coefficient  $\alpha_e$  in the cross-section (134). A noticeable decrease of the conversion coefficient could lead to the increase of the cross-section (134), and, consequently, to the increase of the spectrum area (135). The decrease of the conversion coefficient could be due to the suppression of the conversion channel in favour of the  $\gamma$ -radiation channel whose weight could be increased by the enhancement of the  $\gamma$ -radiation caused by the arising magnetic order [80].

To estimate the influence of an effective magnetic field, appearing in magnets, on the radiation intensity of Mössbauer nuclei, we have to consider the switching factor (130) that in our case, takes the form

$$S(H_0) = 1 + \frac{\gamma_2^2}{\omega_0^2} \left| \frac{\vec{\mu} \cdot \vec{H}_0}{\vec{\mu} \cdot \vec{H}_1} \right|^2.$$

For the characteristic Mössbauer nucleus  $^{57}\text{Fe}$ , we have  $\omega_0 = 1.44 \times 10^4$  eV and  $\gamma_2 = \gamma_1 = 0.67 \times 10^{-8}$  eV, which can be transformed to the frequency units as  $\omega_0 \sim 10^{19}\text{s}^{-1}$  and  $\gamma_2 \sim \gamma_1 \sim 10^7\text{s}^{-1}$ . The corresponding wavelength is  $\lambda \sim 10^{-8}$  cm. Let us take for the effective magnetic field  $H_0 \sim 10^5$  G and for the alternating source field  $H_1 \sim 10^{-5}$  G. The transition magnetic dipole  $\mu_0 \sim 0.1\mu_n$ , where  $\mu_n$  is the nuclear magneton, hence  $\mu_0 \sim 10^{-13}$  eV/G. This gives  $\mu_0 H_0 \sim 10^7\text{s}^{-1}$  and  $\mu_0 H_1 \sim 10^{-3}\text{s}^{-1}$ . From here we obtain  $\gamma_2^2 H_0^2 / \omega_0^2 H_1^2 \sim 10^{-4}$ , which tells us that the switching factor  $S(H_0)$  changes too little. Therefore, although the arising magnetic order does enhance the radiation of Mössbauer nuclei, this enhancement is not sufficient for causing such a drastic increase of the spectrum area.

The last quantity that could be blamed to be responsible for the Mössbauer magnetic anomaly is the absorption width  $\Gamma_{abs}$ . The latter can be presented as the sum

$$\Gamma_{abs} = \gamma_2 + \gamma_2^* \quad (150)$$

of the homogeneous line width  $\gamma_2$  and of the inhomogeneous line width  $\gamma_2^*$ . The inhomogeneous width can be due to the variation of local magnetic fields resulting in the random shift of the Mössbauer transition frequency [81]. Returning to Section 3, we see that, really, an external magnetic field shifts the transition frequency as  $\omega_0 + (\vec{\mu}_{22} - \vec{\mu}_{11}) \cdot \vec{H}_0$ . Therefore, the inhomogeneous width can be of order  $\gamma_2^* \sim (\vec{\mu}_{22} - \vec{\mu}_{11}) \cdot \vec{H}_0$  or  $\gamma_2^* \sim \mu_0 H_0$ . From here,  $\gamma_2^* \sim 10^7 \text{s}^{-1}$ , that is,  $\gamma_2^* \sim \gamma_2$ . In this way, the anomalous increase of the Mössbauer spectrum area (135) below the magnetic transition temperature can be explained by the increase of the absorption width (150) caused by the increasing inhomogeneous width  $\gamma_2^* \sim \mu_0 H_0$ .

## 9. PROBLEM OF PATTERN SELECTION

Nonequilibrium cooperative phenomena are often described by nonlinear differential or integro-differential equations in partial derivatives. The solutions to such equations are in many cases nonuniform in space exhibiting the formation of different spatial structures. It happens that a given set of equations possesses several solutions corresponding to different spatial patterns [13]. In such a case, the question arises which of these solutions, and respectively patterns, to prefer? The problem of pattern selection has no general solution [13]. A possible way of selecting spatial structures, by minimizing the average energy, was delineated in subsection 2.5. Here we advance another method of pattern selection.

Assume that the considered differential equations in partial derivatives can be reduced to a  $d$ -dimensional system of ordinary equations; the dimensionality  $d$  may equal infinity. Suppose also that admissible patterns are parametrized by a multiparameter  $\beta$ . Let the state of the dynamical system be defined by the set

$$y(t) = \{y_i(t) = y_i(\beta, t) \mid i = 1, 2, \dots, d\} \quad (151)$$

of solutions to the system of differential equations

$$\frac{d}{dt} y(t) = v(y, t). \quad (152)$$

For different parameters  $\beta$  there are different sets (151) corresponding to different spatial structures. All admissible values of  $\beta$  form a manifold  $\mathcal{B} = \{\beta\}$ . Each particular value of  $\beta$  can be considered as a realization of the random variable

from the manifold  $\mathcal{B}$ . The classification of the states (151) can be done by defining a probability measure on  $\mathcal{B}$ .

To introduce the probability distribution  $p(\beta, t)$  of patterns at time  $t$ , we resort to the ideas of statistical mechanics [82], where a probability  $p$  can be connected with entropy  $S$  by the relation  $p \sim e^{-S}$ . The entropy at time  $t$  may be expressed as

$$S(t) \equiv \ln |\Delta\Phi(t)| \quad (153)$$

through the elementary phase volume

$$\Delta\Phi(t) \equiv \prod_i \delta y_i(t). \quad (154)$$

Let us count the entropy from its initial value  $S(0)$ , thus, considering the entropy variation

$$\Delta S(t) \equiv S(t) - S(0). \quad (155)$$

Then the probability distribution  $p \sim e^{-\Delta S}$ , normalized by the condition

$$\int p(\beta, t) d\beta = 1$$

takes the form

$$p(\beta, t) = \frac{e^{-\Delta S(\beta, t)}}{Z(t)}, \quad (156)$$

where the normalization factor is

$$Z(t) = \int e^{-\Delta S(\beta, t)} d\beta.$$

The entropy variation (155) writes

$$\Delta S(t) = \ln \left| \frac{\Delta\Phi(t)}{\Delta\Phi(0)} \right|, \quad (157)$$

where the dependence on  $\beta$ , for brevity, is omitted. Define the multiplier matrix [83]

$$M(t) = [M_{ij}(t)], \quad M_{ij}(t) \equiv \frac{\delta y_i(t)}{\delta y_j(0)}, \quad (158)$$

for which at the initial time one has

$$M_{ij}(0) \equiv \frac{\delta y_i(0)}{\delta y_j(0)} = \delta_{ij}. \quad (159)$$

The variation of the state (151) gives

$$\delta y(t) = M(t) \delta y(0), \quad (160)$$

which yields for the elementary phase volume (154)

$$\Delta\Phi(t) = \prod_i \sum_j M_{ij}(t) \delta y_j(0) .$$

Hence, the entropy variation (157) is

$$\Delta S(t) = \ln \left| \prod_i \sum_j M_{ij}(t) M_{ji}(0) \right| .$$

With condition (159), this results in

$$\Delta S(t) = \ln \left| \prod_i M_{ii}(t) \right| = \sum_i \ln |M_{ii}(t)| . \quad (161)$$

Taking the variational derivative of equation (152), we get the equation

$$\frac{d}{dt} M(t) = J(y, t) M(t) \quad (162)$$

for the multiplier matrix (158), where

$$J(y, t) = [J_{ij}(y, t)] , \quad J_{ij}(y, t) \equiv \frac{\delta v_i(y, t)}{\delta y_j(t)} , \quad (163)$$

is the Jacobian matrix. Substituting the entropy variation (161) into Eq. (156), we get

$$p(\beta, t) = \frac{\prod_i |M_{ii}(\beta, t)|^{-1}}{Z(t)} , \quad (164)$$

with

$$Z(t) = \int \prod_i |M_{ii}(\beta, t)|^{-1} d\beta .$$

Expression (164) defines the probability distribution of patterns labelled by a multiparameter  $\beta$ . This expression naturally connects the notion of probability and the notion of stability. Really, the multipliers are smaller by modulus for more stable solutions and, respectively, patterns, for which the probability is higher.

Another form of the distribution (164) can be derived as follows. Introduce the matrix

$$L(t) = [L_{ij}(t)] , \quad L_{ij}(t) \equiv \ln |M_{ij}(t)| . \quad (165)$$

Then the entropy variation (161) becomes

$$\Delta S(t) = \text{Tr } L(t) . \quad (166)$$

Since the trace of a matrix does not depend on its representations, we may perform intermediate transformations of Eq. (166) using one particular representation and returning at the end to the form independent of representations. To this end, let us consider a representation when the multiplier matrix is diagonal. Because of Eq. (162) with the initial condition (159), the matrix  $M$  is diagonal if and only if the Jacobian matrix is also diagonal. Then from the evolution equation (162) it follows that

$$M_{ii}(t) = \exp \left\{ \int_0^t J_{ii}(y(t'), t') dt' \right\}.$$

Hence

$$L_{ii}(t) = \int_0^t \Lambda_i(t') dt', \quad \Lambda_i(t) \equiv \operatorname{Re} J_{ii}(t),$$

from where

$$\operatorname{Tr} L(t) = \int_0^t \Lambda(t') dt', \quad \Lambda(t) \equiv \sum_i \Lambda_i(t).$$

We assume that the state (151) is formed of real functions, so that the velocity field in the evolution equation (152) is also real. Then the eigenvalues of the Jacobian matrix (163) are either real or, if complex, come in complex conjugate pairs. Therefore

$$\sum_i \operatorname{Re} J_{ii}(y, t) = \sum_i J_{ii}(y, t) = \operatorname{Tr} J(y, t).$$

For the entropy variation (166) we obtain

$$\Delta S(t) = \int_0^t \Lambda(t') dt', \quad (167)$$

where

$$\Lambda(t) = \operatorname{Tr} J(y, t) \quad (168)$$

is called [84] the *contraction rate*. The latter is given by the form independent of representations of the Jacobian matrix (163). With the entropy variance (167), the probability distribution (156) becomes

$$p(\beta, t) = \frac{1}{Z(t)} \exp \left\{ - \int_0^t \Lambda(\beta, t') dt' \right\}, \quad (169)$$

where the contraction rate is defined in Eq. (168) and

$$Z(t) = \int \exp \left\{ - \int_0^t \Lambda(\beta, t') dt' \right\} d\beta.$$



The most probable pattern at a time  $t$  corresponds to the maximum of the distribution (169),

$$\text{abs max}_{\beta} p(\beta, t) \rightarrow \beta(t). \quad (170)$$

One may also define the average pattern at  $t$  as corresponding to

$$\bar{\beta}(t) \equiv \int \beta p(\beta, t) d\beta.$$

The most probable and average patterns, in general, do not coincide, although this may happen, especially with increasing time. To illustrate the latter, consider a particular case when the contraction rate  $\Lambda(\beta, t) = \Lambda(\beta)$  does not depend on time. Then, as  $t \rightarrow \infty$ , we have

$$Z(t) = \int e^{-\Lambda(\beta) t} d\beta \simeq \sqrt{\frac{2\pi}{\Lambda''(\beta_0)t}} \exp\{-\Lambda(\beta_0) t\},$$

where  $\beta_0$  is the point of the minimum of  $\Lambda(\beta)$ , so that

$$\frac{d}{d\beta} \Lambda(\beta) = 0, \quad \Lambda''(\beta) \equiv \frac{d^2}{d\beta^2} \Lambda(\beta) > 0 \quad (\beta = \beta_0).$$

In the distribution

$$p(\beta, t) \simeq \sqrt{\frac{\Lambda''(\beta_0)}{2\pi}} t \exp\{-[\Lambda(\beta) - \Lambda(\beta_0)] t\}$$

one may expand  $\Lambda(\beta)$  near  $\beta = \beta_0$ , which gives

$$p(\beta, t) \simeq \frac{1}{\sqrt{2\pi}\sigma(t)} \exp\left\{-\frac{(\beta - \beta_0)^2}{2\sigma^2(t)}\right\}, \quad \sigma(t) \equiv \frac{1}{\sqrt{\Lambda''(\beta_0)t}}.$$

From here one finds

$$\lim_{t \rightarrow \infty} p(\beta, t) = \delta(\beta - \beta_0).$$

In this way, if differential equations describing a nonequilibrium process have several solutions corresponding to different spatial patterns, the latter can be characterized by the probability distribution (169), with the contraction rate (168). In the case when the multiplier matrix (158) can be calculated, one may use the expression (164) of the probability distribution. If all patterns correspond to stable solutions, it is sufficient to analyse only the beginning of the process of pattern formation. Then for the entropy variation (167) we may write

$$\Delta S(\beta, t) \simeq \Lambda(\beta, 0) t \quad (t \rightarrow 0).$$

Consequently, the most probable pattern, defined by the maximum of the probability distribution (169), that is, by the minimum of the entropy variation (167), is now characterized by the minimum of the contraction rate  $\Lambda(\beta, 0)$  at the initial time.

## 10. TURBULENT PHOTON FILAMENTATION

Spatial structures can appear in radiating systems if the radiation wavelength is much shorter than the system characteristic sizes [13]. For instance, electric field in laser cavities can exhibit a state which bears some analogy with a superfluid vortex [85]. The Maxwell–Bloch equations for slowly varying field amplitudes have been shown to be analogous to hydrodynamic equations for compressible viscous fluid [86]. The Fresnel number for optical systems plays the role similar to the Reynolds number for fluids. In the same way as when increasing the Reynolds number, the fluid becomes turbulent, there can appear optical turbulence when increasing the Fresnel number.

Spatial structures emerge from an initially homogeneous state with a break of space-translational symmetry. For small Fresnel numbers  $F \leq 5$ , such structures correspond to the empty-cavity Gauss–Laguerre modes imposed by the cavity geometry. These transverse structures can be described by expanding fields over the modal Gauss–Laguerre functions [87–92], which results in reasonable agreement with experiments for CO<sub>2</sub> and Na<sub>2</sub> lasers. For large Fresnel numbers  $F > 10$ , the appearing structures are very different from those associated with empty-cavity modes. The modal expansion is no longer relevant at large  $F$ , and the boundary conditions have little or no importance. The laser medium looks like divided in a large amount of parallel independently oscillating uncorrelated filaments [93–100] the number of filaments being proportional to  $F$ , contrary to the case of small Fresnel numbers when the number of bright spots is proportional to  $F^2$ . This filamentation was observed in Dye and CO<sub>2</sub> lasers, as well as in other resonance media, even without resonators [101–105]. The same type of patterns arises in active nonlinear media, such as photorefractive Bi<sub>12</sub>SiO<sub>20</sub> crystal pumped by a laser [106–109]. In the latter media there are also two types of pattern formation: for small Fresnel numbers, the symmetry is imposed through the boundary, while for large Fresnel numbers, the symmetry is imposed by the bulk parameters. In the case of large  $F$ , there occurs a kind of self-organization with spontaneous spatial symmetry breaking [110]. It is possible to easily notice a qualitative transition in the behaviour of photorefractive media as well as in that of lasers: In low- $F$  regime there are a few modes of regular arrangement of bright spots corresponding to the peaks of the Gauss–Laguerre functions in cylindrical geometry, the number of modes being proportional to  $F^2$ . And in the high- $F$  regime there are many modes spatially uncorrelated with each other, which is typical for spatiotemporal chaos, the number of the chaotic filaments being proportional to  $F$ . Short-range spatial correlation is characteristic for turbulence, this is why one calls the similar phenomenon in optics the optical turbulence.

The theory of self-organized photon filamentation in high-Fresnel-number resonant media was suggested in Refs. 33,111–116, where the consideration was based on simplified models and only the stationary regime was analysed. The

choice of filament radii was done by means of the variational principle, as is described in subsection 2.5. Here we present a more general and elaborate theory based on the evolution equations (68) to (70), which includes the description of temporal behaviour, and for defining the characteristics of filaments we employ the method of pattern selection developed in Sec. 9.

First, it is convenient to pass in Eqs. (68) to (70) to continuous representation replacing the sums by integrals according to the rule

$$\sum_{i=1}^N = \int \rho(\vec{r}) d\vec{r},$$

where  $\rho(\vec{r})$  is the spatial density of radiators. Wishing to return to the localized representation, one makes the replacement  $\rho(\vec{r}) = \sum_{i=1}^N \delta(\vec{r} - \vec{r}_i)$ . In the case when the structure of matter is of no importance, it can be treated as uniform on average setting  $\rho(\vec{r}) = \rho \equiv N/V$ . Cooperative optical phenomena are often considered in this representation of uniform medium [117]. Let us stress that the uniformity of matter in no case requires the uniformity of fields or polarization. The solutions to Eqs. (68) to (70) can correspond to highly nonuniform structures.

Introduce the notation

$$f(\vec{r}, t) \equiv f_0(\vec{r}, t) + f_{rad}(\vec{r}, t) \quad (171)$$

for an effective field acting on a radiator with the transition dipole  $\vec{d}$ . This field consists of the term

$$f_0(\vec{r}, t) \equiv -i \vec{d} \cdot \vec{E}_0(\vec{r}, t) \quad (172)$$

due to an external electric field and of the term

$$f_{rad}(\vec{r}, t) \equiv k_0 \langle \vec{d} \cdot \vec{A}_{rad}(\vec{r}, t) \rangle \quad (173)$$

responsible for the action of other radiators. Taking into account Eq. (65), we have

$$f_{rad}(\vec{r}, t) = -\frac{3}{4} i\gamma\rho \int \left[ \varphi(\vec{r} - \vec{r}') u(\vec{r}', t) - e_d^2 \varphi^*(\vec{r} - \vec{r}') u^*(\vec{r}', t) \right] d\vec{r}', \quad (174)$$

where the continuous representation is used, and

$$\varphi(\vec{r}) \equiv \frac{e^{ik_0|\vec{r}|}}{k_0|\vec{r}|}, \quad \gamma \equiv \frac{4}{3} k_0^3 d_0^2.$$

Then Eqs. (68) to (70) acquire the form

$$\frac{du}{dt} = -(i\omega_0 + \gamma_2)u + sf, \quad \frac{ds}{dt} = -2(u^*f + f^*u) - \gamma_1(s - \zeta),$$

$$\frac{d|u|^2}{dt} = -2\gamma_2|u|^2 + s(u^*f + f^*u) . \quad (175)$$

Notice that from the latter two equations one has

$$\frac{d}{dt} (s^2 + 4|u|^2) = -2\gamma_1s(s - \zeta) - 8\gamma_2|u|^2 .$$

We consider a sample of the cylindrical shape typical of lasers. The seed laser field defining the cylinder axis is given by the sum of two running waves,

$$\vec{E}_0(\vec{r}, t) = \vec{E}_1 e^{i(kz - \omega t)} + \vec{E}_1^* e^{-i(kz - \omega t)} , \quad (176)$$

which selects a longitudinal mode. The radius,  $R$ , and length,  $L$ , of the cylinder are such that the following inequalities are valid:

$$\frac{a}{\lambda} \ll 1 , \quad \frac{\lambda}{R} \ll 1 , \quad \frac{R}{L} \ll 1 , \quad (177)$$

where  $a$  is the mean distance between radiators and  $\lambda$ , wavelength. There are also the standard small parameters

$$\frac{\gamma_1}{\omega_0} \ll 1 , \quad \frac{\gamma_2}{\omega_0} \ll 1 , \quad \frac{|\Delta|}{\omega_0} \ll 1 , \quad (178)$$

with  $\Delta \equiv \omega - \omega_0$  being detuning.

The solutions to Eqs. (175) are not necessarily uniform in the whole volume  $V = \pi R^2 L$  of the sample, but may have noticeable values only inside narrow regions of filamentary form, while being almost zero outside these filaments. Consider one such filament, and let us surround it by a cylinder of radius  $b$  so that the magnitude of solutions is an order smaller at the surface of this enveloping cylinder than at its axis. If the profile of a filament is close to the Gaussian  $\exp(-r^2/2r_f^2)$ , with  $r_f$  being the filament radius, then

$$b = \sqrt{2 \ln 10} r_f . \quad (179)$$

In what follows we assume this relation between the radius  $b$  of an enveloping cylinder and the radius  $r_f$  of a filament.

Suppose that there are  $N_f$  filaments in the volume of the sample, the axis of each filament being centered at a point  $\{x_n, y_n\}$ , with  $n = 1, 2, \dots, N_f$ . Let us present the solutions to Eqs. (175) as expansions over enveloping cylinders,

$$u(\vec{r}, t) = \sum_{n=1}^{N_f} u_n(\vec{r}, t) \Theta_n(x, y) e^{ikz} , \quad s(\vec{r}, t) = \sum_{n=1}^{N_f} s_n(\vec{r}, t) \Theta_n(x, y) , \quad (180)$$

where

$$\Theta_n(x, y) \equiv \Theta \left( b - \sqrt{(x - x_n)^2 + (y - y_n)^2} \right)$$

is a unit-step function. The filaments are located randomly in the cross-section of the sample, but so that their enveloping cylinders do not intersect with each other. The interaction between filaments is small, which follows from Eq. (174). This is why they do not form a regular lattice but are distributed randomly.

The function  $\varphi(\vec{r})$  in Eq. (174) oscillates at the distance  $\lambda$ , and the solutions  $u_n$  and  $s_n$  essentially change in the radial direction in the interval  $b$ . Assuming that

$$\frac{\lambda}{b} \ll 1, \quad (181)$$

we may say that, in the radial direction, the function  $\varphi(\vec{r})$  is fastly varying in space, as compared to the slow variation of  $u_n$  and  $s_n$ . For the latter, we define the averages

$$u(t) \equiv \frac{1}{V_n} \int_{\mathbf{V}_n} u_n(\vec{r}, t) d\vec{r}, \quad s(t) \equiv \frac{1}{V_n} \int_{\mathbf{V}_n} s_n(\vec{r}, t) d\vec{r} \quad (182)$$

over the corresponding enveloping cylinder of the volume  $V_n \equiv \pi b^2 L$ , where in the left-hand side of Eq. (182) we, for short, do not write the index  $n$ .

The seed field (176) is needed mainly for selecting a longitudinal mode with cylindrical symmetry, but the amplitude of this field is small, so that

$$\frac{|\vec{d} \cdot \vec{E}_1|}{\gamma_2} \ll 1. \quad (183)$$

The excitation of radiators is accomplished by means of pumping characterized by the pumping parameter  $\zeta$  in Eqs. (175).

Defining the effective coupling parameters

$$g \equiv \frac{3\gamma\rho}{4\gamma_2 V_n} \int_{\mathbf{V}_n} \frac{\sin[k_0|\vec{r} - \vec{r}'| - k(z - z')]}{k_0|\vec{r} - \vec{r}'|} d\vec{r} d\vec{r}', \quad (184)$$

$$g' \equiv \frac{3\gamma\rho}{4\gamma_2 V_n} \int_{\mathbf{V}_n} \frac{\cos[k_0|\vec{r} - \vec{r}'| - k(z - z')]}{k_0|\vec{r} - \vec{r}'|} d\vec{r} d\vec{r}', \quad (185)$$

and the collective frequency and width, respectively,

$$\Omega \equiv \omega_0 + g'\gamma_2 s, \quad \Gamma \equiv \gamma_2(1 - gs), \quad (186)$$

for functions (182) we obtain the equations

$$\frac{du}{dt} = -(i\Omega + \Gamma)u - is\vec{d} \cdot \vec{E}_1 e^{-i\omega t},$$

$$\frac{ds}{dt} = -4g\gamma_2|u|^2 - \gamma_1(s - \zeta) - 4\text{Im} \left( u^* \vec{d} \cdot \vec{E}_1 e^{-i\omega t} \right), \quad (187)$$

$$\frac{d|u|^2}{dt} = -2\Gamma|u|^2 + 2s \text{Im} \left( u^* \vec{d} \cdot \vec{E}_1 e^{-i\omega t} \right).$$

Because of the inequalities (178) and (183), the solution  $u$  in Eqs. (187) is fast, while  $s$  and  $|u|^2$  are slow in time. Using the scale separation approach, we find

$$u(t) = u_0 e^{-(i\Omega + \Gamma)t} + \frac{s \vec{d} \cdot \vec{E}_1}{\omega - \Omega + i\Gamma} \left[ e^{-i\omega t} - e^{-(i\Omega + \Gamma)t} \right]. \quad (188)$$

Introduce the parameter

$$\alpha \equiv \lim_{\tau \rightarrow \infty} \frac{\text{Im}}{\tau \Gamma s} \int_0^\tau u^*(t) \vec{d} \cdot \vec{E}_1 e^{-i\omega t} dt, \quad (189)$$

characterizing the coupling of radiators with the seed field. This, with Eq. (188), gives

$$\alpha = \frac{|\vec{d} \cdot \vec{E}_1|^2}{(\omega - \Omega)^2 + \Gamma^2}. \quad (190)$$

The latter, according to inequality (183), is small,

$$|\alpha| \ll 1. \quad (191)$$

Finally, defining the function

$$w \equiv |u|^2 - \alpha s^2, \quad (192)$$

we obtain the equations

$$\frac{ds}{dt} = -4g\gamma_2 w - \gamma_1(s - \zeta), \quad \frac{dw}{dt} = -2\gamma_2(1 - gs) w. \quad (193)$$

The behaviour of solutions to Eqs. (193) essentially depends on the values of the coupling parameters (184) and (185). To evaluate the latter, we may notice that their integrands diminish and fastly oscillate at the distance of the wavelength  $\lambda$ . If condition (181) holds, we may neglect boundary effects in the integrals (184) and (185) writing approximately

$$\int_{\mathbf{V}_n} f(\vec{r} - \vec{r}') d\vec{r} d\vec{r}' \cong V_n \int_{\mathbf{V}_n} f(\vec{r}) d\vec{r}.$$

Then parameter (184) reduces to

$$g = \frac{3\pi\gamma\rho}{2\gamma_2} \int_0^b r dr \int_{-L/2}^{L/2} \frac{\sin(k_0\sqrt{r^2 + z^2} - kz)}{k_0\sqrt{r^2 + z^2}} dz,$$

where  $r$  is the radial variable. Because of the quasiresonance condition  $|\Delta| \ll \omega_0$ , we have  $k_0 \simeq k$ . With the change of the variable  $x \equiv k(\sqrt{r^2 + z^2} - z)$ , we get

$$g = \frac{3\pi\gamma\rho}{2\gamma_2 k} \int_0^b r dr \int_{kr^2/L}^{kL} \frac{\sin x}{x} dx .$$

In this expression, one can replace  $kL \rightarrow \infty$ , thus obtaining

$$g = \frac{3\pi\gamma\rho}{2\gamma_2 k} \int_0^b \left[ \frac{\pi}{2} - \text{Si} \left( \frac{kr^2}{L} \right) \right] r dr ,$$

where the integral sine appears,

$$\text{Si}(x) \equiv \int_0^x \frac{\sin t}{t} dt = \frac{\pi}{2} + \text{si}(x) , \quad \text{si}(x) \equiv \int_\infty^x \frac{\sin t}{t} dt .$$

Introducing the dimensionless quantity

$$\beta \equiv \frac{kb^2}{L} = \frac{2\pi b^2}{\lambda L} , \quad (194)$$

we come to the coupling parameter

$$g = g(\beta) = \frac{3\pi\gamma\rho L}{4\gamma_2 k^2} \int_0^\beta \left[ \frac{\pi}{2} - \text{Si}(x) \right] dx . \quad (195)$$

This can be integrated explicitly by means of the property

$$\int \text{Si}(x) dx = x \text{Si}(x) + \cos x ,$$

which results in

$$g(\beta) = \frac{3\pi\gamma\rho L}{4\gamma_2 k^2} \left\{ \beta \left[ \frac{\pi}{2} - \text{Si}(\beta) \right] + 1 - \cos \beta \right\} . \quad (196)$$

For the coupling parameter (185), one similarly finds

$$g' = g'(\beta) = - \frac{3\pi\gamma\rho L}{4\gamma_2 k^2} \int_0^\beta \text{Ci}(x) dx , \quad (197)$$

where the integral cosine occurs,

$$\text{Ci}(x) \equiv \int_\infty^x \frac{\cos t}{t} dt .$$

Integrating

$$\int \text{Ci}(x) dx = x \text{Ci}(x) - \sin x ,$$

we finally get

$$g'(\beta) = \frac{3\pi\gamma\rho L}{4\gamma_2 k^2} [\sin \beta - \beta \text{Ci}(\beta)] . \quad (198)$$

To better understand the properties of the coupling parameters, we consider two limiting cases. When  $x \ll 1$ , then

$$\text{Si}(x) \simeq x - \frac{x^3}{18} , \quad \text{Ci}(x) \simeq \gamma_E + \ln x - \frac{x^2}{4} ,$$

where  $\gamma_E = 0.577216$  being the Euler constant. From here

$$g(x) \simeq \frac{3\pi\gamma\rho L}{4\gamma_2 k^2} \left( \frac{\pi}{2} x - \frac{1}{2} x^2 \right) , \quad g'(x) \simeq \frac{3\pi\gamma\rho L}{4\gamma_2 k^2} x |\ln x| .$$

In the opposite case, when  $x \gg 1$ , using

$$\text{Si}(x) \simeq \frac{\pi}{2} - \frac{\cos x}{x} - \frac{\sin x}{x^2} , \quad \text{Ci}(x) \simeq \frac{\sin x}{x} - \frac{\cos x}{x^2} ,$$

we find

$$g(x) \simeq \frac{3\pi\gamma\rho L}{4\gamma_2 k^2} \left( 1 + \frac{\sin x}{x} \right) , \quad g'(x) \simeq \frac{3\pi\gamma\rho L}{4\gamma_2 k^2} \left( \frac{\cos x}{x} \right) .$$

These asymptotic expressions help to analyse the dependence of the coupling parameters on the variable (194) changing in the interval

$$0 < \beta \leq 2F \quad \left( F \equiv \frac{\pi R^2}{\lambda L} \right) . \quad (199)$$

The stability analysis of Eqs. (193), similarly to that given in Ref. 58, shows that, for  $g\zeta < 1$ , the solutions tend to the stationary stable point  $s_1^* = \zeta$ ,  $w_1^* = 0$ , while for  $g\zeta > 1$ , the stable fixed point is

$$s_2^* = \frac{1}{g} , \quad w_2^* = \frac{\gamma_1 (g\zeta - 1)}{4g^2\gamma_2} .$$

In this way, for all  $\beta$  from the interval (199), except the sole case when  $g\zeta = 1$ , there exists a stable fixed point, that is, almost all solutions are stable, independently of the value of  $\beta$ . Following the method of pattern selection from Sec. 9, we can equip the solutions labelled by  $\beta$  with the probabilistic weights (169).



The most probable, among all stable solutions, is that providing the minimum of the initial contraction rate, which for this case is

$$\Lambda(\beta, 0) = -\gamma_1 - 2\gamma_2 (1 - gs_0) . \quad (200)$$

The minimum of this rate requires that

$$\frac{dg}{d\beta} = 0 , \quad s_0 \frac{d^2g}{d\beta^2} > 0 . \quad (201)$$

For  $s_0 > 0$ , one needs the minimum of  $g$ , which gives  $\beta = 4.9$ . From Eq. (194), one has  $b = 0.88\sqrt{\lambda L}$ . And the relation (179) yields

$$r_f = 0.41\sqrt{\lambda L} \quad (s_0 > 0) . \quad (202)$$

When  $s_0 < 0$ , conditions (201) imply the maximum of  $g$ , for which  $\beta = 1.92$ ,  $b = 0.55\sqrt{\lambda L}$ , and the filament radius is

$$r_f = 0.26\sqrt{\lambda L} \quad (s_0 < 0) . \quad (203)$$

This is practically the same value as found for the filaments radius in Refs. 33,111–115 by using the variational principle of subsection 2.5. When the system of radiators is not inverted at the initial time and becomes excited by means of a pulse characterized by the pumping parameter  $\zeta$ , one has to consider the filament radius (203) as corresponding to the most probable pattern. The number of filaments can be defined from the normalization condition

$$\frac{1}{V} \int s(\vec{r}, t) d\vec{r} = \zeta , \quad (204)$$

assuming that the population difference equals +1 inside each filament of radius  $r_f$  and  $-1$  outside of the filaments. Then the number of filaments is

$$N_f = \frac{1}{2} (1 + \zeta) \left( \frac{R}{r_f} \right)^2 . \quad (205)$$

The most probable filament radius (203) and the number of filaments (205) are in good agreement with the values observed in experiments [93–99,101–105]. The considered phenomenon of filamentation can be termed turbulent since the filaments are chaotically distributed in space and for sufficiently strong pumping, when  $g\zeta > 1 + \gamma_1/8\gamma_2$ , each filament is aperiodically flashing in time. The turbulent photon filamentation is a self-organized phenomenon due to the bulk properties of interacting radiators. It practically does not depend on boundary conditions and exists in both types of lasers, the resonator–cavity lasers, such as CO<sub>2</sub> and Dye lasers [93–99], as well as in the resonatorless discharge–tube

lasers, such as lasers on Ne, Tl, Pb, N<sub>2</sub>, and N<sub>2</sub><sup>+</sup> vapors [101–105]. The turbulent filamentation is also principally nonlinear phenomenon. Thus, in low-Fresnel-number lasers ( $F \leq 5$ ) the number of light spots is proportional to  $F^2$ . The same dependence of the number of coherent rays on  $F$  is typical of the initial linearized stage of superfluorescence [118]. However, for high- $F$  lasers ( $F \gg 10$ ) the number of filaments is proportional to  $F$ , which is in agreement with formula (205) giving  $N_f \sim F$ .

## 11. SUPERRADIANT SPIN RELAXATION

When the initial state of a spin system is strongly nonequilibrium, different kinds of spin relaxation can occur. If there are no transverse external fields acting on spins, they relax to an equilibrium state by an exponential law with a longitudinal relaxation time  $T_1$ . When the motion of spins is triggered by a transverse magnetic field, the relaxation is again exponential but with a transverse relaxation time  $T_2$  that is usually much shorter than  $T_1$ . A rather special relaxation regime arises, if the spin system is coupled to a resonator. This can be done by inserting the sample into a coil connected with a resonance electric circuit. Because of the action of resonator feedback field, the motion of spins can become highly coherent resulting in their ultrafast relaxation during a characteristic collective relaxation time much shorter than  $T_2$  [119]. This latter type of collective spin relaxation from a strongly nonequilibrium state in the presence of coupling with a resonator is the most difficult to realize experimentally and to describe theoretically. Experimental difficulties have been overcome in a series of observations of this phenomenon for a system of nuclear spins inside different paramagnetic materials [120–127]. The collective relaxation time of this ultrafast coherent process is inversely proportional to the number of spins,  $N$ , and the intensity of magnetodipole radiation is proportional to  $N^2$ , in the same way as cooperative radiation time and radiation intensity of  $N$  resonant atoms depend on this number in optic superradiance [1,29,30,42,45,59]. This is why the process of collective coherent relaxation of spins has been called *superradiant spin relaxation* or, for short, *spin superradiance*. In the case of spin systems, what is usually measured is not the radiation intensity itself, which is rather weak, but the power of current induced in the resonant circuit [128]. The enhancement of generated pulses by using resonators is, actually, well known in laser optics and is important for realizing superradiance of Rydberg atoms [129] and recombination superradiance in electron-hole or electron-positron plasmas [130]. Resonators can be employed for modifying radiated pulses in optical superradiance [131]. Note also the usage of resonators for amplifying the nuclear spin echo signals in magnets [132,133].

The appearance of strong correlations between spins is due to the resonator feedback field, but not to the photon exchange as it happens for atomic systems. Hence, various quantum effects existing in the interaction of electromagnetic field with atoms [32,134–137] seem to be absent in the case of spin systems. Therefore it looked natural to try, for the theoretical description of relaxation in a spin system coupled with a resonator, to invoke the classical Bloch equations complimented by the Kirchhoff equation for the resonant electric circuit [1,119,138–140]. However, these equations can provide a description of coherent spin relaxation only when the latter is triggered by a coherent pulse, similarly to the semiclassical Bloch – Maxwell equations in optics [1,141,142]. The phenomenon of the self-organized coherent spin relaxation cannot be described by the Bloch – Kirchhoff equations. Then, what initiates spin motion leading to the appearance of purely self-organized spin superradiance? This problem of the origin of pure spin superradiance was posed by Bloembergen and Pound [119]. They also noticed that the thermal Nyquist noise of resonator cannot be a mechanism triggering the motion of spins, since the thermal relaxation time is proportional to the number of spins in the sample and, thus, the thermal damping is to be negligibly small for macroscopic samples. Nevertheless, this notice was forgotten by the following researchers who assumed that it is just the thermal noise of resonator which triggers the spin motion.

To resolve this controversy and to discover the genuine mechanisms originating the spin motion, it was necessary to turn to microscopic models. The system of nuclear spins is characterized [143] by the Hamiltonian

$$\hat{H} = \frac{1}{2} \sum_{i \neq j} H_{ij} - \mu_n \sum_i \vec{B} \cdot \vec{I}_i, \quad (206)$$

in which spins interact through the dipole potential

$$H_{ij} = \frac{\mu_n^2}{r_{ij}^3} \left[ \vec{I}_i \cdot \vec{I}_j - 3 \left( \vec{I}_i \cdot \vec{n}_{ij} \right) \left( \vec{I}_j \cdot \vec{n}_{ij} \right) \right],$$

where  $\mu_n$  is the nuclear magnetic moment,  $\vec{I}_i$  is a nuclear spin operator,  $r_{ij} = |\vec{r}_{ij}|$ ,  $\vec{r}_{ij} = \vec{r}_i - \vec{r}_j$ ,  $\vec{n}_{ij} = \vec{r}_{ij}/r_{ij}$ . The total magnetic field

$$\vec{B} = H_0 \vec{e}_z + H \vec{e}_x$$

contains an external magnetic field  $H_0$  and a resonator feedback field  $H$  defined by the Kirchhoff equation.

The temporal behaviour of a finite number of spins, with  $27 \leq N \leq 343$ , was analysed numerically by computer simulations [144–149]. From various cases studied, we present here some that give the general qualitative understanding of the whole picture. In Figs. 1–4,  $K_{\text{coh}} \equiv P_{\text{coh}}/P_{\text{inc}}$  is a coherence coefficient,

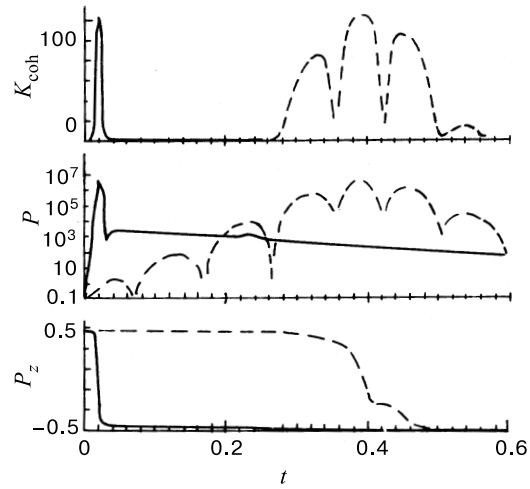


Fig. 1. Coherence coefficient  $K_{\text{coh}}$ , current power  $P$ , and spin polarization  $p_z$  as functions of time for two different coupling parameters defined in Eq. (207),  $g_1$  (solid line) and  $g_2$  (dashed line), with the relation  $g_1/g_2 = 10$

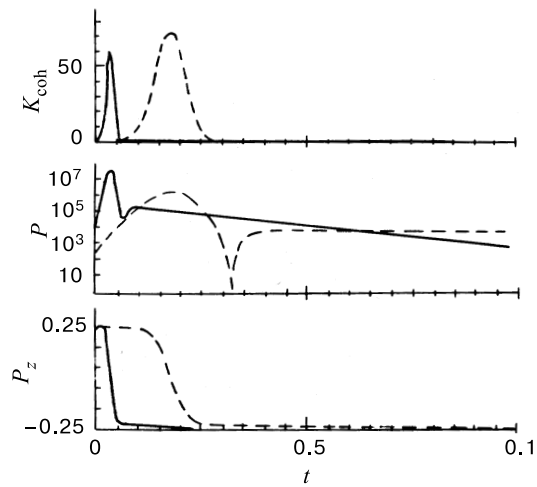


Fig. 2. The same as in Fig. 1 for two different Zeeman frequencies,  $\omega_{01}$  (solid line) and  $\omega_{02}$  (dashed line), related by the ratio  $\omega_{01}/\omega_{02} = 5$

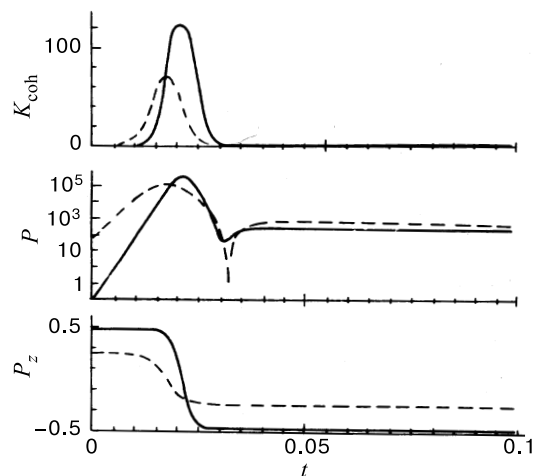


Fig. 3. The same functions as in Fig. 1 for different initial polarizations,  $p_{z1}(0)$  (solid line) and  $p_{z2}(0)$  (dashed line), with the relation  $p_{z1}/p_{z2}(0) = 2$

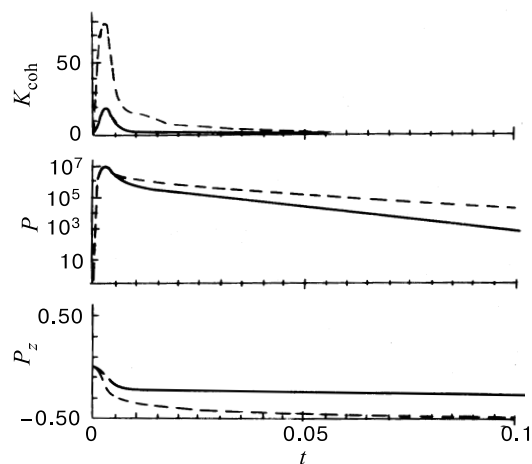


Fig. 4. The same functions as in Fig. 1 for different initial transverse polarizations,  $p_{x1}(0)$  (solid line) and  $p_{x2}(0)$  (dashed line), with the relation  $p_{x1}/p_{x2}(0) = 0.5$

being the ratio of the coherent part of the current power  $P$  to its incoherent part, and  $p_z$  is the negative spin polarization. In Figs. 5–11,  $C_{\text{coh}} \equiv I_{\text{coh}}/I_{\text{inc}}$  is the coherence coefficient of the average magnetodipole radiation defined as in Eq. (90), with respect to the total radiation intensity  $I$ . The current power and radiation intensity are given in dimensionless units and time is measured in

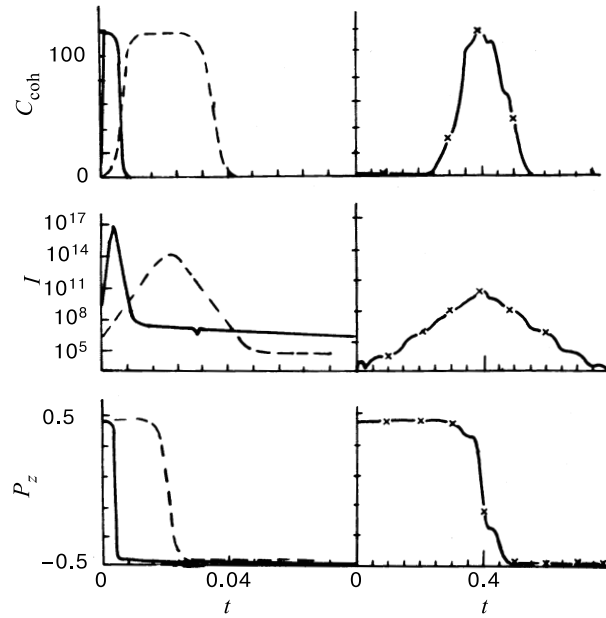


Fig. 5. Coherence coefficient  $C_{\text{coh}}$ , radiation intensity  $I$ , and spin polarization  $p_z$  versus time for  $p_z(0) = 0.48$  and different parameters:  $\omega_0 = 200$ ,  $g = 25$  (solid line);  $\omega_0 = 40$ ,  $g = 25$  (dashed line); and  $\omega_0 = 40$ ,  $g = 2.5$  (solid line with crosses)

units of  $T_2$ . In the figure captions,  $p_z(0)$  and  $p_x(0)$  mean the corresponding polarization components at the initial time,  $\omega_0$  is the Zeeman frequency,  $\omega$  is the natural frequency of the resonant electric circuit and also a frequency of an alternating magnetic field, if any, the amplitude of the latter being denoted by  $h_0$ . The quantity

$$g \equiv \pi^2 \eta \frac{\rho_n \mu_n^2 \omega_0}{\hbar \Gamma_2 \omega} \quad (207)$$

is the effective coupling parameter, in which  $\eta$  is a filling factor;  $\rho_n$ , nuclear density; and  $\Gamma_2 = T_2^{-1}$  is a line width. Computer simulations proved that pure spin superradiance does exist with no thermal noise involved.

However, computer simulations can provide only a qualitative picture, as the number of spins considered in such simulations is incomparably smaller than what one has in real macroscopic samples. Moreover, these simulations give no analytical formulas, making it difficult, if possible, to classify all relaxation regimes occurring when varying the numerous parameters of the system. Simplified models [150] can also provide only a qualitative understanding.

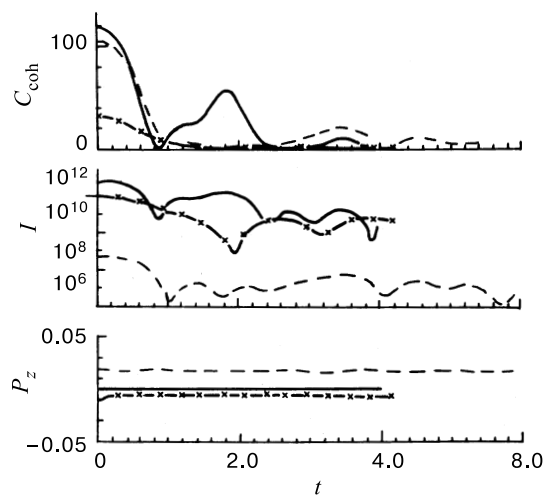


Fig. 6. Coherence coefficient  $C_{\text{coh}}$ , radiation intensity  $I$ , and spin polarization  $p_z$  as functions of time in the case of switched-off resonator-spin coupling ( $g = 0$ ). The varied parameters are:  $\omega_0 = 200$ ,  $p_x(0) = 0.48$  (solid line);  $\omega_0 = 20$ ,  $p_x(0) = 0.48$  (dashed line); and  $\omega_0 = 200$ ,  $p_x(0) = 0.20$  (solid line with crosses)

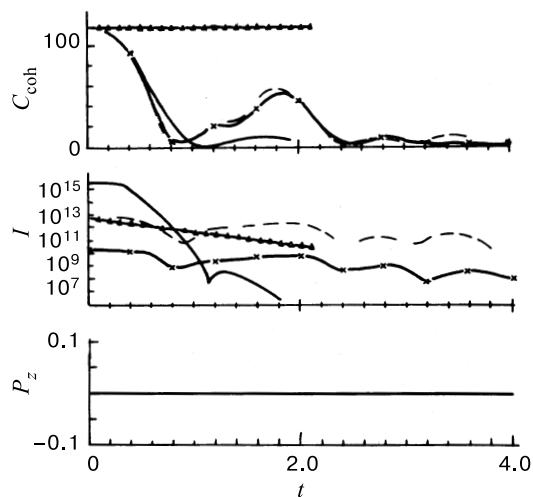


Fig. 7. The same as in Fig. 6 for  $p_x(0) = 0.48$  and for different Zeeman frequencies:  $\omega_0 = 1000$  (solid line);  $\omega_0 = 200$  (dashed line);  $\omega_0 = 50$  (solid line with crosses); and  $\omega_0 = 200$  with switched-off dipole interaction (solid line with triangles)

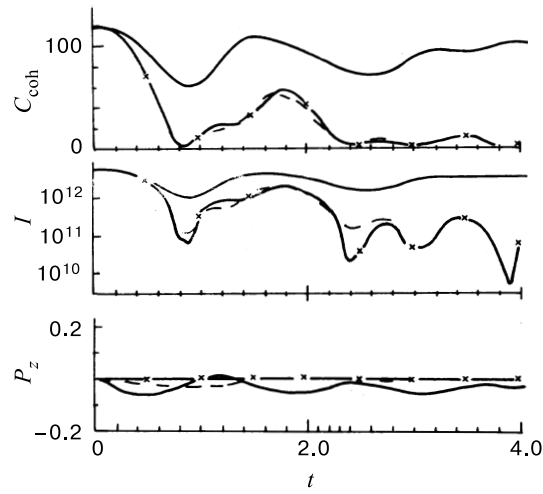


Fig. 8. The same as in Fig. 6 for  $p_x(0) = 0.48$  but in the presence of an alternating magnetic field with the frequency  $\omega = \omega_0$  and different amplitudes:  $h_{01}$  (solid line);  $h_{02}$  (dashed line); where  $h_{01}/h_{02} = 10$ ; and  $h_{03} = 0$  (solid line with crosses)

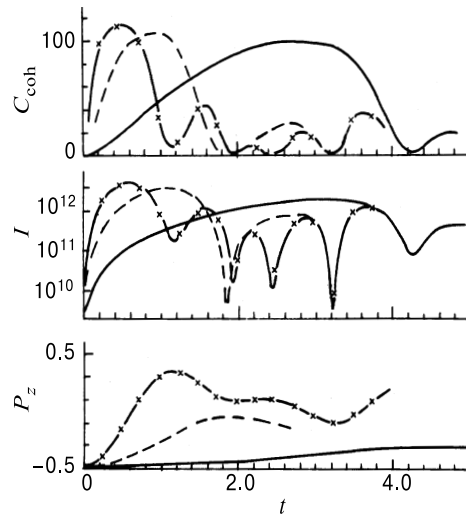


Fig. 9. The same as in Fig. 8 but for  $p_x(0) = -0.48$  and different amplitudes of the alternating field:  $h_{01}$  (solid line);  $h_{02}$  (dashed line); and  $h_{03}$  (solid line with crosses), where the amplitude relations are  $h_{01}/h_{02} = 0.25$  and  $h_{01}/h_{03} = 0.1$



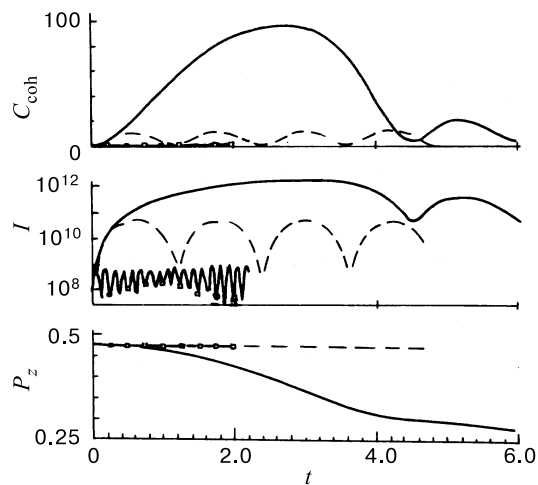


Fig. 10. The same as in Fig. 8 for a varying relative detuning from the resonance  $\delta \equiv (\omega - \omega_0)/\omega_0$  taking the values:  $\delta = 0$  (solid line);  $\delta = 0.025$  (dashed line); and  $\delta = 0.25$  (solid line with squares)

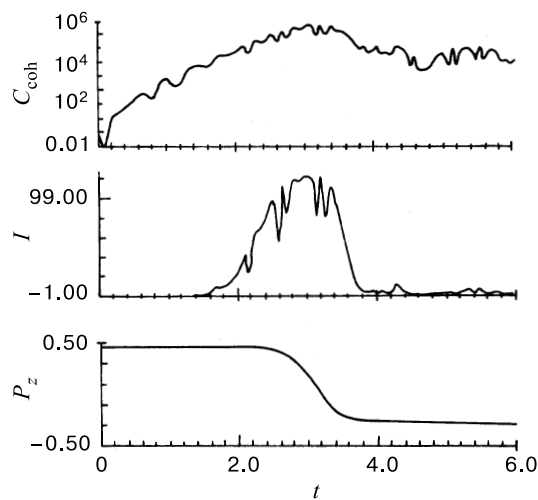


Fig. 11. Radiation intensity  $I$ , coherence coefficient  $C_{\text{coh}}$ , and spin polarization  $p_z$  versus time, in the absence of alternating external fields and with a weak coupling with a resonator,  $g \sim 1$

An analytical solution of the evolution equations corresponding to the microscopic Hamiltonian (206) and a complete analysis of different relaxation regimes of nonequilibrium nuclear magnets coupled with a resonator has been done [25,26,151-158] by employing the scale separation approach. The evolution equations are written for the averages

$$u \equiv \frac{1}{N} \sum_i \langle S_i^- \rangle, \quad s \equiv \frac{1}{N} \sum_i \langle S_i^z \rangle, \quad (208)$$

where  $S_i^- = S_i^x - iS_i^y$ . Presenting local fluctuating fields through stochastic variables  $\xi_0$  and  $\xi$ , one comes [25,26] to the evolution equations

$$\begin{aligned} \frac{du}{dt} &= i(\omega_0 - \xi_0 + i\Gamma_2) u - i(\gamma_3 h + \xi) s, \\ \frac{ds}{dt} &= \frac{i}{2} (\gamma_3 h + \xi) u^* - \frac{i}{2} (\gamma_3 h + \xi^*) u - \Gamma_1 (s - \zeta), \\ \frac{d}{dt} |u|^2 &= -2\Gamma_2 |u|^2 - i(\gamma_3 h + \xi) s u^* + i(\gamma_3 h + \xi^*) s u, \end{aligned} \quad (209)$$

in which the resonator feedback field,  $h$ , in dimensionless units, satisfies the Kirchhoff equation

$$\frac{dh}{dt} + 2\gamma_3 h + \omega^2 \int_0^t h(t') dt' = -2\kappa \frac{d}{dt} (u^* + u) + \gamma_3 f, \quad (210)$$

in which  $f$  is an electromotive force,  $\gamma_3$  is the resonator ringing width, and  $\kappa \equiv \pi\eta\rho_n\mu_n^2/\hbar\gamma_3$ . The random local fields are defined as Gaussian stochastic variables with the stochastic averages

$$\langle\langle \xi_0^2 \rangle\rangle = \langle\langle |\xi|^2 \rangle\rangle = \Gamma_*^2, \quad (211)$$

where  $\Gamma_*$  is the inhomogeneous dipole broadening. Because of the existence of the small parameters

$$\frac{\Gamma_1}{\omega_0} \ll 1, \quad \frac{\Gamma_2}{\omega_0} \ll 1, \quad \frac{\Gamma_*}{\omega_0} \ll 1, \quad \frac{\gamma_3}{\omega_0} \ll 1, \quad \frac{|\Delta|}{\omega_0} \ll 1, \quad (212)$$

where  $\Delta \equiv \omega - \omega_0$ , the functions  $u$  and  $h$  can be classified as fast while  $s$  and  $|u|^2$  as slow.

Solving Eqs. (209) and (210), it was shown [25,26,151] that the role of the thermal Nyquist noise in starting the relaxation process is negligible. But the main cause triggering the motion of spins, leading to coherent self-organization, is the action of *nonsecular dipole interactions*. This gives the answer to the question posed by Bloembergen and Pound [119]: what is the origin of self-organized

coherent relaxation in spin systems? All possible regimes of nonlinear spin dynamics have been analysed. When the nonresonant external pumping is absent, that is  $\zeta > 0$ , there are seven qualitatively different transient relaxation regimes: *free induction*, *collective induction*, *free relaxation*, *collective relaxation*, *weak superradiance*, *pure superradiance*, and *triggered superradiance* [25,26,151,155].

In the presence of pumping, realized, e.g., by means of dynamical nuclear polarization directing nuclear spins against an external constant magnetic field, one has  $\zeta \leq 0$ . Then, as was shown using phenomenological equations [139], two stationary solutions can appear. In our approach, the behaviour of the system is as follows [158]. When  $\zeta \leq 0$ , three dynamical regimes can be observed, depending on the value of  $\zeta$  with respect to the pump thresholds

$$\zeta_1 = -\frac{1}{g}, \quad \zeta_2 = -\frac{1}{g} \left( 1 + \frac{\Gamma_1}{8\Gamma_2} \right). \quad (213)$$

Analysing the equations for the slow variables  $s$  and  $w$ , where

$$w \equiv |u|^2 - \frac{\Gamma^2}{\omega_0^2} s^2, \quad (214)$$

we find two fixed points

$$s_1^* = \zeta, \quad w_1^* = 0; \quad s_2^* = -\frac{1}{g}, \quad w_2^* = -\frac{\Gamma_1(1+g\zeta)}{\Gamma_2 g^2}. \quad (215)$$

When  $\zeta_1 < \zeta \leq 0$ , the first fixed point is a stable node and the second one is a saddle point. For  $\zeta = \zeta_1$ , both points merge together, being neutrally stable. After the bifurcation at  $\zeta = \zeta_1$ , in the region  $\zeta_2 \leq \zeta < \zeta_1$ , the first fixed point loses its stability becoming a saddle point while the second fixed point becomes a stable node. Finally, when  $\zeta < \zeta_2$ , the second fixed point turns into a stable focus, and the first one continues to be a saddle point. In this way, there are three qualitatively different lasting relaxation regimes induced by the pumping [158]. The first one is the *incoherent monotonic relaxation* to the first stationary solution  $s_1^*$ ,  $w_1^*$ . The second regime is the *coherent monotonic relaxation* to the second stationary solution  $s_2^*$ ,  $w_2^*$ , although the level of coherence may be rather low. And the third case is the *coherent pulsing relaxation* to the second fixed point. This unusual regime of pulsing relaxation was observed experimentally [159]. Here we present the results of numerical solution of the evolution equations for the slow variables  $s = z(t)$  and  $w(t)$  defined in Eq. (214). Different cases of the pulsing regime are clearly demonstrated in Figs. 12 to 18. In the corresponding figure captions we use the notation  $z_0 = z(0)$ ,  $w_0 = w(0)$ , and  $\gamma \equiv \gamma_1/\gamma_2$ . Everywhere in Figs. 12 to 17, the pump parameter is  $\zeta = -0.5$ , and in Fig. 18 this parameter is varied. The coupling parameter (207) is always  $g = 10$ .

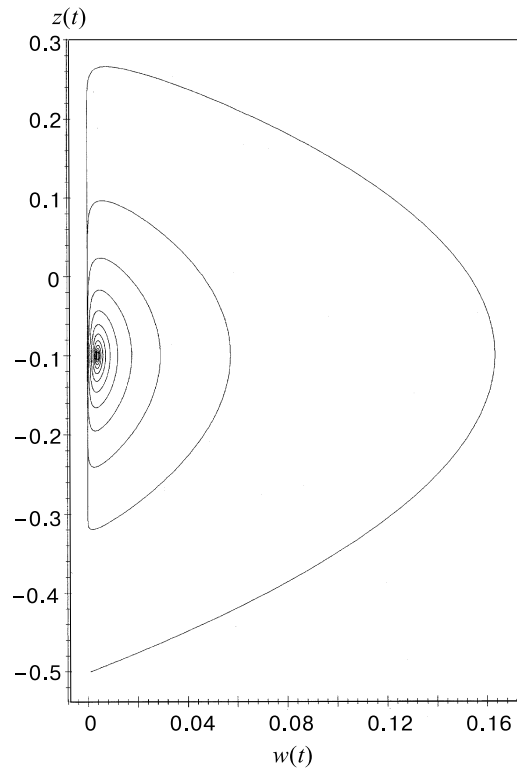


Fig. 12. Phase portrait demonstrating a stable focus for the parameters  $z_0 = -0.5$ ,  $w_0 = 0.001$ ,  $g = 10$ , and  $\gamma = 0.1$

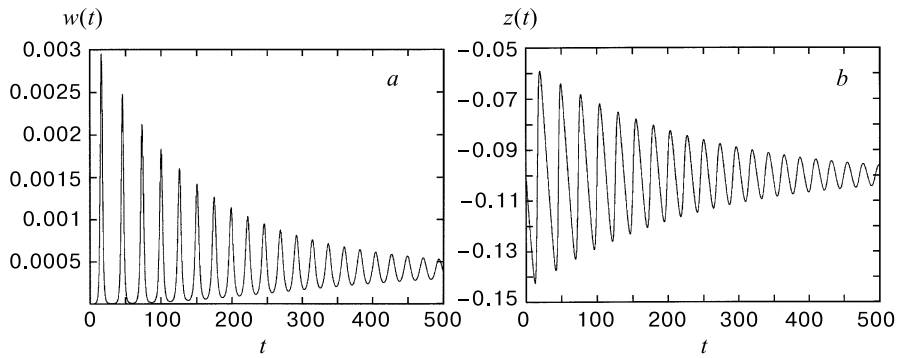


Fig. 13. Pulsing regime of spin relaxation with the parameters  $z_0 = -0.1$ ,  $w_0 = 10^{-6}$  and  $\gamma = 0.01$  for the functions: (a)  $w(t)$ ; (b)  $z(t)$

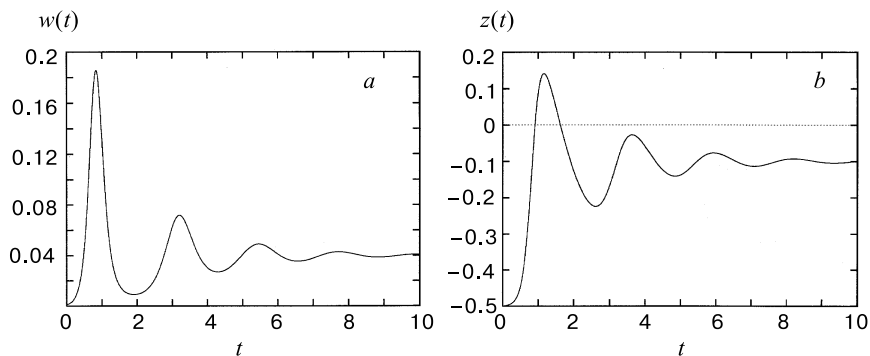


Fig. 14. The time dependence of the functions: (a)  $w(t)$ ; (b)  $z(t)$ , for the parameters  $z_0 = -0.5$ ,  $w_0 = 0.001$ , and  $\gamma = 1$

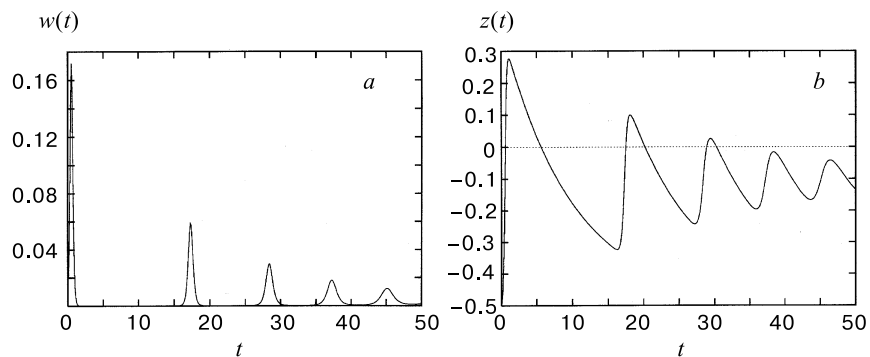


Fig. 15. Dynamics of slow solutions: (a)  $w(t)$ ; (b)  $z(t)$ , for the parameters  $z_0 = -0.5$ ,  $w_0 = 0.01$ , and  $\gamma = 0.1$

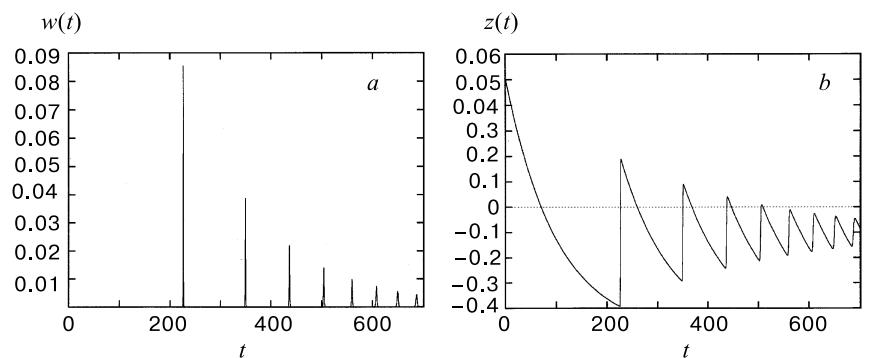


Fig. 16. Evolution of slow solutions: (a)  $w(t)$ ; (b)  $z(t)$ , for the parameters  $z_0 = 0.5$ ,  $w_0 = 0.01$ , and  $\gamma = 0.01$

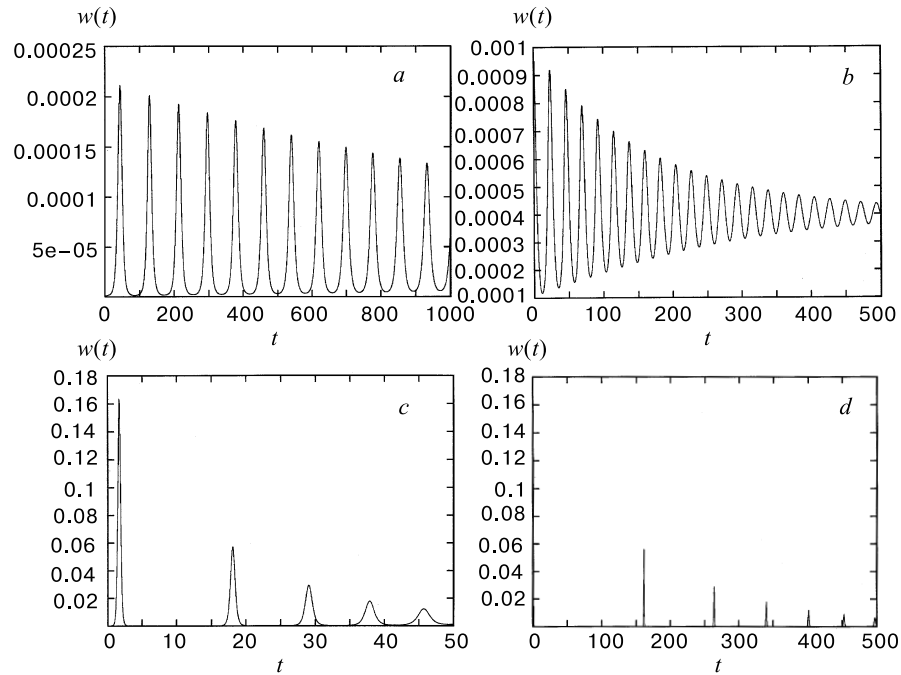


Fig. 17. Temporal behaviour of the function  $w(t)$  for different sets of parameters: (a)  $z_0 = -0.1$ ,  $w_0 = 10^{-6}$ ,  $\gamma = 0.001$ ; (b)  $z_0 = -0.1$ ,  $w_0 = 0.001$ ,  $\gamma = 0.01$ ; (c)  $z_0 = -0.5$ ,  $w_0 = 10^{-6}$ ,  $\gamma = 0.1$ ; (d)  $z_0 = -0.5$ ,  $w_0 = 0.001$ ,  $\gamma = 0.01$

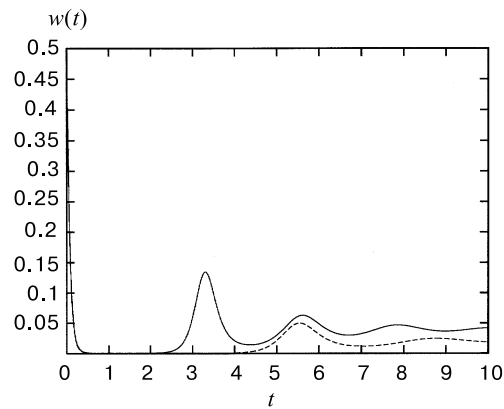


Fig. 18. Function  $w(t)$  for  $z_0 = 0.5$ ,  $w_0 = 0.5$ ,  $\gamma = 1$ , and varying pump parameters:  $\zeta = -0.5$  (solid line);  $\zeta = -0.3$  (dashed line)

The problem of superradiant spin relaxation can be generalized to the case of nuclei incorporated into a ferromagnetic matrix, where nuclear and electron spins interact through hyperline forces. Some model studies of this case have been undertaken [160–162], and a general microscopic theory has also been developed [163]. The latter theory makes it possible to discover all feasible causes triggering the process of self-organized coherent relaxation. The most important such causes are the *dipole hyperfine interactions*, *dipole nuclear interactions*, and the *transverse magnetocrystalline anisotropy*.

## 12. NEGATIVE ELECTRIC CURRENT

The study of electric processes in semiconductors is important for describing and modelling semiconductor devices [164]. One of the most difficult problems is the consideration of strongly nonequilibrium phenomena in essentially nonuniform semiconductors. Nonequilibrium and nonuniform distributions of charge carriers can be formed in several ways, for instance, by means of external irradiation [165,166]. Transport properties of semiconductors with essentially nonuniform distribution of charge carriers can be rather specific. For example, in a sample, biased with an external constant voltage, the resulting electric current may turn against the latter displaying the transient effect of negative electric current [3,166–168].

Transport properties of semiconductors are usually described by the semi-classical drift-diffusion equations [164]. In what follows a plane device, of area  $A$  and length  $L$  is considered, which is biased with a constant voltage  $V_0$ . It is convenient to pass to dimensionless quantities, measuring the space variable  $x$  in units of  $L$ , time in units of the transit time

$$\tau_0 \equiv \frac{L^2}{\mu V_0}, \quad \mu \equiv \min\{|\mu_i|\},$$

where  $\mu_i$  is a mobility of the  $i$ -type carriers. And the characteristic quantities

$$\begin{aligned} \rho_0 &\equiv \frac{Q_0}{AL}, & Q_0 &\equiv \varepsilon AE_0, & E_0 &\equiv \frac{V_0}{L}, \\ j_0 &\equiv \frac{Q_0}{A\tau_0}, & D_0 &\equiv \mu V_0, & \xi_0 &\equiv \frac{\rho_0}{\tau_0}, \end{aligned}$$

are employed for measuring other physical values which are used below.

The drift-diffusion equations consist of the continuity equations

$$\frac{\partial \rho_i}{\partial t} + \mu_i \frac{\partial}{\partial x} (\rho_i E) - D_i \frac{\partial^2 \rho_i}{\partial x^2} + \frac{\rho_i}{\tau_i} = \xi_i, \quad (216)$$

for each type of charge carriers, and of the Poisson equation

$$\frac{\partial E}{\partial x} = 4\pi \sum_i \rho_i \quad (217)$$

for the electric field  $E(x, t)$ . Here  $\rho_i(x, t)$  is a charge density;  $\mu_i$ ,  $D_i$ , and  $\tau_i$  are mobility, diffusion coefficient, and relaxation time, respectively;  $\xi_i$  is a generation–recombination noise [169]. The sample is biased with an external constant voltage, which in our dimensionless notation implies that

$$\int_0^1 E(x, t) dx = 1. \quad (218)$$

At the initial time, the distribution of charge carriers

$$\rho_i(x, 0) = f_i(x) \quad (219)$$

is assumed to be nonuniform.

The total electric current through the semiconductor sample is

$$J(t) \equiv \int_0^1 j(x, t) dx, \quad (220)$$

where the density of current

$$j = \sum_i \left( \mu_i E - D_i \frac{\partial}{\partial x} \right) \rho_i + \frac{1}{4\pi} \frac{\partial E}{\partial t}. \quad (221)$$

Because of the voltage integral (218), one has

$$\int_0^1 \frac{\partial}{\partial t} E(x, t) dx = 0. \quad (222)$$

It is also possible to show that

$$\lim_{\tau \rightarrow \infty} \ll \frac{1}{\tau} \int_0^\tau \frac{\partial}{\partial x} E(x, t) dt \gg = 0. \quad (223)$$

This means that the function  $E$  can be considered as slow on average in time and in space. Then, treating  $E$  as a quasi-invariant, one may find the solutions to Eqs. (216) and (217) in order to analyse their general space-time behaviour and to find conditions when the effect of negative electric current could arise. Such negative current can appear only when the initial charge distribution is essentially nonuniform. For example, if this initial charge distribution forms a narrow layer located at the point  $x = a$ , then the total current (220) becomes negative for



a transient interval of time in the vicinity of  $t = 0$ , if one of the following conditions holds true:

$$a < \frac{1}{2} - \frac{1}{4\pi Q} \quad \left( Q > \frac{1}{2\pi} \right), \quad \text{or} \quad a > \frac{1}{2} + \frac{1}{4\pi|Q|} \quad \left( Q < -\frac{1}{2\pi} \right), \quad (224)$$

where

$$Q \equiv \sum_i Q_i, \quad Q_i \equiv \int_0^1 \rho_i(x, 0) dx.$$

The effect of the negative electric current can be employed for various purposes, as is discussed in Refs. 3,168. For instance, when the initial charge layer is formed by an ion beam irradiating the semiconductor sample, the location  $a$  corresponds to the ion mean free path. In this case, by measuring the negative current  $J(0)$ , one can define this mean free path

$$a = \frac{1}{2} - \frac{1}{4\pi Q} \left[ 1 - \frac{J(0)}{\sum_i \mu_i Q_i} \right]. \quad (225)$$

This formula is valid for both positive and negative values of  $Q$ .

Equations (216) and (217) have also been solved numerically [3,168], which confirmed the appearance of the negative electric current. Two cases were analysed, with one layer of charge carriers and with two such layers. Here we present the results of calculations for the double-layer case. The initial charge distributions (219) are given by the Gaussians

$$f_i(x) = \frac{Q_i}{Z_i} \exp \left\{ -\frac{(x - a_i)^2}{2b_i} \right\},$$

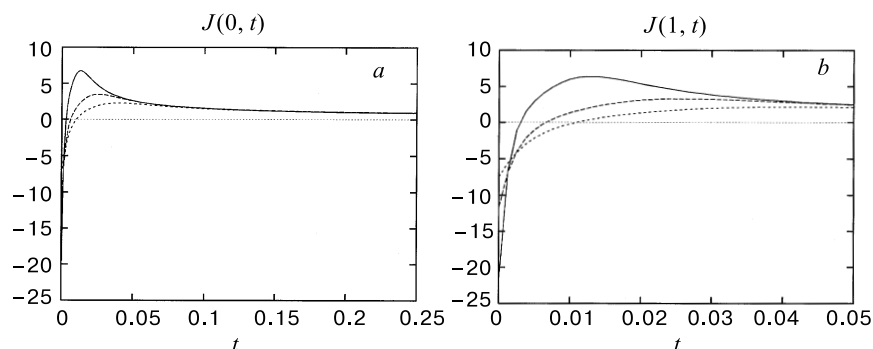


Fig. 19. Electric current through the semiconductor surfaces in the case of  $a = 0.1$ ,  $Q_2 = -1$ ,  $\gamma = 1$  and different mobilities:  $\mu_2 = -10$  (solid line);  $\mu_2 = -5$  (dashed line);  $\mu_2 = -3$  (short-dashed line). (a) Left-surface current  $J(0, t)$ ; (b) Right-surface current  $J(1, t)$

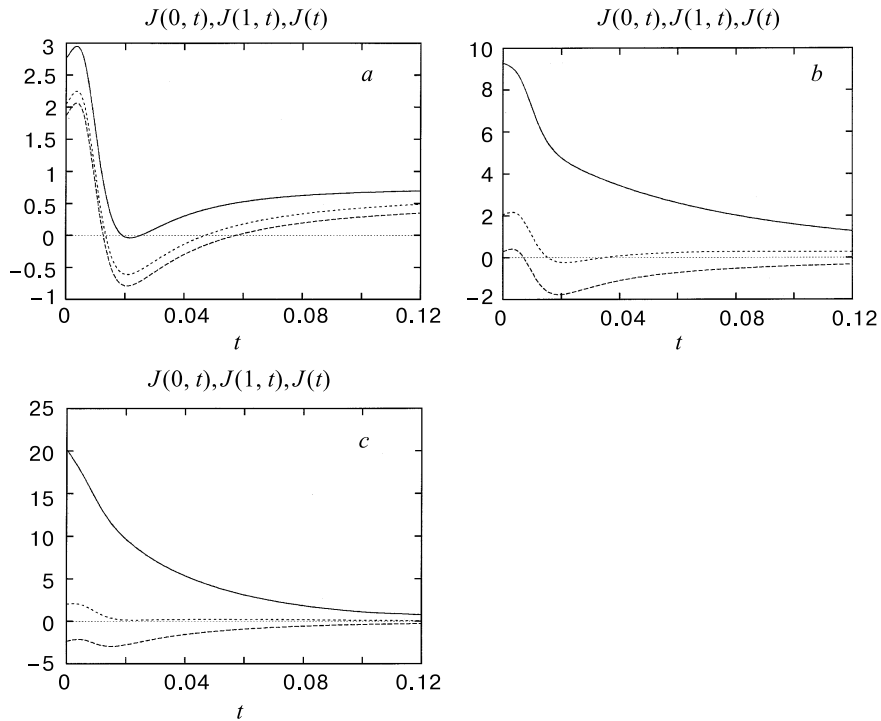


Fig. 20. Left-surface current  $J(0, t)$  (solid line), right-surface current  $J(1, t)$  (dashed line), and the total current  $J(t)$  (short-dashed line) for  $a = 0.25$ ,  $Q_2 = -0.1$ ,  $\mu_2 = -10$  and different relaxation parameters: (a)  $\gamma = 1$ ; (b)  $\gamma = 10$ ; (c)  $\gamma = 25$

in which  $0 \leq a_i \leq 1$  and

$$Q_i = \int_0^1 f_i(x) dx, \quad Z_i = \int_0^1 \exp \left\{ -\frac{(x - a_i)^2}{2b_i} \right\} dx.$$

The positive charge carriers, with  $\mu_1 = 1$  and  $Q_1 = 1$ , form the left layer centered at  $a_1 = a$ , while the negative charge carriers form the layer centered at  $a_2 = 1 - a$ . We keep in mind the relation  $D_2 = 3D_1$  for the diffusion coefficients, typical for holes and electrons, and we set  $D_1 = 10^{-3}$ . For short, we use the notation  $\tau_1^{-1} = \tau_2^{-1} = \gamma$  and  $b_1 = b_2 = b$ . The generation–recombination noise is neglected, which is admissible at the initial stage of the process. As the boundary conditions, we accept the absence of diffusion through the semiconductor surface,

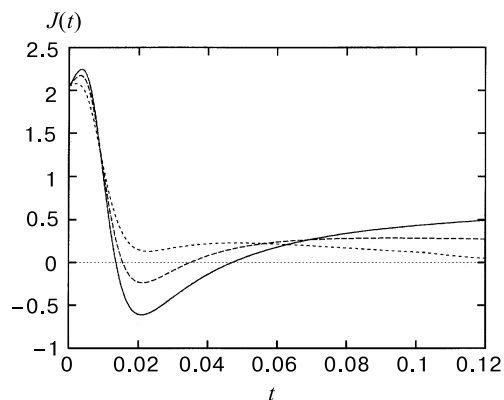


Fig. 21. Total electric current  $J(t)$  for  $a = 0.25$ ,  $Q_2 = -0.1$ ,  $\mu_2 = -10$  and varying relaxation parameters:  $\gamma = 1$  (solid line);  $\gamma = 10$  (dashed line);  $\gamma = 25$  (short-dashed line)

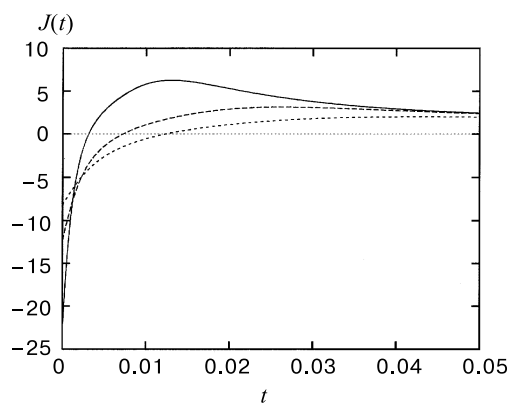


Fig. 22. Electric current through semiconductor for the parameters  $a = 0.1$ ,  $Q_2 = -1$ ,  $\gamma = 1$  and different mobilities:  $\mu_2 = -10$  (solid line);  $\mu_2 = -5$  (dashed line);  $\mu_2 = -3$  (short-dashed line)

which implies the Neumann boundary condition

$$\frac{\partial}{\partial x} \rho_i(x, t) = 0 \quad (x = 0, x = 1).$$

In Figs. 19 to 24, we present the total current (220) as well as the electric current through the left surface,  $J(0, t) \equiv j(0, t)$  and through the right surface,  $J(1, t) \equiv j(1, t)$ , defined by the current (221) at  $x = 0$  or  $x = 1$ , respectively.

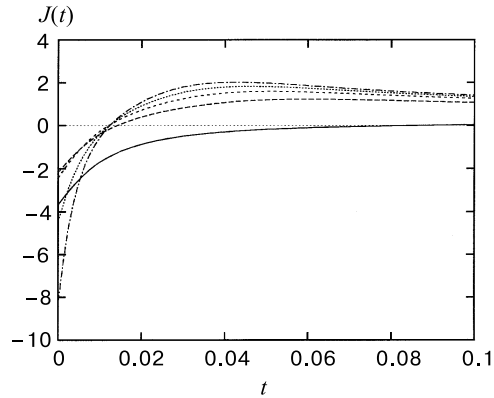


Fig. 23. Electric current  $J(t)$  as a function of time for  $a = 0.1$ ,  $\mu_2 = -3$ ,  $\gamma = 1$  and different initial charges:  $Q_2 = 0$  (solid line);  $Q_2 = -0.25$  (dashed line);  $Q_2 = -0.5$  (short-dashed line);  $Q_2 = -0.75$  (dotted line);  $Q_2 = -1$  (dashed-dotted line)

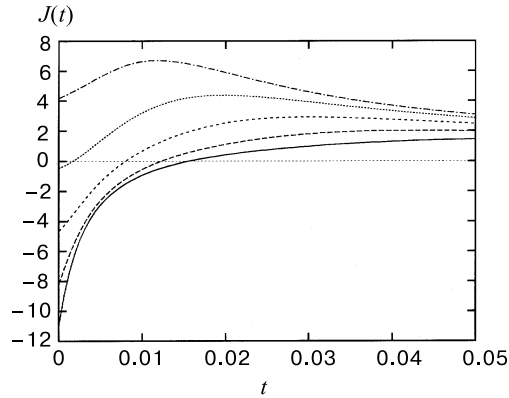


Fig. 24. Electric current  $J(t)$  for  $Q_2 = -1$ ,  $\mu_2 = -3$ ,  $\gamma = 1$ , and different locations of initial charge layers:  $a = 0.05$  (solid line);  $a = 0.1$  (dashed line);  $a = 0.15$  (short-dashed line);  $a = 0.2$  (dotted line);  $a = 0.25$  (dashed-dotted line)

### 13. MAGNETIC SEMICONFINEMENT OF ATOMS

Dynamics of neutral atoms in nonuniform magnetic fields concerns problems of current experimental and theoretical interest. By means of such fields, atoms can be confined inside magnetic traps, which allows to accomplish various experiments with the systems of trapped atoms. Recently, Bose – Einstein condensation has been attained in a dilute gas of trapped atoms of  $^{87}\text{Rb}$  [170],  $^7\text{Li}$  [171],  $\text{Na}$  [172], and  $\text{H}\downarrow$  [173]. The details on theory and experiment can be

found in reviews [174–176]. The Bose – Einstein condensate is believed to form, at least partially, a coherent state. If it would be possible to construct a device emitting a coherent atomic beam, this would be analogous to a laser radiating a coherent photon ray. This is why one may call the device, emitting a coherent atomic beam, an atom laser [177–184]. An output coupler, coherently extracting condensed atoms from a trap, was demonstrated recently [185–187]. But in these demonstrations, the atoms, when escaping from a trap, fly out more or less in all directions, with anisotropy formed only by the gravitational force. While the very first condition on a laser is that its output is highly directional, with the possibility of varying the beam direction [183].

A mechanism for creating well-collimated beams of neutral atoms was advanced in Refs. 188–192. This mechanism suggests an output coupler that extracts trapped atoms in the form of a directed beam.

The motion of neutral atoms in magnetic fields can be described by the semiclassical equations for the quantum-mechanical average of the real-space coordinate  $\vec{r} = \{r_\alpha\}$ , where  $\alpha = x, y, z$ , and for the average  $\vec{S} = \{S_\alpha\}$  of the spin operator [193–195]. The first equation writes

$$m \frac{d^2 r_\alpha}{dt^2} = \mu_0 \vec{S} \cdot \frac{\partial \vec{B}}{\partial r_\alpha} + m g_\alpha + f_\alpha, \quad (226)$$

where  $m$  and  $\mu_0$  are mass and magnetic moment of an atom;  $\vec{B}$  is a magnetic field;  $g_\alpha$  is a component of the standard gravitational acceleration; and  $f_\alpha$  is a collision force component. The equation for the average spin is

$$\hbar \frac{d\vec{S}}{dt} = \mu_0 \vec{S} \times \vec{B}. \quad (227)$$

The total magnetic field

$$\vec{B} = \vec{B}_1 + \vec{B}_2,$$

$$\vec{B}_1 = B'_1 (x \vec{e}_x + y \vec{e}_y + \lambda z \vec{e}_z), \quad \vec{B}_2 = B_2 (h_x \vec{e}_x + h_y \vec{e}_y), \quad (228)$$

where  $|\vec{h}| = 1$ , consists of the quadrupole field  $\vec{B}_1$ , typical of quadrupole magnetic traps, and of a transverse field, e.g., of a rotating field [196,197]. In the quadrupole field,  $\lambda$  is the anisotropy parameter.

It is convenient to pass to the dimensionless space variable, measuring the components of  $\vec{r}$  in units of the characteristic length

$$R_0 \equiv \frac{B_2}{B'_1}. \quad (229)$$

Introduce the characteristic frequencies by the relations

$$\omega_1^2 \equiv \frac{\mu_0 B'_1}{m R_0}, \quad \omega_2 \equiv \frac{\mu_0 B_2}{\hbar}, \quad \omega \equiv \max_t \left| \frac{d\vec{h}}{dt} \right|. \quad (230)$$

Also, we define

$$\delta_\alpha \equiv \frac{g_\alpha}{R_0 \omega_1^2}, \quad \gamma \xi_\alpha \equiv \frac{f_\alpha}{m R_0}, \quad (231)$$

where  $\gamma$  is a collision rate and  $\xi_\alpha$  can be treated as a random variable with the stochastic averages

$$\ll \xi_\alpha(t) \gg = 0, \quad \ll \xi_\alpha(t) \xi_\beta(t') \gg = 2D_\alpha \delta_{\alpha\beta} \delta(t - t'),$$

in which  $D_\alpha$  is a diffusion rate. Then Eq. (226) can be written as the stochastic differential equation

$$\frac{d^2 \vec{r}}{dt^2} = \omega_1^2 (S_x \vec{e}_x + S_y \vec{e}_y + \lambda S_z \vec{e}_z + \vec{\delta}) + \gamma \vec{\xi}, \quad (232)$$

and Eq. (227) acquires the form

$$\frac{d\vec{S}}{dt} = \omega_2 \hat{A} \vec{S}, \quad (233)$$

in which the antisymmetric matrix  $\hat{A} = [A_{\alpha\beta}]$  has the elements

$$A_{\alpha\beta} = -A_{\beta\alpha}, \quad A_{\alpha\alpha} = 0, \\ A_{12} = \lambda z, \quad A_{23} = x + h_x, \quad A_{31} = y + h_y.$$

Assuming the occurrence of the small parameters

$$\left| \frac{\gamma}{\omega_1} \right| \ll 1, \quad \left| \frac{\omega_1}{\omega_2} \right| \ll 1, \quad \left| \frac{\omega}{\omega_2} \right| \ll 1, \quad (234)$$

we may classify the variables  $\vec{r}$  and  $\vec{h}$  as slow, compared to the fast spin variable  $\vec{S}$ . Then Eq. (233) can be solved yielding

$$\vec{S}(t) = \sum_{i=1}^3 a_i \vec{b}_i(t) \exp\{\beta_i(t)\}, \quad (235)$$

where

$$a_i = \vec{S}(0) \cdot \vec{b}_i(0),$$

$$\vec{b}_i(t) = \frac{1}{\sqrt{C_i}} [(A_{12}A_{23} - \alpha_i A_{31}) \vec{e}_x + (A_{12}A_{31} + \alpha_i A_{23}) \vec{e}_y + (A_{12}^2 + \alpha_i^2) \vec{e}_z],$$

$$C_i = (A_{12}^2 - |\alpha_i|^2)^2 + (A_{12}^2 + |\alpha_i|^2) (A_{23}^2 + A_{31}^2),$$

$$\alpha_{1,2} = \pm i\alpha, \quad \alpha_3 = 0, \quad \alpha^2 \equiv A_{12}^2 + A_{23}^2 + A_{31}^2, \quad \beta_i(t) = \omega_2 \int_0^t \alpha_i(t') dt'.$$

Substituting Eq. (235) into the right-hand side of Eq. (232) and averaging the latter over time and over stochastic variables, we obtain

$$\frac{d^2 \vec{r}}{dt^2} = \vec{F} + \omega_1^2 \vec{\delta}, \quad (236)$$

where

$$\begin{aligned} \vec{F} &\equiv \omega_1^2 a_3 \langle b_3^x \vec{e}_x + b_3^y \vec{e}_y + \lambda b_3^z \vec{e}_z \rangle, \\ a_3 &= \frac{(x + h_x^0) S_x^0 + (y + h_y^0) S_y^0 + \lambda z S_z^0}{[(x + h_x^0)^2 + (y + h_y^0)^2 + \lambda^2 z^2]^{1/2}}, \\ \vec{b}_3 &= \frac{(x + h_x) \vec{e}_x + (y + h_y) \vec{e}_y + \lambda z \vec{e}_z}{[(x + h_x)^2 + (y + h_y)^2 + \lambda^2 z^2]^{1/2}}, \end{aligned}$$

angle brackets imply time averaging and  $h_\alpha^0 \equiv h_\alpha(0)$ ,  $S_\alpha^0 \equiv S_\alpha(0)$ . For the rotating transverse field, with

$$h_x = \cos \omega t, \quad h_y = \sin \omega t, \quad (237)$$

we find

$$\vec{F} = \frac{\omega_1^2 [(1+x) S_x^0 + y S_y^0 + \lambda z S_z^0] (x \vec{e}_x + y \vec{e}_y + 2\lambda^2 z \vec{e}_z)}{2[(1+2x+x^2+y^2+\lambda^2 z^2)(1+x^2+y^2+\lambda^2 z^2)]^{1/2}}.$$

The motion of atoms, described by Eq. (236), essentially depends on the initial state, which, as is known [198,199], can be prepared in an arbitrary way. Suppose that atoms, after being laser cooled in a magneto-optical trap [200], are loaded into a magnetic trap where they are further cooled by evaporative cooling down to sufficiently low temperatures, so that there is a portion of atoms with low velocities, which are located close to the trap center. If the initial spin condition for these atoms is such that  $S_x^0 < 0$  and  $S_y = S_z = 0$ , then the atoms are confined inside the trap moving in an approximately harmonic potential. The gradient of the quadrupole field supplies the levitating force to support atoms against gravity. The combination of the magnetic field and gravity produces a very nearly harmonic confining potential within the trap volume in all three dimensions [201].

The semiconfining regime of motion [188–192] can be realized by preparing for the spin variable nonadiabatic initial conditions

$$S_x^0 = S_y^0 = 0, \quad S_z^0 \equiv S \neq 0. \quad (238)$$

Such conditions can be arranged in several ways. One possibility could be to confine atoms in a trap, where all atoms are polarized having their spins in the  $z$  direction, as, e.g., in the trap of Ref. 201, being a quadrupole trap with a bias field along the  $z$  axis. Then the longitudinal bias field is quickly switched off, and at the same time, a transverse field is switched on, which would correspond to the sudden change of potential [202]. Another way could be to prepare spin polarized atoms in one trap and quickly load them into another trap with the required field configuration. Atoms can be prepared practically 100% polarized [203], with the spin–spin relaxation time reaching 100 s [204]. The possibility of realizing two ways of transferring atoms from one trap to another, by means of sudden transfer as opposed to adiabatic transfer, is discussed in Ref. 205. The third way of organizing the nonadiabatic initial conditions (238) could be by acting on the trapped atoms with a short pulse of strong magnetic field, polarizing atomic spins in the desired way.

With the initial conditions (238), the motion of atoms becomes axially restricted from one side, depending on the sign of  $\lambda S$ . Atoms fly out of the trap predominantly in one direction, forming a well-collimated beam [188–192]. This mechanism can be used for atom lasers. Another possibility could be to study the dynamics of binary mixtures of Bose systems, where the effect of conical stratification [206] can arise. The mixtures of two condensates have been realized for rubidium [207] and sodium [208], and the dynamics of two rubidium condensates was observed in Ref. 209.

When solving equation (236) for the realistic case of a finite trap, one should take into account the trap shape factor, which can be written in the Gaussian form

$$\varphi(\vec{r}) = \exp\left(-\frac{x^2 + y^2}{R^2} - \frac{z^2}{L^2}\right),$$

where  $R$  and  $L$  are the trap radius and length. The relation between the latter can be quite different for different traps, starting from almost spherical traps, where  $R \approx L$ , to needle-shape traps, with  $R/L \sim 10^{-3}$ , as for Ioffe–Pritchard magnetic traps [210]. Accepting the initial spin conditions (238), and using the notation

$$f(\vec{r}) \equiv \frac{\varphi(\vec{r})}{[(1 + 2x + x^2 + y^2 + \lambda^2 z^2)(1 + x^2 + y^2 + \lambda^2 z^2)]^{1/2}},$$

from Eq. (236) we obtain

$$\frac{d^2 x}{dt^2} = \omega_1^2 \left( \frac{\lambda}{2} S f z x + \delta_x \right), \quad \frac{d^2 z}{dt^2} = \omega_1^2 (\lambda^3 S f z^2 + \delta_z), \quad (239)$$



where the equation for  $y$ , being similar to that for  $x$ , is not written down. Note that instead of the Gaussian shape factor for the trap, one could opt for

$$\varphi(\vec{r}) = 1 - \Theta(x - R) \Theta(y - R) \Theta\left(|z| - \frac{1}{2}L\right),$$

with  $\Theta(\cdot)$  being the unit-step function.

Equations (239) were analysed both analytically and numerically [188–192]. Their solutions display the semiconfined regime of motion. Taking into account random pair collisions in Eq. (232) shows that atomic collisions do not disturb the semiconfined motion provided that temperature  $T$  is sufficiently low, satisfying the condition

$$\frac{k_B T \hbar \rho^2 a_s^2}{m^2 \omega_1^3} \ll 1, \quad (240)$$

in which  $\rho$  is the density of atoms and  $a_s$ , their scattering length. The semiconfined regime of motion makes it possible to form well-collimated beams on neutral atoms by means of only magnetic fields.

#### 14. NUCLEAR MATTER LASING

The natural question that arises after talking about atom lasers is whether there can be produced matter waves corresponding to other Bose particles, which could be employed for lasing. One such possibility is related to the creation of a large number of pions in hadronic, nuclear, and heavy-ion collisions. If the density of pions appearing in the course of these collisions is sufficiently high, then correlations between pions can result in the formation of coherent state and in the feasibility of realizing a pion laser [211]. Pions are not the sole type of Bose particles arising in nuclear matter under extreme conditions characteristic of fireballs produced in high-energy collisions [212,213]. There are plenty of reviews devoted to the state of nuclear matter at extreme conditions, including the region of deconfinement transition. Here we cite only some recent of such reviews [214–217].

The very first necessary condition that is required for lasing is to be able to generate Bose particles with sufficiently high density. Therefore, in order to answer the question what kind of Bose particles appearing in nuclear matter under extreme conditions could be used for lasing, one has, first of all, to find out what are these Bose particles and under what conditions their density is maximal. In this section, we give a very brief account of an analysis based on the multichannel model of nuclear matter [217–221]. The main idea in constructing this model goes back to the Weinberg approach for describing composite particles [222–224], with effective Hamiltonians that are assumed to be a result of the

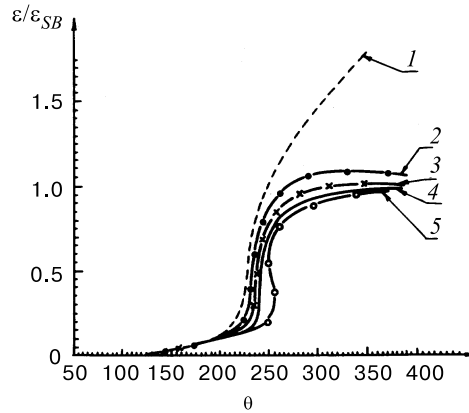


Fig. 25. Relative energy density as a function of temperature in MeV for the  $SU(3)$  gluon–glueball mixture of different glueball radii: 0 (line 1); 0.5 fm (line 2); 0.7 fm (line 3); 0.8 fm (line 4); 1 fm (line 5)

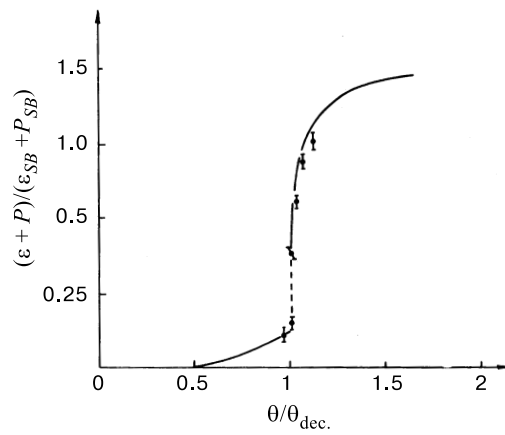


Fig. 26. Relative enthalpy for the gluon–glueball mixture as a function of temperature reduced to the deconfinement temperature, in the case of the glueball radius 0.82 fm, compared with the lattice numerical calculations

Fock–Tani transformation [225]. Now we shall not plunge into the details of the multichannel model, which can be found in Refs. 217,219, but we shall present some figures and will formulate the conclusion of an analysis [221] with regard to the most probable candidates for nuclear matter lasing.

When rising temperature or density, nuclear matter exhibits a transition from hadron state to quark–gluon state. This transition is often assumed to be a sharp

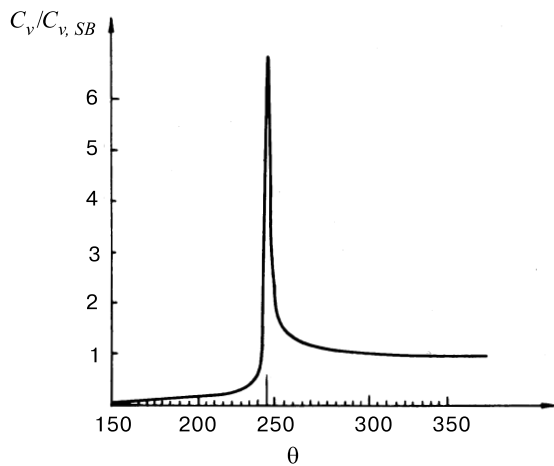


Fig. 27. Relative specific heat for the gluon–glueball mixture, for the glueball radius 0.82 fm, as a function of temperature in MeV

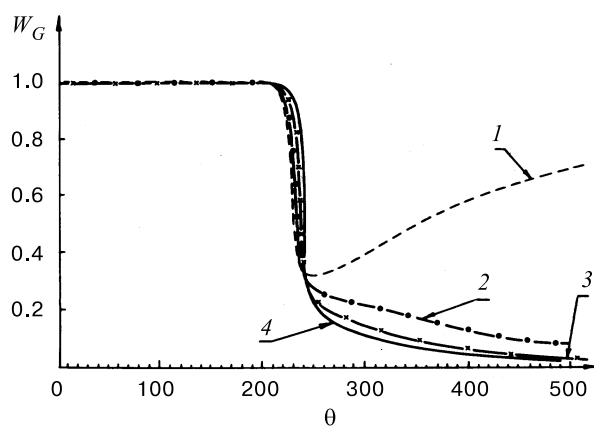


Fig. 28. Glueball channel probability versus temperature in MeV for the glueball radii as in Fig. 25

first-order transition. Lattice numerical simulations for the quarkless  $SU(3)$  gauge model show that deconfinement is really a first-order phase transition [226], which is in agreement with the multichannel model. Figures 25 to 27 illustrate the behaviour of some thermodynamic characteristics, normalized to the corresponding Stefan–Boltzmann limits, for the case of the  $SU(3)$  gluon–glueball mixture. Figure 28 shows the related glueball channel probability. The sharpness on the

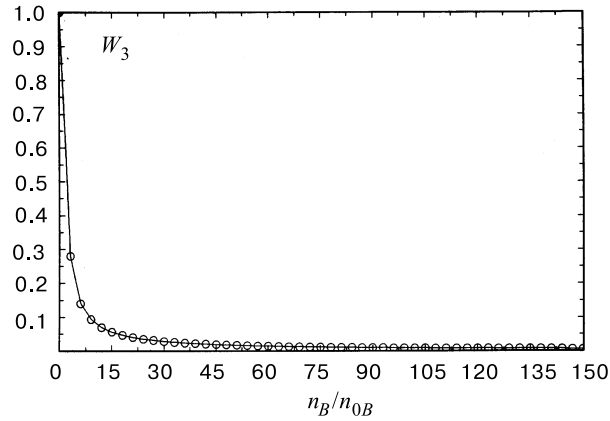


Fig. 29. Nucleon channel probability as a function of relative baryon density

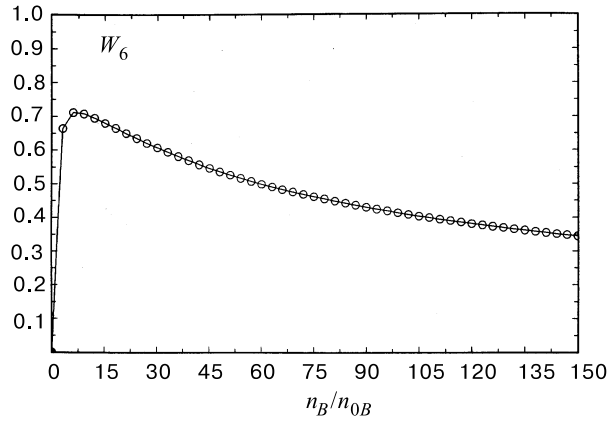


Fig. 30. Dibaryon channel probability versus relative baryon density

deconfinement transition essentially depends on the interactions between particles or on their radii, when the composite particles are treated as bags [227].

In the case of realistic nuclear matter, deconfinement is rather a gradual crossover but not a genuine phase transition [217]. Then all thermodynamic characteristics change continuously, without jumps. This concerns as well the channel probabilities. Thus, in Figs. 29, 30 the channel probabilities of nucleons and dibaryons are shown as functions of baryon density normalized to the normal baryon density of nuclear matter  $n_{0B} = 0.167 \text{ fm}^{-3}$ . The possible appearance of dibaryons is of special interest since they, being bosons, can form a Bose condensate [217,228–230].

Summarizing the results of the analysis [221], three types of Bose particles can appear in nuclear matter in large quantities: pions, dibaryons, and gluons. The maximum of the pion channel probability, reaching  $w_\pi = 0.6$ , occurs in the vicinity of the deconfinement transition at  $T \approx 160$  MeV and low baryon densities  $n_B < n_{0B}$ . Dibaryons can appear mainly at low temperatures  $T < 20$  MeV and relatively high baryon densities  $n_B \sim 10 n_{0B}$ , where their channel probability  $w_6 \approx 0.7$ . Large amount of gluons emerges only at high temperatures  $T > 160$  MeV. In addition, one should keep in mind that gluons cannot be observed as free particles.

Talking about possible pion, dibaryon, or gluon lasing from nuclear matter, we have touched here just one necessary condition, trying to find out when these Bose particles can appear in large quantities. To realize such a lasing in reality will, certainly, require to solve a number of other problems. But, anyway, to understand the conditions when this lasing could be plausible in principle is the necessary first step.

## 15. CONCLUSION

We have described a general method for treating strongly nonequilibrium processes in statistical systems. This method is called the *Scale Separation Approach* since its basic idea is to try to separate different characteristic scales of time and space variables. The idea itself is, of course, not new and we have employed some known techniques. What is original in our approach is: (i) The combination of several methods and their adjustment to the problems of nonequilibrium statistical mechanics. (ii) The generalization of the averaging method to stochastic and partial differential equations. (iii) Probabilistic solution of the problem of pattern selection.

The scale separation approach has been shown to be very useful for describing cooperative phenomena in the interaction of radiation with matter. To emphasize the generality of the approach, it is illustrated here by several different physical examples, whose common feature is that the related evolution equations are non-linear differential or integro-differential stochastic equations. Such equations, as is known, are difficult to solve. The scale separation approach makes it possible to find accurate approximate solutions. The accuracy of these solutions has been confirmed by numerical calculations and by comparison with experiment, when available. Using this approach, several interesting physical problems have been solved and new effects are predicted. Among the most interesting applications we would like to emphasize the following.

*Collective Liberation of Light* happens when an ensemble of resonant atoms is doped into a medium with polariton band gap. If the transition frequency of an atom is inside this prohibited gap, then atomic spontaneous emission is strongly

suppressed, which is termed localization of light. Although spontaneous emission of a single atom is prohibited, a collective of such atoms can radiate due to their coherent interactions. As a result of this coherent radiation, light becomes partially liberated. We have advanced dynamical theory of this light liberation for the realistic situation when the radiation wavelength is smaller than the linear sizes of the sample (see Sec. 6).

*Mössbauer Magnetic Anomaly* has puzzled researchers for many years. This anomaly consists in a strong increase of the area under the Mössbauer spectrum, below the temperature of magnetic phase transition. Several explanations of this anomaly have been suggested. We have thoroughly analysed this phenomenon and concluded that previously suggested mechanisms cannot explain this anomaly but that its origin is rather in the increase of inhomogeneous broadening of Mössbauer nuclei, which is due to the arising magnetic field (see Sec. 8).

*Turbulent Photon Filamentation* in resonant media is an intriguing example of self-organization in a strongly nonequilibrium system, whose dynamical theory was absent. We have developed such a theory, based on the probabilistic approach to pattern selection, and showed that it gives agreement with experiment (see Sec. 10).

*Superradiant Spin Relaxation* occurs in a system of spins coherently interacting with each other through resonator feedback field. This ultrafast coherent relaxation is similar to superradiance in optical systems, because of which the term spin superradiance was coined. Contrary to its optical counterpart, the origin of purely self-organized spin superradiance has not been understood for about 40 years, after Bloembergen and Pound posed this problem in 1954. We have developed a theory of nonlinear spin dynamics, based on a microscopic Hamiltonian, elucidated the origin of pure spin superradiance, and described all main regimes of spin relaxation, without pumping as well as in the presence of the latter (see Sec. 11).

*Negative Electric Current* is a rather unusual effect, when electric current flows against an applied voltage. This is a transient effect that can occur in nonuniform semiconductors. We have predicted this effect and suggested its theory (see Sec. 12).

*Magnetic Semiconfinement of Atoms* is another effect we predict. This effect can serve as a mechanism for creating well-collimated beams of neutral atoms by means of magnetic fields. It can be used to form coherent beams of Bose atoms from atom lasers. We have presented a theory of this effect (see Sec. 13).

The possibility of treating nonequilibrium processes in nonlinear systems of quite different nature has become possible owing to the Scale Separation Approach, which provides accurate approximate solutions to complicated systems of differential and integro-differential equations.

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