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A SHORT REVIEW OF NONCOMMUTATIVE FIELD THEORY

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Some properties of noncommutative field theories are reviewed. The emphasis is in particular on renormalization and anomalies.

INTRODUCTION

Theories defined in noncommutative spaces have been considered over the years both by mathematicians and by physicists. Recently there has been a renewal of interest in this subject due to the realization of physical models of noncommutative spaces based on open strings on D branes, see [1] and references therein for early contributions to the subject. The possibility of pursuing this research with two different languages, that of (noncommutative) field theory and the one of string theory, with the possibility to compare the results, has naturally attracted a lot of activity. One could mention several lines of research. Among others an extremely interesting one, which however will not be covered in this review, is the search for classical solutions in a noncommutative field theory, i. e., solitons and instantons. Noncommutative solitons in particular turn out to be particularly interesting because of their mimicking solutions of String Field Theory and their connection with tachyon condensation. The present short review concerns instead the properties of noncommutative gauge theories. Open strings attached to D branes contain in their spectrum a massless vector field. It is a standard matter to find the amplitudes of the corresponding vertex operators. In the field theory limit ($\alpha' \to 0$) these amplitudes coincide with those of an ordinary $U(1)$ gauge field theory. If, however, we switch on a constant $B_{\mu\nu}$ field with nonzero components only in the space directions parallel to the D brane, and repeat the above calculation, we find that, in the field theory limit, the amplitudes have changed. They are not the amplitudes of an ordinary gauge field theory, rather they correspond to the amplitudes of a *noncommutative field theory*, in

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which the noncommutative parameter is precisely related to the value of the B field.

It is natural to ask whether this new field theory, which is nonlocal in the ordinary sense, enjoys the same properties as the ordinary field theories. In particular, is it renormalizable and unitary? Does it have the same chiral anomalies as ordinary gauge theories? There has been intense research on these subjects and the previous questions have been answered at least in part. This is a short review of such results.

1. NONCOMMUTATIVE $U(N)$ GAUGE FIELD THEORIES

In the introduction it was pointed out that noncommutative gauge field theories can be embedded in string theory. However these theories can also be defined on their own, without reference to string theory. Let us consider the Euclidean space $\mathbf{R}^d$ and define on it the complex algebra $\mathcal{A}_\theta$ endowed with the Moyal–Weyl product

$$f \star g(x) \equiv f(x) \exp\left(\frac{i}{2}\theta^{\mu\nu}\overleftarrow{\partial}_\mu \overrightarrow{\partial}_\nu\right) g(x), \tag{1}$$

where $\theta^{\mu\nu}$ is the deformation parameter. This implies in particular noncommutativity in $\mathbf{R}^d$

$$x^\mu \star x^\nu - x^\nu \star x^\mu = i\theta^{\mu\nu}. \tag{2}$$

It is natural to try to define a field theory in which the ordinary product is replaced by the Moyal product. In particular for a gauge theory we will assume the existence of a noncommutative connection A_μ with curvature

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + iA_\nu \star A_\mu - iA_\mu \star A_\nu \tag{3}$$

and gauge transformation

$$\delta A_\mu = \partial_\mu \lambda + i\lambda \star A_\mu - iA_\mu \star \lambda. \tag{4}$$

It is understood that both A and λ are hermitean: $A_\mu = A_\mu^B t^B$ and $\lambda = \lambda^B t^B$, where t^B is a complete set of $N \times N$ hermitean matrices: $(t^B)^\dagger = t^B$. In other words the connection A and the infinitesimal gauge transformations λ are $u(n)$-valued functions on $\mathbf{R}^d$. The theory we are introducing can therefore be called noncommutative $U(N)$ ($NCU(N)$) gauge theory. Its action is

$$S = -\frac{1}{4g^2}\int \mathrm{Tr}\,(F \star F)\,. \tag{5}$$

We notice that if we expand the integrand in this action in power series of θ, we obtain an infinite series in the ordinary field $A_\mu(x)$ and its derivatives. The presence of higher and higher derivative terms would render an ordinary field theory nonlocal and untreatable. The remarkable thing of noncommutative theories is that, although they are nonlocal, the Moyal–Weyl product organizes such infinite series of terms so that they often behave like ordinary local theories.

For instance, the Feynman rules for the theory (5) can be extracted in the usual way. The propagator is the same as in the ordinary theory, but the vertices are different. As an example, the vertex for three gluons with momenta p_1, p_2, p_3 and polarizations ξ_1, ξ_2, ξ_3 is given, in the $NCU(1)$ case, by

$$(\xi_1 \cdot \xi_2 \, p_2 \cdot \xi_3 + \xi_1 \cdot \xi_3 \, p_1 \cdot \xi_2 + \xi_2 \cdot \xi_3 \, p_3 \cdot \xi_1) \exp\left(-\frac{i}{2} p_1 \theta p_2\right), \tag{6}$$

where $p\theta q$ means $p_\mu \theta^{\mu\nu} q_\nu$.

As a matter of notation, we will use a basis of hermitean matrices $t^A = (t^A)^j{}_i$ (capital letters $A, B, \ldots = 0, \ldots N^2 - 1$ will denote indices in the Lie algebra $u(N)$, while $i, j = 1, \ldots, N$ are the indices in the fundamental representation), with the normalization

$$\mathrm{Tr}\,(t^A t^B) = \frac{1}{2}\,\delta^{AB}. \tag{7}$$

This can be done, for example, by using a basis of hermitean matrices for the Lie algebra of $SU(N)$, t^a (whenever necessary, lower case letters $a, b, \ldots = 1, \ldots N^2 - 1$ will denote indices in the adjoint of $su(N)$), and adjoining $t^0 = (1/\sqrt{2N})\mathbf{1}_N$. The basis t^A satisfies

$$[t^A, t^B] = i f_{ABC} t^C, \quad \{t^A, t^B\} = d_{ABC} t^C, \tag{8}$$

where f_{ABC} is completely antisymmetric; f_{abc} is the same as for $su(N)$ and $f_{0BC} = 0$, while d_{ABC} is completely symmetric; d_{abc} is the same as for $su(N)$, $d_{0BC} = \sqrt{2/N}\delta_{BC}$, $d_{00c} = 0$ and $d_{000} = \sqrt{2/N}$, see [5].

2. STRING THEORY EMBEDDING

As explained in the introduction, the noncommutative gauge theory introduced in the previous section (let us consider for the time being the $NCU(1)$ theory) can be immersed in a string theory. Let us follow the approach of N. Seiberg and E. Witten [1]. Think of a closed string theory in the presence of a D brane. The closed string theory contains in the gravity spectrum an antisymmetric massless B field, which always appears in the equations of motion under the differentiation symbol. Therefore, it is always possible to add a constant part to B without affecting the field equations. In particular the vacuum configuration of such a theory is always defined up to a constant B field. In the absence of any D brane, a constant B field can always be gauged away. But if the vacuum configuration contains a D brane, this operation is not possible anymore along the D-brane world-volume. The reason is that in the D-brane world-volume there exists a $U(1)$ gauge field A which together with B form a gauge invariant combination $B - dA$. Therefore a constant B field can be eliminated by means of a gauge transformation on the bulk, but not along the D-brane world-volume. The upshot of this discussion is that when we are in the presence of such a configuration we should allow for a constant B field, rather than put it to zero.

Therefore let us suppose that we have a constant B field along a D brane*. Now let us see the consequences of having a nonvanishing B field. The D-brane dynamics is represented by the open strings attached to it. These open strings interact with B by the endpoints. This can be seen by looking at the sigma model action for such open strings:

$$
\begin{aligned}
S = \frac{1}{4\pi\alpha'} \int_\Sigma \left(g_{\mu\nu}\partial_a X^\mu \partial^a X^\nu - 2\pi\alpha' B_{\mu\nu}\epsilon^{ab}\partial_a X^\mu \partial_b X^\nu \right) = \\
= \frac{1}{4\pi\alpha'} \int_\Sigma \left(g_{\mu\nu}\partial_a X^\mu \partial^a X^\nu \right) - \frac{i}{2}\int_{\partial\Sigma} B_{\mu\nu} X^\mu \partial_t X^\nu
\end{aligned}
\tag{9}
$$

after partial integration. Here $g_{\mu\nu}$ is the closed string metric, i. e., the metric background of the ambient space where the closed string lives; Σ is the world-sheet of the open string and $\partial\Sigma$ its boundary. The boundary conditions for the theory (9) are

$$
g_{\mu\nu}\,\partial_n X^\nu + 2\pi\alpha'\, B_{\mu\nu}\,\partial_t X^\nu|_{\partial\Sigma} = 0, \tag{10}
$$

where ∂_n, ∂_t are the normal and tangential derivatives to $\partial\Sigma$. We see that B interpolates between the Neumann boundary conditions ($B = 0$) and the Dirichlet ones ($B = \infty$).

Now, at tree level, the relevant world-sheet is the disk or the upper half plane. The string propagator in the upper half plane is

$$
\begin{aligned}
&\langle X^\mu(z) X^\nu(z')\rangle = \\
&= -\,\alpha' \left(g^{\mu\nu}\ln|z-z'| - g^{\mu\nu}\ln|z-\bar{z}'| + G^{\mu\nu}\ln|z-\bar{z}'|^2 + \frac{1}{2\pi\alpha'}\theta^{\mu\nu}\ln\frac{z-\bar{z}'}{\bar{z}-z'} \right),
\end{aligned}
$$

*We suppose throughout the review that only $B_{\mu\nu}$ space components are nonvanishing. A nonvanishing time component would lead not to noncommutative field theory, but rather to an open string theory in a noncommutative ambient space, the so-called NCOS theories, which will not be considered in this review.

where

$$G^{\mu\nu} = \left(\frac{1}{g + 2\pi\alpha' B} g \frac{1}{g - 2\pi\alpha' B}\right)^{\mu\nu},$$
$$\theta^{\mu\nu} = -(2\pi\alpha')^2 \left(\frac{1}{g + 2\pi\alpha' B} B \frac{1}{g - 2\pi\alpha' B}\right)^{\mu\nu}. \tag{11}$$

For open string amplitudes the relevant propagator is evaluated at the insertion points, i. e., on the real axis. This is

$$\langle X^\mu(\tau) X^\nu(\tau')\rangle = -\alpha' G^{\mu\nu}\, \ln(\tau - \tau')^2 + \frac{i}{2}\,\theta^{\mu\nu}, \epsilon(\tau - \tau'), \tag{12}$$

where τ, τ' are the real part of z, z', respectively.

Now, if we take $\alpha \to 0$ keeping B and θ fixed, we get

$$G^{\mu\nu} = -\frac{1}{(2\pi\alpha')^2}\left(\frac{1}{B} g \frac{1}{B}\right)^{\mu\nu}, \qquad \theta^{\mu\nu} = \left(\frac{1}{B}\right)^{\mu\nu}, \tag{13}$$

and

$$\langle X^\mu(\tau) X^\nu(\tau')\rangle = \frac{i}{2}\,\theta^{\mu\nu} \epsilon(\tau - \tau'). \tag{14}$$

Now we can move on to compute amplitudes for the string modes. In particular the gluon is represented by the vertex operator

$$V(\xi, p) = \int_{\partial\Sigma} \xi \partial X\, \mathrm{e}^{ipX}, \tag{15}$$

where the momentum p and polarization ξ satisfy the relations: $p^2 = p\xi = 0$. The vertices are inserted on the real axis (if Σ is the upper half plane) in a definite order and then their positions are integrated over the entire real axis. The calculation from the string theory point of view is completely standard, and, once the limit $\alpha' \to 0$ is taken, the result is

$$\begin{aligned} &\langle V(\xi_1, p_1) V(\xi_2, p_2) V(\xi_3, p_3)\rangle = \\ &\quad = (\xi_1 \cdot \xi_2\, p_2 \cdot \xi_3 + \xi_1 \cdot \xi_3\, p_1 \cdot \xi_2 + \xi_2 \cdot \xi_3\, p_3 \cdot \xi_1)\, \mathrm{e}^{-(i/2) p_1 \theta p_2} \end{aligned} \tag{16}$$

which coincides with the vertex (6) above, which was obtained from the $NCU(1)$ gauge field theory.

It takes more work, but it can be proven that the generic n-gluon amplitudes obtained in the same way from string theory coincide with the tree-level vertices derived from the $NCU(1)$ gauge theory. The generalization to $U(N)$ is straightforward. String amplitudes in this case are simply multiplied by the appropriate

Chan–Paton (CP) factors. For instance, in the 3-gluon case, the CP factor is $\mathrm{Tr}\,(t^{A_1}\,t^{A_2}t^{A_3})$, where t^{A_i} belong to the basis of $N \times N$ hermitean matrices introduced above. It is easy to see that this makes the amplitudes coming from strings coincide with those derived from $NCU(N)$ gauge theory.

In conclusion, at the tree level, there is a perfect correspondence between the gluon amplitudes obtained via string theory in the $\alpha' \to 0$ limit, and the analogous amplitudes obtained from $NCU(N)$ gauge theory. Then a question arises immediately: is this pattern going to persist also at one loop? This means, on the field theory side, renormalizing $NCU(N)$ gauge theory at one loop and calculating the relevant renormalized amplitudes. On the string theory side, it means calculating the string theory one-loop corrected amplitudes after taking the field theory limit of the latter. Finally one has to compare the two results and see whether they coincide.

This is what we are going to see in the next section.

2.1. Renormalization of $NCU(N)$. In this section we study one-loop renormalization of $NCU(N)$ theory. The theory we have to renormalize is specified by the gauge-fixed action

$$S = \int d^4x\,\mathrm{Tr}\left(-\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \frac{1}{2\alpha}(\partial_\mu A^\mu)^2 + \frac{1}{2}(i\bar{c}\star\partial_\mu D^\mu c - i\partial_\mu D^\mu c\star\bar{c})\right), \quad (17)$$

where $c = c^A t^A$ is the Faddeev–Popov ghost field. The notation is as in the previous sections. To simplify the calculations we will choose $\alpha = 1$. The Feynman rules for this theory are given in the Appendix of [5], see also [2–4]. For instance, the propagators are the same as in the corresponding ordinary gauge theory, while the 3-gluon vertex is given by

$$\begin{aligned} &-g\Big(f_{ABC}\,\cos(p\times q) + d_{ABC}\,\sin(p\times q)\Big)\times \\ &\qquad\qquad \times\,(g_{\mu\nu}\,(p-q)_\lambda + g_{\nu\lambda}\,(q-k)_\mu + g_{\lambda\mu}(k-p)_\nu), \end{aligned} \quad (18)$$

where the external gluons carry labels (A, μ, p), (B, ν, q), and (C, λ, k) for the Lie algebra, momentum and Lorentz indices and are ordered in anticlockwise sense. Moreover we use the notation $p \times q = (1/2)p_\mu \theta^{\mu\nu} q_\nu$.

Evaluating the one-loop contributions is lengthy but straightforward. The contributions split into two distinguished sets: *planar* and *nonplanar*. The first are characterized by the fact that the noncommutative factors (which are quadratic exponentials of the momenta) contain only external momenta, while in the nonplanar ones the noncommutative factors contain also the momentum running along the loop. Since eventually we integrate over the running momentum, it follows that in the latter case the noncommutative factors become smoothing factors for ultraviolet singularities. Therefore we should not expect ultraviolet divergences from nonplanar diagram contributions. As a consequence in the following we limit ourselves only to planar diagrams. In [5] the planar part of the 2-, 3- and 4-point functions were evaluated adopting the dimensional regularization ($\epsilon = 4 - D$, as usual). Here we write down some of the results. For instance for the 2-point function we have two nonvanishing contributions to the UV divergent part:

— gluons circulating inside the loop:

$$i\frac{1}{(4\pi)^2}\frac{2}{\epsilon}\delta_{AB}N\left[\frac{19}{12}g_{\mu\rho}p^2 - \frac{11}{6}p_\mu p_\nu\right], \tag{19}$$

— ghosts circulating inside the loop:

$$i\frac{1}{(4\pi)^2}\frac{2}{\epsilon}\delta_{AB}N\left[\frac{1}{12}g_{\mu\rho}p^2 + \frac{1}{6}p_\mu p_\nu\right]. \tag{20}$$

Their sum is:

$$i\frac{1}{(4\pi)^2}\frac{2}{\epsilon}\delta_{AB}N\frac{5}{3}\left[g_{\mu\rho}p^2 - p_\mu p_\nu\right] \tag{21}$$

which entails the usual renormalization constant

$$Z_3 = 1 + \frac{5}{3}g^2N\frac{1}{(4\pi)^2}\frac{2}{\epsilon}. \tag{22}$$

The same happens for the 3- and 4-point functions. They give rise to renormalization constants

$$Z_1 = 1 + \frac{2}{3}g^2N\frac{1}{(4\pi)^2}\frac{2}{\epsilon} \tag{23}$$

and

$$Z_4 = 1 - \frac{1}{3}\,g^2\,N\,\frac{1}{(4\pi)^2}\frac{2}{\epsilon}. \tag{24}$$

These are the same renormalization constants that occur in ordinary $U(N)$ Yang–Mills theories. Therefore, *the noncommutative $U(N)$ Yang–Mills theories are one-loop renormalizable*.

Now, having examined the field theory side, let us have a look at the string theory side. The one-loop calculation from the string theory point of view is much more complicated and we will limit ourselves to a short summary. The calculation of the one-loop corrections to the amplitudes considered in the previous section are the annulus amplitudes, i. e., the relevant world-sheet is the annulus. The latter can be represented as a unit disk from which a smaller disk centred at the origin has been cut out. This two-dimensional surface has one modulus, which can be chosen to be the radius q of the smaller disk. The vertices relevant for the amplitudes in question are inserted at the border of the annulus. So they can be inserted either on the same (internal or external) boundary circle, giving rise to *planar* amplitudes, or are inserted on both boundary circles, giving rise to *non-planar* amplitudes. The field theory limit, in this situation, is not simply the limit $\alpha' \to 0$, because we have to put a condition also on the modulus. As intuition suggests, this corresponds to $q \to 1$, i. e., to the annulus being squeezed to a circle, in which case the string diagrams look as skinny as Feynman diagrams.

In the field theory limit the string diagrams that may give rise to divergences are only the planar ones. Since we are interested in a comparison with the divergent parts that appear in the one-loop renormalization just considered, we will limit ourselves to planar diagrams. Now the modification of the latter when a constant B field is switched on is particularly simple. If we denote by $A_{(1)}(p_1, \dots, p_n)$ and $\mathcal{A}_{(1)}(p_1, \dots, p_n)$ the n-point one-loop planar amplitudes with and without B field, respectively, the relation is, [7],

$$\mathcal{A}_{(1)}(p_1, \dots, p_n) = \prod_{i<j} \mathrm{e}^{p_i \theta p_j} \, A_{(1)}(p_1, \dots, p_n). \tag{25}$$

This result was extended to higher loops in [8]. Now what we have to do is to find the field theory limit in the theory without B field and to multiply it by the noncommutative factor as in (25). Finding a field theory limit of a one-loop amplitude is a nontrivial exercise, but the result can be found in the literature, see [6] and references therein. Once multiplied by the relevant noncommutative factor it reproduces exactly the divergences found in the renormalization of the $NCU(N)$ gauge theory above.

In conclusion we can therefore quote the striking result that the $NCU(N)$ gauge theory and the field theory limit of string theory with $U(N)$ CP factors in the presence of B field do exactly correspond, at least up to one-loop.

2.2. Other Noncommutative Gauge Field Theories. In addition to $NCU(N)$ gauge theories there have been some attempts at defining noncommutative gauge field theories based on subalgebras of $u(N)$. Up to tree level there seems to be no obstruction to defining field theories of orthogonal or symplectic type, [14]. These theories can also be embedded in some particular configurations of string theory, that is, they can be viewed as effective field theories living on D branes immersed in some kind of string theory in the presence of B field. However the one-loop situation is still not clear, [16]. On the one hand, a consistent one-loop renormalization procedure has not yet been found; on the other hand, the string theory one-loop calculations are not entirely unambiguous in the presence of B-field. Until these question marks have been removed, one cannot safely rely on these new theories.

For a different approach to noncommutative gauge theories based on a generic Lie algebra, see [15].

3. CONSISTENCY PROBLEMS IN NONCOMMUTATIVE FIELD THEORIES

Renormalizability is a first test of quantum consistency of a theory, but there are others. After considering renormalization of noncommutative gauge theories in the previous section, a fundamental consistency test is the absence of chiral anomalies. The presence of chiral anomalies is a fatal disease for a theory because it prevents us from defining the fermion functional integral. It is therefore of utmost importance to find out whether a theory is plagued by chiral anomalies. What we are interested in in the following is of course whether noncommutativity brings in any difference as far as anomalies are concerned. The answer will be that it does.

3.1. Chiral Anomalies. Let us couple a $NCU(N)$ gauge theory to fermionic matter and let us start from the simplest situation in which chiral anomalies are relevant, i. e., an action with chiral spinor in the fundamental representation in D dimensions:

$$S = \int d^D x \left(\bar{\psi}_i \star \gamma^\mu (i\partial_\mu \psi^i + A_\mu^i{}_k \star P_+ \psi^k) \right). \tag{26}$$

Here $P_\pm = (1/2)(1 \pm \hat{\gamma})$ and $\hat{\gamma} = \gamma_0 \gamma_1 \ldots \gamma_{D-1}$, and we have used the correspondence

$$A_\mu^j{}_i \equiv A_\mu^B (t^B)^j{}_i, \qquad A_\mu = A_\mu^B t^B, \qquad A_\mu^B = 2\,\mathrm{tr}\,(t^B A_\mu),$$

where tr denotes the trace in the fundamental representation.

There are basically two ways to calculate chiral anomalies. One is based on Feynman diagram techniques [9,10,12], the second on the WZ consistency conditions, [11,13]. The method we review here is the second. It relies on the concept of nc *locality*, which means that the space of cochains (i. e., field theory monomials such as action terms) we consider is the same as in ordinary local field theories with the ordinary product replaced by the Weyl–Moyal product. This principle of nc locality is suggested by one-loop renormalization of noncommutative field theories, where counterterms are precisely of the above type, and, in the cases in which the noncommutative field theories can be embedded in string theory in the presence of B field, can be traced back to the properties of string amplitudes, precisely to the fact that such string amplitudes factorize into noncommutative factors and ordinary string amplitudes (see previous section). The advantage of using this method is that, once the formalism is established, many conclusions are evident without resorting to explicit Feynman-diagram calculations.

To write down the WZ consistency conditions we consider a matrix-valued one-form $A = A_\mu dx^\mu$, with gauge field strength two-form $F = dA + iA \star A$ and gauge transformation parameter c (it is the same as λ above but we take it to be a

Grassmann-odd, i. e., the Faddeev–Popov ghost with ghost number 1). All these quantities are valued in the Lie algebra generated by the t^A. They are therefore hermitean matrices. The gauge (BRST) transformations are:

$$sA = dc - iA \star c + ic \star A, \qquad sc = -\, c \star c, \tag{27}$$

d and s are assumed to commute. As a consequence, the transformations (27) are nilpotent like in the ordinary case.

Now, following [11, 13], we can write down the descent equations relevant to $D = 2n$ dimensions, starting from a closed and BRST invariant $2n+2$-form Ω_{2n+2}, constructed as a polynomial of F and referred to as the *top form*:

$$\Omega_{2n+2} = d\Omega^0_{2n+1}, \quad s\Omega^0_{2n+1} = d\Omega^1_{2n}, \quad s\Omega^1_{2n} = d\Omega^2_{2n-1}, \tag{28}$$

where the upper index is the ghost number and the lower index is the form order. Ω^1_{2n} is the (unintegrated) anomaly. Upon integrating the last equation, one gets

$$s\left(\int d^D x\, \Omega^1_{2n}\right) = 0 \tag{29}$$

which precisely says that $\int d^D x\, \Omega^1_{2n}$ satisfies the Wess–Zumino consistency conditions. The virtue of the discent equations formalism is that it provides explicit expressions for anomalies and one has spared the details of the complicated verification that Ω^1_{2n} does indeed satisfy the Wess–Zumino consistency conditions. The latter is an automatic consequence of the top form Ω_{2n+2} being closed (and nontrivial).

In noncommutative gauge theories there is however a complication. This method does not work straightforwardly, because there exists no closed invariant polynomial that can be built with the noncommutative curvature F. There is however a way out that was pointed out in [11]: the differential space of cochains must be constituted by forms that are defined up to an overall cyclic permutations of the Moyal product factors involved. So, keeping this specification in mind, we can easily obtain the anomaly expression from the top form $\Omega_{2n+2} = \mathrm{tr}\,(F \star F \star F \star \ldots \star F)$ even though the latter, strictly speaking, is neither closed nor invariant. The anomaly is

$$\Omega^2_{2n-1} = n\int_0^1 dt \frac{(t-1)^2}{2} \mathrm{Tr}\,(dc \star dc \star A \star F_t \star \ldots \star F_t + \ldots), \tag{30}$$

where the dots represent $(n-1)(n-2)-1$ terms obtained from the first by permuting in all distinct ways dc, A and $F_t = tdA + it^2 A \star A$, keeping track of the grading and keeping dc fixed in the first position.

In four dimensions the anomaly takes the form

$$\Omega^1_4 = -\frac{1}{2}\mathrm{Tr}\,(dc \star A \star dA + dc \star dA \star A + dc \star A \star A \star A). \tag{31}$$

If we integrate it over space-time, it coincides with the result obtained via Feynman diagram methods, see [10], Eq. (24).

Before we discuss this equation let us consider other situations which may be potentially anomalous. There are not many. In fact it is well known by now that, beside the fundamental representation of $U(N)$, only the antifundamental and the adjoint representations of $u(N)$ extend to linear representations of the Lie algebra of noncommutative $u(N)$ gauge transformations. So we can build noncommutative gauge theories only with the latter representations (or direct sums of them). Now as long as we stick to $D = 4$, in ordinary gauge theories with chiral fermions in the adjoint representation the chiral anomaly identically vanishes. In such theories one can resort to the well-known argument of reality of the adjoint representation to reach the relevant conclusion. However, this conclusion can be easily extended to noncommutative theories, [12, 13]. Therefore, in $D = 4$, the only (potentially) anomalous action is the one specified by (26). The anomaly corresponding to it is given by Eq. (31). Let us discuss this anomaly in detail.

The main question is of course whether this anomaly may vanish under specific conditions. The first two terms in (31) are proportional to $\mathrm{Tr}\,(T^A T^B T^C)$. Therefore the anomaly (31) vanishes only if $\mathrm{Tr}\,(T^A T^B T^C) = 0$. Notice that in ordinary theories the anomaly is proportional to $\mathrm{Tr}\,(T^A[T^B, T^C])$, which vanishes for instance in the case of $SU(2)$. Is this possible for $NCU(N)$? The answer is no. In fact $\mathrm{Tr}\,(T^A T^B T^C) = (1/2)\mathrm{Tr}\,(T^A\{T^B, T^C\}) + (1/2)\mathrm{Tr}\,(T^C[T^B, T^C])$. The first term in the RHS is the usual symmetric adinvariant third order tensor; the second term, which is absent in the commutative case, is proportional to the structure constant and vanishes only when all the structure constants do. Therefore we see that (31) cannot vanish. Therefore the only possibility for a noncommutative theory to be chiral anomaly free is to be nonchiral.

This is a very drastic conclusion, which can be extended to any even dimension (see [13]), and prevents, for instance, a simple extension of the Standard Model to the noncommutative case. Whether we can get around it and define anomaly free chiral noncommutative gauge theories is not a minor problem we have to face in the construction of noncommutative theories.

3.2. Unitarity and IR/UV Mixing. The previous subsection tells us that consistent noncommutative gauge theories can be found in the realm of theories without fermions or with nonchiral fermionic matter. Let us limit ourselves to such theories. However even in these theories there are new problems (as compared to ordinary theories) due to the overlap between IR and UV properties, [17].

We have noticed above that nonplanar diagrams do not give rise to UV singularities. This is the beneficial effect of IR/UV mixing, and it is due to the smoothing effect of the noncommutative factors in the high momentum region. This can be seen most clearly in the simple case of a noncommutative ϕ^4 theory in 4D, with mass m and coupling constant g. The two-point vertex at one-loop can be easily evaluated. It splits, as usual, into a planar and a nonplanar part, which, after introducing a regulator Λ, are respectively given by

$$\Gamma^{(2)}_{\text{planar}} = \frac{g^2}{48\pi^2}\left(\Lambda^2 - m^2 \ln\left(\frac{\Lambda^2}{m^2}\right) + \dots\right), \tag{32}$$

$$\Gamma^{(2)}_{\text{nonplanar}} = \frac{g^2}{96\pi^2}\left(\Lambda^2_{\text{eff}} - m^2 \ln\left(\frac{\Lambda^2_{\text{eff}}}{m^2}\right) + \dots\right), \tag{33}$$

where

$$\Lambda^2_{\text{eff}} = \frac{1}{1/\Lambda^2 + p \circ p}, \qquad p \circ p = -p_\mu \theta^{\mu\lambda} \theta_\lambda{}^\nu p_\nu,$$

the metric signature being $(-1, 1, \dots, 1)$.

From (32) above, we see that, as long as $p \circ p \neq 0$, we can safely remove the regulator Λ ($\Lambda \to \infty$).

However this beneficial effect in the UV is compensated by an increasing singularity pattern in the IR. To see this it is enough to look at the one-loop $1PI$ quadratic effective action, which takes the form (forgetting logarithms for simplicity)

$$S^{(2)}_{1PI} = \int d^4p \, \frac{1}{2}, \phi(p) \left(p^2 + M^2 + \frac{g^2}{96\pi^2(1/\Lambda^2 + p \circ p)} + \dots\right) \phi(-p), \tag{34}$$

where $M^2 = m^2 + (g^2\Lambda^2)/(48\pi^2) + \dots$ is the renormalized mass. From (34), we see that besides the usual propagator pole in p^2, we have another pole in $p \circ p$. This is not in the original spectrum of the theory.

The feature shown in this simple example is actually characteristic of all the noncommutative field theories, including the gauge theories. These modes, which appear dynamically in the spectrum, have been recognized as due to the lack of decoupling between the modes of the field theory leaving on the D brane (i. e., the massless open string modes) and the closed string modes that live on the bulk.

3.3. Comments. The above-mentioned new modes, which are not present in the original spectrum of the theory, represent a problem for noncommutative field theories. They may affect renormalization and unitarity of noncommutative field theories at higher loops. But they also raise more radical questions. In fact it is clear that in general the noncommutative field theories represent an intermediate species between ordinary field and string theory. They are interesting in themselves, but they might represent a serious opportunity as a new method to extract string theory results in a simplified way, without having to resort to full-fledged string theory. However the new modes, as well as the chiral anomaly problem we have seen above, represent an obstruction in this direction. The questions we would like to be able to answer are: can noncommutative field theories have an autonomous formulation, much in the same way ordinary field theories do? or, at some point, do we have to resort to our knowledge of string theory to understand otherwise incomprehensible facts about noncommutative field theories? Some answers to these issues have already been put forward, but the discussion is open.

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REFERENCES

1. *Seiberg N., Witten E.* String Theory and Noncommutative Geometry // JHEP. 1999. V. 09. P. 032; hepth/9908142.
2. *Sheikh-Jabbari M.* One-Loop Renormalizability of Supersymmetric Yang–Mills Theories on Noncommutative Two–Torus // Ibid. V. 9906. P. 015; hepth/9903107.
3. *Krajewski T., Wulkenhaar R.* Perturbative Quantum Gauge Fields on the Noncommutative Torus // Int. J. Mod. Phys. A. 2000. V. 15. P. 1011.
4. *Armoni A.* Comments on Perturbative Dynamics of Noncommutative Yang–Mills Theory. hepth/0005208.
5. *Bonora L., Salizzoni M.* Renormalization of Noncommutative $U(N)$ Gauge Theories // Phys. Lett. B. 2001. V. 504. P. 80–88; hepth/0011088.
6. *Di Vecchia P. et al.* String Techniques for the Calculation of Renormalization Constants in Field Theory // Nucl. Phys. B. 1996. V. 469. P. 235–286; hepth/9601143.
7. *Bilal A., Chu C.-S., Russo R.* String Theory and Noncommutative Field Theories at One-Loop // Nucl. Phys. B. 2000. V. 582. P. 65; hepth/0003180.
8. *Chu C.-S., Russo R., Sciuto S.* Multiloop String Amplitudes with B Field and Noncommutative QFT // Ibid. V. 585. P. 193; hepth/0004183.
9. *Ardalan F., Sadooghi N.* Axial Anomaly in Noncommutative QED on R^4. hepth/0002143.
10. *Gracia-Bondia J. M., Martin C. P.* Chiral Gauge Anomalies on Noncommutative R^4. hepth/0002171.
11. *Bonora L., Schnabl M., Tomasiello A.* A Note on Consistent Anomalies in Noncommutative YM Theories // Phys. Lett. B. 2000. V. 485. P. 311.
12. *Martin C. P.* The UV and IR Origin of Non-Abelian Chiral Gauge Anomalies on Noncommutative Minkowski Space-Time // J. Phys. A. 2001. V. 34. P. 9037–9056; hepth/008126.
13. *Bonora L., Sorin A.* Chiral Anomalies in Noncommutative YM Theories // Phys. Lett. B. 2001. V. 521. P. 421–428; hepth/0109204.
14. *Bonora L. et al.* Noncommutative $SO(n)$ and $Sp(n)$ Gauge Theories // Nucl. Phys. B. 2000. V. 589. P. 461; hepth/0006191.
15. *Madore J. et al.* Gauge Theory on Noncommutative Spaces. hepth/0001203.
16. *Bonora L., Salizzoni M.* Nonorientable String One-Loop Corrections in the Presence of a B Field. hepth/0105307.
17. *Minwalla S., Van Raamsdonk M., Seiberg N.* Noncommutative Perturbative Dynamics // JHEP. 2000. V. 0002. P. 020; hepth/9912072;
   Van Raamsdonk M., Seiberg N. Comments on Noncommutative Perturbative Dynamics // Ibid. V. 0003. P. 035; hepth/0002186.