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ON FRACTIONAL SUPERSYMMETRIC QUANTUM MECHANICS

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Two approaches of $\mathcal{N}=2$ fractional supersymmetric quantum mechanics of order k are studied in a complementary way. The first one, based on a generalized Weyl–Heisenberg algebra W_k (that comprizes the affine quantum algebra $U_q(sl_2)$ with $q^k=1$ as a special case), apparently contains solely one bosonic degree of freedom. The second one uses generalized bosonic and k-fermionic degrees of freedom. As an illustration, a particular emphasis is put on the fractional supersymmetric oscillator of order k.

INTRODUCTION

Supersymmetric quantum mechanics (SSQM) needs two degrees of freedom: one bosonic degree (described by a complex variable) and one fermionic degree (described by a Grassmann variable). From a mathematical point of view, we then have a Z_2 grading of the Hilbert space of physical states (involving bosonic and fermionic states). Fractional supersymmetric quantum mechanics (FSSQM) of order k is an extension of ordinary SSQM for which the Z_2 grading is replaced by a Z_k grading with $k \in \mathbf{N} \setminus \{0,1\}$. The Z_k grading corresponds to a bosonic degree of freedom (described again by a complex variable) and a para-fermionic or k-fermionic degree of freedom (described by a generalized Grassmann variable of order k). In other words, to pass from ordinary supersymmetry or SSQM to fractional supersymmetry or FSSQM of order k, we retain the bosonic variable and replace the fermionic variable by a para-fermionic or k-fermionic variable.

A possible approach to FSSQM of order k thus amounts to replace fermions by para-fermions of order $k-1$. This yields para-supersymmetric quantum mechanics as first developed, with one boson and one para-fermion of order 2, by Rubakov and Spiridonov [1] and extended by various authors [2–7]. An alternative approach to FSSQM of order k consists in replacing fermions by k fermions which are objects interpolating between bosons (for $k \to \infty$) and

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fermions (for $k = 2$) and which satisfy a generalized Pauli exclusion principle according to which one cannot put more than $k-1$ particles on a given quantum state [8]. The k fermions take their origin in a pair of q- and $\bar{q}$-oscillator algebras (or q- and $\bar{q}$-uon algebras) with

$$q = \frac{1}{\bar{q}} := \exp\left(\frac{2\pi \mathrm{i}}{k}\right), \tag{1}$$

where $k \in \mathbf{N} \setminus \{0, 1\}$. Along this line, a fractional supersymmetric oscillator was derived in terms of boson and k-fermion operators in Ref. 9.

Fractional supersymmetric quantum mechanics was also developed without an explicit introduction of k-fermionic degrees of freedom [10, 11]. In this respect, FSSQM of order $k = 3$ was worked out by Quesne and Vansteenkiste [11] owing to the introduction of the C_λ-extended oscillator algebra. Their work is an extension of the construction by Plyushchay [10] of SSQM, viz., FSSQM of order $k = 2$, with one bosonic degree of freedom only.

The connection between FSSQM (and thus SSQM) and quantum groups has been worked out by several authors [12–20] mainly with applications to exotic statistics in mind. In particular, LeClair and Vafa [12] studied the isomorphism between the affine quantum algebra $U_q(sl_2)$ and $\mathcal{N} = 2$ FSSQM in $D = 1 + 1$ dimensions when q^2 goes to a root of unity ($\mathcal{N}$ is the number of supercharges); in the special case where $q^2 \to -1$, they recovered ordinary SSQM.

It is the aim of this paper to approach $\mathcal{N} = 2$ FSSQM of order k from different routes: (i) first, from a generalized Weyl–Heisenberg algebra W_k (defined in Sec. 1) and (ii) second, in terms of generalized bosonic and k-fermionic operators (Secs. 3 and 4). In Sec. 2, a fractional supersymmetric Hamiltonian is derived from the generators of W_k and specialized to the case of a fractional supersymmetric oscillator. In Sec. 4, this fractional supersymmetric oscillator is further investigated on the basis of a Q-uon approach to the algebra W_k, with Q going to a kth root of unity. Finally, differential realizations, involving bosonic and generalized Grassmannian variables, of FSSQM are given in Sec. 5 for some particular cases of W_k.

1. A GENERALIZED WEYL–HEISENBERG ALGEBRA W_k

1.1. The Algebra. For fixed k, with $k \in \mathbf{N} \setminus \{0, 1\}$, we define a generalized Weyl–Heisenberg algebra, denoted as W_k, as an algebra spanned by four linear operators X_- (annihilation operator), X_+ (creation operator), N (number operator) and K (grading operator) acting on some Hilbert space and satisfying the following relations:

$$[X_-, X_+] = \sum_{s=0}^{k-1} f_s(N)\, \Pi_s, \tag{2a}$$

$$[N, X_-] = -X_-, \quad [N, X_+] = +X_+, \tag{2b}$$

$$[K, X_+]_q = [K, X_-]_{\bar{q}} = 0, \tag{2c}$$

$$[K, N] = 0, \tag{2d}$$

$$K^k = 1, \tag{2e}$$

where q is the kth root of unity given by (1). In Eq. (2a), the f_s are reasonable functions (see below) and the operators Π_s are polynomials in K defined by

$$\Pi_s := \frac{1}{k} \sum_{t=0}^{k-1} q^{-st} \, K^t \tag{3}$$

for $s = 0, 1, \ldots, k-1$. Furthermore, we suppose that the operator K is unitary ($K^\dagger = K^{-1}$), the operator N is self-adjoint ($N^\dagger = N$), and the operators X_- and X_+ are connected via Hermitean conjugation ($X_-^\dagger = X_+$). The functions f_s: $N \mapsto f_s(N)$ must satisfy the constrain relation $f_s(N)^\dagger = f_s(N)$ (with $s = 0, 1, \ldots, k-1$) in order that $X_+ = X_-^\dagger$ be verified.

1.2. Projection Operators. It is clear that we have the resolution of the identity operator

$$\sum_{s=0}^{k-1} \Pi_s = 1$$

and the idempotency relation

$$\Pi_s \Pi_t = \delta(s, t) \Pi_s,$$

where δ is the Kronecker symbol. Consequently, the k self-adjoint operators Π_s are projection operators for the cyclic group $Z_k = \{1, K, \ldots, K^{k-1}\}$ of order k spanned by the generator K. In addition, these projection operators satisfy

$$\Pi_s X_+ = X_+ \Pi_{s-1} \Leftrightarrow X_- \Pi_s = \Pi_{s-1} X_- \tag{4}$$

with the convention $\Pi_{-1} \equiv \Pi_{k-1}$ and $\Pi_k \equiv \Pi_0$ (more generally, $\Pi_{s+kn} \equiv \Pi_s$ for $n \in \mathbf{Z}$). Note that Eq. (3) can be reversed in the form

$$K^t = \sum_{s=0}^{k-1} q^{ts} \, \Pi_s$$

with $t = 0, 1, \ldots, k-1$.

1.3. Representation. We now consider an Hilbertean representation of the algebra W_k. Let $\mathcal{F}$ be the Hilbert–Fock space on which the generators of W_k act. Since K obeys the cyclicity condition $K^k = 1$, the operator K admits the set $\{1, q, \dots, q^{k-1}\}$ of eigenvalues. It thus makes it possible to graduate, via a Z_k-grading, the representation space $\mathcal{F}$ of the algebra W_k as

$$\mathcal{F} = \bigoplus_{s=0}^{k-1} \mathcal{F}_s, \tag{5a}$$

where

$$\mathcal{F}_s := \{|kn + s\rangle : n \in \mathbf{N}\} \tag{5b}$$

with

$$K|kn + s\rangle = q^s |kn + s\rangle.$$

Therefore, to each eigenvalue q^s (with $s = 0, 1, \dots, k-1$) we associate a subspace $\mathcal{F}_s$ of $\mathcal{F}$. It is evident that

$$\Pi_s |kn + t\rangle = \delta(s, t)\, |kn + s\rangle$$

and, thus, the application Π_s: $\mathcal{F} \to \mathcal{F}_s$ yields a projection of $\mathcal{F}$ onto its subspace $\mathcal{F}_s$.

The action of $X_\pm$ and N on $\mathcal{F}$ can be taken to be

$$N|kn + s\rangle = n|kn + s\rangle$$

and

$$X_-|kn + s\rangle = \sqrt{F(n)}\, |k(n-1) + s - 1\rangle, \quad s \neq 0, \tag{6a}$$

$$X_-|kn\rangle = \sqrt{F(n)}\, |k(n-1) + k - 1\rangle, \quad s = 0, \tag{6b}$$

$$X_+|kn + s\rangle = \sqrt{F(n+1)}\, |k(n+1) + s + 1\rangle, \quad s \neq k - 1, \tag{6c}$$

$$X_+|kn + k - 1\rangle = \sqrt{F(n+1)}\, |k(n+1)\rangle, \quad s = k - 1. \tag{6d}$$

The function F is a structure function that fulfills the initial condition $F(0) = 0$ (see Refs. 21, 22). Furthermore, it satisfies

$$X_- X_+ = F(N + 1), \quad X_+ X_- = F(N)$$

and

$$F(N + 1) - F(N) = \sum_{s=0}^{k-1} f_s(N)\, \Pi_s$$

which admits the classical solution $F(N) = N$ for $f_s = 1$ $(s = 0, 1, \dots, k - 1)$.

1.4. Particular Cases. The algebra W_k covers a great number of situations encountered in the literature [9–11, 23, 24]. These situations differ by the form given to the right-hand side of (2a) and can be classified as follows.

(i) As a particular case, the algebra W_2 for $k = 2$ with

$$[X_-, X_+] = 1 + cK, \quad [N, X_\pm] = \pm X_\pm,$$

$$[K, X_\pm]_+ = 0, \quad [K, N] = 0, \quad K^2 = 1,$$

where c is a real constant ($f_0 = 1 + c$, $f_1 = 1 - c$), corresponds to the Calogero–Vasiliev [23] algebra considered by Gazeau [24] for describing a system of two anyons, with an $Sl(2, \mathbf{R})$ dynamical symmetry, subjected to an intense magnetic field and by Plyushchay [10] for constructing SSQM without fermions. Of course, for $k = 2$ and $c = 0$ we recover the algebra describing the ordinary or Z_2-graded supersymmetric oscillator.

If we define

$$c_s = \frac{1}{k} \sum_{t=0}^{k-1} q^{-ts}\, f_t(N), \tag{7}$$

with the functions f_t chosen in such a way that c_s is independent of N (for $s = 0, 1, \ldots, k-1$), the algebra W_k defined by

$$[X_-, X_+] = \sum_{s=0}^{k-1} c_s K^s, \tag{8}$$

together with Eqs. (2b)–(2e), corresponds to the C_λ-extented harmonic oscillator algebra introduced by Quesne and Vansteenkiste [11] for formulating FSSQM of order 3. The latter algebra was explored in great detail in the case $k = 3$ [11].

(ii) Going back to the general case where $k \in \mathbf{N} \setminus \{0, 1\}$, if we assume in Eq. (2a) that $f_s = G$ is independent of s with $G(N)^\dagger = G(N)$, we get

$$[X_-, X_+] = G(N). \tag{9}$$

We refer the algebra W_k defined by Eq. (9) together with Eqs. (2b)–(2e) to as a nonlinear Weyl–Heisenberg algebra. The latter algebra was considered by the authors as a generalization of the Z_k-graded Weyl–Heisenberg algebra describing a generalized fractional supersymmetric oscillator [9].

(iii) As a particular case, for $G = 1$ we have

$$[X_-, X_+] = 1, \tag{10}$$

and here we can take

$$N := X_+ X_-. \tag{11}$$

The algebra W_k defined by Eqs. (10) and (11) together with Eqs. (2b)–(2e) corresponds to the Z_k-graded Weyl–Heisenberg algebra connected to the fractional supersymmetric oscillator studied in Ref. 9.

(iv) Finally, it is to be noted that the affine quantum algebra $U_q(sl_2)$ with $q^k = 1$ can be considered as a special case (with the generators $J_- \equiv X_-$, $J_+ \equiv X_+$, $q^{-J_3} \equiv K^{-1}$, and $q^{+J_3} \equiv K$, where $J_3 \equiv N$) of the generalized Weyl–Heisenberg algebra W_k. This result is valid for all the representations (studied in Ref. 25) of the algebra $U_q(sl_2)$.

2. A GENERAL SUPERSYMMETRIC HAMILTONIAN

2.1. Supercharges. We are now in a position to introduce supercharges which are basic operators for the formulation of FSSQM. We define the supercharge operators Q_- and Q_+ by

$$Q_- := X_-(1 - \Pi_1), \tag{12a}$$

$$Q_+ := X_+(1 - \Pi_0), \tag{12b}$$

or alternatively

$$Q_- := X_-(\Pi_2 + \ldots + \Pi_{k-2} + \Pi_{k-1} + \Pi_0), \tag{13a}$$

$$Q_+ := X_+(\Pi_1 + \Pi_2 + \ldots + \Pi_{k-2} + \Pi_{k-1}). \tag{13b}$$

Indeed, we have here one of k, with $k \in \mathbf{N}\backslash\{0, 1\}$, possible equivalent definitions of the supercharges Q_- and Q_+ corresponding to the k circular permutations of the indices $0, 1, \ldots, k-1$. Obviously, we have the Hermitean conjugation relation $Q_-^\dagger = Q_+$. Thus, our approach corresponds to a $\mathcal{N} = 2$ formulation of FSSQM of order k ($\mathcal{N}/2$ is the number of independent supercharges). By making use of the commutation relations between the projection operators Π_s and the shift operators X_- and X_+ (see Eqs. (4)), we easily get

$$Q_-^m = X_-^m(\Pi_0 + \Pi_{m+1} + \Pi_{m+2} + \ldots + \Pi_{k-1}), \tag{14a}$$

$$Q_+^m = X_+^m(\Pi_1 + \Pi_2 + \ldots + \Pi_{k-m-1} + \Pi_{k-m}) \tag{14b}$$

for $m = 0, 1, \ldots, k-1$. By combining Eqs. (12) or (13) and (14), we obtain $Q_-^k = Q_+^k = 0$. Hence, the supercharge operators Q_- and Q_+ are nilpotent operators of order k.

We continue with some relations at the basis of the derivation of a supersymmetric Hamiltonian. The central relations are

$$Q_+Q_-^m = X_+X_-^m(1 - \Pi_m)(\Pi_0 + \Pi_{m+1} + \ldots + \Pi_{k-1}), \tag{15a}$$

$$Q_-^m Q_+ = X_-^m X_+ (1 - \Pi_0)(\Pi_m + \Pi_{m+1} + \ldots + \Pi_{k-1}) \tag{15b}$$

with $m = 0, 1, \ldots, k-1$. From Eqs. (15), we can derive the following identities giving $Q_-^m Q_+ Q_-^\ell$ with $m + \ell = k - 1$.

(i) We have

$$Q_+ Q_-^{k-1} = X_+ X_-^{k-1} \Pi_0, \tag{16a}$$

$$Q_-^{k-1} Q_+ = X_-^{k-1} X_+ \Pi_{k-1} \tag{16b}$$

in the limiting cases corresponding to $(m = 0, \ell = k-1)$ and $(m = k-1, \ell = 0)$.

(ii) Furthermore, we have

$$Q_-^m Q_+ Q_-^\ell = X_-^m X_+ X_-^\ell (\Pi_0 + \Pi_{k-1}) \tag{16c}$$

with the conditions $(m \neq 0, \ell \neq k-1)$ and $(m \neq k-1, \ell \neq 0)$.

2.2. The Hamiltonian. Following Rubakov and Spiridonov [1], we consider the multilinear relation

$$Q_-^{k-1} Q_+ + Q_-^{k-2} Q_+ Q_- + \ldots + Q_+ Q_-^{k-1} = Q_-^{k-2} H,$$

where H is an operator that depends on the algebra W_k. The operator H defines the Hamiltonian for a supersymmetric system associated to W_k. This dynamical system, that we shall refer to a fractional or Z_k-graded supersymmetric system, depends on the functions f_s occurring in the definition (1) of W_k. By repeated use of Eqs. (1) and (16), we find that the most general expression of H is

$$\begin{aligned} H = (k-1) X_+ X_- - \sum_{s=k-2}^{k} \sum_{t=2}^{s-1} (t-1)\, f_t(N - s + t) \Pi_s - \\ - \sum_{s=1}^{k-1} \sum_{t=s}^{k-1} (t-k)\, f_t(N - s + t) \Pi_s \end{aligned} \tag{17}$$

in terms of the product $X_+ X_-$, the operators Π_s and the functions f_s. In the general case, we can check that

$$H^\dagger = H \tag{18}$$

and

$$[H, Q_-] = [H, Q_+] = 0. \tag{19}$$

Equations (18) and (19) show that the two supercharge operators Q_- and Q_+ are two (nonindependent) constants of the motion for the Hamiltonian system described by the self-adjoint operator H. From Eqs. (17)–(19), it can be seen that the Hamiltonian H is a linear combination of the projection operators Π_s with coefficients corresponding to isospectral Hamiltonians (or supersymmetric partners) associated to the various subspaces $\mathcal{F}_s$ with $s = 0, 1, \ldots, k-1$.

2.3. Particular Cases. The general, expression (17) for the Hamiltonian H can be particularized to some interesting cases. These cases correspond to the above-mentioned forms of the generalized Weyl–Heisenberg algebra W_k.

(i) In the particular case $k = 2$, by taking $f_0 = 1 + c$ and $f_1 = 1 - c$, where c is a real constant, the Hamiltonian (17) gives back the one derived by Plyushchay [10] for SSQM.

More generally, by restricting the functions f_t in Eq. (17) to constants (independent of N) defined by

$$f_t = \sum_{s=0}^{k-1} q^{ts}\, c_s$$

in terms of the constants c_s (cf. Eq. (7)), the so-obtained Hamiltonian H corresponds to the C_λ oscillator fully investigated for $k = 3$ in Ref. 11.

(ii) In the case $f_s = G$ (independent of $s = 0, 1, \ldots, k-1$), i.e., for a generalized Weyl–Heisenberg algebra W_k defined by (2b)–(2e) and (9), the Hamiltonian H can be written as

$$H = (k-1)X_+X_- - \sum_{s=2}^{k-1}\sum_{t=1}^{s-1} G(N-t)(1 - \Pi_1 - \Pi_2 - \ldots - \Pi_s) + \\ + \sum_{s=1}^{k-1}\sum_{t=0}^{k-s-1} (k-s-t)G(N+t)\,\Pi_s. \quad (20)$$

The latter expression was derived in Ref. 9.

(iii) If $G = 1$, i.e., for a Weyl–Heisenberg algebra defined by (2b)–(2e) and (10), Eq. (20) leads to the Hamiltonian

$$H = (k-1)X_+X_- + (k-1)\sum_{s=0}^{k-1}\left(s + 1 - \frac{1}{2}k\right)\Pi_{k-s} \quad (21)$$

for a fractional supersymmetric oscillator. The energy spectrum of H is made of equally spaced levels with a ground state (singlet), a first excited state (doublet), a second excited state (triplet), ..., a $(k-2)$-th excited state ($(k-1)$-plet) followed by an infinite sequence of further excited states (all k-plets).

(iv) In the case where the algebra W_k is restricted to $U_q(sl_2)$, the corresponding Hamiltonian H is given by Eq. (17) where the f_t are simple coefficients. This yields

$$H = (k-1)J_+J_- + \frac{1}{\sin(2\pi/k)}\sum_{s=k-2}^{k}\sum_{t=2}^{s-1}(t-1)\,\sin\frac{4\pi t}{k}\Pi_s + \\ + \frac{1}{\sin(2\pi/k)}\sum_{s=1}^{k-1}\sum_{t=s}^{k-1}(t-k)\,\sin\frac{4\pi t}{k}\Pi_s. \quad (22)$$

Alternatively, Eq. (22) can be rewritten in the form (20) where $X_\pm \equiv J_\pm$ and $N \equiv J_3$ and where the function G is defined by

$$G(X) := -[2X]_q,$$

where the symbol $[\ ]_q$ is defined by

$$[2X]_q := \frac{q^{2X} - q^{-2X}}{q - q^{-1}}$$

with X an arbitrary operator or number. The quadratic term J_+J_- can be expressed in the term of the Casimir operator J^2 of $U_q(sl_2)$. Thus, the so-obtained expression for the Hamiltonian H is a simple function of J^2 and J_3.

3. A DEFORMED-BOSON + k-FERMION APPROACH TO FRACTIONAL SUPERSYMMETRY

3.1. A Deformed-Boson + k-Fermion Realization of W_k. In this section, the main tools consist of k pairs $(b(s)_-, b(s)_+)$ with $s = 0, 1, \ldots, k-1$ of deformed bosons and a pair (f_-, f_+) of k fermions. The operators $f_\pm$ satisfy

$$[f_-, f_+]_q = 1, \quad f_-^k = f_+^k = 0,$$

and the operators $b(s)_\pm$ satisfy the commutation relation

$$[b(s)_-, b(s)_+] = f_s(N), \tag{23}$$

where the functions f_s with $s = 0, 1, \ldots, k-1$ and the operator N occur in Eq. (2). In addition, the pairs (f_-, f_+) and $(b(s)_-, b(s)_+)$ are two pairs of commuting operators and the operators $b(s)_\pm$ commute with the projection operators Π_t with $s, t = 0, 1, \ldots, k-1$. We also introduce the linear combinations

$$b_- := \sum_{s=0}^{k-1} b(s)_-\, \Pi_s, \quad b_+ := \sum_{s=0}^{k-1} b(s)_+\, \Pi_s.$$

It is immediate to verify that we have the commutation relation

$$[b_-, b_+] = \sum_{s=0}^{k-1} f_s(N)\, \Pi_s, \tag{24}$$

a companion of Eq. (23).

We are now in a situation to find a realization of the generators X_-, X_+ and K of the algebra W_k in terms of the b's and f's. Let us define the shift operators X_- and X_+ by

$$X_- := b_- \left(f_- + \frac{f_+^{k-1}}{[[k-1]]_q!} \right), \tag{25}$$

$$X_+ := b_+ \left(f_- + \frac{f_+^{k-1}}{[[k-1]]_q!} \right)^{k-1}, \tag{26}$$

where the new symbol $[[\ ]]_q$ is defined by

$$[[X]]_q := \frac{1-q^X}{1-q}$$

with X an arbitrary operator or number and where the q-deformed factorial is given by $[[n]]_q! := [[1]]_q\, [[2]]_q\, \ldots\, [[n]]_q$ for $n \in \mathbf{N}^*$ (and $[[0]]_q! := 1$). It is also always possible to find a representation for which the relation $X_-^\dagger = X_+$ holds. Furthermore, we define the grading operator K by

$$K := [f_-, f_+]. \tag{27}$$

In view of the remarkable property

$$\left(f_- + \frac{f_+^{k-1}}{[[k-1]]_q!} \right)^k = 1,$$

we obtain

$$[X_-, X_+] = [b_-, b_+]. \tag{28}$$

Equations (24) and (28) show that Eq. (2a) is satisfied. It can be checked also that the operators X_-, X_+ and K satisfy Eqs. (2c) and (2e). Of course, Eqs. (2b) and (2d) have to be considered as postulates. However, note that the operator N is formally given in terms of the b's by

$$F(N+1) = b_- b_+ = \sum_{s=0}^{k-1} b(s)_- b(s)_+ \, \Pi_s,$$

$$F(N) = b_+ b_- = \sum_{s=0}^{k-1} b(s)_+ b(s)_- \, \Pi_s,$$

with the help of the structure function F introduced in Sec. 1. We thus have a realization of the generalized Weyl–Heisenberg algebra W_k by multilinear forms involving k pairs $(b(s)_-, b(s)_+)$ of deformed-boson operators ($s = 0, 1, \ldots, k-1$) and one pair (f_-, f_+) of k-fermion operators.

3.2. The Resulting Hamiltonian. The supercharges Q_- and Q_+ can be expressed by means of the deformed-bosons and k fermions. By using the identity

$$\Pi_s \left(f_- + \frac{f_+^{k-1}}{[[k-1]]_q!} \right)^n = \left(f_- + \frac{f_+^{k-1}}{[[k-1]]_q!} \right)^n \Pi_{s+n},$$

with $s = 0, 1, \ldots, k-1$ and $n \in \mathbf{N}$, Eqs. (12) can be rewritten as

$$Q_- = \left(f_- + \frac{f_+^{k-1}}{[[k-1]]_q!} \right) \sum_{s=1}^{k-1} b(s)_- \, \Pi_{s+1},$$

$$Q_+ = \left(f_- + \frac{f_+^{k-1}}{[[k-1]]_q!} \right)^{k-1} \sum_{s=1}^{k-1} b(s+1)_+ \, \Pi_s,$$

with the convention $b(k)_+ = b(0)_+$. Then, the supersymmetric Hamiltonian H given by Eq. (17) assumes the form

$$H = (k-1) \sum_{s=0}^{k-1} F_s(N) \Pi_s - \sum_{s=k-2}^{k} \sum_{t=2}^{s-1} (t-1) \, f_t(N-s+t) \Pi_s - \\ - \sum_{s=1}^{k-1} \sum_{t=s}^{k-1} (t-k) \, f_t(N-s+t) \Pi_s,$$

in terms of the operators $b(s)_\pm$, the projection operators Π_s (that may be written with k-fermion operators), the structure functions F_s and the structure constants f_s with $s = 0, 1, \ldots, k-1$.

4. THE FRACTIONAL SUPERSYMMETRIC OSCILLATOR

4.1. A Special Case of W_k. In this section, we deal with the particular case where $f_s = 1$ and the deformed bosons $b(s)_\pm \equiv b_\pm$ are independent of s with $s = 0, 1, \ldots, k-1$. We thus end up with a pair (b_-, b_+) of ordinary bosons, satisfying $[b_-, b_+] = 1$, and a pair (f_-, f_+) of k-fermions. The ordinary bosons $b_\pm$ and the k fermions $f_\pm$ may be considered as originating from the decomposition of a pair of Q-uons when Q goes to the root of unity q.

Here, the two operators X_- and X_+ are given by Eqs. (25) and (26), where now $b_\pm$ are ordinary boson operators. They satisfy the commutation relation $[X_-, X_+] = 1$. Then, the number operator N may be defined by

$$N := X_+ X_-, \tag{29a}$$

which is amenable to the form

$$N = b_+ b_-. \tag{29b}$$

Finally, the grading operator K is still defined by Eq. (27). We can check that the operators X_-, X_+, N and K so-defined generate the generalized Weyl–Heisenberg algebra W_k defined by Eq. (2) with $f_s = 1$ for $s = 0, 1, \ldots, k-1$. The latter algebra W_k can thus be realized with multilinear forms involving ordinary boson operators $b_\pm$ and k-fermion operators $f_\pm$.

4.2. The Resulting Fractional Supersymmetric Oscillator. The supercharge operators Q_- and Q_+ as well as the Hamiltonian H associated to the algebra W_k introduced in Sec. 3.2 (in terms of the operators b_-, b_+, f_- and f_+) can be constructed according to the prescriptions given in Sec. 2. This leads to the expression

$$H = (k-1) b_+ b_- + (k-1) \sum_{s=0}^{k-1} \left(s + 1 - \frac{1}{2} k \right) \Pi_{k-s}$$

to be compared with Eq. (21).

Most of the properties of the Hamiltonian H are essentially the same as the ones given above for the Hamiltonian (21). In particular, we can write

$$H = \sum_{m=1}^{k} H_m \Pi_m, \quad H_m := (k-1) \left(b_+ b_- + \frac{1}{2} k + 1 - m \right)$$

and thus H is a linear combination of projection operators with coefficients H_m corresponding to isospectral Hamiltonians (remember that $\Pi_k := \Pi_0$).

To close this section, let us mention that the fractional supercoherent state $|z, \theta)$ introduced in Ref. 9 is a coherent state corresponding to the Hamiltonian H. As a point of fact, we can check that the action of the evolution operator $\exp(-\mathrm{i}Ht)$ on the state $|z, \theta)$ gives

$$\exp(-\mathrm{i}Ht)\, |z, \theta) = \exp\left[-\frac{\mathrm{i}}{2} (k-1)(k+2) t \right] \, |\mathrm{e}^{-\mathrm{i}(k-1)t} z, \mathrm{e}^{+\mathrm{i}(k-1)t} \theta),$$

i. e., another fractional supercoherent state.

5. DIFFERENTIAL REALIZATIONS

In this section, we consider the case of the algebra W_k defined by Eqs. (2b)–(2e) and Eq. (8) with $c_0 = 1$ and $c_s = c\delta(s, 1)$, $c \in \mathbf{R}$, for $s = 1, 2, \ldots, k-1$. In other words, we have

$$[X_-, X_+] = 1 + cK, \quad K^k = 1, \tag{30a}$$

$$[K, X_+]_q = [K, X_-]_{\bar{q}} = 0, \tag{30b}$$

which corresponds to the C_λ-extended oscillator. The operators X_-, X_+, and K can be realized in terms of a bosonic variable x and its derivative d/dx satisfying $[d/dx, x] = 1$ and a k-fermionic variable (or generalized Grassmann variable) θ and its derivative $d/d\theta$ satisfying [1, 12] (see also Refs. 2–8)

$$\left[\frac{d}{d\theta}, \theta\right]_{\bar{q}} = 1, \quad \theta^k = \left(\frac{d}{d\theta}\right)^k = 0.$$

Of course, the sets $\{x, d/dx\}$ and $\{\theta, d/d\theta\}$ commute. It is a simple matter of calculation to derive the two following identities

$$\left(\frac{d}{d\theta} + \frac{\theta^{k-1}}{[[k-1]]_{\bar{q}}!}\right)^k = 1$$

and

$$\left(\frac{d}{d\theta}\theta - \theta\frac{d}{d\theta}\right)^k = 1,$$

which are essential for the realizations given below.

As a first realization, we can show that the shift operators

$$X_- = \frac{d}{dx}\left(\frac{d}{d\theta} + \frac{\theta^{k-1}}{[[k-1]]_{\bar{q}}!}\right)^{k-1} - \frac{c}{x}\theta, \quad X_+ = x\left(\frac{d}{d\theta} + \frac{\theta^{k-1}}{[[k-1]]_{\bar{q}}!}\right),$$

and the Witten grading operator

$$K = \left[\frac{d}{d\theta}, \theta\right]$$

satisfy Eqs. (30). This realization of X_-, X_+ and K clearly exibits the bosonic and k-fermionic degrees of liberty via the sets $\{x, d/dx\}$ and $\{\theta, /d/d\theta\}$, respectively. In the particular case $k = 2$, the k-fermionic variable θ is an ordinary Grassmann variable and the supercharge operators Q_- and Q_+ take the simple form

$$Q_- = \left(\frac{d}{dx} - \frac{c}{x}\right)\theta, \quad Q_+ = x\frac{d}{d\theta}. \tag{31}$$

(Note that the latter realization for Q_- and Q_+ is valid for $k = 3$, too.)

Another possible realization of X_- and X_+ for arbitrary k is

$$X_- = P\left(\frac{d}{d\theta} + \frac{\theta^{k-1}}{[[k-1]]_{\bar{q}}!}\right)^{k-1} - \frac{c}{x}\theta, \quad X_+ = X\left(\frac{d}{d\theta} + \frac{\theta^{k-1}}{[[k-1]]_{\bar{q}}!}\right),$$

where P and X are the two canonically conjugated quantities

$$P := \frac{1}{\sqrt{2}}\left(x + \frac{d}{dx} - \frac{c}{2x}K\right), \quad X := \frac{1}{\sqrt{2}}\left(x - \frac{d}{dx} + \frac{c}{2x}K\right).$$

This realization is more convenient for a Schrödinger type approach to the supersymmetric Hamiltonian H. According to Eq. (17), we can derive an Hamiltonian H involving bosonic and k-fermionic degrees of freedom. To illustrate this point, let us continue with the particular case $k = 2$. It can be seen that the supercharge operators (31) must be replaced by

$$Q_- = \left(P - \frac{c}{X}\right)\theta, \quad Q_+ = X\frac{d}{d\theta}.$$

(Note the formal character of Q_- since the definition of Q_- lies on the existence of an inverse for the operator X.) Then, we obtain the following Hamiltonian

$$H = -\frac{1}{2}\left[\left(\frac{d}{dx} - \frac{c}{2x}K\right)^2 - x^2 + K + c(1+K)\right].$$

For $c = 0$, we have

$$H = -\frac{1}{2}\frac{d^2}{dx^2} + \frac{1}{2}x^2 - \frac{1}{2}K$$

that is the Hamiltonian for an ordinary super-oscillator, i. e., a Z_2-graded supersymmetric oscillator. Here, the bosonic character arises from the bosonic variable x and the fermionic character from the ordinary Grassmann variable θ in K.

6. CONCLUDING REMARKS

A first facet of this work concerns an approach of $\mathcal{N} = 2$ FSSQM of order k ($k \in \mathbf{N} \setminus \{0, 1\}$) in $D = 1+1$ dimensions through a generalized Weyl–Heisenberg algebra W_k which is an extension of the Calogero–Vasiliev algebra [23]. We have seen how the algebra W_k is connected to the quantum algebra $U_q(sl_2)$ with $q^k = 1$. This approach of FSSQM, in the spirite of the pioneer works in Refs. 10,11, differs from the one developed in Refs. 1–7 via the introduction of two degrees of freedom, a bosonic one and a para-fermionic one. At first glance, our approach seems to be of an entirely bosonic character. However, the para-fermionic or k-fermionic character is hidden behind the (Klein–Witten) operator K. This operator ensures a Z_k-grading of the Hilbert space $\mathcal{F}$ of the physical states according to the decomposition $\mathcal{F} = \bigoplus_{s=0}^{k-1} \mathcal{F}_s$. The generators of W_k (and consequently of $U_q(sl_2)$) have been used for constructing a general fractional supersymmetric Hamiltonian H which is a linear combination of projection operators on the subspaces $\mathcal{F}_s$ ($s = 0, 1, \ldots, k-1$), the coefficients of which being isospectral Hamiltonians. The general Hamiltonian H covers the particular case of the fractional supersymmetric oscillator.

A second facet of this paper is devoted to a Q-uon approach of $\mathcal{N} = 2$ FSSQM of order k with Q going to $q = \exp(2\pi \mathrm{i}/k)$. The bosonic and k-fermionic degrees of freedom are present since the very beginning, a situation which parallels the *à la* Rubakov and Spiridonov [1, 2] construction of para-supersymmetric quantum mechanics. Indeed, the *Q-uon $\to$ boson + k-fermion decomposition* was obtained when $Q \rightsquigarrow q$ has been exploited for building a realization of W_k corresponding to the fractional supersymmetric oscillator. This approach of FSSQM is especially appropriate for deriving the fractional supercoherent states associated to this fractional supersymmetric oscillator. In addition, it is appropriate to the writing of supercharges and fractional supersymmetric Hamiltonians in terms of ordinary bosonic variables and generalized Grassmann variables, as shown with the specific differential realizations of Sec. 5.

The two approaches of FSSQM developed in this paper are obviously complementary. In this direction, it is to be emphasized that this work might be useful for generating isospectral Hamiltonians for exactly integrable potentials and for constructing their coherent states.

Finally, two comments of a group-theoretical nature are in order. First, we have shown here that supercharges and fractional supersymmetric Hamiltonians can be expressed from the generators of $U_q(sl_2)$, with q a kth root of unity, in a way independent of the representations chosen for the quantum algebra $U_q(sl_2)$. This approach is different from the one in Refs. 12, 15 where nilpotent and cyclic representations of $U_q(sl_2)$, with q^2 being a root of unity, are separately considered for an investigation of $\mathcal{N} = 2$ FSSQM in $D = 1 + 1$ dimensions. Second, the algebra $U_q(sl_2)$ has not to be confused with the algebra spanned by the supercharges Q_- and Q_+ and the Hamiltonian H. The latter algebra coincides with the Z_2-graded Lie algebra $sl(1/1)$ for $q = -1$, i. e., $k = 2$, in the case of $\mathcal{N} = 2$ SSQM. An open question is to find the algebra spanned by Q_-, Q_+, and H for $k \geq 3$ in the case of $\mathcal{N} = 2$ FSSQM. In another terminology, can $\mathcal{N} = 2$ FSSQM of order k be described by a q-deformed algebra (with $q^k = 1$) that gives back $sl(1/1)$ for $q = -1$? It is hoped that the results of this paper shall shed light on this question.

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