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PHYSICAL INSTANCES OF NONCOMMUTING COORDINATES

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Noncommuting spatial coordinates and fields can be realized in actual physical situations. Plane wave solutions to noncommuting photodynamics exhibit violaton of Lorentz invariance (special relativity).

INTRODUCTION

These days, investigators are probing the validity of Lorentz invariance (special relativity). This activity is documented by the papers presented at the Indiana meeting and submitted to the (recently postponed) Harvard meeting. Experimental and theoretical studies are pursued: experimentalists measure limits on Lorentz-violating processes; theorists build plausible Lorentz-violating extensions of the standard model.

When selecting Lorentz-violating terms, for possible inclusion in a modified standard model, we prefer to use structures that have a preexisting role in physics or mathematics. Thus our old proposal to add to the Maxwell Lagrangian the Lorentz-noninvariant quantity $(m/2)\int d^3r\mathbf{A}\cdot\mathbf{B} = (m/2)\int d^3r\mathbf{A}\cdot(\boldsymbol{\nabla}\times\mathbf{A})$, which leads to birefringence of the vacuum and to a Faraday-like rotation for the polarization of light propagating through the vacuum, makes use of the $\int d^3r\mathbf{A}\cdot\mathbf{B}$ quantity, which was previously known in magnetohydrodynamics as the «magnetic helicity», in fluid mechanics (with the fluid velocity $\mathbf{v}$ replacing the electromagnetic vector potential $\mathbf{A}$) as the «kinetic vorticity», and in mathematics as the «Chern–Simons term». While the inclusion of this term in an electrodynamical theory leads to Lorentz, parity, and CTP violation, experiment conclusively rules out such a modification in Nature [1].

Another mechanism for Lorentz-invariance breaking has become the focus of recent research: the suggestion is made that spatial coordinates need not commute. While present attention to this idea derives from string theory, we shall place this mechanism in the more familiar context of quantum mechanics and quantum field-theory.

Like many interesting quantal ideas, the notion that spatial coordinates may not commute can be traced to Heisenberg who, in a letter to Peierls, suggested that a coordinate uncertainty principle may ameliorate the problem of infinite self-energies. We shall describe later the physical application that Peierls made with Heisenberg's idea. Evidently, Peierls also described it to Pauli, who told it to Oppenheimer, who told it to Snyder, who wrote the first paper on the subject [2].

Let us begin with a physical application of the idea that goes back to Peierls.

1. NONCOMMUTATIVITY IN THE PRESENCE OF STRONG MAGNETIC FIELDS

1.1. Particle Noncommutativity in the Lowest Landau Level. We are interested in a point-particle moving on a plane, with an external magnetic field b perpendicular to the plane. The equation for the 2-vector $\mathbf{r} = (x, y)$ is

$$m\dot{v}^i = \frac{e}{c}\varepsilon^{ij}v^j b + f^i(\mathbf{r}), \tag{1}$$

where v is the velocity $\mathbf{r}$; and f represents other forces, which we take to be derived from a potential V: $\mathbf{f} = -\boldsymbol{\nabla} V$. Absent additional forces, the quantized theory gives rise to the well-known Landau levels, with separations $O(b/m)$. The limit of large b effectively projects onto the lowest Landau level and is equivalent to small m. Setting the mass to zero in (1) leaves a first order equation

$$\dot{r}^i = \frac{c}{eb}\varepsilon^{ij} f^j(\mathbf{r}). \tag{2}$$

This may be obtained by taking Poisson brackets of $\mathbf{r}$ with the Hamiltonian

$$H_0 = V \tag{3}$$

provided the fundamental brackets describe noncommuting coordinates,

$$\{r^i, r^j\} = \frac{c}{eb}\varepsilon^{ij} \tag{4}$$

so that

$$\dot{r}^i = \{H_0, r^i\} = \{r^j, r^i\}\partial_j V = \frac{c}{eb}\varepsilon^{ij} f^i(\mathbf{r}). \tag{5}$$

The noncommutative algebra (4) and the associated dynamics can be derived in the following manner. The Lagrangian for the equation of motion (1) is

$$L = \frac{1}{2}mv^2 + \frac{e}{c}\mathbf{v}\cdot\mathbf{A} - V, \tag{6}$$

where we choose the gauge $\mathbf{A} = (0, bx)$. Setting m to zero leaves

$$L_0 = \frac{eb}{c}x\dot{y} - V(x, y), \tag{7}$$

which is of the form $p\dot{q} - h(p;q)$, and one sees that $((eb/c)x, y)$ form a canonical pair. This implies (4), and identifies V as the Hamiltonian.

Finally, we give a canonical derivation of noncommutativity in the $m \to 0$ limit, starting with the Hamiltonian

$$H = \frac{\pi^2}{2m} + V. \tag{8}$$

H gives (1) upon bracketing with $\mathbf{r}$ and $\boldsymbol{\pi}$, provided the following brackets hold:

$$\{r^i, r^j\} = 0, \tag{9}$$

$$\{r^i, \pi^j\} = \delta^{ij}, \tag{10}$$

$$\{\pi^i, \pi^j\} = -\frac{eb}{c}\varepsilon^{ij}. \tag{11}$$

Here $\boldsymbol{\pi}$ is the kinematical (noncanonical) momentum, $m\dot{\mathbf{r}}$, related to the canonical momentum $\mathbf{p}$ by $\boldsymbol{\pi} = \mathbf{p} - (e/c)\mathbf{A}$.

We wish to set m to zero in (8). This can only be done provided $\boldsymbol{\pi}$ vanishes, and we impose $\boldsymbol{\pi} = 0$ as a constraint. But according to (11), the bracket of the constraints is nonzero, and the constraints are recognized to be «second-class» in Dirac's terminology. To proceed with the canonical formalism, we must introduce the Dirac brackets. We omit the details of that technology, but merely record the resulting Dirac bracket:

$$\{r^i, r^j\}_D = \frac{c}{eb}\varepsilon^{ij}. \tag{12}$$

In this approach, noncommuting coordinates arise as the Dirac brackets in a system constrained to lie in the lowest Landau level. Notice that the coordinate noncommutativity is already established at the classical level in that the Poisson bracket of coordinates is nonvanishing. Later we shall discuss the quantum version [3].

Peierls observed that when an impurity in the electron system is described by V, one can obtain the first-order energy shift of the lowest Landau level by taking the coordinates of (x, y) on which V depends to be noncommuting [4].

A further interesting subject, which is not discussed here, concerns the behavior of the wave function in the phase-space reductive, $m \to 0$, limit that projects onto the lowest Landau level. Before the reduction, the wave function is a normalized expression depending on the two coordinates. After the reduction, the wave function can depend only on one coordinate, because the other is a conjugate variable. How all this comes about is explained in the literature [3].

1.2. Field Noncommutativity in the Lowest Landau Level. The above demonstrates that spatial coordinates of particles in an intense magnetic field do not (Poisson) commute. But we are interested in fields. To find an example of noncommuting fields, we turn to the equations of a charged fluid, moving on a plane in an external magnetic field perpendicular to plane. The fluid is described by a density ρ and velocity $\mathbf{v}$, both defined on the two-dimensional plane. A mass parameter m is introduced for dimensional reasons, so that the mass density is $m\rho$. The fields ρ and $\mathbf{v}$ are functions of t and $\mathbf{r}$ and give an Eulerian description of the fluid. The equations that are satisfied are the continuity equation

$$\dot{\rho} + \boldsymbol{\nabla}(\rho\mathbf{v}) = 0 \tag{13}$$

which expresses matter conservation, and the Euler equation

$$m\dot{v}^i + m\mathbf{v}\cdot\boldsymbol{\nabla}v^i = \frac{e}{c}\varepsilon^{ij}v^j b + f^i, \tag{14}$$

which is the force equation. Here f^i describes additional forces, e. g., $-(1/\rho)\boldsymbol{\nabla}P$, where P is pressure. We shall take the force to be derived from a potential of the form

$$\mathbf{f}(\mathbf{r}) = -\boldsymbol{\nabla}\frac{\delta}{\delta\rho(\mathbf{r})}\int d\mathbf{r}V. \tag{15}$$

(For isentropic systems, the pressure is only a function of ρ; (15) holds with V a function of ρ, related to the pressure by $P(\rho) = \rho V'(\rho) - V(\rho)$. Here we allow more general dependence of V on ρ (e. g., nonlocality or dependence on derivatives of ρ) and also translation noninvariant, explicit dependence on $\mathbf{r}$ [5].)

Equations (13)–(15) follow by bracketing ρ and $\boldsymbol{\pi}$ with the Hamiltonian

$$H = \int d^2r\left(\rho\frac{\pi^2}{2m} + V\right) \tag{16}$$

provided that fundamental brackets are taken as

$$\{\rho(\mathbf{r}),\ \rho(\mathbf{r}')\} = 0, \tag{17}$$

$$\{\pi(\mathbf{r}),\ \rho(\mathbf{r}')\} = \boldsymbol{\nabla}\delta(\mathbf{r}-\mathbf{r}'), \tag{18}$$

$$\{\pi^i(\mathbf{r}),\ \pi^j(\mathbf{r}')\} = -\varepsilon^{ij}\frac{1}{\rho(\mathbf{r})}\left(m\omega(\mathbf{r}) + \frac{eb}{c}\right)\delta(\mathbf{r}-\mathbf{r}'), \tag{19}$$

where $\varepsilon^{ij}\omega(\mathbf{r})$ is the vorticity $\partial_i v^j - \partial_j v^i$, and $\boldsymbol{\pi} = m\mathbf{v}$.

We now consider a strong magnetic field and take the limit $m \to 0$, which is equivalent to large b. Equations (14) and (15) reduce to

$$v^i = -\frac{c}{eb}\varepsilon^{ij}\frac{\partial}{\partial r^j}\frac{\delta}{\delta\rho(\mathbf{r})}\int d^2 r V. \tag{20}$$

Combining this with the continuity equation (13) gives the equation for the density «in the lowest Landau level»:

$$\dot{\rho}(\mathbf{r}) = \frac{c}{eb}\frac{\partial}{\partial r^i}\rho(\mathbf{r})\varepsilon^{ij}\frac{\partial}{\partial r^j}\frac{\delta}{\delta\rho(\mathbf{r})}\int d^2 r V. \tag{21}$$

(For the right-hand side not to vanish, V must not be solely a function of ρ.)

The equation of motion (21) can be obtained by bracketing with the Hamiltonian

$$H_0 = \int d^2 r V \tag{22}$$

provided the charge density bracket is nonvanishing, showing noncommutativity of the ρ's [6]:

$$\{\rho(\mathbf{r}),\ \rho(\mathbf{r}')\} = -\frac{c}{eb}\varepsilon^{ij}\partial_i\rho(\mathbf{r})\partial_j\delta(\mathbf{r}-\mathbf{r}'). \tag{23}$$

H_0 and this bracket may be obtained from (16)–(19) with the same Dirac procedure presented for the particle case: We wish to set m to zero in (16); this is possible only if $\boldsymbol{\pi}$ is constrained to vanish. But the bracket of the $\boldsymbol{\pi}$'s (19) is nonvanishing, even at $m = 0$, because $b \neq 0$. Thus at $m = 0$ we are dealing with a second-class constraint which leads to a nonvanishing Dirac bracket of densities as in (23):

$$\{\rho(\mathbf{r}),\ \rho(\mathbf{r}')\}_D = -\frac{c}{eb}\varepsilon^{ij}\partial_i\rho(\mathbf{r})\partial_j\delta(\mathbf{r}-\mathbf{r}'). \tag{24}$$

The ρ bracket (23), (24) enjoys a more appealing expression in momentum space. Upon defining

$$\tilde{\rho}(\mathbf{p}) = \int d^2 r \mathrm{e}^{i\mathbf{pr}}\rho(\mathbf{r}) \tag{25}$$

we find

$$\{\tilde{\rho}(\mathbf{p}),\ \tilde{\rho}(\mathbf{q})\} = -\frac{c}{eb}\varepsilon^{ij}p^i q^j\tilde{\rho}(\mathbf{p}+\mathbf{q}). \tag{26}$$

The form of the charge density bracket (23), (24), (26) can be understood by reference to the particle substructure for the fluid. Take

$$\rho(\mathbf{r}) = \sum_n \delta(\mathbf{r}-\mathbf{r}_n), \tag{27}$$

where n labels the individual particles. When the coordinates of each particle satisfy the nonvanishing bracket (4), (12), the $\{\rho(\mathbf{r}),\ \rho(\mathbf{r}')\}$ bracket takes the form (23), (24), (26).

1.3. Quantization. Quantization before the reduction to the lowest Landau level is straightforward. For the particle case (9)–(11) and for the fluid case (17)–(19) we replace brackets with $i/\hbar$ times commutators. After reduction to the lowest Landau level we do the same for the particle case thereby arriving at the «Peierls substitution», which (as mentioned previously) states that the effect of an impurity [V in (6)] on the lowest Landau energy level can be evaluated to the lowest order by viewing the $(x,\ y)$ arguments of V as noncommuting variables [4].

For the fluid, quantization presents a choice. On the one hand, we can simply promote the bracket (23), (24), (26) to a commutator by multiplying by $i/\hbar$.

$$[\rho(\mathbf{r}),\ \rho(\mathbf{r}')] = i\hbar\frac{c}{eb}\varepsilon^{ij}\partial_i\rho(\mathbf{r}')\partial_j\delta(\mathbf{r}-\mathbf{r}'), \tag{28}$$

$$[\tilde{\rho}(\mathbf{p}),\ \tilde{\rho}(\mathbf{q})] = i\hbar\frac{c}{eb}\varepsilon^{ij}p^i q^j\tilde{\rho}(\mathbf{p}+\mathbf{q}). \tag{29}$$

Alternatively we can adopt the expression (27), for the operator $\rho(\mathbf{r})$, where $\mathbf{r}_n$ now satisfy the noncommutative algebra

$$[r_n^i, r_{n'}^j] = -i\hbar\frac{c}{eb}\varepsilon^{ij}\delta_{nn'} \tag{30}$$

and calculate the ρ commutator as a derived quantity.

However, once $\mathbf{r}_n$ is a noncommuting operator, functions of $\mathbf{r}_n$, even δ functions, have to be ordered. We choose the Weyl ordering, which is equivalent to defining the Fourier transform as

$$\tilde{\rho}(\mathbf{p}) = \sum_n \mathrm{e}^{i\mathbf{p}\mathbf{r}_n}. \tag{31}$$

With the help of (30) and the Baker–Hausdorff lemma, we arrive at the «trigonometric algebra»

$$[\tilde{\rho}(\mathbf{p}),\ \tilde{\rho}(\mathbf{q})] = 2i\sin\left(\frac{\hbar c}{2eb}\varepsilon^{ij}p^i q^j\right)\tilde{\rho}(\mathbf{p}+\mathbf{q}). \tag{32}$$

This reduces to (29) for small $\hbar$.

This form for the commutator, (32), is connected to a Moyal star product in the following fashion. For an arbitrary c-number function $f(\mathbf{r})$ define

$$\langle f\rangle = \int d^2r\rho(\mathbf{r})f(\mathbf{r}) = \frac{1}{(2\pi)^2}\int d^2p\tilde{\rho}(\mathbf{p})\tilde{f}(-\mathbf{p}). \tag{33}$$

Multiplying (32) by $\tilde{f}(-\mathbf{p})\tilde{g}(-\mathbf{q})$ and integrating gives

$$[\langle f\rangle,\ \langle g\rangle] = \langle h\rangle \tag{34}$$

with

$$h(\mathbf{r}) = (f\star g)(\mathbf{r}) - (g\star f)(\mathbf{r}), \tag{35}$$

where the «⋆» product is defined as

$$(f\star g)(\mathbf{r}) = \exp\left(\frac{i}{2}\frac{\hbar c}{eb}\varepsilon^{ij}\partial_i\partial'_j\right) f(\mathbf{r})g(\mathbf{r}')|_{\mathbf{r}'=\mathbf{r}}. \tag{36}$$

Note however that only the commutator is mapped into the star commutator. The product $\langle f\rangle\langle g\rangle$ is not equal to $\langle f\star g\rangle$.

The lack of consilience between (29) and (32) is an instance of the Groenwald–Van Hove theorem which establishes the impossibility of taking over into quantum mechanics all classical brackets [7]. Equations (30)–(36) explicitly exhibit the physical occurrence of the star product for fields in a strong magnetic background.

2. VARIOUS ALGEBRAS

Before proceeding with our construction of a noncommutative Maxwell field theory, let us summarize here the various (nontrivial) algebras that we have encountered in the above development.

The fluid velocity algebra (19) at $b = 0$ and $m = 1$ reads in any spatial dimension

$$\{v^i(\mathbf{r}),\ v^j(\mathbf{r}')\} = -\frac{1}{\rho(\mathbf{r})}(\partial_i v^j(\mathbf{r}) - \partial_j v^i(\mathbf{r}))\delta(\mathbf{r}-\mathbf{r}'). \tag{37}$$

This was first given by Landau [8]. In spite of the awkward appearance, the algebra in fact takes a familiar form when we define the momentum density $\mathcal{P} = \rho\mathbf{v}$, and use (17), (18) for the ρ brackets. Then (37), with (17) and (18) implies

$$\{\mathcal{P}^i(\mathbf{r}),\ \mathcal{P}^j(\mathbf{r}')\} = \left(\mathcal{P}^j(\mathbf{r})\frac{\partial}{\partial r^i} + \mathcal{P}^i(\mathbf{r}')\frac{\partial}{\partial r^j}\right)\delta(\mathbf{r}-\mathbf{r}'). \tag{38}$$

This is the usual momentum density algebra, which also describes diffeomorphisms of space in the following fashion. If an infinitesimal coordinate transformation is given by

$$\delta r^i = -f^i(\mathbf{r}), \tag{39}$$

we define the average $\langle f \rangle$ of f^i by integrating with $\mathcal{P}^i$

$$\langle f \rangle \equiv \int d\mathbf{r} f^i(\mathbf{r}) \mathcal{P}^i(\mathbf{r}), \tag{40}$$

then (38) has the consequence that for two such functions f and g we have

$$\{\langle f \rangle, \ \langle g \rangle\} = -\langle h \rangle, \tag{41}$$

where h is the Lie bracket of f and g:

$$h^i = g^j \partial_j f^i - f^j \partial_j g^i. \tag{42}$$

By scaling ρ the noncommutative density algebra (23), (24) may be presented as

$$\{\rho(\mathbf{r}), \ \rho(\mathbf{r}')\} = \varepsilon^{ij} \partial_i \rho(\mathbf{r}) \partial_j \delta(\mathbf{r} - \mathbf{r}'). \tag{43}$$

This intrinsically two-dimensional structure is the area-preserving algebra, studied by Arnold [9]. Area-preserving coordinate transformations (volume preserving in arbitrary dimensionality) possess unit Jacobian. For the infinitesimal form of the transformation (39) this means that f^i is transverse: $\partial_i f^i = 0$. Therefore, in two dimensions, an area-preserving transformation is generated by a scalar:

$$f^i = \varepsilon^{ij} \partial_j f. \tag{44}$$

When an average $\langle f \rangle$ is defined by

$$\langle f \rangle = \int d^2 r f(\mathbf{r}) \rho(\mathbf{r}), \tag{45}$$

equation (43) again implies (41), but now we have

$$h = \varepsilon^{ij} \partial_i f \partial_j g, \tag{46}$$

which also follows from (42) when all three functions take the form (44).

Finally the algebra (32)

$$\{\tilde{\rho}(\mathbf{p}), \ \tilde{\rho}(\mathbf{q})\} = -\frac{2}{\hbar} \sin\left(\frac{\hbar c}{2eb} \varepsilon^{ij} p^i q^j\right) \tilde{\rho}(\mathbf{p} + \mathbf{q}),$$

which also leads to the Moyal-star product (36) for averages (45), is called a trigonometric algebra, which was introduced by D. Fairlie, P. Fletcher, and C. Zachos [10].

3. NONCOMMUTATIVE ELECTRODYNAMICS

Stimulated by the occurrence of the star product in the discussion of charged fluids in an intense magnetic field, we abstract the idea and use it in the new setting of noncommutative Maxwell theory. This theory is described by the vector potential $\widehat{A}_\mu$ (the caret denotes noncommuting quantities) and the theory is built on a gauge-invariance principle. Gauge transformations act on $\widehat{A}_\mu$ according to

$$\widehat{A}_\mu \to \widehat{A}_\mu^\lambda = (\mathrm{e}^{i\lambda})^{-1} \star (\widehat{A}_\mu - i\partial_\mu) \star (\mathrm{e}^{i\lambda}). \tag{47}$$

The star ($\star$) product of two quantities is defined by

$$(O_1 \star O_2)(\mathbf{r}) = \exp\left(\frac{i}{2}\theta^{\mu\nu}\frac{\partial}{\partial r^\mu}\frac{\partial}{\partial r'^\nu}\right) O_1(\mathbf{r}) O_2(\mathbf{r}')|_{\mathbf{r}=\mathbf{r}'}, \tag{48}$$

and we take $\theta^{\mu\nu}$ to have no time components ($\theta^{0i} = 0$, $\theta^{ij} = \varepsilon^{ijk}\theta^k$). The field strength $\widehat{F}_{\mu\nu}$ is constructed from $\widehat{A}_\mu$ in a manner such that the gauge transformation (47) effects a covariant transformation:

$$\widehat{F}_{\mu\nu} \to \widehat{F}_{\mu\nu}^\lambda = (\mathrm{e}^{i\lambda})^{-1} \star F_{\mu\nu} \star (\mathrm{e}^{i\lambda}). \tag{49}$$

This requirement is met, provided $\widehat{F}_{\mu\nu}$ is given by

$$\widehat{F}_{\mu\nu} = \partial_\mu \widehat{A}_\nu - \partial_\nu \widehat{A}_\mu - i[\widehat{A}_\mu,\ \widehat{A}_\nu]_\star, \tag{50}$$

where $[\widehat{A}_\mu,\ \widehat{A}_\nu]_\star = \widehat{A}_\mu \star \widehat{A}_\nu - \widehat{A}_\nu \star \widehat{A}_\mu$. Finally, the action is taken to be

$$\widehat{I} = -\frac{1}{4}\int d^4x \widehat{F}^{\mu\nu} \star \widehat{F}_{\mu\nu} = -\frac{1}{4}\int d^4x \widehat{F}^{\mu\nu}\widehat{F}_{\mu\nu}. \tag{51}$$

One would like to find the equations of motion, calculate physically interesting quantities, and compare them to corresponding quantities in the Maxwell theory. In this way one could assess the effect of noncommutativity and perhaps place experimental limits on it. However, a problem arises: local quantities in noncommutative electrodynamics are gauge variant and no invariant meaning can be assigned to their profiles. Nonlocal, integrated, expressions can be gauge invariant (for example, the action (51) is gauge invariant), but in the ordinary Maxwell theory we deal with local quantities (like profiles of electromagnetic waves) and we would like to compare these classical local disturbances to corresponding quantities in the noncommutative theory.

A way out of this difficulty is provided by Seiberg and Witten's observation that the noncommuting gauge theory may be equivalently described by a commuting gauge theory that is formulated in terms of ordinary (not star) products of a commuting vector potential A_μ, together with an explicit dependence on $\theta^{\alpha\beta}$, which acts as a constant «background». This equivalence is established by expressing the noncommuting vector potential $\widehat{A}_\mu$ as a function of A_μ and $\theta^{\alpha\beta}$ that solves the Seiberg–Witten equation [11]

$$\frac{\partial \widehat{A}_\mu}{\partial \theta^{\alpha\beta}} = -\frac{1}{8}\{\widehat{A}_\alpha,\ \partial_\beta \widehat{A}_\mu + \widehat{F}_{\beta\mu}\}_\star - (\alpha \leftrightarrow \beta), \tag{52}$$

where the bracketed expression denotes the «star» anticommutator. Solutions of this equation are expressed in terms of $\theta^{\alpha\beta}$ and the «initial condition» $\widehat{A}_\mu|_{\theta^{\alpha\beta}=0}$; the latter quantity being just the commuting A_μ.

We work to the lowest order in θ and find

$$\widehat{A}_\mu = A_\mu - \frac{1}{2}\,\theta^{\alpha\beta} A_\alpha(\partial_\beta A_\mu + F_{\beta\mu}). \tag{53}$$

The noncommuting action, expressed in terms of the commuting quantities A_μ, $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$, and $\theta^{\alpha\beta}$, now reads [12]

$$\widehat{I} = -\frac{1}{4}\int d^4x \left(\left(1 - \frac{1}{2}\theta^{\alpha\beta}F_{\alpha\beta}\right)F^{\mu\nu}F_{\mu\nu} + 2\theta^{\alpha\beta}F_{\mu\alpha}F_{\nu\beta}F^{\mu\nu}\right). \tag{54}$$

This is gauge invariant in the conventional sense, and from the equations of motion that are implied by $\widehat{I}$ we can determine the gauge-invariant electric $(E^i = F^{i0})$ and magnetic fields $(B^i = -\varepsilon^{ijk}F_{jk})$.

These fields satisfy the equations, which maintain a Maxwell form.

$$\frac{1}{c}\frac{\partial}{\partial t}\mathbf{B} - \boldsymbol{\nabla} \times \mathbf{E} = 0, \tag{55a}$$

$$\boldsymbol{\nabla} \cdot \mathbf{B} = 0, \tag{55b}$$

$$\frac{1}{c}\frac{\partial}{\partial t}\mathbf{D} - \boldsymbol{\nabla} \times \mathbf{H} = 0, \tag{56a}$$

$$\boldsymbol{\nabla} \cdot \mathbf{D} = 0. \tag{56b}$$

The first set (55) reflects the gauge invariance of the system, namely, that $\mathbf{E}$ and $\mathbf{B}$ are given in terms of potentials. The second set (56) is a consequence of the nonlinear dynamics implied by (54). The constitutive relations relating $\mathbf{D}$ and $\mathbf{H}$ to $\mathbf{E}$ and $\mathbf{B}$ follow from (54):

$$\begin{aligned} \mathbf{D} &= (1 - \boldsymbol{\theta}\cdot\mathbf{B})\mathbf{E} + (\boldsymbol{\theta}\cdot\mathbf{E})\mathbf{B} + (\mathbf{E}\cdot\mathbf{B})\boldsymbol{\theta}, \\ \mathbf{H} &= (1 - \boldsymbol{\theta}\cdot\mathbf{B})\mathbf{B} - (\boldsymbol{\theta}\cdot\mathbf{E})\mathbf{E} + \frac{1}{2}(\mathbf{E}^2 - \mathbf{B}^2)\boldsymbol{\theta}. \end{aligned} \tag{57}$$

Note that parity is preserved — coordinate reflection leaves the constant vector $\boldsymbol{\theta}$ unchanged, hence $\boldsymbol{\theta} \cdot \mathbf{B}$ transforms as a scalar field; and $\boldsymbol{\theta} \cdot \mathbf{E}$, as a pseudoscalar field. Similarly, $(\mathbf{E} \cdot \mathbf{B})\boldsymbol{\theta}$ behaves as a vector field; while $(\mathbf{E}^2 - \mathbf{B}^2)\boldsymbol{\theta}$, as a pseudovector.

We seek plane-wave solutions to (55)–(57) — functions of $\omega t - \mathbf{k} \cdot \mathbf{r}$ — keeping terms to the lowest order in θ. Such solutions indeed exist provided the dispersion relation, relating $\mathbf{k}$ and ω, takes the following form. In the absence of an external magnetic field the dispersion relation is conventional, $\omega = ck$. However, plane wave solutions to our system of equations exist even in the presence of a constant background magnetic induction $\mathbf{b}$. Then the dispersion relation is modified to

$$\omega = ck(1 - \boldsymbol{\theta}_T \cdot \mathbf{b}_T), \tag{58}$$

where $\boldsymbol{\theta}_T$ and $\mathbf{b}_T$ are components transverse to $\mathbf{k}$, the direction of propagation $\mathbf{k} \cdot \boldsymbol{\theta}_T = \mathbf{k} \cdot \mathbf{b}_T = 0$ [6].

The result (58) puts into evidence an explicit violation of Lorentz invariance. Conservation of parity, which we remarked on previously, ensures that both polarizations travel at the same velocity, which generically differs from c by the factor $(1 - \boldsymbol{\theta}_T \cdot \mathbf{b}_T)$, and there is no Faraday rotation. Let us also observe that the effective Lagrange density in (54) possesses two interaction terms proportional to θ, with definite numerical constants. Owing to the freedom of rescaling θ, only their ratio is significant. It is straightforward to verify that if the ratio is different from what is written in (54), the two linear polarizations travel at different velocities. Thus the noncommutative theory is unique in affecting the two polarizations equally, at least to $O(\theta)$.

The change in velocity for motion relative to an external magnetic induction $\mathbf{b}$ allows searching for the effect with a Michelson–Morley experiment. In a conventional apparatus with two legs of length l_1 and l_2 at right angles to each other, a light beam of wavelength λ is split in two, and one ray travels along $\mathbf{b}$ (where there is no effect), while the other, perpendicular to $\mathbf{b}$, feels the change of velocity and interferes with the first. After rotating the apparatus by 90°, the interference pattern will shift by $2(l_1 + l_2)\boldsymbol{\theta}_T \cdot \mathbf{b}_T/\lambda$ fringes. Taking light in the visible range, $\lambda \sim 10^{-5}$ cm, a field strength $b \sim 1$ T, and using the current bound on $\theta \leq (10 \text{ TeV})^{-2}$ obtained in [13]*, one finds that a length $l_1 + l_2 \geq 10^{18}$ cm ~ 1 pc would be required for a shift of one fringe. Galactic magnetic fields are neither that strong nor coherent over such large distances, so another experimental setting needs to be found to test for noncommutativity.

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*Bounds on θ from experimental limits on modifications to a fermion sector are $\theta = O\ (10 \text{ TeV})^{-2}$.

Finally, what about Heisenberg's intuition that noncommuting coordinates will ammeliorate divergences in relativistic field theory? It turns out that that is indeed true as far as ultraviolet divergences are concerned. However, novel infrared divergences appear, so the problem of divergences remains, albeit in another form. Indeed, these infrared effects associated with noncommutative coordinates provide another obstacle to physical applications of this idea.

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