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ABELIAN DEFORMATIONS

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Certain physical problems lead to a need for quantization in a context where a Poisson bracket does not provide the direction. Nambu mechanics on a three-dimensional «phase space» is one example. Another is the problem of quantization on coadjoint orbits, especially on singular orbits. Abelian $*$-products are often governed by Harrison cohomology, but are erroneously said to be trivial. In fact, varieties with singularities, including simple examples of physical relevance, do have a nontrivial Harrison cohomology. Besides, Harrison cohomology is not always decisive. Minkowski space is a smooth manifold, with vanishing Harrison cohomology; the coordinate algebra admits, nevertheless, nontrivial Abelian deformations.

INTRODUCTION

Quantization continues to be a vital subject of research in physics, and recently it has become central to a broad development in mathematics as well. This paper contains only the briefest possible summary of general developments, and then concentrates on a particular problem. What usually characterizes quantization is noncommutativity, the passage from a commutative structure to a noncommutative one. More precisely, quantization is a deformation of algebraic structure, a deformation of the product. Since the pioneering work of M. Gerstenhaber [1], deformation theory is the study of formal series, as in

$$f * g = fg + \sum_{n=1}^{\infty} \hbar^n C_n(f, g), \tag{1}$$

where fg is the original product, and $\hbar$ is a parameter. One requires that the coefficients C_n have properties such as to make the $*$-product associative.

In the historical setting the object under discussion is the algebra of differentiable functions on a symplectic space. The early work of Weyl on quantization consisted in setting up a correspondence between such functions on the one hand, and an algebra of operators in a Hilbert space on the other [2]. The above series first appears in the work of Moyal and Vey [3]; it became the basis for a general investigation in the papers [4]. Later the same series played an important role in the Drinfel'd work on quantum group. In all these developments a dominant role was played by the Poisson bracket $\{f,g\}$. A deformation «in the direction of the Poisson bracket» is a series of the form

$$f * g = fg + (\hbar/2)\{f,g\} + \sum_{n=2}^{\infty} \hbar^n C_n(f,g),$$

where $C_1 = (1/2)\{,\}$ is antisymmetric. Recent developments in mathematics [5, 6] take place in the more general setting of Poisson manifolds*, but the Poisson bracket continues to play a dominant role.

In all these theories that are based on a Poisson structure one assumes that the Poisson bracket accounts for the term of order $\hbar$; which implies that this term is antisymmetric,

$$C_1(f,g) = -C_1(g,f). \tag{2}$$

It has been observed that every deformation (1) is «equivalent» to one that satisfies (2). But that statement holds under certain specific conditions, and this paper is about ways to get around them.

Interest in the problems discussed in this review came to light in the context of quantization of Nambu mechanics. The suggestion of Nambu was to replace the canonical equation of motion

$$\frac{d}{dt}F = \{F,H\}$$

by

$$\frac{d}{dt}F = \{F,G,H\},$$

where a pair of functions G,H replaces the Hamiltonian, and the new bracket is defined by

$$\{F,G,H\} := \det\left(\frac{\partial_i F}{dt}, \frac{\partial_j G}{dt}, \frac{\partial_k H}{dt}\right),$$

which makes sense in a 3-dimensional «phase-space». The problem of quantizing this system has remained unsolved, and in 1996 it led to the realization that the heart of the matter should be an Abelian deformation of the algebra of functions; that is, a deformation with C_1 symmetric [7]. The problem was that all such products were believed to be «trivial».

In Section 1 we shall explain what is meant by saying that an associative deformation of a commutative algebra is trivial, or inessential. The emphasis in Section 1 is on first order deformations, and the appropriate language

*The Poisson bracket takes the form $\{f,g\} = \Lambda^{ij}\partial_i f \partial_j g$. If Λ is invertible, then the inverse mapping is closed and defines a symplectic form. If Λ has constant rank, then the situation is less complicated. The problem was to prove the existence of quantization on an arbitrary Poisson manifold.

is Hochschild cohomology. Triviality of first order Abelian deformations is a property of algebras of functions on smooth manifolds.

In Section 2 we examine certain algebraic varieties with singularities, especially cones, and show that essential, Abelian deformations exist. In Section 3 we study an old example of invariant quantization on singular coadjoint orbits and show that deformations with C_1 essentially nonantisymmetric were known already.

In Section 4 we study some examples of smooth manifolds, going beyond first order deformations to formal or convergent power series. The main result is that deformations that are trivial to first order may be nontrivial when thus extended. An example of particular interest is a deformation of the algebra of functions on Minkowski space that turns out to be equivalent to the ordinary algebra of functions on (anti-) De Sitter space.

1. FIRST ORDER DEFORMATIONS

As a first example, consider the space $\mathbb{R}$, with global coordinates $x_1, \ldots, x_n$, and the commutative and associative algebra

$$A = \mathbb{C}[x_1, \ldots, x_N]$$

of complex polynomials in N variables. Introduce a formal parameter $\hbar$ and consider a formal deformation of A with a new product

$$f * g = fg + \sum_{n=1}^{\infty} \hbar^n C_n(f,g), \tag{3}$$

where, for $n = 1, 2, \ldots$; C_n is a function from $A \otimes A$ to A. We ask that this new structure be associative, namely

$$(f * g) * h = f * (g * h),$$

or

$$\sum_{m,n=0}^{\infty} h^{m+n} \Big(C_m(f, C_n(g,h) - C_m(C_n(f,g),h)) \Big) = 0, \tag{4}$$

where $C_0(f,g) = fg$. In general, (3) makes sense only as a formal series, therefore Eq. (4) must be interpreted as an identity in $\hbar$; thus

$$\sum_{m,n=0}^{\infty} \delta_{m+n,k} \Big(C_m(f, C_n(g,h) - C_m(C_n(f,g),h)) \Big) = 0, \quad k = 1, 2, \ldots \tag{5}$$

A first order associative deformation is a deformation $f * g = fg + \hbar C_1(f,g)$ that is associative to first order in $\hbar$. Taking $k = 1$ in (5) we get

$$fC_1(g,h) - C_1(fg,h) + C_1(f,gh) - C_1(f,g)h = 0. \tag{6}$$

It is necessary to worry about the possibility that an associative deformation is «trivial». By this we mean that it can be interpreted as the result of a mapping of the algebra on itself, of the form

$$f \to E(f) = f + \sum_{n=1}^{\infty} \hbar^n E_n(f),$$

with

$$f * g = E^{-1}\big(E(f)E(g)\big).$$

Again this must be interpreted as an identity in the deformation parameter; in particular, to first order in $\hbar$ it states that

$$C_1(f,g) = fE_1(g) - E_1(fg) + E_1(f)g. \tag{7}$$

Notice that the right-hand side is even in f, g; it is only when $C_1(f,g)$ is symmetric in the two arguments that the $*$-product (3) risks being trivial. The formulas (6) and (7) introduce us to the following topic.

Hochschild Cohomology. The n-cochains of the Hochschild complex Hoch* are maps $A^{\otimes n} \to A$, and the differential is

$$\begin{aligned} dC(f_1,\dots,f_n) = f_1C(f_2,\dots,f_n) - C(f_1f_2,f_3,\dots,f_n) + \\ + C(f_1,f_2f_3,f_4,\dots,f_n) + \dots + (-)^n C(f_1,\dots,f_{n-1})f_n. \end{aligned}$$

An n-chain C is closed if $dC = 0$ and it is exact if there is an $(n-1)$-cochain E such that $C = dE$. The fundamental property of d is that $d \circ d = 0$. The space B^n of exact n-cochains is thus a subspace of the space Z^n of closed n-cochains. Two closed n-cochains are said to be equivalent, or cohomologous, if the difference is exact. The quotient Hoch$^n = B^n/Z^n$ is the nth Hochschild cohomology group. (It is a group under addition.) This space is identified with the space of equivalence classes of first order deformations of the algebra of functions.

We need to be precise about what algebra of functions we are talking about. Until further notice let us consider the space $A = \mathbb{C}[x_1,\dots,x_N]$ of polynomials in N generators. In this case we have

Theorem. (Hochschild, Kostant and Rosenberg [8].) Every closed n-cochain is cohomologous to an alternating n-cochain.

Corollary. Every Abelian first order deformation of the algebra $A = \mathbb{C}[x_1,\dots, x_N]$ is trivial. (This was already proved by Harrison [9].)

The alternating cochains form a subcomplex of the Hochschild complex. The study of Abelian $*$-products leads to another subcomplex, first described by Harrison [9]. Later Barr [10], Fleury [11], and Gerstenhaber [1], showed that there is a complete decomposition of the Hochschild complex of any commutative algebra, into «irreducible» subcomplexes,

$$\mathrm{Hoch}^* = \oplus_n \mathrm{Hoch}^*(n). \tag{8}$$

It is likely that such a decomposition exists in the case of some noncommutative algebras, but this question may never have been explored. In particular, it is likely that the coordinate algebras of Manin's quantum planes have this property, and establish that may be a good way to begin an investigation with larger scope.

Let us describe the Harrison component $\mathrm{Harr}^* = \mathrm{Hoch}^*(1)$, but only 1-, 2- and 3-cochains. We have seen that 1-cochains are mappings of the algebra into itself, Hom (A, A). There are no exact 1-cochains, so $H^1 = Z^1$. The 2-cochains of the Harrison complex are symmetric. Closed 2-cochains are associative, Abelian deformations and exact 2-cochains are trivial deformations. The quotient H^2 is the space of equivalence classes of Abelian deformation [1]. The 3-cochains have the symmetry of (6) with C_1 symmetric, namely

$$C(f, g, h) - C(g, f, h) + C(g, h, f) = 0.$$

In the case of a finitely generated algebra the decomposition (8) is finite, ending with $\mathrm{Hoch}^*(N)$, where N is the number of generators. The cochains of this complex are the alternating maps from $A^\wedge$ to A. The theorem of [8] tells us that, in the case of the coordinate algebra of a smooth manifold of dimension N, only this component has nonvanishing cohomology. It is related by duality to the simplicial homology of the manifold and this is of course a highly developed subject. The other components are related to geometric properties of another kind. Concerning Harr^*, the only thing that appears to be known is that it reveals the existence of singularities [9]. In the next section we shall look at some examples.

2. ABELIAN DEFORMATIONS ON ALGEBRAIC VARIETIES

It will be instructive to give a simple proof of the fact that Abelian $*$-products on $\mathbb{R}^N$ are trivial. The algebra is

$$A = \mathbb{C}[x_1, \ldots, x_N],$$

it has a Poincaré–Witt basis of monomials. The deformed algebra has a basis of $*$-monomials, and the map Φ that takes $x_{i_1} * x_{i_2} * \ldots$ to $x_{i_1} x_{i_2} \ldots$ is an algebra isomorphism.

An algebraic variety is a quotient space

$$\mathbb{R}^N/R,$$

where R is a set of polynomials. The coordinate algebra is

$$A = \mathbb{C}[x_1 \dots x_N]/R,$$

where the polynomials in R are now interpreted as relations among the generators. Consider a formal deformation of the form (1). Let R^* be the relations of the algebra with the $*$-product, and let $R_\hbar$ be these same relations after applying the mapping Φ. The deformation is trivial if there is an invertible mapping of R into R such that the pull-back takes R to $R_\hbar$.

Example. Let $A = \mathbb{C}[x, y]$ and let R be the polynomial $x^2 - y^2 + r^2 = 0$, with r^2 — a real parameter. There is a unique decomposition

$$f(x, y) = f_1(x) + y f_2(x).$$

Let

$$f * g = fg + \hbar f_2 g_2 = fg + \hbar C_1(f, g).$$

This is associative to all orders. We have $x * x = x^2,\ y * y = x^2 + r^2 + \hbar$ and thus $R^* = x * x - y * y + r^2 + \hbar$ = 0, and $R_\hbar = x^2 - y^2 + r^2 + \hbar$. If $r^2 \neq 0$, then there is an invertible map, namely the map $x, y \to x\sqrt{1 + \hbar/r^2}, y\sqrt{1 + \hbar/r^2}$, that takes $R_\hbar$ to R. To first order in $\hbar$ it is generated by

$$E_1(f) = \frac{1}{2r^2}\left(x\frac{\partial}{\partial x}f + f_2 y\right) = \frac{1}{2r^2}x\frac{df}{dx} + \frac{1}{2y}f_2.$$

The first expression shows that $E(f)$ is a polynomial; the first part of the second expression is a derivation, so that

$$dE(f, g) = \frac{1}{2y}\big(fg_2 - (fg)_2 + f_2 g\big) = f_2 g_2 = C_1(f, g).$$

Note, however, that though $C_n = 0$ for $n > 1$, E is an infinite series.

We have thus verified that, when $r^2 \neq 0$, the deformed algebra is the coordinate algebra of the variety (actually manifold) $\mathbb{R}^2/(x^2 - y^2 + r^2 + \hbar)$, and that it is trivial because this manifold is diffeomorphic to the original manifold $\mathbb{R}^2/(x^2 - y^2 + r^2)$. But this is evidently not true if $r^2 = 0$. The original variety is a cone, singular at $x = y = 0$; it is not diffeomorphic to the smooth manifold $\mathbb{R}^2/(x^2 - y^2 + \hbar)$, and the respective coordinate algebras are not isomorphic.

So we have found that the Harrison cohomology of the algebra of polynomials on the cone $x^2 - y^2 = 0$ is not zero, and that in consequence of this there is a nontrivial $*$-product deformation of this algebra. A preliminary study of the cohomology of cones of any dimension may be found in hep-th/0109001.

3. QUANTIZATION ON COADJOINT ORBITS

Some years ago Souriau [12] and Kostant [13] developed the concept of geometric quantization. Briefly, this is the idea. Consider a symplectic space W, with the action of a Lie algebra, via the Poisson bracket. That is, one has a Lie subalgebra L of the Poisson algebra. If we are fortunate (as we shall assume without essential loss of generality), then we can take a basis for L as coordinates for W. Or to put it another way, consider an orbit X of the coadjoint action of the associated Lie group, extend the action to functions on X by the derivation rule; this gives a Poisson bracket and turns X into a phase space, with an action by L via the Poisson bracket.

We have just described, without some of the trimmings, the first part of a program of geometric quantization. The completion of it requires what Kostant calls an invariant polarization. Unfortunately, an invariant polarization is not always available, and this is especially likely to be the case on singular orbits. To avoid this problem we turn to deformation quantization. To illustrate, both the method of invariant quantization and the role of singularities, consider the following example.

The Lie algebra $so(2,1)$ acts on the cone

$$Q := x_1^2 + x_2^2 - x_3^2 = 0,$$

and this action is equivalent to the coadjoint action on the unique singular orbit. We shall construct an invariant $*$-product on the polynomials over this space. An invariant $*$-product is one that satisfies the condition [4, 14]

$$f * g - g * f = \hbar\{f, g\}, \quad f, g \in L.$$

This implies that

$$\{f, g * h\} = \{f, g\} * h + g * \{f, h\}, \quad f \in L,$$

and, in particular, that the Casimir element

$$Q^* = x_1 * x_2 + x_2 * x_2 - x_3 * x_3$$

is in the centre of the deformed algebra.

One can think of a $*$-product as a mapping from polynomials to $*$-polynomials. For $a \in L$, let $P_n(a)$ denote the nth Legendre polynomial, and let $P_n(a*)$ be the $*$-polynomial with the same coefficients. Then every invariant $*$-product takes the form

$$P_n(a*) = c_n P_n,$$

with complex coefficients c_n. If all these coefficients are different from zero, then they can be chosen to be equal to 1 with no essential loss of generality. We shall also fix the centre, so our invariant $*$-product is defined by

$$x_i * x_j - x_j * x_i = \hbar \sum_k \epsilon_{ijk} x_k,$$

$$P_n(a*) = P_n(a), \quad Q^* = \hbar l(l+1).$$

In particular, for $n = 2$,

$$x_i * x_j = x_i x_j + \hbar C_1(x_i, x_j), \quad C_1(x_i, x_j) = \frac{1}{2} \sum_k \epsilon_{ijk} x_k + \frac{1}{3} l(l+1)\ \delta_{ij}.$$

It is obvious that the last term cannot be transformed away; it is a nontrivial Harrison 2-cochain.

The problem of finding the cocycle defined by $f * g = fg + \hbar C_1(f,g) + \dots$ is left for the pleasure of the reader.

It is evident that Harrison cohomolgy enters into quantization on many if not all singular coadjoint orbits. Some of those orbits, as the most singular orbit of $so(4,2)$, that is, the phase space for Kepler motion, are algebraic varieties with very complicated relations. One may ask what Harrison cohomology can teach us about planetary orbits and the Schroedinger H atom.

4. ABELIAN DEFORMATIONS ON SMOOTH MANIFOLDS

We saw that a $*$-product of the form $f * g = fg + \hbar C_1(f,g)$, with no higher order terms, can be exactly associative. It may also be cohomologically trivial, but the associated trivialization map is not always a polynomial in $\hbar$. If $\mathrm{Harr}^2 = 0$, then we can transform away the term linear in $\hbar$, but higher order terms may take its place. Obstructions may appear in higher orders, but the most interesting possibility is that we can push the expression for $f * g$ to arbitrarily high order in $\hbar$, and yet the deformation is not trivial. The point is that if the trivialization map is an infinite power series, then it may have a finite (or even vanishing) radius of convergence. As an example of this interesting phenomenon, consider the following.

Let $M =$ Minkowski space, $A = \mathbb{C}[x_1, \dots, x_4]$. Decompose $f \in A$ into even and odd parts: $f = (f_+, f_-)$, and define*:

$$f * g = fg - \rho x^2 f_- g_- = \Big(f_+ g_+ + f_- g_- (1 - \rho x^2), \quad f_+ g_- + f_- g_+\Big).$$

*The curvature ρ takes the place of Planck's constant $\hbar$.

Now, let $M' = 3 + 2$-dimensional anti-De Sitter space, more precisely the cone in $\mathbb{R}^5$:

$$M' = \mathbb{R}^5/(\rho x^2 + y^2 - 1),$$

and

$$A' = \mathbb{C}[x_1, \ldots, x_4, y]^e/(\rho x^2 + y^2 - 1),$$

where $[\ldots]^e$ means polynomials of even order. Decompose $f \in A'$ as follows, $f = (f_+, f_-) = f_+(x) + y f_-(x)$, then

$$fg = f_+g_+ + f_-g_-(1 - \rho x^2) + y(f_+g_- + f_-g_+) = \\ = \Big(f_+g_+ + f_-g_-(1 - \rho x^2), \quad f_+g_- + f_-g_+\Big).$$

Therefore, the deformed $*$-product algebra of functions on Minkowski space is isomorphic to the ordinary algebra of functions on AdS$/Z_2$. But the two spaces are not diffeomorphic, therefore A and A' are not isomorphic and the $*$-product is not trivial.

This observation suggests that there may be a solution to some problems of interpretation of physics on AdS, notably the fact that one cannot define an S matrix.

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