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q-PLANE WAVE SOLUTIONS OF q-MAXWELL EQUATIONS

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We give new solutions of the quantum conformal deformations of the full Maxwell equations in terms of deformations of the plane wave.

INTRODUCTION

One of the purposes of quantum deformations is to provide an alternative of the regularization procedures of quantum field theory. Applied to Minkowski space-time the quantum deformations approach is also an alternative to Connes' noncommutative geometry [1]. The first problem to tackle in a noncommutative deformed setting is to analyze the behavior of the wave equations analogues. Here we continue the study of hierarchies of deformed equations derived in [2, 3] with the use of quantum conformal symmetry. One hierarchy involves the massless representations of the conformal group and is parametrized by a nonnegative integer r [3]. The case $r = 0$ corresponds to the q-d'Alembert equation, while for each $r > 0$ there are two couples of equations involving fields of conjugated Lorentz representations of dimension $r + 1$. The construction of solutions of the hierarchy was started in [4] with the q-d'Alembert equation. One of the solutions given was a deformation of the plane wave as a formal power series in the noncommutative coordinates of q-Minkowski space-time and four-momenta. (For the latter deformations we use the one from [2] since, unlike the other known examples [5–7], it is related to a deformation of the conformal group.) This q-plane wave has some properties analogous to the classical one but is not an exponent or q exponent. Thus, it differs conceptually from the classical plane wave and may serve as a regularization of the latter. For the equations labelled by $r > 0$ it turned out that one needs a second q deformation of the plane wave in a conjugated basis [8]. The solutions of the hierarchy in terms of the two q-plane waves were given in [8] for $r = 1$ and in [9] for $r > 1$. Later these two q-plane waves were generalized and correspondingly more general solutions of the hierarchy were given in [10]. Another hierarchy is the Maxwell hierarchy [2]. The two hierarchies have only one common member — the Maxwell equations —

they are the lowest member of the Maxwell hierarchy and the $r = 2$ member of the massless hierarchy. The compatibility of the solutions of the free q-Maxwell equations with the q-potential equations was studied [11]. In the present paper we study the full q-Maxwell equations and the compatibility of their solutions with the conservation of the current. The results of Section 2 are all new.

1. PRELIMINARIES

First we introduce new Minkowski variables:

$$x_\pm \equiv x_0 \pm x_3, \qquad v \equiv x_1 - ix_2, \qquad \bar{v} \equiv x_1 + ix_2, \tag{1}$$

which (unlike the x_μ) have definite group-theoretical interpretation as part of a six-dimensional coset of the conformal group $SU(2,2)$ (as explained in [2]). In terms of these variables, e. g., the d'Alembert equation is:

$$\Box\varphi = (\partial_- \partial_+ - \partial_v \partial_{\bar{v}})\varphi = 0. \tag{2}$$

In the q-deformed case we use the noncommutative q-Minkowski space-time of [2] which is given by the following commutation relations (with $\lambda \equiv q - q^{-1}$):

$$x_\pm v = q^{\pm 1} v x_\pm, \quad x_\pm \bar{v} = q^{\pm 1} \bar{v} x_\pm, \quad x_+ x_- - x_- x_+ = \lambda v \bar{v}, \quad \bar{v} v = v \bar{v}, \tag{3}$$

with the deformation parameter being a phase: $|q| = 1$. Relations (3) are preserved by the antilinear anti-involution ω:

$$\omega(x_\pm) = x_\pm, \quad \omega(v) = \bar{v}, \quad \omega(q) = \bar{q} = q^{-1}, \quad (\omega(\lambda) = -\lambda). \tag{4}$$

The solution spaces consist of formal power series in the q-Minkowski coordinates (which we give in two conjugate bases):

$$\varphi = \sum_{j,n,\ell,m \in \mathbb{Z}_+} \mu_{jn\ell m}\ \varphi_{jn\ell m}, \qquad \varphi_{jn\ell m} = \hat{\varphi}_{jn\ell m}, \tilde{\varphi}_{jn\ell m}, \tag{5}$$

$$\hat{\varphi}_{jn\ell m} = v^j x_-^n x_+^\ell \bar{v}^m, \tag{6}$$

$$\tilde{\varphi}_{jn\ell m} = \bar{v}^m x_+^\ell x_-^n v^j = \omega(\hat{\varphi}_{jn\ell m}). \tag{7}$$

The solution spaces (5) are representation spaces of the quantum algebra $U_q(sl(4))$. For the latter we use the rational basis of Jimbo [12]. The action of $U_q(sl(4))$ on $\hat{\varphi}_{jn\ell m}$ was given in [13], and on $\tilde{\varphi}_{jn\ell m}$ in [8]. Further we suppose that q is not a nontrivial root of unity.

In order to write our q-deformed equations in compact form it is necessary to introduce some additional operators. We first define the operators:

$$\hat{M}_\kappa^\pm \varphi = \sum_{j,n,\ell,m\in\mathbb{Z}_+} \mu_{jn\ell m}\ \hat{M}_\kappa^\pm \varphi_{jn\ell m}, \qquad \kappa = \pm, v, \bar{v}, \tag{8}$$

$$T_\kappa^\pm \varphi = \sum_{j,n,\ell,m\in\mathbb{Z}_+} \mu_{jn\ell m}\ T_\kappa^\pm \varphi_{jn\ell m}, \qquad \kappa = \pm, v, \bar{v}, \tag{9}$$

and $\hat{M}_+^\pm$, $\hat{M}_-^\pm$, $\hat{M}_v^\pm$, $\hat{M}_{\bar{v}}^\pm$, resp., act on $\varphi_{jn\ell m}$ by changing by ± 1 the value of j, n, ℓ, m, resp., while $T_+^\pm$, $T_-^\pm$, $T_v^\pm$, $T_{\bar{v}}^\pm$, resp., act on $\varphi_{jn\ell m}$ by multiplication by $q^{\pm j}, q^{\pm n}, q^{\pm \ell}, q^{\pm m}$, resp. We shall use also the «logs» N_κ such that $T_\kappa = q^{N_\kappa}$. Now we can define the q-difference operators:

$$\hat{\mathcal{D}}_\kappa \varphi = \frac{1}{\lambda} \hat{M}_\kappa^{-1} \left(T_\kappa - T_\kappa^{-1}\right) \varphi = \frac{1}{\lambda} \hat{M}_\kappa^{-1} \left(q^{N_\kappa} - q^{-N_\kappa}\right) \varphi. \tag{10}$$

Note that when $q \to 1$, then $\hat{\mathcal{D}}_\kappa \to \partial_k$. Using (8) and (10) the q-d'Alembert equation may be written as [3, 8], respectively,

$$(q\hat{\mathcal{D}}_- \hat{\mathcal{D}}_+ T_v T_{\bar{v}} - \hat{\mathcal{D}}_v \hat{\mathcal{D}}_{\bar{v}})\ T_v T_- T_+ T_{\bar{v}} \hat{\varphi} = 0, \tag{11}$$

$$(\hat{\mathcal{D}}_- \hat{\mathcal{D}}_+ - q\hat{\mathcal{D}}_v \hat{\mathcal{D}}_{\bar{v}} T_v T_{\bar{v}})\ T_- T_+ \tilde{\varphi} = 0. \tag{12}$$

Note that when $q \to 1$ both equations (11), (12) go to (2). Note that the operators in (8), (10)–(12) for different variables commute, i. e., we have passed to commuting variables. However, keeping the normal ordering it is straightforward to pass back to noncommuting variables.

Next we recall that Maxwell's equations are part also of Maxwell's hierarchy of equations. The quantum conformal deformation of the equations of the hierarchy are [2]:

$$ {}_qI_n^+\ {}_qF_n^+ = {}_q\ J^n, \qquad {}_qI_n^-\ {}_qF_n^- = {}_q\ J^n, \tag{13}$$

where in the basis (6) the operators are:

$$\begin{aligned} {}_qI_n^+ = \frac{1}{2}\big((q\hat{\mathcal{D}}_v + \hat{M}_{\bar{z}} \hat{\mathcal{D}}_+ (T_- T_v)^{-1} T_{\bar{v}}) T_- [n+2-N_z]_q - \\ - q^{-n-2} (\hat{\mathcal{D}}_- T_- + q^{-1} \hat{M}_{\bar{z}} \hat{\mathcal{D}}_{\bar{v}} - \lambda \hat{M}_v \hat{M}_{\bar{z}} \hat{\mathcal{D}}_- \hat{\mathcal{D}}_+ T_{\bar{v}}) T_-^{-1} \hat{\mathcal{D}}_z\big) T_+ T_v T_z T_{\bar{z}}^{-1}, \end{aligned} \tag{14}$$

$$\begin{aligned} {}_qI_n^- = \frac{1}{2}(\hat{\mathcal{D}}_{\bar{v}} + q\hat{M}_z \hat{\mathcal{D}}_+ T_{\bar{v}} T_- T_v^{-1} - q\lambda \hat{M}_v \hat{\mathcal{D}}_- \hat{\mathcal{D}}_+ T_{\bar{v}}) T_{\bar{v}} [n+2-N_{\bar{z}}]_q - \\ - \frac{1}{2} q^{n+3} (\hat{\mathcal{D}}_- + q\hat{M}_z \hat{\mathcal{D}}_v T_-) \hat{\mathcal{D}}_{\bar{z}} T_- T_{\bar{v}}, \end{aligned} \tag{15}$$

while in the basis (7) the operators are:

$$
\begin{aligned}
{}_qI_n^+ = \frac{1}{2}q(\hat{\mathcal{D}}_v + \hat{M}_{\bar{z}}\hat{\mathcal{D}}_+ T_- T_{\bar{v}}^{-1} T_v) T_v [n+2-N_z]_q - \\
- \frac{1}{2}q^{n+3}(\hat{\mathcal{D}}_- + \hat{M}_{\bar{z}}\hat{\mathcal{D}}_{\bar{v}} T_- + \lambda q^{-1}\hat{M}_v \hat{M}_{\bar{z}}\hat{\mathcal{D}}_-\hat{\mathcal{D}}_+ T_{\bar{v}}^{-1} T_-)\hat{\mathcal{D}}_z T_- T_v,
\end{aligned} \quad (16)
$$

$$
\begin{aligned}
{}_qI_n^- = \frac{1}{2}\big((\hat{\mathcal{D}}_{\bar{v}} T_{\bar{v}} T_- + \hat{M}_z \hat{\mathcal{D}}_+ T_v + \\
+ q^{-1}\lambda \hat{M}_v \hat{\mathcal{D}}_- \hat{\mathcal{D}}_+ T_-)[n+2-N_{\bar{z}}]_q - \\
- q^{-n-2}(\hat{\mathcal{D}}_- + \hat{M}_z \hat{\mathcal{D}}_v T_-^{-1})\hat{\mathcal{D}}_{\bar{z}} T_{\bar{v}}\big) T_+ T_{\bar{z}} T_z^{-1}.
\end{aligned} \quad (17)
$$

Note that for $q = 1$, (14), (15) coincide with (16), (17), respectively. Maxwell's equations $\partial^\mu F_{\mu\nu} = J_\nu$, $\epsilon_{\mu\nu\rho\sigma}\partial^\mu F^{\rho\sigma} = 0$ are obtained from (13) for $n = 0$, $q = 1$, substituting the fixed helicity constituents $F^\pm$ by: $F^+ = z^2(F_1^+ + iF_2^+) - 2zF_3^+ - (F_1^+ - iF_2^+)$, $F^- = \bar{z}^2(F_1^- - iF_2^-) - 2\bar{z}F_3^- - (F_1^- + iF_2^-)$, $F_k^\pm = F_{k0} \pm (i/2)\varepsilon_{k\ell m}F_{\ell m} = E_k \pm iH_k$, $J^0 = \bar{z}z(J_0 + J_3) + z(J_1 + iJ_2) + \bar{z}(J_1 - iJ_2) + (J_0 - J_3)$, and then comparing the coefficients of the resulting first order polynomials in z and $\bar{z}$.

We shall look for solutions of the full q-Maxwell equations in terms of deformations of the plane wave. Let us first recall these deformations from [10]. The first deformation is given in the basis (6):

$$
\begin{gathered}
\widehat{\exp}_q(k,x) = \sum_{s=0}^{\infty} \frac{1}{[s]_q!}\hat{h}_s, \\
[s]_q! \equiv [s]_q[s-1]_q \ldots [1]_q, \quad [0]_q! \equiv 1, \quad [n]_q \equiv \frac{q^n - q^{-n}}{q - q^{-1}},
\end{gathered} \quad (18)
$$

$$
\begin{aligned}
\hat{h}_s = \beta^s \sum_{a,b,n \in \mathbb{Z}_+} \frac{(-1)^{s-a-b} q^{n(s-2a-2b+2n)+a(s-a-1)+b(-s+a+b+1)} q^{P_s(a,b)}}{\Gamma_q(a-n+1)\Gamma_q(b-n+1)\Gamma_q(s-a-b+n+1)[n]_q!} \times \\
\times k_v^{s-a-b+n} k_-^{b-n} k_+^{a-n} k_{\bar{v}}^n v^n x_-^{a-n} x_+^{b-n} \bar{v}^{s-a-b+n},
\end{aligned} \quad (19)
$$

$$
(\beta^s)^{-1} = \sum_{p=0}^{s} \frac{q^{(s-p)(p-1)+p}}{[p]_q![s-p]_q!},
$$

where the momentum components $(k_v, k_-, k_+, k_{\bar{v}})$ are supposed to be non-commutative between themselves (obeying the same rules (3) as the q-Minkowski coordinates), and commutative with the coordinates. Further, Γ_q is a q deformation of the Γ function, of which here we use only the properties: $\Gamma_q(p) = [p-1]_q!$ for $p \in I\!N$, $1/\Gamma_q(p) = 0$ for $p \in \mathbb{Z}_-$; $P_s(a,b)$ is a polynomial in a, b. Note that $(\hat{h}_s)|_{q=1} = (kx)^s$ and thus $(\widehat{\exp}_q(k,x))|_{q=1} = \exp(kx)$. This q-plane wave has some properties analogous to the classical one but is not an exponent or q-exponent, cf. [14]. This is enabled also by the fact (true also for $q = 1$) that solving the equations may be done in terms of the components $\hat{h}_s$. This deformation of the plane wave generalizes the original one from [4] to obtain which one sets $P_s(a,b) = 0$, in which case we shall use the notation f_s for the components from [4] since:

$$(\hat{h}_s)_{P_s(a,b)=0} = f_s. \tag{20}$$

Each $\hat{h}_s$ satisfies the q-d'Alembert equation (11) on the momentum q cone:

$$\mathcal{L}_q^k \equiv k_- k_+ - q^{-1} k_v k_{\bar{v}} = k_+ k_- - q k_v k_{\bar{v}} = 0. \tag{21}$$

The second deformation is given in the basis (7):

$$\widetilde{\exp}_q(k,x) = \sum_{s=0}^{\infty} \frac{1}{[s]_q!} \tilde{h}_s, \tag{22}$$

$$\tilde{h}_s = \tilde{\beta}^s \sum_{a,b,n} \frac{(-1)^{s-a-b} q^{n(2a+2b-2n-s)+a(a-s-1)+b(s-a-b+1)} q^{Q_s(a,b)}}{\Gamma_q(a-n+1)\Gamma_q(b-n+1)\Gamma_q(s-a-b+n+1)[n]_q!} \times \\ \times k_{\bar{v}}^n k_+^{a-n} k_-^{b-n} k_v^{s-a-b+n} \bar{v}^{s-a-b+n} x_+^{b-n} x_-^{a-n} v^n, \tag{23}$$

$$(\tilde{\beta}^s)^{-1} = \sum_{p=0}^{s} \frac{q^{(p-s)(p-1)+p}}{[p]_q! [s-p]_q!},$$

where $Q_s(a,b)$ are arbitrary polynomials. If the latter are zero, then $\widetilde{\exp}_q(k,x)$ becomes the q-plane wave deformation found in [8]. The $\tilde{h}_s$ have the same properties as the $\hat{h}_s$ but the conjugated basis is used; in particular, they satisfy the q-d'Alembert equation (12) on the momentum q cone (21).

2. SOLUTIONS OF THE q-MAXWELL EQUATIONS

First we shall use the basis (6). The solutions of (13) for $n = 0$ in the homogeneous case ($J = 0$) are:

$$\hat{F}^{h\pm} \doteq \left({}_q F_0^{\pm}\right)_{J=0} = \sum_{m,s=0}^{\infty} \frac{1}{[s]_q!} \hat{F}_{ms}^{h\pm}(k) f_s, \tag{24}$$

$$\hat{F}^{h+}_{ms}(k) = \sum_{i=0}^{m}\Big(\sum_{j=0}^{m-i} \hat{p}^{ms1}_{ij} k_v^i k_-^{m-i-j} k_{\bar{v}}^j (k_v - q^{s+6} z k_-)(k_v - q^{s+3} z k_-) + \\ + \hat{p}^{ms2}_{i} k_v^i k_{\bar{v}}^{m-i} (k_v - q^{s+6} z k_-)(k_+ - q^{s+3} z k_{\bar{v}}) + \\ + \sum_{j=0}^{m-i} \hat{p}^{ms3}_{ij} k_v^i k_+^{m-i-j} k_{\bar{v}}^j (k_+ - q^{s+6} z k_{\bar{v}})(k_+ - q^{s+3} z k_{\bar{v}})\Big), \quad (25)$$

$$\hat{F}^{h-}_{ms}(k) = \sum_{i=0}^{m}\Big(\sum_{j=0}^{m-i} \hat{r}^{ms1}_{ij} k_v^i k_-^{m-i-j} k_{\bar{v}}^j (k_{\bar{v}} - q^{-1} \bar{z} k_-)(k_{\bar{v}} - \bar{z} k_-) + \\ + \hat{r}^{ms2}_{i} k_v^i k_{\bar{v}}^{m-i} (k_+ - q^{-1} \bar{z} k_v)(k_{\bar{v}} - \bar{z} k_-) + \\ + \sum_{j=0}^{m-i} \hat{r}^{ms3}_{ij} k_v^i k_+^{m-i-j} k_{\bar{v}}^j (k_+ - q^{-1} \bar{z} k_v)(k_+ - \bar{z} k_v)\Big), \quad (26)$$

where $\hat{p}^{msa}_{i(j)}, \hat{r}^{msa}_{i(j)}$ are independent constants. The check that these are solutions is done for commutative Minkowski coordinates and noncommutative momenta on the q cone. The terms with $m = 0$ of the solutions (24)–(26), were obtained earlier [9] (later they were generalized using more general q-plane waves [10]). The solution (26) can be written in terms of the deformed plane wave if we suppose that the $\hat{r}^{msa}_{i(j)}$ for different s coincide: $\hat{r}^{msa}_{i(j)} = \hat{r}^{ma}_{i(j)}$. Then we have:

$$\hat{F}^{h-} = \sum_{m=0}^{\infty} \hat{F}^{h-}_{m}(k) \exp_q(k, x), \quad \hat{F}^{h-}_{m}(k) = \hat{F}^{h-}_{ms}(k). \quad (27)$$

In the inhomogeneous case the solutions of (13) for $n = 0$ are:

$$_qJ^0 = \bar{z} z \hat{J}_+ + z \hat{J}_v + \bar{z} \hat{J}_{\bar{v}} + \hat{J}_-, \quad (28)$$

$$\hat{J}_\kappa = \sum_{m,s=0}^{\infty} \frac{1}{[s]_q!}\, \hat{J}^{ms}_\kappa(k) f_{s-1}, \quad \kappa = \pm, v, \bar{v}, \quad (29)$$

$$\begin{aligned} \hat{J}^{ms}_+(k) &= -\hat{K}^s_m(k) k_-, \\ \hat{J}^{ms}_-(k) &= -q^{-s-2} \hat{K}^s_m(k) k_+, \\ \hat{J}^{ms}_v(k) &= \hat{K}^s_m(k) k_{\bar{v}}, \\ \hat{J}^{ms}_{\bar{v}}(k) &= q^{-s-2} \hat{K}^s_m(k) k_v, \\ \hat{K}^s_m(k) &\doteq \hat{\gamma}^s_v k_v^{m+1} + \hat{\gamma}^s_- k_-^{m+1} + \hat{\gamma}^s_+ k_+^{m+1} + \hat{\gamma}^s_{\bar{v}} k_{\bar{v}}^{m+1}, \end{aligned} \quad (30)$$

$$
\begin{aligned}
{}_qF_0^{\pm} &= \hat{F}^{\pm} + \hat{F}^{h\pm}, && (31)\\
\hat{F}^{\pm} &= \sum_{m,s=0}^{\infty} \frac{1}{[s]_q!} \hat{F}^{\pm}_{ms}(k) f_s, && (32)
\end{aligned}
$$

$$
\begin{aligned}
\hat{F}^{+}_{ms}(k) = 2d_s q^{-s}\big(& (q^{-s-5}\hat{\gamma}^s_- k^m_- + z\hat{\gamma}^s_v k^m_v)(k_v - q^{s+3} z k_-) + \\
&+ (q^{-s-5}\hat{\gamma}^s_{\bar{v}} k^m_{\bar{v}} + z\hat{\gamma}^s_+ k^m_+)(k_+ - q^{s+3} z k_{\bar{v}})\big),
\end{aligned}
$$

$$
\begin{aligned}
\hat{F}^{-}_{ms}(k) = 2d_s q^{-2s-2}\big(& (\hat{\gamma}^s_- k^m_- + q^{-2}\bar{z}\hat{\gamma}^s_{\bar{v}} k^m_{\bar{v}})(k_{\bar{v}} - \bar{z} k_-) + \\
&+ (\hat{\gamma}^s_v k^m_v + q^{-2}\bar{z}\hat{\gamma}^s_+ k^m_+)(k_+ - \bar{z} k_v)\big),
\end{aligned}
$$

where $d_s = \beta^s/\beta^{s+1}$. As in the homogeneous case we cannot make $\hat{F}^{+}_{ms}(k)$ independent of s. We can make $\hat{F}^{-}_{ms}(k)$ independent of s by choosing $\hat{\gamma}^s_\kappa \sim q^{2s} d_s^{-1}$, but we cannot make $\hat{J}^{ms}_\kappa(k)$ independent of s.

Since we work with the full Maxwell equations we have also to check the q deformation of the current conservation $\partial^\nu J_\nu = 0$:

$$
I_{13} J = 0, \tag{33}
$$

$$
\begin{aligned}
I_{13} = q^3[N_z - 1]_q T_z \hat{\mathcal{D}}_{\bar{z}} \hat{\mathcal{D}}_v T_v T_- T_+ + q\hat{\mathcal{D}}_z T_z \hat{\mathcal{D}}_{\bar{z}} \hat{\mathcal{D}}_- T_v T_+ +& \\
+ q[N_z - 1]_q T_z [N_{\bar{z}} - 1]_q \hat{\mathcal{D}}_+ T_+ T_{\bar{v}} +& \\
+ q^{-1}[N_{\bar{z}} - 1]_q \hat{\mathcal{D}}_z T_z \hat{\mathcal{D}}_{\bar{v}} T_v T_-^{-1} T_+ -& \\
- \lambda \hat{M}_v [N_{\bar{z}} - 1]_q \hat{\mathcal{D}}_z T_z \hat{\mathcal{D}}_- \hat{\mathcal{D}}_+ T_v T_-^{-1} T_+ T_{\bar{v}}.& \quad (34)
\end{aligned}
$$

Substituting (28), (29) in the above we get:

$$
qJ^s_+(k)k_+ + J^s_v(k)k_v + q^{s+2} J^s_{\bar{v}} k_{\bar{v}} + q^{s+1} J^s_-(k) k_- = 0. \tag{35}
$$

The latter is fulfilled by the explicit expressions in (30), but we should note that these expressions fulfil also the following splittings of (35):

$$
qJ^s_+(k)k_+ + J^s_v(k)k_v = 0, \quad qJ^s_{\bar{v}}(k)k_{\bar{v}} + J^s_-(k), k_- = 0, \tag{36}
$$

$$
J^s_+(k)k_+ + q^{s+1} J^s_{\bar{v}}(k)k_{\bar{v}} = 0, \quad J^s_v(k)k_v + q^{s+1} J^s_-(k)k_- = 0. \tag{37}
$$

Furthermore the expressions from (30) fulfil also:

$$
qJ^s_+(k)k_{\bar{v}} + J^s_v(k)k_- = 0, \quad qJ^s_{\bar{v}}(k)k_+ + J^s_-(k)k_v = 0, \tag{38}
$$

$$J_+^s(k)k_v + q^{s+1}J_{\bar{v}}^s(k)k_- = 0, \quad J_v^s(k)k_+ + q^{s+1}J_-^s(k)k_{\bar{v}} = 0. \tag{39}$$

Now we shall use the basis (7). Then solutions of (13) for $n = 0$ in the homogeneous case ($J = 0$) are:

$$\tilde{F}^{h\pm} \doteq \left({}_qF_0^{\pm}\right)_{J=0} = \sum_{m,s=0}^{\infty} \frac{1}{[s]_q!} \tilde{F}_{ms}^{h\pm}(k)\tilde{h}_s, \tag{40}$$

$$\begin{aligned}\tilde{F}_{ms}^{h+}(k) = \sum_{i=0}^{m}\Big(&\sum_{j=0}^{m-i} \tilde{p}_{ij}^{ms1} k_{\bar{v}}^i k_-^{m-i-j} k_v^j (k_v - zk_-)(k_v - qzk_-) + \\ &+ \tilde{p}_i^{ms2} k_{\bar{v}}^i k_v^{m-i} (k_+ - zk_{\bar{v}})(k_v - qzk_-) + \\ &+ \sum_{j=0}^{m-i} \tilde{p}_{ij}^{ms3} k_{\bar{v}}^i k_+^{m-i-j} k_v^j (k_+ - zk_{\bar{v}})(k_+ - qzk_{\bar{v}})\Big),\end{aligned} \tag{41}$$

$$\begin{aligned}\tilde{F}_{ms}^{h-}(k) = \sum_{i=0}^{m}\Big(&\sum_{j=0}^{m-i} \tilde{r}_{ij}^{ms1} k_{\bar{v}}^i k_-^{m-i-j} k_v^j (k_{\bar{v}} - q^{s+1}\bar{z}k_-)(k_{\bar{v}} - q^{s+2}\bar{z}k_-) + \\ &+ \tilde{r}_i^{ms2} k_v^i k_{\bar{v}}^{m-i} (k_{\bar{v}} - q^{s+1}\bar{z}k_-)(k_+ - q^{s+2}\bar{z}k_v) + \\ &+ \sum_{j=0}^{m-i} \tilde{r}_{ij}^{ms3} k_v^i k_+^{m-i-j} k_{\bar{v}}^j (k_+ - q^{s+1}\bar{z}k_v)(k_+ - q^{s+2}\bar{z}k_v)\Big),\end{aligned} \tag{42}$$

where $\tilde{p}_{i(j)}^{msa}$, $\tilde{r}_{i(j)}^{msa}$ are independent constants; $Q_s(a,b) = 0$ in $\tilde{h}_s$. The terms with $m = 0$ of the solutions (40)–(42) were obtained earlier in [9] (and using the generalized q-plane wave in [10]). The solution (41) can be written in terms of the deformed plane wave if we suppose that the $\tilde{p}_{i(j)}^{msa}$ for different s coincide: $\tilde{p}_{i(j)}^{msa} = \tilde{p}_{i(j)}^{ma}$. Then we have:

$$\tilde{F}^{h+} = \sum_{m=0}^{\infty} \tilde{F}_m^{h+}(k)\widetilde{\exp}_q(k,x), \quad \tilde{F}_m^{h+}(k) = \tilde{F}_{ms}^{h+}(k). \tag{43}$$

In the inhomogeneous case the solutions of (13) for $n = 0$ are:

$$ {}_qJ^0 = \bar{z}z\tilde{J}_+ + z\tilde{J}_v + \bar{z}\tilde{J}_{\bar{v}} + \tilde{J}_-, \tag{44}$$

$$\tilde{J}_\kappa = \sum_{m,s=0}^{\infty} \frac{1}{[s]_q!}\, \tilde{J}_\kappa^{ms}(k)\tilde{h}_{s-1}, \quad \kappa = \pm, v, \bar{v}, \tag{45}$$

$$\begin{aligned}
\tilde{J}^{ms}_{+}(k) &= -q^{s+1}\tilde{K}^{s}_{m}(k)k_{-}, \\
\tilde{J}^{ms}_{-}(k) &= -q^{-1}\tilde{K}^{s}_{m}(k)k_{+}, \\
\tilde{J}^{ms}_{v}(k) &= \tilde{K}^{s}_{m}(k)k_{\bar{v}}, \\
\tilde{J}^{ms}_{\bar{v}}(k) &= q^{s}\tilde{K}^{s}_{m}(k)k_{v}, \\
\tilde{K}^{s}_{m}(k) &\doteq \tilde{\gamma}^{s}_{v}k_{v}^{m+1} + \tilde{\gamma}^{s}_{-}k_{-}^{m+1} + \tilde{\gamma}^{s}_{+}k_{+}^{m+1} + \tilde{\gamma}^{s}_{\bar{v}}k_{\bar{v}}^{m+1},
\end{aligned} \tag{46}$$

$$ {}_{q}F^{\pm}_{0} = \tilde{F}^{\pm} + \tilde{F}^{h\pm}, \tag{47}$$

$$\tilde{F}^{\pm} = \sum_{m,s=0}^{\infty} \frac{1}{[s]_{q}!}\tilde{F}^{\pm}_{ms}(k)\tilde{h}_{s}, \tag{48}$$

$$\begin{aligned}
\tilde{F}^{+}_{ms}(k) = 2\tilde{d}_{s}q^{s-2}\big((\tilde{\gamma}^{s}_{-}k_{-}^{m} + q^{-1}z\tilde{\gamma}^{s}_{v}k_{v}^{m})(k_{v} - qzk_{-}) + {} \\
+ (\tilde{\gamma}^{s}_{\bar{v}}k_{\bar{v}}^{m} + q^{-1}z\tilde{\gamma}^{s}_{+}k_{+}^{m})(k_{+} - qzk_{\bar{v}})\big),
\end{aligned}$$

$$\begin{aligned}
\tilde{F}^{-}_{ms}(k) = 2\tilde{d}_{s}\big((q^{-s-3}\tilde{\gamma}^{s}_{-}k_{-}^{m} + q\bar{z}\tilde{\gamma}^{s}_{\bar{v}}k_{\bar{v}}^{m})(k_{\bar{v}} - q^{s+2}\bar{z}k_{-}) + {} \\
+ (q^{-s-3}\tilde{\gamma}^{s}_{v}k_{v}^{m} + q\bar{z}\tilde{\gamma}^{s}_{+}k_{+}^{m})(k_{+} - q^{s+2}\bar{z}k_{v})\big),
\end{aligned}$$

where $\tilde{d}_s = \tilde{\beta}^s/\tilde{\beta}^{s+1}$, $Q_s(a,b) = 0$ in $\tilde{h}_s$. We cannot make $\tilde{F}^{-}_{ms}(k)$ or $\tilde{J}^{ms}_{\kappa}(k)$ independent of s. We can make $\tilde{F}^{+}_{ms}(k)$ independent of s by choosing $\tilde{\gamma}^{s}_{\kappa} \sim q^{-s}\tilde{d}_{s}^{-1}$.

Also here we shall check whether the q deformation of the current conservation (33) is fulfilled. The analog of (34) in the basis (7) is:

$$\begin{aligned}
I_{13} = [N_z - 1]_q\hat{\mathcal{D}}_{\bar{z}}T_{\bar{z}}\hat{\mathcal{D}}_{v}T_{\bar{v}}T_{+}T_{-}^{-1} + q\hat{\mathcal{D}}_{\bar{z}}T_{\bar{z}}\hat{\mathcal{D}}_{z}\hat{\mathcal{D}}_{-}T_{\bar{v}}T_{+} + {} \\
+ q[N_{\bar{z}} - 1]_qT_{\bar{z}}[N_z - 1]_q\hat{\mathcal{D}}_{+}T_{+}T_{v} + q^2[N_{\bar{z}} - 1]_q\hat{\mathcal{D}}_{z}T_{\bar{z}}\hat{\mathcal{D}}_{\bar{v}}T_{\bar{v}}T_{-}T_{+} - {} \\
- \lambda q\hat{M}_{v}[N_{\bar{z}} - 1]_q\hat{\mathcal{D}}_{z}T_{\bar{z}}\hat{\mathcal{D}}_{-}\hat{\mathcal{D}}_{+}T_{-}T_{+}.
\end{aligned} \tag{49}$$

Then the analog of (35) is:

$$J^{s}_{+}(k)k_{+} + q^{s}J^{s}_{v}(k)k_{v} + J^{s}_{\bar{v}}k_{\bar{v}} + q^{s}J^{s}_{-}(k)k_{-} = 0. \tag{50}$$

The latter is fulfilled by the explicit expressions in (46), but we should note that these expressions fulfil also the following splittings of (50):

$$J^{s}_{+}(k)k_{+} + q^{s}J^{s}_{v}(k)k_{v} = 0, \quad J^{s}_{\bar{v}}(k)k_{\bar{v}} + q^{s}J^{s}_{-}(k)k_{-} = 0, \tag{51}$$

$$J^s_+(k)k_+ + J^s_{\bar{v}}(k)k_{\bar{v}} = 0, \quad J^s_v(k)k_v + J^s_-(k)k_- = 0. \tag{52}$$

Furthermore the expressions from (46) fulfil also:

$$J^s_+(k)k_{\bar{v}} + q^s J^s_v(k)k_- = 0, \qquad J^s_{\bar{v}}(k)k_+ + q^s J^s_-(k)k_v = 0, \tag{53}$$

$$J^s_+(k)k_v + J^s_{\bar{v}}(k)k_- = 0, \quad J^s_v(k)k_+ + J^s_-(k)k_{\bar{v}} = 0. \tag{54}$$

3. SUMMARY AND OUTLOOK

We have given new solutions of the full q-Maxwell equations in two conjugated bases (6) and (7). The solutions of the homogeneous equations are also new (the old solutions are special cases). We see that the roles of the solutions F^+ and F^- are exchanged in the two conjugated bases. We note also that the currents components are different: $\hat{J}^{ms}_\kappa \neq \tilde{J}^{ms}_\kappa$ (for $q \neq 1$, $\kappa \neq v$), and in both cases they cannot be made independent of s. Thus, there is no advantage of choosing either of the bases (6) or (7). It may be also possible to use both in a Connes–Lott type model [15].

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