ФИЗИКА ЭЛЕМЕНТАРНЫХ ЧАСТИЦ И АТОМНОГО ЯДРА
2002. Т. 33. ВЫП. 7

УДК 539.12.01

ALGEBRAIC STRUCTURES ASSOCIATED TO NAMBU DYNAMICS

S. Codriansky

Departamento de Matemáticas y Física, Instituto Pedagógico de Caracas, Venezuela

The role of Clifford algebra in the geometric description of the Nambu dynamical system is discussed.

INTRODUCTION

The study of the geometric description of the dynamical system proposed by Nambu [4] some time ago has attracted attention lately. The developments along this line of research have concentrated in the construction of structures that are similar to the ones found in the geometric description of the Hamiltonian dynamical system. Phase space is m-dimensional (m even or odd). Let F be the set of smooth functions over phase space. A Nambu manifold is defined once a multilinear operation $\{A_1, \dots, A_n\}$ ($2 \le n \le m$) over F is postulated which satisfies

$$\begin{aligned}
&\{A_1, \dots, A_i, A_{i+1}, \dots, A_n\} = -\{A_1, \dots, A_{i+1}, A_i, \dots, A_n\},\\
&\{A_1, \dots, A_i + B_i, \dots, A_n\} =\\
&\qquad = \{A_1, \dots, A_i, \dots, A_n\} + \{A_1, \dots, B_i, \dots, A_n\},\\
&\{A_1, \dots, A_i B_i, \dots, A_n\} =\\
&\qquad = \{A_1, \dots, A_i, \dots, A_n\} B_i + A_i \{A_1, \dots, B_i, \dots, A_n\},\\
&\{\{A_1, \dots, A_n\}, B_2, \dots, B_n\} =\\
&\qquad = \sum_{i=1}^{n} \{A_1, \dots, A_{i-1}, \{A_i, B_2, \dots, B_n\}, A_{i+1}, \dots, A_n\}.
\end{aligned} \tag{1}$$

The last line in (1) is called the Fundamental Identity (FI); it is the generalization of the Jacobi identity to which it reduces when $n = 2$. The dynamical system proposed in [4] did not include FI as part of its definition; this identity was introduced in [6–8]. In [7] an algebraic description of the Nambu dynamical system is presented and called of type I if FI is not included and of type II if FI is included.

Remark: FI is identically satisfied only if there is a single multiplet in phase space; in this case the Nambu bracket is a Jacobian of order m

$$[F_1, \dots, F_m] = \frac{\partial(F_1, \dots, F_m)}{\partial(x_1, \dots, x_m)}. \tag{2}$$

If phase space is spanned by N multiplets of dimension S, FI is not an identity and is an extra condition to be satisfied.

The geometric study of the Nambu system has, however, overlooked some difficulties that follow from the antisymmetry of the exterior product. The source of this difficulty stems from the fact that if $m = 3N$ and phase space is considered as spanned by a set of N triplets, then the evolution equations of the coordinates $\mathbf{x} = (x_1, \dots, x_{3N})$ should be obtained from the canonical 3-form

$$\omega = \sum_{i=0}^{N-1} dx_{i+1} \wedge dx_{i+2} \wedge dx_{i+3}, \tag{3}$$

but due to the antisymmetry of the exterior product all powers of ω vanish. As a consequence, the Liouville condition $\sum \partial \dot{x}_i / \partial x_i = 0$ [2] which describes the invariance of the volume form cannot be related to ω. The requirement that the Nambu equations of motion be derived from

$$\mathbf{i}_{\mathbf{v}} \omega = dF \wedge dG, \tag{4}$$

where F and G are the generalized Hamilton functions, called in the sequel Nambu functions, is inconsistent since on the right-hand side there are terms that include $dx_i \wedge dx_j$, $i \neq j+1, j+2, j+3$, while on the left-hand side there are no such terms. These problems have been partially solved in [3] introducing a «partial diferential» $d_{(i)}$, $(d = d_{(1)} + \ldots + d_{(N)})$, which acts on the ith triplet indices $(i+1, i+2, i+3)$. The vector field is also considered as a sum of terms $v_{(i)}$ each of which is a sum of derivatives with respect to $(i+1, i+2, i+3)$; the «partial» vector field $v_{(i)}$ reproduces, after contraction with the fundamental 3-form, the Nambu dynamical equations for the ith triplet coordinates through

$$i_{\mathbf{v_{(i)}}}(\omega) = d_{(i)} H_1 \wedge d_{(i)} H_2. \tag{5}$$

The Lie derivative of $\omega_{(i)} = dx_{i+1} \wedge dx_{i+2} \wedge dx_{i+3}$ does not vanish and therefore neither the Lie derivative of the canonical 3-form. To compute higher powers the procedure in [3] goes as follows: define $\omega^2 = \omega_{(1)} \wedge \omega_{(2)}$ and so on until the volume form $\omega^N = \omega_{(1)} \wedge \omega_{(2)} \ldots \wedge \omega_{(N)}$ is obtained. It turns out that the Lie derivative of $\omega^{\mathbf{k}}$ is different from zero for $k = 1, \dots, N-1$ while it vanishes for $k = N$ as a consequence of the Liouville condition.

In order to recover the property that the different powers of the canonical 3-form do not vanish and that its powers define constants of the motion, a modification of the exterior product has been proposed in [5] and [1] where a partial reformulation of the differential operation, the contraction and the Lie derivation has been proposed with the result that a consistent scheme has emerged in the sense that the desired properties are present: all powers of the canonical form are nonzero, they define integral invariants and the maximum nonzero form is the volume form. The modification of the exterior product adds a symmetric part and this feature extends to all operations (details in Sec. 1).

In this paper a particular realization of the scheme constructed along the lines described previously is presented. This particular realization is accomplished defining a set of variables that are noncommutative. The noncommutativity is associated to a set of algebraic operators that satisfy the Clifford algebra commutation relations.

1. EXTENDED EXTERIOR CALCULUS

To proceed with the modification of the exterior calculus two sets of variables are introduced: those that span phase are $\mathbf{x} = ((x_1^\alpha, x_2^\alpha, x_3^\alpha),\ \alpha = 1, \ldots, N)$, where the upper index labels a particular triplet; and the lower, its place within the triplet; these variables are real and evolve in time according to the Nambu equations of motion [4]

$$\frac{dx_i^\alpha}{dt} = \frac{\partial(x_i^\alpha, F, G)}{\partial(x_1^\alpha, x_2^\alpha, x_3^\alpha)}, \tag{6}$$

where $\partial(\ldots)/\partial(\ldots)$ is a third order Jacobian; F and G are the Nambu functions for the dynamical system which are given as input data.

If the variables are grouped in N multiplets of order S ($m = NS$), then the Jacobians are of order S and $S-1$ Nambu functions have to be provided. The evolution equations for the phase space variables are

$$\frac{dx_i^\alpha}{dt} = \frac{\partial(x_i^\alpha, N_1, \ldots, N_{S-1})}{\partial(x_1^\alpha, \ldots, x_S^\alpha)} \tag{7}$$

and for an arbitrary function K

$$\frac{dK}{dt} = \sum_{\alpha=1}^{N} \frac{\partial(K, N_1, \ldots, N_{S-1})}{\partial(x_1^\alpha, \ldots, x_S^\alpha)} = \{K, N_1, \ldots, N_{S-1}\} \tag{8}$$

which defines the Nambu bracket $\{K_1, \ldots, K_S\}$. In the particular case of N triplets

$$\{A, B, C\} = \sum_{\alpha=1}^{N} \frac{\partial(A, B, C)}{\partial(x_1^\alpha, x_2^\alpha, x_3^\alpha)}. \tag{9}$$

Remark: the properties listed in (1) do not require that the dimension of the multiplets divide the dimension of phase space. If phase space is four-dimensional a terciary bracket can be constructed defining it as in (9) but summing over all four subsets of three variables.

A second set $\mathbf{y} = ((y_1^\alpha, y_2^\alpha, y_3^\alpha),\ \alpha = 1, \dots, N),\ y_i^\alpha = P^\alpha z_i^\alpha$, is introduced where the P^α satisfy

$$P^\alpha P^\beta + P^\beta P^\alpha = 2\delta^{\alpha\beta} \tag{10}$$

and the z_i^α are real variables; $\mathbf{z} = (z_1^1, z_2^1, z_3^1, \dots, z_3^N)$. The y_i^α are noncommutative; in fact, $y_i^\alpha y_j^\beta = (-1)^{\delta_{\alpha\beta}+1} y_j^\beta y_i^\alpha$. The manifold considered in this case is the set of functions over $R^{3N} \otimes C(N)$, where $C(N)$ is the Clifford algebra generated by the P^α.

An arbitrary function $F(\mathbf{y})$ is written in terms of the real variables $\mathbf{z}$ in the form $F(\mathbf{y}) = \sum_{\overline{A}} F_{\overline{A}}(\mathbf{z})\mathbf{P}^{\overline{\mathbf{A}}}$, where $F_{\overline{A}}(\mathbf{z})$ is a real function labelled by the multi-index $\overline{A} = (\alpha_1, \dots, \alpha_A)$ with $1 \le \alpha_i < \alpha_j \le N$ if $i < j$ (this defines strict ordering); $P^{\overline{A}} = P^{\alpha_1} \dots P^{\alpha_A}$ and if $A = 0$, $P^{\overline{0}}$ is the identity I. $\sum_{\overline{A}}$ is a sum over all $\binom{N}{A}$ sets of A α's and a sum over $A = 0, 1, \dots, N$. As a result $F(\mathbf{y})$ is represented by the set of 2^N real functions $(F_0, (F)_{\overline{1}}, (F)_{\overline{2}}, \dots, (F)_{\overline{N}})$, where $(F)_{\overline{A}}$ is the set of $\binom{N}{A}$ functions with a fixed value of A. It is in terms of the y_i^α that the modification of the exterior calculus will be realized.

The extended exterior product (eproduct) is defined for 1-forms (called extended 1-forms or eforms) θ^α, θ^β by

$$\theta^\alpha \,\overline{\wedge}\, \theta^\beta = (-1)^{\delta_{\alpha\beta}} \theta^\beta \,\overline{\wedge}\, \theta^\alpha, \tag{11}$$

where α and β are multiplet indices. It is straightforward to give a precise definition of this product antisymmetrizing and symmetrizing tensor products.

The basic 1-eforms are $\overline{d}y_i^\alpha = P^\alpha dz_i^\alpha$; for these (11) is clearly satisfied. The extended differential of a function $f(\mathbf{y})$ is defined by

$$\overline{d}f(\mathbf{y}) = \overline{\partial}_\alpha^i f(\mathbf{y}) \overline{d}y_i^\alpha = \sum_{\overline{A}} df_{\overline{A}}(\mathbf{z}) P^{\overline{A}} = \sum_{\overline{A}} \partial_\alpha^i f_{\overline{A}}(\mathbf{z}) dz_i^\alpha P^{\overline{A}}. \tag{12}$$

If $f(\mathbf{y})$ reduces to a function along the identity ($f_{\overline{A}} = 0$ if $A \neq 0$), it follows from (12)

$$\overline{\partial}_\alpha^i f(\mathbf{y}) = P^\alpha \partial_\alpha^i f_0(\mathbf{z}), \tag{13}$$

this relation between the two operations will be taken to be valid in general.

With these definitions it follows that $\overline{\partial}_\alpha^i \overline{\partial}_\beta^j f(\mathbf{y}) = (-1)^{\delta_{\alpha\beta}+1} \overline{\partial}_\beta^j \overline{\partial}_\alpha^i f(\mathbf{y})$ and $\overline{d}(\overline{d}f(\mathbf{y})) = 0$. To prove the relation between the second partials consider a monomial $(y_i^\alpha)^\mu (y_j^\beta)^\nu$; then

$$\begin{aligned}\overline{\partial}^i_\alpha(\overline{\partial}^j_\beta(y_i^\alpha)^\mu(y_j^\beta)^\nu) = \mu\nu(-1)^{\mu(\delta_{\alpha\beta}+1)}(y_i^\alpha)^{\mu-1}(y_j^\beta)^{\nu-1},\\ \overline{\partial}^j_\beta(\overline{\partial}^i_\alpha(y_i^\alpha)^\mu(y_j^\beta)^\nu) = \mu\nu(-1)^{(\mu-1)(\delta_{\alpha\beta}+1)}(y_i^\alpha)^{\mu-1}(y_j^\beta)^{\nu-1}\end{aligned} \tag{14}$$

which proves the result for this particular monomial; if other factors multiply it, the result is the same after moving $(y_i^\alpha)^\mu (y_j^\beta)^\nu$ to the left and finally, by linearity the result follows for an arbitrary function which is represented by a series expansion. The same is obtained when the computation is performed with $\mathbf{z}$ instead of $\mathbf{y}$. Once this is proved, $\overline{dd} = 0$ follows after acting on a function $f(\mathbf{y})$

$$\overline{d}(\overline{d}f(\mathbf{y})) = \overline{\partial}^i_\alpha \overline{\partial}^j_\beta f(\mathbf{y}) \overline{d}y_j^\beta \,\overline{\wedge}\, \overline{d}y_i^\alpha \tag{15}$$

again, the same follows using $\mathbf{z}$ instead of $\mathbf{y}$.

Consider two vector fields $U^i_\alpha = f(\mathbf{y})\overline{\partial}^i_\alpha$, $V^j_\beta = g(\mathbf{y})\overline{\partial}^j_\beta$ with $f(\mathbf{y})$ and $g(\mathbf{y})$ functions along the identity. These satisfy, when acting on a function $F = F(\mathbf{y})$,

$$\begin{aligned}[V^j_\beta, U^i_\alpha]F &= V^j_\beta(U^i_\alpha(F)) - (-1)^{\delta_{\alpha\beta}+1}U^i_\alpha(V^j_\beta(F))\\ &= P^\beta P^\alpha \left(g\partial^j_\beta f \partial^i_\alpha - f\partial^i_\alpha g \partial^j_\beta\right) F\end{aligned} \tag{16}$$

which shows that the vector fields satisfy an algebra defined by the bracket

$$[V^j_\beta, U^i_\alpha]F = V^j_\beta(U^i_\alpha(F)) - (-1)^{\delta_{\alpha\beta}+1}U^i_\alpha(V^j_\beta(F)). \tag{17}$$

If $\alpha = \beta$, the bracket is a commutator. In the general case $f(\mathbf{y}) = f_{\overline{A}}(\mathbf{z})P^{\overline{A}}$, $g(\mathbf{y}) = g_{\overline{B}}(\mathbf{z})P^{\overline{B}}$, $\overline{A} = (a_1, \ldots, a_A)$, $\overline{B} = (b_1, \ldots, b_B)$ with $\overline{A}$ and $\overline{B}$ fixed multi-indices; then the actions of U^i_α and V^j_β on $F(\mathbf{y}) = F_{\overline{C}}(\mathbf{z})P^{\overline{C}}$ are

$$\begin{aligned}V^j_\beta[U^i_\alpha(F)] = g_{\overline{B}}\partial^j_\beta(f_{\overline{A}}\partial^i_\alpha F_{\overline{C}})P^{\overline{B}}P^\beta P^{\overline{A}}P^\alpha P^{\overline{C}},\\ U^i_\alpha[V^j_\beta(F)] = f_{\overline{A}}\partial^i_\alpha(g_{\overline{B}}\partial^j_\beta F_{\overline{C}})P^{\overline{A}}P^\alpha P^{\overline{B}}P^\beta P^{\overline{C}}.\end{aligned} \tag{18}$$

The commutation of $P^{\overline{A}}$ and $P^{\overline{B}}$ gives

$$P^{\overline{A}}P^{\overline{B}} = (-1)^{\sigma(\overline{AB})}P^{\overline{B}}P^{\overline{A}}, \tag{19}$$

where $\sigma(\overline{AB}) = \sum_{i=1}^A \sum_{j=1}^B (\delta_{a_i b_j} + 1)$. With this result it follows that

$$\begin{aligned}[V^j_\beta, U^i_\alpha]F &= V^j_\beta[U^i_\alpha(F)] - (-1)^{\sigma(\overline{AB})+\sigma(\alpha\overline{B})+\sigma(\beta\overline{A})+\sigma(\alpha\beta)}U^i_\alpha[V^j_\beta(F)]\\ &= P^{\overline{B}}P^\beta P^{\overline{A}}P^\alpha \left(g_{\overline{B}}\partial^j_\beta f_{\overline{A}}\partial^i_\alpha - f_{\overline{A}}\partial^i_\alpha g_{\overline{B}}\partial^j_\beta\right) F.\end{aligned} \tag{20}$$

The generalized bracket (20) is a commutator if $\sigma(\overline{AB}) + \sigma(\alpha\overline{B}) + \sigma(\beta\overline{A})+$ $\sigma(\alpha\beta) = 0$ (mod 2), which holds if $P^{\overline{A}}P^{\alpha}P^{\overline{B}}P^{\beta} -$ $P^{\overline{B}}P^{\beta}P^{\overline{A}}P^{\alpha} = 0$.

The contraction is defined by $\bar{\mathrm{i}}_{v_\gamma} = P^{\gamma}\mathrm{i}_{v_\gamma}$ so that

$$\bar{\mathrm{i}}_v(\overline{d}y_i^{\alpha}) = v_i^{\alpha}(\mathbf{z}) \tag{21}$$

and when acting with $\bar{\mathrm{i}}_v$ on a 2-eform

$$\bar{\mathrm{i}}_{v_\gamma}(\overline{d}y_i^{\alpha}\,\overline{\wedge}\,\overline{d}y_j^{\beta}) = \bar{\mathrm{i}}_{v_\gamma}(\overline{d}y_i^{\alpha})\,\overline{\wedge}\,\overline{d}y_j^{\beta} + (-1)^{\delta_{\alpha\gamma}}\overline{d}y_i^{\alpha}\,\overline{\wedge}\,\bar{\mathrm{i}}_{v_\gamma}(\overline{d}y_j^{\beta}). \tag{22}$$

It is seen that it is necessary to define the action of the extended contraction, or econtraction, on up to a 2-eform. In the same way as before, the action of the contraction on a 1-eform $\theta = \theta_{\alpha}^{\overline{A}i}P^{\overline{A}}\overline{d}x_i^{\alpha}$ with the vector field $\mathbf{v}_\gamma$ includes the sign $(-1)^{\sigma(\gamma\overline{A})}$ in each of the summands in θ.

2. THE VECTOR FIELD FOR NAMBU DYNAMICS

The determination of the vector field $\mathbf{v}$ is done once the canonical 3-eform $\omega = \sum_{\alpha} P^{\alpha}dz_1^{\alpha} \wedge dz_2^{\alpha} \wedge dz_3^{\alpha}$ is given. It has to satisfy

$$\bar{\mathrm{i}}_v\omega = \frac{1}{2}(\overline{d}H\,\overline{\wedge}\,\overline{d}G - \overline{d}G\,\overline{\wedge}\,\overline{d}H) \tag{23}$$

and its components are obtained requiring that the summands in the right-hand side of (23), that involve $dx_i^{\alpha} \wedge dx_j^{\beta}$ with $\alpha \neq \beta$, vanish. Using $\overline{d}H = \partial_{\alpha}^{i}H_{\overline{A}}dz_i^{\alpha}P^{\overline{A}}$, $\overline{d}G = \partial_{\beta}^{j}G_{\overline{B}}dz_j^{\beta}P^{\overline{B}}$ it is found

$$\frac{1}{2}(\overline{d}H\,\overline{\wedge}\,\overline{d}G - \overline{d}G\,\overline{\wedge}\,\overline{d}H) = \frac{1}{2}(P^{\overline{A}}P^{\overline{B}} + P^{\overline{B}}P^{\overline{A}})\partial_{\alpha}^{i}H_{\overline{A}}\partial_{\beta}^{j}G_{\overline{B}}dz_i^{\alpha} \wedge dz_j^{\beta}. \tag{24}$$

The particular case of two triplets is exhibited as an illustration. In this case

$$\begin{aligned} H &= H_0 + H_1P^1 + H_2P^2 + H_{12}P^1P^2, \\ G &= G_0 + G_1P^1 + G_2P^2 + G_{12}P^1P^2 \end{aligned} \tag{25}$$

which leads to

$$\begin{aligned} &\frac{1}{2}(\overline{d}H\,\overline{\wedge}\,\overline{d}G - \overline{d}G\overline{\wedge}\overline{d}H) = \\ &\quad = [(\partial_{\alpha}^{i}H_0\partial_{\beta}^{j}G_0 + \partial_{\alpha}^{i}H_1\partial_{\beta}^{j}G_1 + \partial_{\alpha}^{i}H_2\partial_{\beta}^{j}G_2 - \partial_{\alpha}^{i}H_{12}\partial_{\beta}^{j}G_{12}) + \\ &\quad +P^1(\partial_{\alpha}^{i}H_0\partial_{\beta}^{j}G_1 + \partial_{\alpha}^{i}H_1\partial_{\beta}^{j}G_0) + P^2(\partial_{\alpha}^{i}H_0\partial_{\beta}^{j}G_2 + \partial_{\alpha}^{i}H_2\partial_{\beta}^{j}G_0) + \\ &\qquad + P^1P^2(\partial_{\alpha}^{i}H_0\partial_{\beta}^{j}G_{12} + \partial_{\alpha}^{i}H_{12}\partial_{\beta}^{j}G_0)]dz_i^{\alpha} \wedge z_j^{\beta} \end{aligned} \tag{26}$$

which for $\alpha \neq \beta$ gives the following set of equations

$$\partial^i_\alpha H_0 \partial^j_\beta G_0 - \partial^j_\beta H_0 \partial^i_\alpha G_0 + \partial^i_\alpha H_1 \partial^j_\beta G_1 - \partial^j_\beta H_1 \partial^i_\alpha G_1 + \partial^i_\alpha H_2 \partial^j_\beta G_2 - \\ - \partial^j_\beta H_2 \partial^i_\alpha G_2 - \partial^i_\alpha H_{12} \partial^j_\beta G_{12} + \partial^j_\beta H_{12} \partial^i_\alpha G_{12} = 0, \quad (27)$$

$$\partial^i_\alpha H_0 \partial^j_\beta G_1 - \partial^j_\beta H_0 \partial^i_\alpha G_1 + \partial^i_\alpha H_1 \partial^j_\beta G_0 - \partial^j_\beta H_1 \partial^i_\alpha G_0 = 0, \quad (28)$$

$$\partial^i_\alpha H_0 \partial^j_\beta G_{12} - \partial^j_\beta H_0 \partial^i_\alpha G_{12} + \partial^i_\alpha H_{12} \partial^j_\beta G_0 - \partial^j_\beta H_{12} \partial^i_\alpha G_0 = 0, \quad (29)$$

$$\partial^i_\alpha H_0 \partial^j_\beta G_1 - \partial^j_\beta H_0 \partial^i_\alpha G_1 + \partial^i_\alpha H_1 \partial^j_\beta G_0 - \partial^j_\beta H_1 \partial^i_\alpha G_0 = 0. \quad (30)$$

It is possible to take $H_1 = H_2 = G_1 = G_2 = 0$ and consider the functions H_0 and G_0 as given. The various derivatives of H_{12} and G_{12} are determined by the above set of equations. Replacing this result in the set obtained by equating α and β it is found that the unique form of the components of the vector field is

$$v^\alpha_i = \frac{\partial(H_0, G_0)}{\partial(x^\alpha_j, x^\alpha_k)} \quad (31)$$

$(i,\ j,\ k)$ cyclic. The result is that the vector field is along the identity and its components are precisely the ones corresponding to the Nambu dynamical system. The Nambu vector fields satisfy, therefore, a Lie algebra and moreover, they do not take a particular $P^{\overline{A}}$ out of $C(A)$. The action of the Nambu vector field separates the set of monomials with a fixed value of A in a subspace that is invariant under evolution and under the action of the orthogonal group in the Clifford space. The subset $R^{3N} \otimes C(A)$ corresponds to a leaf of the Nambu dynamical system; its dimension is $3N\binom{N}{A}$.

Acknowledgments. It is a pleasure to thank the Organizing Committee of the IX International Conference on Symmetry Methods in Physics and Vicerrectorado de Investigación y Posgrado Universidad Pedagógica Experimental Libertador for financial support.

REFERENCES

1. *Codriansky S., Navarro R., Gonzalez C.* // Proc. of the XXIII Colloquium on Group Theoretical Methods in Physics, Dubna, 2000. To appear.
2. *Estabrook F. B.* // Phys. Rev. D. 1973. V. 8. P. 2740.
3. *Fecko M.* // J. Math. Phys. 1992. V. 33. P. 926; 930.
4. *Nambu Y.* // Phys. Rev. D. 1973. V. 7. P. 2405.

5. *Navarro R., Codriansky S., Gonzalez C.* // Proc. of Dynamic Systems and Applications. 2001. V. 3. P. 453.

6. *Ruggeri G. J.* // Int. J. Theor. Phys. 1975. V. 12. P. 287.

7. *Sahoo D., Valsakumar M. C.* // Phys. Rev. A. 1992. V. 46. P. 4410.

8. *Takhtajan L.* // Commun. Math. Phys. 1994. V. 160. P. 295.