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RECURRENCE TIMES IN DYNAMICAL SYSTEMS VIA A QUASICRYSTAL APPROACH

R. Twarock

Department of Mathematics, City University, London, UK

Integrable Hamiltonian system can be characterized via the recurrence time spectra of their Poincaré maps. We show here that the gaps in the recurrence time spectra can be determined via techniques from the theory of cut-and-project quasicrystals and we thus derive new results for the gap-problem.

INTRODUCTION

A standard tool in the analysis of the orbit structure of nonlinear dynamical systems are Poincaré maps (first return maps) [1]. These are discrete maps which arise via a restriction of the continuous system to a suitably chosen hypersurface in phase space.

In the case of integrable Hamiltonian systems with two degrees of freedom and two integrals of motion, it is known that the phase space trajectories lie on a two-dimensional manifold which is topologically equivalent to a 2-torus, and the flow on the surface of these tori can be analyzed via a Poincaré map with domain of definition (Poincaré section) given by the manifold S^1, that is a circle. Quasiperiodic solutions are characterized by the fact that they cut the manifold S^1 densely, whereas periodic solutions cut it only a finite number of times.

A typical Poincaré map is

$$T_\beta x = (x + \beta) \bmod 1, \quad \beta \in R, \tag{1}$$

which with the choice $x_0 = 0$ as a starting value leads to the following sequence of points

$$x_n = n\beta \bmod 1 \tag{2}$$

on the Poincaré section S^1.

It has been suggested [6] to use this sequence as a means to characterize the Hamiltonian system it is associated to, in particular, to characterize a general integrable Hamiltonian system by the recurrence time spectrum of its Poincaré map.

Definition 1. *Let $x_n = n\beta \bmod 1$ with $x_0 = 0$ be the sequence associated to an orbit $x(t)$, $t \geq 0$, with $x(0) = 0$ of a Hamiltonian system.*

For any interval $I \subset [0,1)$, we call

$$N_\beta(I) := \{n \in N | n\beta \bmod 1 \in I\} \tag{3}$$

*the set of **recurrence times** or the **recurrence times spectrum** of $x(t)$ in I.*

Furthermore, we call

$$G_\beta(I) := \{\Delta_i = m_{i+1} - m_i | m_i \in N_\beta(I)\} \tag{4}$$

*the **gaps in the recurrence time spectrum**.*

For any interval $I = [a, a+d) \subset [0,1)$ and β irrational it is known (for example [2, 8, 9]) that one has at most three different recurrence times («3-gap theorem») given by:

$$\begin{array}{rcl} \Delta_1 & = & \min\limits_{n \in N} \{n\beta \bmod 1 < d\}, \\ \Delta_2 & = & \min\limits_{n \in N} \{n\beta \bmod 1 > 1 - d\}, \\ \Delta_3 & = & \Delta_1 + \Delta_2 \quad \text{not always realized.} \end{array} \tag{5}$$

This result has been generalized [3, 4] to *disconnected* intervals of the form $I = [0,a) \cup (b,1)$, $0 < a < b < 1$, thus proving that at most three different gaps exist in the recurrence time spectrum for any connected subset $I \subset S^1$.

It is shown here that the gap problem for recurrence times is related to a problem in the theory of cut-and-project quasicrystals and that techniques developed in the theory of cut-and-project quasicrystals can successfully be implemented to derive new results for the gap-problem. In particular, apart from providing a simple alternative proof for known results, we obtain in this way

• Information on the sequence of recurrence times other than the nearest-neighbour distances.

• A generalization to higher dimensional Hamiltonian systems, for which at present only numerical results exist. For the gaps $G_\beta(I^k)$ in the sequence (T-denoting transposition)

$$x_n = n(\beta_1, \beta_2, \ldots, \beta_k)^T \bmod 1, \quad \beta_j \text{ irrational} \tag{6}$$

in an arbitrary convex set I^k in the unit cube in R^k, numerical results for $k \leq 6$ suggest [7] the existence of $k+1$ basic gaps from which all other can be obtained by integer-valued linear combinations. An analytical proof for this fact is provided in [10] for a large set of irrational numbers using the correspondence between cut-and-project quasicrystals and recurrence times, which is given in this contribution.

1. CUT-AND-PROJECT QUASICRYSTALS AS MODELS FOR APERIODIC POINT SETS

To each irrationality which is a *Pisot number*, we can associate an aperiodic point set or cut-and-project quasicrystal.

Definition 2. *A* ***Pisot number*** *is a real solution x of an algebraic equation with integer coefficients, such that $|x| > 1$ and $|x_j| < 1$ for all other solutions x_j.*

An example for a Pisot number is *the golden mean* $\tau = (1/2)(1 + \sqrt{5})$. Together with $\tau' := (1/2)(1 - \sqrt{5})$, it is a solution of

$$x^2 = x + 1 \tag{7}$$

with $|\tau| > 1$ and $|\tau'| < 1$. Using the ring of integers in the algebraic extension of the rational numbers $Q[\sqrt{5}]$, that is

$$Z[\tau] := \{a + \tau b | a, b \in Z\}, \tag{8}$$

and the Galois-automorphism in $Q[\sqrt{5}]$, that is

$$' : c + \sqrt{5}d \mapsto c - \sqrt{5}d \ \text{ for } c, d \in Q, \tag{9}$$

which restricted to $Z[\tau]$ gives

$$' : a + \tau b \mapsto a + \tau' b \ \text{ for } a, b \in Z, \tag{10}$$

we obtain

Definition 3. *A one-dimensional* ***cut-and-project quasicrystal*** *related to the irrationality τ is the aperiodic point set*

$$\Sigma(\Omega) := \{x \in Z[\tau] | x' \in \Omega\},$$

where Ω is a bounded interval in R, called ***acceptance window****.*

Ω controls how many points are admitted in the set. For example, if $\Omega = [0, 1]$, the condition imposed by the acceptance window implies

$$\Sigma([0,1]) := \left\{ \left[1 + \frac{b}{\tau} \right] + b\tau | b \in Z \right\} \cup \{0\}. \tag{11}$$

The points nearest to the origin 0 are:

$$\ldots, -\tau, 0, 1, 1 + \tau, 2 + 2\tau, 2 + 3\tau, 3 + 4\tau, \\ 4 + 5\tau, 4 + 6\tau, 5 + 7\tau, 5 + 8\tau, 6 + 9\tau, \ldots \tag{12}$$

and it is easy to see from (11) or (12) that the distances between adjacent points are of the length 1, τ or τ^2, where 1 is an exceptional tile which occurs precisely once between 0 and 1.

2. QUASICRYSTAL LANGUAGE FOR RECURRENCE TIMES

We connect here the quasicrystal picture with the recurrence time spectrum of the sequence $x_n = \tau n \bmod 1$.

Definition 4. *For $x = a + \tau b \in \Sigma(I)$ we call b the τ* ***component*** *of x.*

Then we have:

Lemma 1. *The τ components of the nearest-neighbour distances in $\Sigma(I)$ correspond to the gaps $G_\tau(I)$ in the recurrence time spectrum $N_\tau(I)$ of $x_n = n\tau \bmod 1$ for any interval $I \subset [0,1]$.*

Proof. Due to $\tau + \tau' = 1$ we have

$$\begin{aligned} n\tau \bmod 1 &= n(1-\tau') \bmod 1 \\ &= -n\tau' \bmod 1\,. \end{aligned} \tag{13}$$

Thus there exists $a \in N_0$ with $0 \leq -a - \tau' n < 1$ such that

$$n\tau \bmod 1 = -a - \tau' n. \tag{14}$$

Since $0 \leq -a - \tau' n < 1 \Leftrightarrow 0 \geq a + \tau' n > -1$, we obtain furthermore

$$\begin{aligned} \{n\tau \bmod 1 | n \in N_0\} =& \{a + \tau' n | n, a \in N_0 \text{ with } a + \tau' n \in (-1, 0]\} \\ =& (\Sigma((-1,0])_{N_0})'. \end{aligned} \tag{15}$$

Here, the index N_0 at $\Sigma((-1,0])$ means a restriction to quasicrystal points $x = a + \tau b$ with a, $b \in N_0$, and $'$ denotes conjugation of the points in $\Sigma((-1,0])_{N_0}$ under the (Galois-)automorphism.

Similarly, for any (not necessarily connected) $I \subset [0,1)$ we have

$$\begin{aligned} &\{n\tau \bmod 1 | n \in N_0\} \cap I \\ =& \{a + \tau' n | a, n \in N_0, a + \tau' n \in -I\} \\ =& (\Sigma(-I)_{N_0})'. \end{aligned} \tag{16}$$

In order to obtain the gaps in the recurrence time spectrum $N_\tau(I)$, we use that $\Sigma(-I)_{N_0} = -\Sigma(I)_{N_0}$, which implies that the nearest-neighbour distances in these two cut-and-project quasicrystals coincide and thus have the same τ components.

Finally, the fact that the τ components of the nearest-neighbour distances of the quasicrystal points in $\Sigma(I)_{N_0}$ correspond to the gaps in the recurrence time spectrum is a consequence of the fact that for any $I \subset [0,1)$ and $a + \tau b$, $c + \tau d \in \Sigma(I)$ we have

$$a + \tau b < c + \tau d \Leftrightarrow b < d. \tag{17}$$

■

In particular, any information about the sequence of tiles in $\Sigma(I)$ automatically relates to an information on the corresponding sequence of recurrence times.

Based on Lemma 1, it is possible to compute the gaps in the recurrence time spectrum for two-dimensional Hamiltonian systems related to the irrationality τ. For this, we will need the following results on the minimal distances, that is the nearest-neighbour distances, in cut-and-project quasicrystals:

If $\Sigma(\Omega)$ denotes a quasicrystal with open acceptance window Ω of length d with $d \in (\tau^{k-1}, \tau^k]$ then we have [5]:

• Information on the number and shape of minimal distances:

— The minimal distance in $\Sigma(\Omega)$ is equal to τ^{1-k}.

— If $d \neq \tau^k$, then $\Sigma(\Omega)$ has three types of tiles of length τ^{1-k}, τ^{2-k}, and τ^{3-k}.

— If $d = \tau^k$, then the quasicrystal $\Sigma(\Omega)$ has only two types of tiles with lengths τ^{1-k} and τ^{2-k}. (An exception occurs if Ω has boundary points in $Z[\tau]$. Then there is one exceptional tile of length τ^{3-k}.)

• Information on the sequence formed by the minimal distances:

— In $\Sigma(\Omega)$, two tiles with the length equal to the minimal distance are never adjacent.

— In $\Sigma(\Omega)$, a nontrivial string formed only by tiles of the type τ^{2-k} has length at most two and occurs if $d \in (2\tau^{k-2}, \tau^k]$.

— In $\Sigma(\Omega)$, a nontrivial string formed only by tiles of the type τ^{3-k} has length at most two and occurs if $d \in (\tau^{k-1}, 2\tau^{k-2})$.

Then, using furthermore

Definition 5. *The sequence A_k given by*

$$A_{k+2} = A_{k+1} + A_k, \quad A_1 = 1, \quad A_0 = 0$$

*is called the **Fibonacci sequence**,*
we obtain

Theorem. *For any interval $I = [a, a+d) \subset [0,1]$ the sequence $x_n = n\tau \bmod 1$, $n \in N_0$, has at most three different gaps in the recurrence time spectrum $G_\tau(I)$ which are given in dependence on d.*

In particular, for $d \in (\tau^{k-1}, \tau^k]$ with $k \in -N_0$ these are A_{1-k}, A_{2-k} and A_{3-k}, where A_k denotes the Fibonacci sequence. If $d \neq \tau^k$, all three are realized; for $d = \tau^k$, A_{3-k} does not appear.

Proof. According to Lemma 1, the recurrence times correspond exactly to the tiles in $\Sigma((a, a+d))$. Furthermore, due to the results on the tiles of cut-and-project quasicrystals (3), these are

— for $d \in (\tau^{k-1}, \tau^k]$ with $d \neq \tau^k$: τ^{1-k}, τ^{2-k}, and τ^{3-k};

— for $d = \tau^k$: τ^{1-k} and τ^{2-k}.

Then, using that $\tau^k = A_k\tau + A_{k-1}$ where A_k denotes the Fibonacci sequence, the claim follows. ∎

This result is generalized to disconnected intervals in the following

Lemma 2. *The recurrence times for* $x_n = n\tau \bmod 1$ *in the interval* $I = [0,a) \cup (b,1) \subset [0,1)$, $0 < a < b < 1$, *and* $\hat{I} = (b-1,a)$ *coincide, that is*

$$G_\tau(I) = G_\tau(\hat{I}). \tag{18}$$

Proof. Since $\Sigma(\Omega+1) = 1+\Sigma(\Omega)$, $\Sigma(b-1,a)$ differs from $\Sigma([0,a)\cup(b,1))$ by the fact that some of its points are translated by 1. Thus, calling S, M, and L the three minimal distances of $\Sigma((b-1),a)$, it follows that $\Sigma([0,a)\cup(b,1))$ has minimal distances in the set $\{S, M, L, S\pm 1, M\pm 1, L\pm 1\}$.

Since the addition or subtraction of 1 to or from the length of a minimal distance does not change the τ component of the minimal distance, the claim follows. ∎

With this, it has been shown that for any connected subset I of S^1 there exist at most three gaps in the recurrence time spectrum of $x_n = n\tau \bmod 1$ in I, and that their size depends on the size of I.

On top of the *types of tiles*, one also obtains *information on the tiling sequence* from the quasicrystal approach:

1. The results of Florek and Slater state that the sum $\Delta_1 + \Delta_2$ may appear as a third gap in the recurrence time spectrum. From the quasicrystal approach we know exactly *under which condition* it happens. It is the case precisely if the length of the interval representing the Poincaré section is given by $d \neq \tau^k$.

2. Since the sequence of recurrence times corresponds to the τ component of the quasicrystal points in consecutive order, information on the sequence of minimal distances translates one-to-one into information about the recurrence times. Denoting as S (=small), M (=medium), and L (=large) the three minimal distances in the quasicrystal, or, the three gaps in the recurrence time sequence, we have

(a) The smallest entry S does never occur twice in a row.

(b) The medium entry M occurs at most twice in a row, and only if $d \in (2\tau^{k-2}, \tau^k]$.

(c) The largest entry L (sum of S and M) occurs at most twice in a row and only if $d \in (\tau^{k-1}, 2\tau^{k-2})$.

3. GENERALIZATIONS

The toy model situation in the previous section, which is characterized by a restriction to the irrationality τ and to two-dimensional Hamiltonian systems, allows for various generalizations. In particular, more general classes of irrational numbers (Pisot numbers) can be treated in two dimensions along similar lines, and furthermore the setting can be generalized to higher dimensional Hamiltonian systems. Both approaches are treated in detail in [10]; here, we sketch the corresponding results only briefly.

3.1. Generalization to Other Irrationalities in Two Dimensions. Similar considerations as for the case of the irrational number τ apply also to Pisot numbers given by the so-called **+ and – families**. These are solutions of the following equations

$$\begin{aligned} x^2 &= mx+1, \text{ for } m = 1,2,3,\dots, \\ x^2 &= mx-1, \text{ for } m = 3,4,5,\dots, \end{aligned} \tag{19}$$

and are given by

$$\beta = \frac{m+\sqrt{m^2 \pm 4}}{2}, \qquad \beta' = \frac{m-\sqrt{m^2 \pm 4}}{2} \tag{20}$$

with

$$\beta + \beta' = m, \quad \beta\beta' = \pm 1. \tag{21}$$

As before, the gaps in the recurrence time spectrum for $x_n = n\beta \bmod 1$ in $I \subset [0,1)$ follow from the minimal distances of a cut-and-project quasicrystal, in this case $\Sigma_\beta(I)$, and results about the nearest-neighbour distances in $\Sigma_\beta((c, c+d])$ [5] can be used along similar lines, leading to the fact that there are three gaps of the form

$$1, \quad m, \quad m+1, \tag{22}$$

where the exact value of m depends on the length of the interval I as explained in [10].

3.2. Generalization to Higher Dimensional Hamiltonian Systems. In the case of an N-dimensional Hamiltonian system with N integrals of motion, the corresponding Poincaré map leads to sequences of the form

$$x_n = n \begin{pmatrix} \beta_1 \\ \cdot \\ \cdot \\ \beta_{N-1} \end{pmatrix} \bmod 1, \tag{23}$$

where $\beta_1, \dots, \beta_{N-1}$ denote any irrational numbers, and where the Poincaré section $I_{\mathbf{a}}^{N-1} := [0, a_1] \times \dots \times [0, a_{N-1}]$ is a subset of the unit cube in R^{N-1}.

If β_j, $j = 1, \dots, N-1$ are Pisot numbers from the $+$- and $-$-families, then it can be shown analytically that all gaps are (integer-)linear combinations of a set of N basic gaps [10]. This proves the numerical results and conjecture of Slater [7] for the class of irrationalities which are related to $+$- and $-$-families.

CONCLUSION

It has been shown that quasicrystal techniques can be applied for the study of the recurrence time spectrum of Hamiltonian systems. This correspondence has lead to a new and simple proof for known results about recurrence times. Furthermore, this technique, presented here, leads to new results on the sequence formed by the recurrence times, as well as to analytical results for the open problem of a generalization to higher dimensional Hamiltonian systems, thus confirming a conjecture of Slater, which is based on numerical results.

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REFERENCES

1. *Drazin P.* Nonlinear Systems: Cambridge Texts in Applied Mathematics. 1992.
2. *Florek K.* Une remarque sur la repartition des nombres $m\xi$ mod 1 // Coll. Math. Wroclaw. 1951. V. 2. P. 323–324.
3. *Lohöfer G., Mayer D.* On the Gap Problem for the Sequence $||m\beta||$ // Lett. Math. Phys. 1988. V. 16. P. 145–149.
4. *Lohöfer G., Mayer D.* On a Theorem by Florek and Slater on Recurrence Properties of Circle Maps // Publ. RIMS. Kyoto Univ. 1990. V. 26. P. 335–357.
5. *Masáková Z., Patera J., Pelantová E.* Quadratic Irrationalities and Geometric Properties of One-Dimensional Quasicrystals. CRM-2565.
6. *Mayer D. H.* On the Distribution of Recurrence Times in Nonlinear Systems // Lett. Math. Phys. 1988. V. 16. P. 139–143.
7. *Slater N.* // Comp. Math. 1964. V. 16. P. 176–183.
8. *Slater N. B.* The Distribution of the Integers n for Which $\{\theta n\} < \phi$ // Proc. Camb. Phil. Soc. 1950. V. 46. P. 525–534.
9. *Slater N. B.* Gaps and Steps for the Sequence $n\theta$ mod 1 // Proc. Camb. Phil. Soc. 1967. V. 63. P. 1115–1123.
10. *Twarock R.* Gaps in the Recurrence Time Spectrum of Hamiltonian Systems via a Quasicrystal Approach. In preparation.