AN ALGEBRAIC METHOD
TO SOLVE THE TAVIS–CUMMINGS PROBLEM

I. P. Vadeiko [a,1], *G. P. Miroshnichenko* [a,2], *A. V. Rybin* [b,3], *J. Timonen* [b,4]

[a] Department of Mathematics, Saint-Petersburg Institute of Fine Mechanics and Optics, St. Petersburg, Russia

[b] Department of Physics, University of Jyväskylä, Jyväskylä, Finland

We study cooperative behaviour of the system of two-level atoms coupled to a single mode of the electromagnetic field in the resonator. We have developed a general procedure allowing one to rewrite a polynomial deformed $SU(2)$ algebra in terms of another polynomial deformation. Using these methods, we have constructed a perturbation series for the Tavis–Cummings Hamiltonian and diagonalized it in the third order. Based on the zero-order Hamiltonian we calculate an intensity of spontaneous emission of $\mathcal{N}$ two-level atoms inside a cavity, which are in thermal equilibrium with the reservoir. The atom–atom correlation determining superradiance in the system is analyzed.

INTRODUCTION

We consider here the collective behaviour of the system of $\mathcal{N}$ two-level atoms coupled to a single mode of the electromagnetic field in a resonator. The useful form for the atom–field interaction was proposed in the rotating wave approximation (RWA) by Tavis and Cummings [1]. In their model $\mathcal{N}$ identical two-level atoms interact via dipole coupling with a single-mode quantized radiation field at resonance, so that the Hamiltonian is given by

$$H = H_0 + V, \quad H_0 = \omega a^\dagger a + \omega_0 \left(S_3 + \frac{\mathcal{N}}{2} \right),$$
$$V = g \left(a^\dagger S_- + a S_+ \right). \tag{1}$$

Here, ω is a frequency of the electromagnetic field and ω_0 is the level splitting of the two-level atoms. The operators $S_3, S_\pm$ are collective spin variables of $\mathcal{N}$ two-level atoms. These operators are defined as

$$S_3 = \sum_{j=1}^{\mathcal{N}} \sigma_3^j, \quad S_\pm = \sum_{j=1}^{\mathcal{N}} \sigma_\pm^j, \tag{2}$$

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[1]e-mail: vadeiko@mkk.ifmo.ru

[2]e-mail: mirosh@mkk.ifmo.ru

[3]e-mail: rybin@phys.jyu.fi

[4]e-mail: jussi.timonen@phys.jyu.fi

where σ's are Pauli matrices. They satisfy the $SU(2)$ algebra. $a, a^\dagger$ are the annihilation and creation operators of the field. Due to historical reasons, the Tavis–Cummings (TC) model is often called the Dicke model [2]. We concentrate here on the case of exact resonance, i.e., $\omega = \omega_0$. In this case, the system exhibits a most interesting collective behaviour. For simplicity, time will be measured in units of the coupling constant g; i.e., we assume in the following that $g = 1$.

The Hamiltonian (1) belongs to a class of operators that can be expressed in the form:

$$H = f(A_0) + g\,(A_+ + A_-).\tag{3}$$

Here, $f(x)$ is an analytic function of x with real coefficients, while the operators $A_\pm$ satisfy the commutation relations

$$[A_0, A_\pm] = \pm A_\pm.\tag{4}$$

We also assume that the operators $A_0, A_\pm$ must satisfy the conditions

$$(A_0)^\dagger = A_0, \ (A_-)^\dagger = A_+.\tag{5}$$

We reformulate the Hamiltonian in terms of an algebra that allows one to diagonalize it in terms of perturbation series with some small parameter which should be introduced. This idea was already used by Holstein and Primakoff [3]. They expressed the generators $S_3, S_\pm$ of the $SU(2)$ algebra in terms of boson operators $b, b^\dagger$:

$$S_3 = r - b^\dagger b, \quad S_+ = \sqrt{2r}\sqrt{1 - \frac{b^\dagger b}{2r}}\,b, \quad S_- = (S_+)^\dagger.\tag{6}$$

Here, r is an index that characterizes the irrep of $SU(2)$.

However, in [3] the square root in the transformation (6) was in the end replaced by unity, which amounts to applying the so-called «weak field» approximation ($\langle b^\dagger b \rangle \ll 2r$). Obviously, this approximation corresponds to zeroth order in the expansion of the problem with respect to parameter $1/2r$. Transfromation (6) has also been applied [4] with expansion up to second order.

We consider here the case when the operators $A_0, A_\pm$ in Eq. (3) are generators of a polynomial deformation $SU_n(2)$ of the Lie algebra $SU(2)$ [5–7]. Numerous physical applications exist for polynomially deformed algebras [3, 8–18]. A particularly interesting, in view of the present problem, application of deformed algebras was developed by Karassiov (see [8] and references therein). The method to be introduced below is an extension of Karassiov's method.

We introduce here the notion of a Polynomial Algebra of Excitations (PAE). In this algebra, the coefficients of the structure polynomials are c numbers. We derive an exact mapping between isomorphic representations of two arbitrary PAE. We formulate then an analytical approach that allows us to expand the Hamiltonian, when expressed in terms of PAE, as a perturbation series.

For completely symmetric states of atoms our results agree with those reported in [19, 20]. Our formalism provides however a solution of the problem for any value of r, which allows us to discuss new physical effects in the Dicke model.

The article is organized as follows. In Section 1, we discuss the irreducible representations of PAE and apply the general approach to the Tavis–Cummings model in order to construct

the perturbation theory. We solve eigenvalue problem of the Hamiltonian up to third order. The generalized $\mathcal{N}$-atom quantum Rabi frequency is defined for arbitrary quantum states of the system. In Section 2, we use the zero-order approximation for the TC Hamiltonian to calculate the intensity of spontaneous emission of atoms prepared in the state of thermal equilibrium with the resonator mode. We show that the correlation of the atoms due to interaction with the field gives rise to the enhancement of spontaneous emission as compared to the atoms in the absence of resonator. In conclusion, we discuss possible further applications of the methods developed here.

1. TAVIS–CUMMINGS MODEL

The coefficients of the structure polynomial of a polynomially deformed algebra are usually expressed through the Casimir operators of the algebra. We discuss representations of a special class (PAE) of polynomially deformed algebras when the coefficients of the structure polynomial are c numbers. We denote a PAE with a structure polynomial of order κ as $\mathcal{U}_\kappa$. Formally $\mathcal{U}_\kappa$ is defined by three generators $A_\pm, A_0$. These operators satisfy two basic commutation relations, Eq. (4). As can be readily seen from these commutation relations, $[A_0, A_+A_-] = 0$. We can thus assume that

$$A_+A_- = p_\kappa\left(A_0\right) = c_0 \prod_{i=1}^{\kappa} (A_0 - q_i). \tag{7}$$

Here, $p_\kappa(x)$ is a structure polynomial of order κ, whose coefficients are generally complex numbers. The terminology is chosen in analogy to the structure functions of quantum algebras (q-deformed algebras) [21], and the structure constants of the linear Lie algebras. The set of κ real roots of the structure polynomial is denoted as $\{q_i\}_{i=1}^{\kappa}$. In physical applications the operators $A_\pm$ of Eq. (4) often play the role of creation and annihilation operators of collective excitations. Therefore, hereafter the algebra $\mathcal{U}_\kappa$ will be referred to as the polynomial algebra of excitations (PAE) of order κ.

To begin, we assume, without loss of generality, that $c_0 = \pm 1$. Indeed, in the case $|c_0| \neq 1$, it is always possible to renormalize the generators of $\mathcal{U}_\kappa$,

$$A_\pm \to |c_0|^{-1/2} A_\pm, \tag{8}$$

such that the commutation relations (4) remain intact. The most simple and important example of a PAE of first order, $\mathcal{U}_1$, is provided by the well-known Heisenberg–Weyl–Lie algebra, viz.

$$b^\dagger \to A_+, \quad b \to A_-, \quad b^\dagger b \to A_0, \quad c_0 = 1, q_1 = 0. \tag{9}$$

Here, $b, b^\dagger$ are the usual boson operators. For the sake of simplicity, in what follows we will denote the generators of $\mathcal{U}_1$ by $b, b^\dagger$. The algebra $\mathcal{U}_1$ allows us to construct the irrep of any other PAE of higher order $\kappa > 1$ as a multiple tensor product of $\mathcal{U}_1$.

An irrep of PAE is characterized by a set of parameters $\{k_-, k_+, d\}$, where d is a dimension of the invariant subspace and k_- is an order of the left and k_+ of the right roots defining the corresponding irreducible representation (see Ref. [22]). We will denote such

irrep by $R(k_-, k_+, d)$. For instance, $R(1, 0, \infty)$ means the representation of $\mathcal{U}_1$, while the irrep of $\mathbb{S}_r$ is $R(1, 1, 2r + 1)$.

An isomorphism between irreps of $\mathcal{U}_\kappa$ and $\mathcal{U}'_{\kappa'}$ that belong to the same class $R(k_-, k_+, d)$ is given by

$$A_0 = A'_0 + (q_j - q'_{j'}),$$

$$A_+ = \sqrt{\frac{c_0 \prod_{i=1}^{\kappa} (A'_0 + q_j - q'_{j'} - q_i)}{c'_0 \prod_{i'=1}^{\kappa'} (A'_0 - q'_{i'})}} A'_+, \quad A_- = A'_- \sqrt{\frac{c_0 \prod_{i=1}^{\kappa} (A'_0 + q_j - q'_{j'} - q_i)}{c'_0 \prod_{i'=1}^{\kappa'} (A'_0 - q'_{i'})}}, \tag{10}$$

or by

$$A_0 = (q'_{j'} + q_j - 1) - A'_0,$$

$$A_+ = A'_- \sqrt{\frac{c_0 \prod_{i=1}^{\kappa} (q'_{j'} + q_j - A'_0 - q_i)}{c'_0 \prod_{i'=1}^{\kappa'} (A'_0 - q'_{i'})}}, \quad A_- = \sqrt{\frac{c_0 \prod_{i=1}^{\kappa} (q'_{j'} + q_j - A'_0 - q_i)}{c'_0 \prod_{i'=1}^{\kappa'} (A'_0 - q'_{i'})}} A'_+. \tag{11}$$

In Eqs. (10), (11) the operator argument of the square root function should be taken after identical multipliers in the nominator and denominator are cancelled. The roots q_j and $q'_{j'}$, which we call *pivotal roots*, define a vacuum vector of the irrep.

Now, the interaction part of the Hamiltonian (1) can be expressed in terms of third-order PAE. The generators $M_0, M_\pm$ of this algebra are realized as

$$M_- = aS_+, \quad M_+ = a^\dagger S_-, \quad M_0 = \frac{a^\dagger a - S_3}{2}. \tag{12}$$

It is plain that these generators satisfy the commutation relations (4). The generators of the algebra $M_0, M_\pm$ commute with the operators

$$M = a^\dagger a + S_3 + r, \quad S^2 = S_3^2 + \frac{1}{2}(S_+ S_- + S_- S_+). \tag{13}$$

Hereafter we use the same notation M both for the Casimir operator and its eigenvalue, if no confusion arises. We show below that the eigenvalues $M, r(r+1)$ of the operators of Eq. (13) parameterize the PAE in question. We thus denote this PAE as $\mathbb{M}_{M,r}$. The structure polynomial of $\mathbb{M}_{M,r}$ can be expressed in the form

$$p_3(M_0) = M_+ M_- = a^\dagger a \left(S^2 - S_3^2 - S_3\right) = a^\dagger a \left(r - S_3\right)\left(r + S_3 + 1\right) =$$
$$= -\left(M_0 + \frac{M - r}{2}\right)\left(M_0 - \frac{M - 3r}{2}\right)\left(M_0 - \frac{M + r + 2}{2}\right). \tag{14}$$

The parameters of this structure polynomial are

$$c_0 = -1, \quad q_1 = -\frac{M-r}{2}, \quad q_2 \frac{M-r}{2} - r, \quad q_3 = \frac{M-r}{2} + r + 1. \qquad (15)$$

We turn next to the description of finite dimensional irrep of $\mathbb{M}_{M,r}$. In physical applications the parameter r has the meaning of collective Dicke index. This index runs from $\varepsilon(\mathcal{N}) = \frac{1 - (-1)^{\mathcal{N}}}{4}$ to $\frac{\mathcal{N}}{2}$ with unit steps, while M can be any natural number including zero. Thus, q_3 is the biggest positive root of first order. If $M < 2r$, then $q_1 > q_2$; if $M > 2r$ then $q_1 < q_2$; the case $M = 2r \Rightarrow q_1 = q_2$ corresponds to a root of second order. The different values of M and r define different algebras $\mathbb{M}_{M,r}$, whose single physical finite dimensional representation we will call a *zone*. The case $M < 2r$ corresponds to *nearby* zones. The two largest roots are q_1 and q_3 and the irrep has the type $R(1, 1, M + 1)$. Consequently, the well-known weak-field limit corresponds to nearby zones.

The case $M > 2r$ corresponds to *remote* zones. The two largest roots are q_2 and q_3, and the corresponding irrep is of the type $R(1, 1, 2r + 1)$. Notice that the region $M \gg 2r$ is usually called the strong-field limit.

In the special case $2r = M$, referred to *intermediate* zone, the algebra $\mathbb{M}_{M,r}$ possesses an irrep of the type $R(2, 1, 2r + 1)$. It is the only irreducible representation that principally differs from all the others.

The simplest PAE with irrep of the type $R(1, 1, d)$ is $\mathbb{S}_{\tilde{r}}$ (we use here $\tilde{r}$ to distinguish it from the (physical) collective index r). It would be convenient to solve the eigenvalue problem in terms of the simplest algebra $\mathbb{S}_{\tilde{r}}$.

To begin with, we consider the transformation of $\mathbb{M}_{M,r}$ to $\mathbb{S}_{\tilde{r}}$ for the case of remote zones. The dimension of a remote zone is $2r + 1$, and the algebra $\mathbb{S}_{\tilde{r}}$ should be characterized by $\tilde{r} = r$. The finite dimensional irrep of $\mathbb{S}_{\tilde{r}}$ is isomorphic to the corresponding irreducible representation of the algebra of Eq. (2) of the atomic subsystem. For the pivotal root q_j, we choose the largest root that bounds the irrep of $\mathbb{M}_{M,r}$ from the right (the root q_3 in (15)), while as the root $q'_{j'}$ we take the root $q_1 = -r$ of $\mathbb{S}_r$. Applying the mapping (11), we obtain

$$M_0 = \frac{M-r}{2} - \tilde{S}_3, \quad M_+ = \tilde{S}_- \sqrt{\left(M - r + 1 - \tilde{S}_3\right)},$$

$$M_- = \sqrt{\left(M - r + 1 - \tilde{S}_3\right)} \tilde{S}_+. \qquad (16)$$

The spectrum $\{\tilde{m}\}$ of the operator $\tilde{S}_3$ belongs to the region $-r \le \tilde{m} \le r$, consequently, the argument of the square root function in Eq. (16) is positive in the remote zones $(M - r > r)$. The relations (16) express the generators of algebra $\mathbb{M}_{M,r}$ as analytic function of the generators of the $\mathbb{S}_r$ algebra. Thus, they allow us to approximate the more complex algebra $\mathbb{M}_{M,r}$ of third order by a simpler algebra of second order.

We turn now to the nearby zones $M < 2r$. For this region, the mapping of the algebra $\mathbb{M}_{M,r}$ to the algebra $\mathbb{S}_{\tilde{r}}$ is realized through procedure similar to that described above for remote zones. Notice that the dimension of nearby zones is $d = q_3 - q_1 = M + 1$, and therefore $\tilde{r} = M/2$. Applying Eq. (11), we obtain

$$M_0 = \frac{r}{2} - \tilde{S}_3, \quad M_+ = \tilde{S}_- \sqrt{\left(\frac{4r - M}{2} + 1 - \tilde{S}_3\right)}, \quad M_+ = (M_-)^\dagger. \qquad (17)$$

Since all the eigenvalues of the operator $\tilde{S}_3$ belong to the interval $-\tilde{r}$ to $\tilde{r}$, the argument of the square root function does not have zero eigenvalues in the nearby zones.

Let us introduce an $\mathcal{N}$-atom generalization for arbitrary values r and M of the well-known quantum Rabi frequency such that

$$\Omega_R \equiv \begin{cases} 2\sqrt{M - r + \dfrac{1}{2}}, \ M \geq 2r \\[3mm] 2\sqrt{\dfrac{4r - M + 1}{2}}, \ M < 2r \end{cases} . \qquad (18)$$

For $r = \mathcal{N}/2$, our definition agrees with that used in [19, 20]. Introducing a small parameter $\alpha \equiv (1/2\Omega_R)^{-2}$, we can rewrite the realizations of $M_\pm$ in nearby and remote zones (see Eqs. (16), (17)) in the form

$$M_+ = \frac{\Omega_R}{2}\tilde{S}_- \sqrt{1 - \alpha\left(\tilde{S}_3 - \frac{1}{2}\right)}, \quad M_- = M_+^\dagger. \qquad (19)$$

The diagonalization problem for the operator in Eq. (1) can now be solved in each zone by means of perturbation theory with respect to the small parameter α. One can show that the eigenvalues of the argument of the square root function in Eq. (19) are less than unity. Hence, we can expand the square with respect to α and find thereby the interaction part of the Hamiltonian. In the interaction representation, the Hamiltonian coincides with V, we only need to diagonalize the latter. Up to third order in α, we find that

$$V = \frac{\Omega_R}{2}\left(V^{(0)} + V^{(1)} + V^{(2)} + V^{(3)}\right),$$

where the $V^{(n)}$ are terms of nth order in α and are given in the Appendix 2. In the Appendix, we also show the unitary transformations $\tilde{U}_k$, $k = 0, 1, 2, 3$, which bring the interaction operator into diagonal form

$$\bar{V} \equiv \tilde{U}V\tilde{U}^{-1} = \Omega_R\tilde{S}_3\left\{1 + \left(\frac{\alpha}{4}\right)^2\left[5\tilde{S}_3^2 - 3\tilde{r}\left(\tilde{r} + 1\right) + 1\right]\right\}, \quad \tilde{U} \equiv \tilde{U}_3\,\tilde{U}_2\,\tilde{U}_1\,\tilde{U}_0. \qquad (20)$$

The spectrum of the operator V as given by Eq. (20) agrees with the results of [19, 20] for the symmetric states of the atoms.

We compared the third-order solution of Eq. (20) with the exact numerical diagonalization of V and found that the result (20) is very accurate, especially for increasing values of $|M - 2r|$. The results of this comparison are shown in Fig. 1.

Figure 2 compares the energies calculated numerically and in accordance with the analytical solution of Eq. (20). In the intermediate region of M, the curves for nearby and remote zones overlap and coincide, thus providing still satisfactory correspondence to the exact solution. However evidently the expansion for the remote zone breaks down in the nearby zone and vice versa. This means that the classification of zones introduced in this paper is indeed adequate.
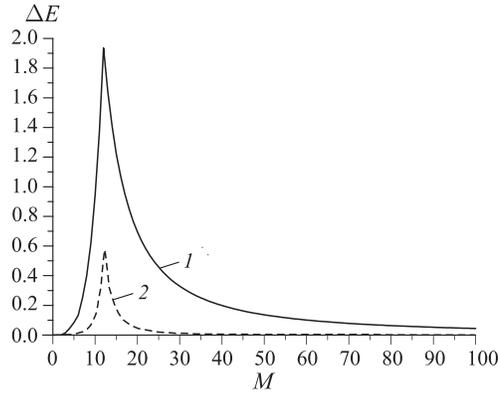
Fig. 1. Deviation of the eigenvalues of V from their numerical values in zero (curve *1*) and second (curve *2*) order in α, for $r = 6$
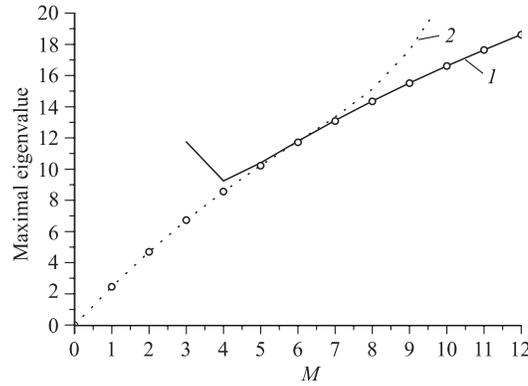

Fig. 2. The dependence on M of the maximal eigenvalue of V for $r = 3$, as calculated numerically and from Eq. (20): *1* — remote zones; *2* — neaby zones; ○ — numerical values

2. ENHANCEMENT OF SPONTANEOUS EMISSION IN THE RESONATOR DUE TO COLLECTIVE EFFECTS

In the previous section we developed an algebraic approach to the Tavis–Cummings model. We introduced the operators $\tilde{S}_{\pm}$ describing collective excitations in the atom–field system. In terms of these operators, we constructed a perturbation series for the Tavis–Cummings Hamiltonian (1). The derived perturbation series gives us a tool to distinguish and classify cooperative (multiparticle) effects of different orders that are involved in calculations of different physical observables characterizing the atom–field system. In this section, we study a contribution of cooperative effects into the rate of spontaneous emission generated by the atom–field system. The atom–field system is assumed to be prepared in the state of thermal equilibrium. This state is described by the canonical Gibbs ensemble with the thermostat temperature T. We demonstrate that nontrivial physical results for the intensity of spontaneous emission of $\mathcal{N}$ two-level atoms placed inside the cavity can be already obtained

for the zero-order approximation of the exact Hamiltonian (1). The thermal state is given by

$$\rho_{\text{therm}} = \frac{1}{Z} \exp \left[-\frac{H_0 + \Omega_R/2 \left(\tilde{S}_+ + \tilde{S}_- \right)}{kT} \right]. \tag{21}$$

Here Z is a normalization factor.

When calculating the rate of spontaneous emission (or the intensity proportional to this quantity) we merely follow the ideas of Dicke's paper (see, e. g., [2]). According to this theory, the rate of spontaneous emission in the system is proportional to the average of the square of the atomic dipole, viz.,

$$I = I_0 \langle S_+ S_- \rangle = I_0 \operatorname{Tr} \left\{ \rho_{\text{therm}} S_+ S_- \right\}. \tag{22}$$

It can be shown (Ref. [22]) that the intensity of spontaneous emission is given by

$$I = \frac{I_0}{Z} \sum_{M=0}^{\infty} \sum_{r=\varepsilon}^{\mathcal{N}/2} G(r) \times$$
$$\times \sum_{\tilde{m}=-\tilde{r}}^{\tilde{r}} \exp \left(-\frac{\omega \left(M - r + \mathcal{N}/2 \right) + \Omega_R \tilde{m}}{kT} \right) \left[\tilde{r} \left(2r - \frac{3}{2} \tilde{r} + \frac{1}{2} \right) + \frac{1}{2} \tilde{m}^2 \right]. \tag{23}$$

Here $\varepsilon \equiv \dfrac{1 - (-1)^{\mathcal{N}}}{4}$, and $G(r) = \dfrac{\mathcal{N}!(2r+1)}{(\mathcal{N}/2 + r + 1)!(\mathcal{N}/2 - r)!}$ is the number of equivalent representations with the same r.
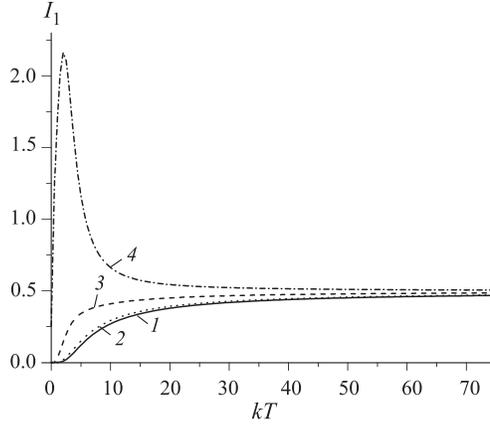

Fig. 3. The intensity of spontaneous emission per atom (in units of I_0) versus cavity temperature. The curve *1* is the classical result given by Eq. (25), the curves *2, 3, 4* correspond to $\mathcal{N} = 10$, $\mathcal{N} = 50$, $\mathcal{N} = 100$, respectively, and $\omega/g = 10$

Let us consider the intensity per atom, i. e., $I_1 \equiv I/\mathcal{N}$. This intensity consists of two terms; i. e., the first is given by a single-particle contribution I_{single} and the second one is

proportional to the two-particle correlation function Cor, $I_1 = I_{\text{single}} + I_0(\mathcal{N} - 1)$Cor. They are found to be

$$I_{\text{single}} \equiv I_0 \frac{1}{\mathcal{N}} \langle \sum_i \sigma_+^i \sigma_-^i \rangle = I_0 \left(\frac{1}{2} + \frac{1}{\mathcal{N}} \langle S_3 \rangle \right),$$

$$\text{Cor} \equiv \frac{1}{\mathcal{N}(\mathcal{N} - 1)} \langle \sum_{i \neq j} \sigma_+^i \sigma_-^j \rangle... \tag{24}$$

It is plain that in the absence of the cavity, the correlation function vanishes and the only contribution to I_1 is given by the first term,

$$I_{\text{single}} = I_{\text{cl}} \equiv I_0 \left(1 + \mathrm{e}^{\omega/kT} \right)^{-1}. \tag{25}$$

Notice that if the number of atoms is big enough, the intensity of radiation exhibits a high maximum. In a cavity at low T, the cluster of $\mathcal{N}$ two-level atoms emits much more intensively than it does in the free space. It should be possible to drive the system to thermal equilibrium at the temperature where the spontaneous emission exhibits maximum. The marked amplification of spontaneous emission should be observed in cavity experiments.

Concluding this section, we recapitulate our main results. We consider spontaneous emission of the system comprised of $\mathcal{N}$ two-level atoms strongly coupled to the cavity mode and prepared in the state of thermal equilibrium. In the absence of the cavity the atoms in the thermal equilibrium would be uncorrelated. In this case, the spontaneous emission would be described by the conventional formula (25). For high-Q resonators the strong coupling to the resonator mode should necessarily be taken into account. We demonstrate that in this case the intensity of spontaneous emission can be greatly enhanced. This phenomenon can be explained by additional correlation between atoms established by the cavity mode. Analytically, we have replaced the exact Tavis–Cummings Hamiltonian (1) by its zero-order approximation derived in the previous sections. This allowed us to represent the intensity of spontaneous emission in the simple analytical form Eq. (23). It is appropriate to emphasize once again that the zero-order approximation of the Hamiltonian contains strong coupling and, thus, describes cooperative effects in the atomic subsystem. This is the consequence of the fact that the operators $\tilde{S}_\pm$ describe collective excitations in the atom–field system.

CONCLUSION

In this work, we solved the Tavis–Cummings problem by applying the technique of polynomially deformed algebras. We constructed the transformations that map one polynomial algebra of operators onto another. This allowed us to reformulate the problem in terms of a simpler algebra of second order, $\mathbb{S}_r$, and develop a specific perturbation theory. We were able to find analytical expressions for all the eigenvalues of the Hamiltonian up to third order in the small parameter α. For the nearby zones, we showed explicitly how the collective quantum Rabi frequency depends on Dicke index r. Since this index characterizes the symmetry of atomic states, the result has significant physical implications. The dependence on atomic symmetry is revealed already in zeroth order in the perturbation expansion. Employing our methods, we found an interesting new effect, which is amplification of spontaneous emission of thermal $\mathcal{N}$-atom states due to collective effects. We expect that this phenomenon can be observed in cavity experiments.

APPENDIX

1. Similarity Transformations. We look for similarity transformations $\tilde{U}_k$ that diagonalize the Hamiltonian in different orders of α,

$$V_k^{(n)} \equiv \tilde{U}_k V_{k-1}^{(n)} \tilde{U}_k^{-1}, \tag{A1}$$

where $k = 0, 1, 2, 3$. In Eq. (A1) only the terms of order n in the small parameter α are present. For $k = 0$, the term $V_{k-1}^{(n)}$ should be replaced by the corresponding term in expansion of V in the series. Regrouping the terms, we obtain

$$V^{(0)} = 2\tilde{S}_x \left(1 - \frac{1}{2} \left(\frac{\alpha}{4} \right)^2 \right), \tag{A2}$$

$$V^{(1)} = -\frac{\alpha}{2} \left(1 + \frac{1}{4} \left(\frac{\alpha}{4} \right)^2 \right) B, \tag{A3}$$

$$V^{(2)} = -\left(\frac{\alpha}{2} \right)^2 \left[\tilde{S}_3 \frac{\tilde{S}_+ + \tilde{S}_-}{2} \tilde{S}_3 \right], \tag{A4}$$

$$V^{(3)} = -\frac{1}{2} \left(\frac{\alpha}{2} \right)^3 \left[\tilde{S}_3 B \tilde{S}_3 \right], \tag{A5}$$

where

$$B \equiv \left[\tilde{S}_3 \tilde{S}_x + \tilde{S}_x \tilde{S}_3 \right], \tag{A6}$$

and

$$\tilde{S}_x \equiv \frac{\tilde{S}_+ + \tilde{S}_-}{2}. \tag{A7}$$

2. The Zero-Order Transformation $\tilde{U}_0$. As is known from the theory of $SU(2)$ algebra, the operator $\tilde{S}_x$ can be diagonalized by the transformation

$$\tilde{U}_0 = \exp \left[\frac{\pi}{4} \left(\tilde{S}_+ - \tilde{S}_- \right) \right] = \exp \left[-i \frac{\pi}{2} \tilde{S}_y \right]. \tag{A8}$$

Employing this transformation, we obtain

$$\begin{aligned} V_0^{(0)} &= 2\tilde{S}_3 \left(1 - \frac{1}{2} \left(\frac{\alpha}{4} \right)^2 \right), \quad V_0^{(1)} = -V^{(1)}, \\ V_0^{(2)} &= -\left(\frac{\alpha}{2} \right)^2 \left[\tilde{S}_x \tilde{S}_3 \tilde{S}_x \right], \quad V_0^{(3)} = \frac{1}{2} \left(\frac{\alpha}{2} \right)^3 \left[\tilde{S}_x B \tilde{S}_x \right]. \end{aligned} \tag{A9}$$

3. The First-Order Transformation $\tilde{U}_1$. It can readily be seen that the transformation

$$\tilde{U}_1 = \exp\left[\alpha x D1\right], \quad D1 \equiv -i\left[\tilde{S}_3\tilde{S}_y + \tilde{S}_y\tilde{S}_3\right] \tag{A10}$$

diagonalizes the operators in the first order. In the diagonalization, one needs the commutators

$$\left[\tilde{S}_3, D1\right] = B, \quad \left[\left[\tilde{S}_3, D1\right], D1\right] = 4\tilde{S}_3\left(\tilde{S}^2 - 2\tilde{S}_3^2 - \frac{1}{4}\right),$$

$$\left[\left[\left[\tilde{S}_3, D1\right], D1\right], D1\right] = \left(4\tilde{S}^2 - 1\right)B - 8\left[\tilde{S}_3 B + B\tilde{S}_3 - \tilde{S}_3\tilde{S}_x\tilde{S}_3\right], \tag{A11}$$

$$\left[\tilde{S}_3\tilde{S}_x\tilde{S}_3, D1\right] = 2\tilde{S}_3^2\left(\tilde{S}^2 - 2\tilde{S}_3^2\right) + \left(\tilde{S}_3\tilde{S}_x\right)^2 + \left(\tilde{S}_x\tilde{S}_3\right)^2.$$

Then up to third order in α,

$$\tilde{U}_1 V_0^{(0)}\left(\tilde{U}_1\right)^{-1} = V_0^{(0)} - 2\alpha x\left(1 - \frac{1}{2}\left(\frac{\alpha}{4}\right)^2\right)B + 4(\alpha x)^2\tilde{S}_3\left(\tilde{S}^2 - 2\tilde{S}_3^2 - \frac{1}{4}\right) -$$

$$-\frac{(\alpha x)^3}{3}\left\{\left(4\tilde{S}^2 - 1\right)B - 8\tilde{J}\right\}, \tag{A.12}$$

and

$$\tilde{U}_1 V_0^{(1)}\left(\tilde{U}_1\right)^{-1} = \frac{\alpha}{2}\left(1 + \frac{1}{4}\left(\frac{\alpha}{4}\right)^2\right)B - \alpha^2 x 2\tilde{S}_3\left(\tilde{S}^2 - 2\tilde{S}_3^2 - \frac{1}{4}\right) +$$

$$+\alpha^3\left(\frac{x}{2}\right)^2\left\{\left(4\tilde{S}^2 - 1\right)B - 8\tilde{J}\right\}, \tag{A.13}$$

where $\tilde{J} \equiv \tilde{S}_3 B + B\tilde{S}_3 - \tilde{S}_3\tilde{S}_x\tilde{S}_3$.

$$\tilde{U}_1 V_0^{(2)}\left(\tilde{U}_1\right)^{-1} = V_0^{(2)} + \alpha^3\left(\frac{x}{2}\right)^2\left\{\tilde{S}_x\tilde{S}_3\left(2\tilde{S}^2 - 4\tilde{S}_3^2 - \tilde{S}_x^2\right) +\right.$$

$$\left. + \left(2\tilde{S}^2 - 4\tilde{S}_3^2 - \tilde{S}_x^2\right)\tilde{S}_3\tilde{S}_x\right\}. \tag{A.14}$$

In the third order, the operator $V_0^{(3)}$ remains unchanged after the transformation, i. e., $V_1^{(3)} = V_0^{(3)}$. In order to calculate $V_0^{(2)}$ up to third order, we take into account that

$$x = \frac{1}{4}\frac{\left(1 + (\alpha/8)^2\right)}{\left(1 - 2\left(\alpha/8\right)^2\right)} \approx \frac{1}{4}\left(1 + 3\left(\frac{\alpha}{8}\right)^2\right),$$

and find then that

$$V_1^{(0)} = V_0^{(0)} - 4\left(\frac{\alpha}{4}\right)^2\tilde{S}_3\left(\tilde{S}^2 - 2\tilde{S}_3^2 - \frac{1}{4}\right), \quad V_1^{(1)} = 0, \quad V_1^{(2)} = V_0^{(2)},$$

$$V_1^{(3)} = V_0^{(3)} - \frac{1}{2}\left(\frac{\alpha}{4}\right)^3 B + \frac{2}{3}\left(\frac{\alpha}{4}\right)^3\left\{\left(4\tilde{S}^2 - 1\right)B - 8\tilde{J}\right\} + \tag{A15}$$

$$+\left(\frac{\alpha}{4}\right)^3\left\{\tilde{S}_x\tilde{S}_3\left(2\tilde{S}^2 - 4\tilde{S}_3^2 - \left(\tilde{S}_x\right)^2\right) + \left(2\tilde{S}^2 - 4\tilde{S}_3^2 - \left(\tilde{S}_x\right)^2\right)\tilde{S}_3\tilde{S}_x\right\}.$$

4. The Second- and the Third-Order Transformations. To find the second-order transformation, we rewrite $V_1^{(2)}$ in a symmetrized form,

$$V_1^{(2)} = -\frac{1}{2} \left(\frac{\alpha}{2}\right)^2 \left[\tilde{L}_x \tilde{S}_3 + \tilde{S}_3 \tilde{L}_x + \tilde{S}_3 \left(\tilde{S}^2 - \tilde{S}_3^2 - 1\right)\right], \tag{A16}$$

where $\tilde{L}_x \equiv \dfrac{\tilde{S}_+^2 + \tilde{S}_-^2}{4}$. The diagonalizing transformation is then given by

$$\tilde{U}_2 \equiv \exp\left\{i\frac{1}{2} \left(\frac{\alpha}{4}\right)^2 \left[\tilde{L}_y \tilde{S}_3 + \tilde{S}_3 \tilde{L}_y\right]\right\}, \tag{A17}$$

where $\tilde{L}_y \equiv \dfrac{\tilde{S}_+^2 - \tilde{S}_-^2}{4}$. Keeping the terms up to third order we obtain (see (A13))

$$\tilde{U}_2 V_1^{(0)} \left(\tilde{U}_2\right)^{-1} = V_1^{(0)} + \frac{1}{2} \left(\frac{\alpha}{2}\right)^2 \left[\tilde{L}_x \tilde{S}_3 + \tilde{S}_3 \tilde{L}_x\right]. \tag{A18}$$

The transformation (A17) does not change the expressions given above for $V_1^{(2)}$ and $V_1^{(3)}$, and we find that

$$V_2^{(0)} = V_1^{(0)} - \left(\frac{\alpha}{4}\right)^2 2\tilde{S}_3 \left(\tilde{S}^2 - \tilde{S}_3^2 - 1\right),$$
$$V_2^{(1)} = V_2^{(2)} = 0, \quad V_2^{(3)} = V_1^{(3)}. \tag{A19}$$

Diagonalization of $V_2^{(3)}$ can be performed in a similar way with an operator $\tilde{U}_3 = \exp\left[-(\alpha/4)^3 \mathcal{O}\right]$. As there are no diagonal terms in $V_2^{(3)}$, which would contribute to the spectrum of the Hamiltonian, we do not give here the fairly complicated form of operator $\mathcal{O}$. The final diagonal form for the interaction V is thus given by

$$\bar{V} = \Omega \tilde{S}_3 \left\{1 + \left(\frac{\alpha}{4}\right)^2 \left[5\tilde{S}_3^2 - 3\tilde{r}\left(\tilde{r} + 1\right) + 1\right]\right\}. \tag{A20}$$

To recapitulate, we introduced four transformations $\tilde{U}_k$, $k = 0, 1, 2, 3$, which successively diagonalize the interaction operator in the Tavis–Cummings Hamiltonian up to third order with respect to the small parameter $\alpha = (1/2\Omega_R)^{-2}$, with Ω_R the generalized Rabi frequency of Eq. (18), such that

$$\bar{V} \equiv \tilde{U} V \tilde{U}^{-1}, \quad \tilde{U} \equiv \tilde{U}_3 \tilde{U}_2 \tilde{U}_1 \tilde{U}_0. \tag{A21}$$

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