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THE GOTO–IMAMURA–SCHWINGER TERM AND RENORMALIZATION GROUP

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In connection with the question of color confinement the origin of the Goto–Imamura–Schwinger term has been studied with the help of renormalization group. An emphasis has been laid on the difference between theories with and without a cut-off.

1. INTRODUCTION

Field theory is full of ghosts and bugs, and we have to bring divergences, anomalies and ambiguities under control. Among others we shall concentrate on the origin of the so-called Goto–Imamura–Schwinger (GIS) term [1,2] in field theory, since it bears a close connection with the question of color confinement [3–5].

In evaluating the equal-time commutator (ETC) between two local operators we sometimes encounter a result in conflict with that obtained by a naive application of the canonical commutation relations (CCRs). The deviation from the naive expectation is referred to as the Goto–Imamura–Schwinger term hereafter. Such a term does not arise, however, when we evaluate the ETC between two fundamental fields, and it indicates that the origin of the GIS term must be sought in the definition of the singular product of field operators at the same space-time point.

In many examples it is possible to find a renormalization group (RG) equation controlling the GIS term in question, but then the next question is raised of how to formulate the initial or boundary condition for this equation. In the RG approach we introduce running parameters such as the running coupling constant and they tend to the bare or nonrenormalized ones in the high energy limit provided that we introduce a cut-off in the unrenormalized version of the theory as we shall see in Sec. 2. Then we can introduce boundary conditions in the high energy limit

into the cut-off theory by assuming the CCRs. In some cases it is possible to formulate the boundary condition kinematically, namely, without reference to the dynamics of the system but often it is necessary to refer to the dynamics of the system by evaluating higher order corrections. In Sec.3 we shall illustrate these statements in quantum electrodynamics (QED). Then, we find that the origin of the GIS terms may be attributed to one of the following causes: (1) operator-mixing under renormalization [3,5], (2) nonlocal character of the product of field operators at the same space-time point, and (3) divergences induced by lifting the cut-off. In Sec.4 we shall proceed to quantum chromodynamics (QCD) in connection with the question of color confinement.

2. RENORMALIZATION GROUP

In introducing the RG approach [6–8] we shall employ the neutral scalar theory for illustration. We assume the quartic interaction of the scalar field $\phi(x)$ with the coupling constant g . The unrenormalized Green function is given by

$$G_0^{(n)}(x_1, \dots, x_n) = \langle 0|T [\phi^{(0)}(x_1) \cdots \phi^{(0)}(x_n)] |0\rangle, \quad (1)$$

where the subscript 0 and the superscript (0) denote unrenormalized quantities. The Fourier transform of the renormalized n-point Green function is denoted by

$$G^{(n)}(p_1, \dots, p_n; g(\mu), \mu), \quad (2)$$

where μ denotes the renormalization point defined below; and $g(\mu)$, the running coupling constant defined at the renormalization point as seen from

$$(p^2 + m^2)G^{(2)}(p^2; g(\mu), \mu) = 1, \quad \text{for } p^2 = \mu^2, \quad (3)$$

$$G_{\text{conn}}^{(4)}(p_1, \dots, p_4; g(\mu), \mu) = g(\mu) \prod_{i=1}^4 G^{(2)}(p_i^2; g(\mu), \mu) \cdot \Gamma(p_1, \dots, p_4; g(\mu), \mu), \quad (4)$$

$$\Gamma(p_1, \dots, p_4; g(\mu), \mu) = 1, \quad \text{for } p_i \cdot p_j = \frac{\mu^2}{3}(4\delta_{ij} - 1), \quad (5)$$

where $G_{\text{conn}}^{(4)}$ denotes the 4-point Green function for connected Feynmann diagrams alone. These are the normalization conditions for the Green functions and specify the renormalization point in the Pauli metric.

The generator of the RG is given by

$$\mathcal{D} = \mu \frac{\partial}{\partial \mu} + \beta(g) \frac{\partial}{\partial g}, \quad (6)$$

and the RG equation for the n -point Green function is given by

$$[\mathcal{D} + n\gamma_\phi(g)] G^{(n)}(p_1, \dots, p_n; g, \mu) = 0, \quad (7)$$

where we write g for $g(\mu)$ and γ_ϕ denotes the anomalous dimension of the scalar field ϕ . For the two-point Green function or the propagator we may assume the Lehmann representation [9],

$$G^{(2)}(p^2; g, \mu) = \int d\kappa^2 \frac{\rho(\kappa^2; g, \mu)}{p^2 + \kappa^2 - i\epsilon}, \quad (8)$$

and we have

$$[\mathcal{D} + 2\gamma_\phi(g)] \rho(\kappa^2; g, \mu) = 0. \quad (9)$$

Then Eq. (2.3) in the limit $\mu \rightarrow \infty$ yields

$$\lim_{\mu \rightarrow \infty} (\mu^2 + m^2) G^{(2)}(\mu^2; g, \mu) = \int d\kappa^2 \rho(\kappa^2; g(\infty), \infty) = 1, \quad (10)$$

in the cut-off theory where m denotes the mass of the quantum of the scalar field.

Lehmann's theorem [9] on the ETC for the field operator normalized at μ readily yields the relation

$$\delta(x_0 - y_0) [\phi(x; g, \mu), \dot{\phi}(y; g, \mu)] = i\delta^4(x - y) \int d\kappa^2 \rho(\kappa^2; g, \mu), \quad (11)$$

and Eq. (2.10) then implies that the field operators are identified with the unrenormalized ones in the limit $\mu \rightarrow \infty$ since they satisfy the CCR. At the same time we can show that $g(\mu)$ also tends to the bare coupling constant g_0 in the same limit.

In order to define the running parameters we introduce

$$R(\rho) = \exp(\rho \mathcal{D}), \quad (12)$$

where ρ denotes the parameter of the RG, then $R(\rho)$ obeys the composition law

$$R(\rho_1) \cdot R(\rho_2) = R(\rho_1 + \rho_2), \quad (13)$$

and the RG is literally a group identified with $GL(1, R)$.

The running parameters in the scalar theory are defined by

$$\bar{g}(\rho) = R(\rho) \cdot g, \quad (14)$$

$$\bar{\mu}(\rho) = R(\rho) \cdot \mu = \mu \exp(\rho), \quad (15)$$

then we readily obtain

$$R(\rho) G^{(n)}(p_1, \dots, p_n; g, \mu) = G^{(n)}(p_1, \dots, p_n; \bar{g}(\rho), \bar{\mu}(\rho)). \quad (16)$$

We differentiate this equation with respect to ρ and combine it with Eq. (2.7) to obtain

$$\begin{aligned} \frac{\partial}{\partial \rho} G^{(n)}(p_1, \dots, p_n; \bar{g}(\rho), \bar{\mu}(\rho)) &= R(\rho) \mathcal{D}G^{(n)}(p_1, \dots, p_n; g, \mu) \\ &= -nR(\rho)\gamma_\phi(\rho) G^{(n)}(p_1, \dots, p_n; g, \mu) \\ &= -n\gamma_\phi(\bar{g}(\rho)) G^{(n)}(p_1, \dots, p_n; \bar{g}(\rho), \bar{\mu}(\rho)). \end{aligned} \quad (17)$$

We have to introduce a boundary condition to this differential equation. In a cut-off theory we may set

$$\lim_{\mu \rightarrow \infty} G^{(n)}(p_1, \dots, p_n; g(\mu), \mu) = G_0^{(n)}(p_1, \dots, p_n; g_0), \quad (18)$$

where g_0 denotes the bare coupling constant.

By integrating Eq. (2.17) we find

$$G^{(n)}(p_1, \dots, p_n; g, \mu) = \exp \left[n \int_0^\rho d\rho \gamma_\phi(\bar{g}(\rho)) \right] \cdot G^{(n)}(p_1, \dots, p_n; \bar{g}(\rho), \bar{\mu}(\rho)). \quad (19)$$

In the limit $\rho \rightarrow \infty$ and consequently $\bar{\mu}(\rho) \rightarrow \infty$ we have

$$G^{(n)}(p_1, \dots, p_n; g, \mu) = \exp \left[n \int_0^\infty d\rho \gamma_\phi(\bar{g}(\rho)) \right] \cdot G_0^{(n)}(p_1, \dots, p_n; g_0). \quad (20)$$

In a cut-off theory all the vertex corrections to $g(\mu)$ for $\mu \rightarrow \infty$ tend to vanish leaving only the bare one, namely,

$$\lim_{\mu \rightarrow \infty} g(\mu) = \lim_{\rho \rightarrow \infty} \bar{g}(\rho) = g_0. \quad (21)$$

The fundamental field ϕ is multiplicatively renormalized as

$$\phi^{(0)}(x) = Z_\phi^{1/2} \phi(x), \quad (22)$$

where Z_ϕ is the renormalization constant of the scalar field ϕ , and it is a function of g . Comparison of Eqs. (2.20) and (2.22) yields

$$Z_\phi^{-1} = \exp \left[2 \int_0^\infty d\rho \gamma_\phi(\bar{g}(\rho)) \right]. \quad (23)$$

The running renormalization constant is given by

$$\begin{aligned} Z_\phi^{-1}(\rho) &= R(\rho) Z_\phi^{-1}(g) \\ &= \exp \left[2 \int_\rho^\infty d\rho' \gamma_\phi(\bar{g}(\rho')) \right]. \end{aligned} \quad (24)$$

When Z_ϕ depends not only on g but also on μ , $\gamma_\phi(\rho)$ must be replaced by $\gamma_\phi(\rho, \mu)$.

In a *cut-off theory* we have

$$\lim_{\rho \rightarrow \infty} Z_\phi^{-1}(\rho) = 1, \quad (25)$$

but this is not true when the integral in the exponent of Eq.(2.23) does not converge and as we shall see later this feature is a possible cause of the emergence of the GIS terms.

Although the RG approach has been introduced for the scalar theory we can easily extend it to gauge theories. In QED the generator of the RG is given by

$$\mathcal{D} = \mu \frac{\partial}{\partial \mu} + \beta(e) \frac{\partial}{\partial e} - 2\alpha \gamma_V(e) \frac{\partial}{\partial \alpha}, \quad (26)$$

where α denotes the gauge parameter. The $\gamma_V(e)$ denotes the anomalous dimension of the electromagnetic field and is related to $\beta(e)$ through the Ward identity

$$\beta(e) = e \gamma_V(e). \quad (27)$$

Furthermore in QCD the generator is given by

$$\mathcal{D} = \mu \frac{\partial}{\partial \mu} + \beta(g) \frac{\partial}{\partial g} - 2\alpha \gamma_V(g, \alpha) \frac{\partial}{\partial \alpha}, \quad (28)$$

where g denotes the gauge coupling constant; and γ_V , the anomalous dimension of the color gauge field. The running parameters in QCD satisfy the following equations:

$$\frac{d\bar{g}}{d\rho} = \beta(\bar{g}), \quad (29)$$

$$\frac{d\bar{\alpha}}{d\rho} = -2\bar{\alpha} \gamma_V(\bar{g}, \bar{\alpha}). \quad (30)$$

Then we introduce their asymptotic values by

$$g_\infty = \lim_{\rho \rightarrow \infty} \bar{g}(\rho), \quad \alpha_\infty = \lim_{\rho \rightarrow \infty} \bar{\alpha}(\rho). \quad (31)$$

This is possible since the RG is a group $GL(1, R)$ but not $U(1)$. Asymptotic freedom [13,14] of QCD implies

$$g_\infty = 0. \quad (32)$$

By integrating Eq. (2.30) we immediately find a sum rule,

$$2 \int_0^\infty d\rho \bar{\gamma}_V(\rho) = \ln\left(\frac{\alpha}{\alpha_\infty}\right) \quad (33)$$

and hence we also have [4,5,12]

$$Z_3^{-1} = \exp\left[2 \int_0^\infty d\rho \bar{\gamma}_V(\rho)\right] = \frac{\alpha}{\alpha_\infty}, \quad (34)$$

where $\bar{\gamma}_V(\rho) \equiv \gamma_V(\bar{g}(\rho), \bar{\alpha}(\rho))$.

In QCD it is known that α_∞ can take three possible values [4,5,12]

$$\alpha_\infty = 0, \quad \alpha_0, \quad -\infty, \quad (35)$$

where α_0 is a constant which depends only on the number of quark flavors. These three values are related to the integral of γ_V as

$$\int_0^\infty d\rho \bar{\gamma}_V(\rho) = \begin{cases} \infty, & \text{for } \alpha_\infty = 0 \\ \text{finite}, & \text{for } \alpha_\infty = \alpha_0 \\ -\infty, & \text{for } \alpha_\infty = -\infty \end{cases} \quad (36)$$

and Z_3^{-1} vanishes when $\alpha_\infty = -\infty$.

3. QUANTUM ELECTRODYNAMICS

Quantum electrodynamics is a suitable ground to exercise the analysis of the GIS terms. The Lagrangian density for QED is given by

$$\mathcal{L} = \mathcal{L}_{\text{em}} + \mathcal{L}_{\text{matter}}, \quad (37)$$

where the unrenormalized version of the Lagrangian density for the electromagnetic field is given by

$$\mathcal{L}_{\text{em}} = -\frac{1}{4} F_{\mu\nu}^{(0)} \cdot F_{\mu\nu}^{(0)} + \partial_\mu B^{(0)} \cdot A_\mu^{(0)} + \frac{\alpha_0}{2} B^{(0)} \cdot B^{(0)}, \quad (38)$$

where B denotes the Nakanishi–Lautrup auxiliary field [10] and the interactions are included in the matter Lagrangian. The resulting field equations are given by

$$\partial_\mu F_{\mu\nu}^{(0)} + \partial_\nu B^{(0)} = -J_\nu^{(0)}, \quad (39)$$

$$\partial_\mu A_\mu^{(0)} = \alpha_0 B^{(0)}, \quad (40)$$

and the renormalized version of these equations can be expressed as

$$\partial_\mu F_{\mu\nu} + \partial_\nu B = -J_\nu, \quad (41)$$

$$\partial_\mu A_\mu = \alpha B. \quad (42)$$

The fundamental fields A_μ and B as well as the gauge parameter α are renormalized multiplicatively,

$$\begin{aligned} A_\mu^{(0)} &= Z_3^{1/2} A_\mu, \\ B^{(0)} &= Z_3^{-1/2} B, \\ \alpha_0 &= Z_3 \alpha. \end{aligned} \quad (43)$$

Apparently renormalization of the composite current operator J_ν is not multiplicative, but its execution requires operator mixing [3,5] as illustrated by

$$J_\nu^{(0)} = Z_3^{1/2} [J_\nu + (1 - Z_3^{-1})\partial_\nu B], \quad (44)$$

or

$$J_\nu = Z_3^{-1/2} [J_\nu^{(0)} + (1 - Z_3)\partial_\nu B^{(0)}].$$

Operator mixing is one of the sources of the GIS terms, and in order to illustrate this statement we shall evaluate the ETC

$$\delta(x_0 - y_0) [A_j(x), J_0(y)] \quad (45)$$

for $j = 1, 2, 3$. In the unrenormalized version we have

$$\delta(x_0 - y_0) [A_j^{(0)}(x), J_0^{(0)}(y)] = 0. \quad (46)$$

As has been mentioned before we can rely on the ETCs only between two fundamental fields, so that we shall express J in terms of A and B by using Eqs. (3.5) and (3.7),

$$\begin{aligned} [A_j(x), J_4(y)] &= -[A_j(x), \partial_\mu F_{\mu 4}(y) + \partial_4 B(y)] \\ &= -Z_3^{-1} [A_j^{(0)}(x), \partial_k F_{k4}^{(0)}(y)] - [A_j^{(0)}(x), \partial_4 B^{(0)}(y)] \\ &= (-Z_3^{-1} + 1)\partial_j \delta^3(x - y) \end{aligned}$$

for $x_0 = y_0$. Thus we have

$$\begin{aligned} \delta(x_0 - y_0) [A_j(x), J_0(y)] &= i(Z_3^{-1} - 1)\partial_j\delta^4(x - y) \\ &\equiv is\partial_j\delta^4(x - y), \end{aligned} \quad (47)$$

where s is the coefficient of the GIS term. In this case it is clear that the origin of the GIS term is the operator mixing. Then s satisfies the RG equation

$$[\mathcal{D} + 2\gamma_V(e)](s + 1) = [\mathcal{D} + 2\gamma_V(e)]Z_3^{-1} = 0, \quad (48)$$

where \mathcal{D} is given by Eq. (2.26), and the running GIS coefficient $\bar{s}(\rho)$ satisfies the differential equation

$$\left[\frac{\partial}{\partial \rho} + 2\gamma_V(\bar{e}(\rho)) \right] (\bar{s}(\rho) + 1) = 0. \quad (49)$$

In a cut-off theory the GIS term is absent in the unrenormalized version as expressed by Eq. (3.10), and the boundary condition for $\bar{s}(\rho)$ is given by

$$\bar{s}(\infty) = 0. \quad (50)$$

By combining the boundary condition (3.14) with Eq. (3.13) we find the solution

$$Z_3^{-1}(\rho) = 1 + \bar{s}(\rho) = \exp \left[2 \int_{\rho}^{\infty} d\rho' \gamma_V(\bar{e}(\rho')) \right]. \quad (51)$$

In the absence of the cut-off we do not know what kind of boundary condition we should impose on $\bar{s}(\rho)$ so that we take this solution (3.15) for granted even in this case.

In QED we assume that $Z_3^{-1} = Z_3^{-1}(0)$ is divergent so that we have

$$1 + \bar{s}(\infty) = \lim_{\rho \rightarrow \infty} \exp \left[2 \int_{\rho}^{\infty} d\rho' \gamma_V(\bar{e}(\rho')) \right] = \infty, \quad (52)$$

and the boundary condition (3.14) is no longer satisfied in the absence of the cut-off. This is another source of the GIS terms, and the field operators do not necessarily tend to the unrenormalized ones in the limit $\rho \rightarrow \infty$ and hence $\mu \rightarrow \infty$ when the cut-off is lifted.

Finally we shall turn our attention to the ETC between two components of the current density. This is precisely the original problem in which the GIS term was recognized [1,2]. We shall make use of the field equations (3.5) to express the current density as a linear combination of the fundamental fields, and then we can make use of the commutativity of B with $F_{\mu\nu}$ and B itself [10],

$$\begin{aligned}
 [J_\mu(x), J_\nu(y)] &= [\partial_\alpha F_{\alpha\mu}(x) + \partial_\mu B(x), \partial_\beta F_{\beta\nu}(y) + \partial_\nu B(y)] \\
 &= [\partial_\alpha F_{\alpha\mu}(x), \partial_\beta F_{\beta\nu}(y)],
 \end{aligned} \tag{53}$$

and we introduce the GIS coefficient s by

$$\delta(x_0 - y_0) \langle 0 | [J_j(x), J_0(y)] | 0 \rangle = i s \partial_j \delta^4(x - y). \tag{54}$$

As a matter of fact, the ETC on the left-hand side of Eq.(3.18) is known to be a c number before taking its vacuum expectation value in spinor electrodynamics [15]. Here we are aware of the fact that the GIS term can be expressed in terms of the ETC between derivatives of field strengths. In order to evaluate the ETC by making use of the CCRs it is necessary to express derivatives of field strengths in terms of canonical variables by making use of canonical field equations. Therefore, we are taking commutators between those operators that are nonlocal in time and then taking the local limit. The way in which this limit is taken is dictated in the evaluation of the higher order corrections as we shall see below.

This is in a sharp contrast to the original naive way of evaluating the commutator between two bilinear forms of the Dirac fields by making use of only the CCRs without taking the possibility of nonlocality into consideration. This gap generates the GIS term.

By combining Eqs. (3.17) and (3.18) we find that the GIS coefficient s satisfies the RG equation

$$[\mathcal{D} + 2\gamma_V(e)] s = 0. \tag{55}$$

In this case we cannot give the boundary condition for this equation since it requires the information about the dynamics of the system such as the photon propagator. The Lehmann representation of the electromagnetic field is given in the following form:

$$\langle 0 | T [A_\mu(x), A_\nu(y)] | 0 \rangle = \frac{-i}{(2\pi)^4} \int d^4 k e^{ik \cdot (x-y)} D_{F\mu\nu}(k), \tag{56}$$

$$D_{F\mu\nu}(k) = \left(\delta_{\mu\nu} - \frac{k_\mu k_\nu}{k^2 - i\epsilon} \right) \int dM^2 \frac{\rho(M^2; e, \mu)}{k^2 + M^2 - i\epsilon} + \alpha \frac{k_\mu k_\nu}{(k^2 - i\epsilon)^2}. \tag{57}$$

Then inserting this expression into Eq.(3.17) we find

$$s = \int dM^2 \rho(M^2; e, \mu) M^2. \tag{58}$$

This expression certainly satisfies Eq. (3.19) since we have

$$[\mathcal{D} + 2\gamma_V(e)]\rho(M^2; e, \mu) = 0. \quad (59)$$

It is clear that Z_3^{-1} also satisfies Eq. (3.19) since it is given by

$$Z_3^{-1} = \int dM^2 \rho(M^2; e, \mu). \quad (60)$$

We may conclude that the GIS terms are controlled by RG if not completely.

4. COLOR CONFINEMENT IN QCD

In QCD the GIS term plays an important role in connection with color confinement [3–5]. The field equation in QCD corresponding to Eq. (3.5) is given by

$$\partial_\mu F_{\mu\nu}^a + J_\nu^a = i\delta\bar{\delta}A_\nu^a, \quad (61)$$

where δ and $\bar{\delta}$ denote two kinds of Becchi–Rouet–Stora (BRS) transformations [11], respectively, and the superscript a represents the color index. Since we are not entering the subject of BRS transformations here we shall refer to other references [3–5] for their definitions.

We are interested in ETC

$$\begin{aligned} & \partial_\mu \langle 0|T [i\delta\bar{\delta}A_\mu^a(x), A_j^b(y)] |0\rangle \\ &= \delta(x_0 - y_0) \langle 0| [i\delta\bar{\delta}A_0^a(x), A_j^b(y)] |0\rangle \\ &= i\delta_{ab}C \partial_j \delta^4(x - y), \end{aligned}$$

or

$$\begin{aligned} & \delta(x_0 - y_0) \langle 0| [\partial_k F_{k4}^a(x) + J_4^a(x), A_j^b(y)] |0\rangle \\ &= -\delta_{ab}C \partial_j \delta^4(x - y). \end{aligned} \quad (62)$$

The constant C is gauge-dependent, and a sufficient condition for color confinement is the existence of a gauge in which the following equality holds:

$$C = 0. \quad (63)$$

In order to determine C we have to evaluate the ETC in Eq. (4.2), and for that purpose we introduce the RG equation satisfied by C [3–5],

$$(\mathcal{D} - 2\gamma_{FP})C = 0, \quad (64)$$

where \mathcal{D} is given by Eq. (2.28) and γ_{FP} denotes the anomalous dimension of the Faddeev–Popov ghost fields. Then the renormalization constant of the ghost fields denoted by \tilde{Z}_3 also satisfies the same RG equation,

$$(\mathcal{D} - 2\gamma_{FP}) \tilde{Z}_3 = 0. \quad (65)$$

We are going to study the relationship between C and \tilde{Z}_3 in this section. They satisfy the same RG equation, but their normalizations are different.

The unrenormalized version of Eq. (4.1) reads as

$$i\delta\bar{\delta}A_\nu^{(0)}(x) = \partial_\mu A_{\mu\nu}^{(0)} + g_0\partial_\mu(A_\mu^{(0)} \times A_\nu^{(0)}) + J_\nu^{(0)}, \quad (66)$$

where $A_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ denotes the linear part of $F_{\mu\nu}$ and we have suppressed the color index. The cross product denotes the antisymmetric product in the color space defined in terms of the structure constants of the algebra $su(3)$. When we insert the r.h.s. of Eq. (4.6) into the ETC (4.2) in the unrenormalized version, we find that only the first term $\partial_\mu A_{\mu\nu}^{(0)}$ gives a nonvanishing canonical commutator and the rest would give only a vanishing result provided that the naive CCRs are employed. However, this is true only in a cut-off theory or in a convergent theory and in general we should not discard the possibility of a nonvanishing GIS term so that the unrenormalized constant C_0 would be given by

$$C_0 = 1 + s. \quad (67)$$

The first term is a result of the CCR and is equal to unity. Thus the renormalized C is given by

$$C = C_0\tilde{Z}_3 = (1 + s)\tilde{Z}_3. \quad (68)$$

Then a question is raised of how to evaluate the GIS coefficient s . For this purpose we introduce a cut-off theory and we write $\bar{a}(\rho)$ for $Z_3^{-1}(\rho)$, and we shall rewrite Eq. (4.7) in the form

$$\bar{C}(\infty) = \bar{a}(\infty) \quad (69)$$

based on the argument developed in Sec. 2. In a cut-off theory the GIS coefficient s vanishes, but it does not vanish when the cut-off is lifted. The running parameters $\bar{C}(\rho)$, $\bar{a}(\rho)$ and $\tilde{Z}_3(\rho)$ satisfy the following differential equations,

respectively,

$$\left[\frac{\partial}{\partial \rho} - 2\bar{\gamma}_{FP}(\rho) \right] \bar{C}(\rho) = 0, \quad (70)$$

$$\left[\frac{\partial}{\partial \rho} + 2\bar{\gamma}_V(\rho) \right] \bar{a}(\rho) = 0, \quad (71)$$

$$\left[\frac{\partial}{\partial \rho} - 2\bar{\gamma}_{FP}(\rho) \right] \tilde{Z}_3(\rho) = 0. \quad (72)$$

Among them the last two are renormalization constants, and they are immediately given by

$$\bar{a}(\rho) = Z_3^{-1}(\rho) = \exp \left[2 \int_{\rho}^{\infty} d\rho' \bar{\gamma}_V(\rho') \right], \quad (73)$$

$$\tilde{Z}_3^{-1}(\rho) = \exp \left[2 \int_{\rho}^{\infty} d\rho' \bar{\gamma}_{FP}(\rho') \right]. \quad (74)$$

We should be aware of the following relations:

$$Z_3^{-1} = Z_3^{-1}(0), \quad \tilde{Z}_3^{-1} = \tilde{Z}_3^{-1}(0), \quad (75)$$

Then $\bar{C}(\rho)$ should be determined by solving Eq. (4.10) under the boundary condition (4.9) and we obtain

$$\bar{C}(\rho) = \lim_{\rho' \rightarrow \infty} \exp \left[2 \int_{\rho'}^{\infty} d\rho'' \bar{\gamma}_V(\rho'') - 2 \int_{\rho}^{\rho'} d\rho'' \bar{\gamma}_{FP}(\rho'') \right], \quad (76)$$

and, in particular, we have

$$C = \lim_{\rho' \rightarrow \infty} \exp \left[2 \int_{\rho'}^{\infty} d\rho'' \bar{\gamma}_V(\rho'') - 2 \int_0^{\rho'} d\rho'' \bar{\gamma}_{FP}(\rho'') \right]. \quad (77)$$

From now on we lift the cut-off while keeping these formulas. With recourse to Eqs. (2.34) and (2.36) we find that C vanishes when Z_3^{-1} vanishes as claimed before [3–5]. Then we may express Eq. (4.17) as

$$C = \lim_{\rho \rightarrow \infty} \exp \left[2 \int_{\rho}^{\infty} d\rho' \bar{\gamma}_V(\rho') \right] \cdot \tilde{Z}_3, \quad (78)$$

and with reference to Eq.(4.8) we find

$$1 + s = \lim_{\rho \rightarrow \infty} \exp \left[2 \int_{\rho}^{\infty} d\rho' \bar{\gamma}_V(\rho') \right]$$

$$= \begin{cases} \infty, & \text{for } \alpha_{\infty} = 0 \\ 1, & \text{for } \alpha_{\infty} = \alpha_0 \\ 0, & \text{for } \alpha_{\infty} = -\infty. \end{cases} \quad (79)$$

Only in the case $\alpha_{\infty} = \alpha_0$ do we find the vanishing GIS coefficient s , and this is precisely what happens when the integration of $\bar{\gamma}_V$ converges just as in the cut-off theory. Now we shall summarize the relationship between C and \tilde{Z}_3 as follows:

$$C = \begin{cases} \infty, & \alpha_{\infty} = 0 \\ \tilde{Z}_3, & \alpha_{\infty} = \alpha_0 \\ 0 & \alpha_{\infty} = -\infty. \end{cases} \quad (80)$$

As we have seen above we formulate the boundary condition for a given RG equation by introducing a cut-off, but when the cut-off is lifted in the solution the GIS term appears as a manifestation of the divergent character of the theory.

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