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HIDDEN SYMMETRIES OF QCD AT HIGH ENERGY

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Different applications of the Wilson-line formalism to the high-energy QCD are reviewed.

Обзор посвящен различным применениям формализма вильсоновских линий к высокоэнергетической КХД.

1. MY TEACHER

It is a special pleasure and a privilege for me to contribute to this Festschrift to celebrate 70th anniversary of Anatolij Vasil'evich Efremov as I consider him as one of my teachers who deeply influenced my scientific interests and from whom I have learned many important lessons both in physics and in maintaining high moral standards back to the USSR time.

My first meeting with Anatolij Vasil'evich (or simply AV) goes back to the beginning of eighties. At that time I have been a fourth-year student at the Rostov-on-Don State University studying theoretical high-energy physics. I was very much interested in doing research in QCD and my supervisor Sergey Ivanov proposed to start with reading the series of papers written by AV together with Anatolij Radyushkin on «Field Theoretic Treatment of High Momentum Transfer Processes» [1–3]. This was a tough job and it took me quite some time to accomplish it. I cannot say that I understood every word in their papers but I got the main message — the papers presented an elegant framework for investigating various hadronic processes in high-energy QCD. They contained a proof that the contribution to hard scattering amplitudes (deep-inelastic scattering, Drell–Yan process, pion form factor, ...) from long and short distances can be factored out into nonpertubative distributions and perturbatively calculable partonic cross sections. One of the crucial points in proving the factorization was a demonstration that contribution of soft gluons cancels completely in the sum of all Feynman diagrams. Reading the papers I was struck at some point by a comment that soft gluons and, in general, infrared asymptotics in QCD deserve additional studies and are ultimately related to solving the Sudakov problem in

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QCD. I knew very little about this subject and I decided to learn more about soft gluons and all that. Together with my supervisor we started to look into the problem and this is how my first scientific contacts with AV have occurred. Our discussions were later materialized in publishing my first paper [4].

My discussions with AV have continued after I came to Dubna as an aspirant. This was also the beginning of my collaboration with another AV's former student — Anatolij Radyushkin. Later, I was proud to have both of them as supervisors of my PhD which was devoted to investigation of infrared asymptotics in QCD. I would come to AV office and write a few formulae on the blackboard. He would listen quietly, smoke his famous pipe and make very deep comments at the end with his soft voice. After finishing my PhD in the late 80's, I had to face an eternal problem of finding a job and being a jew created additional complications at that time. This was AV who took the problem personally and who put tremendous efforts for making it possible for me to get a position in Dubna. I am deeply grateful to AV for this as well as for generosity that he extended to me through all these years. Since our first meeting almost twenty years ago many things have changed, our scientific interests have evolved in different directions and we are not meeting as often as before. But I never forget the role that AV has played in my life and I am proud to be associated with the school created by AV.

In this, I would like to describe the project, the so-called Wilson-loop formalism, which was initiated in the above-mentioned paper written together with AV and which arose from numerous discussions with AV. The formalism of path-ordered exponentials, or Wilson loops, is an indispensable tool in QCD. It allows one to formulate complicated QCD dynamics in terms of gauge invariant degrees of freedom [5] and express correlation functions as a sum over random walks, e.g.,

$$\langle 0|J_\mu(x)J_\nu(0)|0\rangle = \sum_C \mathrm{e}^{-mL[C]}\,\Phi_{\mu\nu}[C]\;\langle 0|\,\mathrm{tr}\,P\exp\left(i\oint_C dx_\mu A^\mu(x)\right)|0\rangle\,, \quad (1)$$

where $J_\mu(x) = \overline{\Psi}(x)\gamma_\mu\Psi(x)$ is the electromagnetic current of a quark with mass m; $L[C]$ is the length of a closed path $C = C[0,x]$ that passes through the points x and 0; $\Phi_{\mu\nu}[C]$ is a geometrical phase, the so-called Polyakov spin factor, that takes into account the variation of the quark spin upon parallel transport along the path C. To evaluate (1), one has to calculate the (nonperturbative) expectation value of the Wilson loop for an arbitrary path C and perform resummation in the right-hand side of (1). Both tasks are extremely difficult and cannot be performed in full at the current stage.

There exists a special class of QCD observables, for which the sum over paths in the right-hand side of (1) can be performed exactly. As a relevant physical example, let us consider a propagation of an energetic quark through a cloud of soft gluons. In the limit when its energy goes to infinity, the quark behaves as a point-like charged particle that moves along a straight line and interacts with soft gluons. This means that the sum over all paths in (1) is dominated in that case by a saddle point describing a propagation of a quark along its classical path. The Wilson loop corresponding to this path has the meaning of the eikonal phase acquired by the quark field upon interaction with gluons. In this way, the Wilson loop encodes universal features of soft radiation in QCD. Let us point out two important QCD observables, in which similar semiclassical regime occurs: the Isgur–Wise heavy-meson form factor, $\xi(\theta)$, and parton distributions in a hadron, $f(x)$, at the edge of the phase space, $x \to 1$. As we will demonstrate below, both observables are given by an expectation value of a Wilson loop with the integration contour C fixed by the kinematics of the process. A unique feature of the contour C is that it contains a few cusps at points in Minkowski space-time where the interaction with a large momentum has occurred in the underlying process. Thus, the Wilson loops with cusps, being fundamental objects in gauge theories, have a direct relevance for QCD phenomenology. Their calculation in the strong coupling (nonperturbative) regime is one of the prominent problems in gauge theories.

2. ISGUR–WISE FORM FACTOR

The Isgur–Wise form factor $\xi(\theta)$ describes the electromagnetic transition of a heavy meson $|M(v)\rangle$ with mass m and momentum $p_\mu \equiv mv_\mu$, built from a heavy quark and a light component, to the same meson with momentum $p'_\mu \equiv mv'_\mu$ (with $v_\mu^2 = {v'_\mu}^2 = 1$) [6]

$$\langle M(v')|\overline{\Psi}(0)\gamma_\mu \Psi(0)|M(v)\rangle = \xi(\theta)(v+v')_\mu. \tag{2}$$

In the heavy-quark limit, $m \to \infty$, it depends only on the product of velocities $v' \cdot v = \cosh\theta$, or equivalently on the angle θ between them in Minkowski space-time. The operator $\Psi(0)$ annihilates the heavy quark inside the meson $|M(v)\rangle$. For $m \to \infty$, the heavy quark behaves as a classical particle with the velocity v_μ interacting with the light component of the meson through its eikonal current, $J_\mu^{a,\text{eik}}(x) = \int_{-\infty}^0 d\tau\, v_\mu t^a \delta^{(4)}(x - v\tau)$ with t^a being the quark color charge. This allows one to replace

$$\begin{aligned} \Psi(x) &\to \mathrm{e}^{-im(vx)}\, b_v\, \Phi_v[x;-\infty], \\ \Phi_v[x;-\infty] &\equiv P\exp\left(i\int_{-\infty}^0 d\tau\, vA(x+v\tau)\right), \end{aligned} \tag{3}$$

where $\Phi_v[x;-\infty]$ is the eikonal phase of a heavy quark in the fundamental representation of the $SU(N_c)$, and b_v amputates this quark inside the heavy meson. Applying similar transformation to the quark field in the final state meson $|M(v')\rangle$, one obtains the following expression for the form factor [7]

$$\xi(\theta) = \langle\widetilde{M}(v')|\Phi_{v'}[\infty;0]\Phi_v[0;-\infty]|\widetilde{M}(v)\rangle \equiv \langle P\exp\left(i\oint_\wedge dx_\mu A^\mu(x)\right)\rangle, \quad (4)$$

with $|\widetilde{M}(v)\rangle = b_v|M(v)\rangle$ standing for the light component of the meson with the amputated heavy quark. Here the net effect of nonperturbative interaction with the light component of the heavy meson is accumulated only via the Wilson line evaluated along the contour consisting of two rays that run along the meson velocities v_μ and v'_μ. It is important to notice that the contour has a cusp at the point 0, in which the interaction with the external probe has occurred.

The Isgur–Wise form factor $\xi(\theta)$ is a nonperturbative observable in QCD [8]. It depends on hadronic, long-distance scales as well as on the ultraviolet cutoff $\mu \sim m$, which sets up the maximal energy of soft gluons. Although $\xi(\theta)$ cannot be calculated at present in QCD from the first principles, its dependence on μ can be found from the renormalization group equation [5, 9]]

$$\left(\mu\frac{\partial}{\partial\mu} + \beta(g)\frac{\partial}{\partial g} + \Gamma_{\rm cusp}(\theta;\alpha_s)\right)\xi(\theta) = 0, \quad (5)$$

where $\alpha_s = g^2/(4\pi)$ is the QCD coupling constant and $\Gamma_{\rm cusp}(\theta;\alpha_s)$ is the cusp anomalous dimension. To the lowest order in α_s

$$\Gamma_{\rm cusp}(\theta;\alpha_s) = \frac{\alpha_s C_F}{\pi}\left(\theta\coth\theta - 1\right) + \mathcal{O}(\alpha_f^\in), \quad (6)$$

where $C_F = (N_c^2-1)/(2N_c)$ is the Casimir operator of the $SU(N_c)$ group in the fundamental representation. The two-loop correction to (6) has been calculated in [9] and its dependence on θ is more involved.

Equation (5) follows from renormalization properties of the Wilson line in the right-hand side of (4). It acquires the anomalous dimension due to the presence of a cusp on the integration contour. The cusp anomalous dimension $\Gamma_{\rm cusp}(\theta;\alpha_s)$ determines universal features of soft-gluon radiation and is known as the *QCD bremsstrahlung function*. As such, it is a positive definite function of the cusp angle (for real Minkowski angle θ) at arbitrary value of the coupling constant

$$\Gamma_{\rm cusp}(\theta;\alpha_s) \geqslant 0. \quad (7)$$

To see this we recall that at the cusp point the heavy quark suddenly changes its velocity from v_μ to v'_μ and, due to instantaneous acceleration, it starts to emit soft (virtual and real) gluons with momentum $k < \mu$ with a cut-off $\mu \sim m$. Denoting the eikonal phase of the heavy quark as $\Phi \equiv \Phi_{v'}[\infty;0]\Phi_v[0;-\infty]$ and using its unitarity, $\Phi^\dagger\Phi = 1$, one calculates the total probability for the heavy quark to undergo the scattering (the Bjorken sum rule) as

$$1 = \langle \widetilde{M}(v)|\Phi^\dagger\Phi|\widetilde{M}(v)\rangle = |\xi(\theta)|^2 + \sum_X \left|\langle \widetilde{M}_X(v')|\Phi|\widetilde{M}(v)\rangle\right|^2, \qquad (8)$$

where in the right-hand side we inserted the decomposition of the unity operator over the physical hadronic states and separated the contribution of the ground state meson, $|\widetilde{M}(v)\rangle$, from excited states $|\widetilde{M}_X(v)\rangle$. The Wilson line (4) defines the probability of the elastic transition, $|\xi|^2 \sim \exp(-w)$ with $w = 2\int^\mu (dk/k)\,\Gamma_{\rm cusp}(\theta;\alpha_s(k))$. For $\theta \neq 0$, depending on the sign of $\Gamma_{\rm cusp}(\theta;\alpha_s)$, it either vanishes or goes to infinity for $\mu \to \infty$. In order to preserve the unitarity condition $|\xi|^2 \leqslant 1$ that follows from (8), one has to require that $\xi \to 0$ for $\mu \to \infty$ leading to (7). At $\theta = 0$ the cusp vanishes, that is the heavy meson stays intact, the sum in (8) equals zero and $\xi(\theta = 0) = 1$. This implies that the cusp anomalous dimension vanishes for $\theta \to 0$.

3. DEEP-INELASTIC SCATTERING AT $x \to 1$

Our second example is provided by deep-inelastic scattering of a hadron $H(p)$ with momentum p_μ off a virtual photon $\gamma^*(q)$ with momentum $-q_\mu^2 = Q^2 \gg p^2$ in the exclusive limit $x_{\rm Bj} = Q^2/(2pq) \to 1$, i.e., when the invariant mass of the final state system becomes small $(q+p)^2 \ll Q^2$. In the scaling limit, $Q^2 \to \infty$, the cross section of the process is expressed in terms of the twist-two quark distribution function [10]

$$f(x) = \int_{-\infty}^{\infty} \frac{d\xi}{2\pi}\, {\rm e}^{-ix\xi} \langle H(p)|\overline{\Psi}(\xi n)\Gamma\Phi_n[\xi;0]\Psi(0)|H(p)\rangle, \qquad (9)$$

describing the probability to find a quark inside the hadron $H(p)$ with the fraction x of its momentum p. The Wilson line stretched in-between the quark fields makes the bilocal operator gauge invariant. It goes along the light-like direction $n_\mu = (q_\mu + p_\mu x_{\rm Bj})/(pq)$, so that $n^2 = 0$ and $np = 1$.

The matrix $\Gamma = \not{n}$ in (9) serves to select the quark states with opposite helicities. For our purposes, we will not specify Γ and treat it as a free parameter. The Mellin moments of the distribution function (9) are related to the matrix elements of local twist-two operators

$$\int_0^1 dx\, x^J f(x;\mu^2) = \langle H(p)|\overline{\Psi}(0)\Gamma\,(in\mathcal{D})^J\,\Psi(0)|H(p)\rangle \equiv \langle \mathcal{O}_J^\Gamma(\mu^2)\rangle, \qquad (10)$$

where $\mathcal{D}_\mu = \partial_\mu - iA_\mu$ is a covariant derivative. Their dependence on the ultraviolet cutoff μ is described by an evolution equation, whose solution reads in terms of the anomalous dimensions

$$\langle \mathcal{O}_J^\Gamma(\mu^2)\rangle = (\mu/\mu_0)^{-\gamma_J^\Gamma(\alpha_s)}\,\langle \mathcal{O}_J^\Gamma(\mu_0^2)\rangle, \tag{11}$$

where we assumed for simplicity that the coupling constant does not run, $\beta = 0$. The anomalous dimension of the twist-two operators, $\gamma_J^\Gamma(\alpha_s)$, depends on the choice of the matrix Γ. In particular, in the case when Γ selects the same helicities of the quark fields in (10), $\Gamma = (1+\gamma_5)\not{n}\gamma_\perp$, the anomalous dimension is

$$\gamma_J(\alpha_s) = \frac{\alpha_s}{\pi} C_F \Big(2\psi(J+2) + 2\gamma_E - 3/2 \Big) + \mathcal{O}(\alpha_s^2), \tag{12}$$

where $\psi(J) = d\ln\Gamma(J)/dJ$ is the Euler ψ function and γ_E is the Euler constant. For other choices of Γ, the anomalous dimensions have extra (rational in J) terms in addition to the ψ function (see, e.g., [11]). As we will argue below, Eq. (12) has a hidden symmetry which is responsible for integrability of evolution equations for three-quark (baryonic) composite operators.

As follows from (10), the asymptotics of the distribution function for $x \to 1$ is related to the contribution of twist-two operators of large Lorentz spins $J \sim 1/(1-x) \gg 1$. One finds from (12) that the anomalous dimension scales in this limit as

$$\gamma_J(\alpha_s) = \frac{\alpha_s}{\pi} C_F \Bigg\{ \ln(J+2) + \gamma_E - 3/4 - \\ - \frac{1}{2(J+2)} - \sum_{n=1}^{\infty} \frac{B_{2n}}{2n}(J+2)^{-2n} \Bigg\} + \ldots, \tag{13}$$

where B_n's are the Bernoulli numbers; $B_2 = 1/6$, $B_4 = -1/30, \ldots$, and the ellipsis stands for higher order terms in α_s. It turns out that the leading scaling behavior $\gamma_J^\Gamma(\alpha_s) \sim \ln J$ is a universal property of the anomalous dimensions of the twist-two operators (10) for arbitrary Γ. It holds to all orders in α_s and is intrinsically related to the cusp anomaly of the Wilson loops. The reason for this is that analyzing deep-inelastic scattering for $x \to 1$ one encounters the same physical phenomenon as in the case of the Isgur–Wise form factor, i.e., the struck quark carries almost the whole momentum of the hadron and, therefore, it interacts with other partons by exchanging soft gluons. In these circumstances, in complete analogy to the previous case, Eq. (3), the quark field can be approximated by an eikonal phase evaluated along the classical path in the direction of its velocity $p_\mu = m v_\mu$,

$$f(x) = \int_{-\infty}^{\infty} \frac{d\xi}{2\pi} \mathrm{e}^{i(1-x)\,\xi} \langle \widetilde{H}(p)| W_\Pi(v\cdot n\,\xi\mu - i0)|\widetilde{H}(p)\rangle, \tag{14}$$

where $|\widetilde{H}(p)\rangle$ is the state of the target hadron with amputated energetic quark and the causal $-i0$ prescription ensures the correct spectral property; $f(x) = 0$ for

$x > 1$. The Π-shaped Wilson line in Eq. (14) consists out of two rays and one segment: a link from $-\infty$ to 0 along the velocity of the incoming quark, next along the light-cone direction n_μ to the point ξn_μ and, then, along $-v_\mu$ from 0 to ∞,

$$W_\Pi(vn\,\xi\mu) = \Phi_v^\dagger[\xi; -\infty]\,\Phi_n[\xi; 0]\,\Phi_v[0; -\infty]. \tag{15}$$

Substituting (14) into (10), one finds the following relation between the matrix elements of local composite operators at large spin J and the Wilson loop expectation value [12]

$$\langle \mathcal{O}_J^\Gamma(\mu^2)\rangle = \langle \widetilde{H}(p)|W_\Pi(-iJ)|\widetilde{H}(p)\rangle \equiv \langle P\exp\left(i\int_\Pi dx_\mu A^\mu(x)\right)\rangle. \tag{16}$$

Here, the large Lorentz spin of the local operator defines the length of the light-cone segment:

$$v\cdot n\,\xi\mu \to -iJ. \tag{17}$$

We would like to stress that Eq. (16) holds only for $J \gg 1$.

According to Eq. (16), the μ dependence of the twist-two operators follows from the renormalization of the Wilson line (15). The latter has two cusps located at the points 0 and ξn_μ. In distinction with the previous case, one of the segments attached to the cusps lies on the light-cone, $n^2 = 0$, and the corresponding cusp angle is infinite, $\theta \sim \frac{1}{2}\ln[(vn)^2/n^2] \to \infty$. In this limit, the cusp anomalous dimension scales to all orders in α_s as [9]

$$\Gamma_{\rm cusp}(\theta; \alpha_s) = \theta\Gamma_{\rm cusp}(\alpha_s) + \mathcal{O}(\theta'). \tag{18}$$

Here $\Gamma_{\rm cusp}(\alpha_s)$ is a universal anomalous dimension independent of θ. At weak coupling, it has the following form in QCD

$$\Gamma_{\rm cusp}(\alpha_s) = \frac{\alpha_s}{\pi}C_F + \left(\frac{\alpha_s}{\pi}\right)^2 C_F\left\{N_c\left(\frac{67}{36} - \frac{\pi^2}{12}\right) - n_f\frac{5}{18}\right\} + \mathcal{O}(\alpha_s^3), \tag{19}$$

where n_f is the number of quark flavors. This expression was obtained within the dimensional regularization scheme (DREG) by using the $\overline{\rm MS}$-subtraction procedure, $\alpha_s \equiv \alpha_s^{\overline{\rm MS}}$.

The divergence of the anomalous dimension (18) for $\theta \to \infty$ indicates that the Wilson line with a light-like segment satisfies an evolution equation different from (5). The modified equation looks like [12]

$$\left(\mu\frac{\partial}{\partial\mu} + \beta(g)\frac{\partial}{\partial g} + 2\Gamma_{\rm cusp}(\alpha_s)\ln[i(vn)\,\xi\mu] + \Gamma(\alpha_s)\right)\langle W_\Pi(vn\,\xi\mu)\rangle = 0. \tag{20}$$

Here the factor of 2 stems from the presence of two cusps on the Π-shaped line contour and $\Gamma(\alpha_s)$ is a process-dependent anomalous dimension. The explicit dependence of the anomalous dimension on the renormalization scale μ implies the absence of the multiplicative renormalizability of the light-like Wilson line. Combining together Eqs. (20) and (16), we obtain the renormalization group equation for local composite operators $\langle \mathcal{O}_J^\Gamma(\mu^2)\rangle$ at large J. Matching its solution into (11), we find the asymptotic behavior of the anomalous dimensions of the twist-two quark operators for $J \to \infty$

$$\gamma_J^{(qq)}(\alpha_s) = 2\Gamma_{\rm cusp}(\alpha_s)\ln J + \mathcal{O}(J^0). \tag{21}$$

Repeating a similar analysis for the twist-two gluon operators, one can show that their matrix elements satisfy (16) with the Wilson line defined in the adjoint representation. Therefore, their anomalous dimension satisfies (21) upon replacing $C_F \to N_c$ leading to [12]

$$\gamma_J^{(gg)}(\alpha_s) = \frac{N_c}{C_F}\gamma_J^{(qq)}(\alpha_s) + \mathcal{O}(J^0). \tag{22}$$

In general, the quark and gluon operators mix with each other. However, at large J the mixing occurs through the exchange of a soft quark with momentum $\sim 1/J$. Its contribution to the corresponding anomalous dimensions is suppressed by a power of $1/J$ leading to

$$\gamma_J^{(gq)}(\alpha_s) = \mathcal{O}(1/J), \qquad \gamma_J^{(qg)}(\alpha_s) = \mathcal{O}(1/J). \tag{23}$$

We would like to stress that the relations (21)–(23) are valid to all orders in α_s. Remarkably enough, they hold both in QCD and its supersymmetric extensions. In the latter case, the mixing matrix has a bigger size due to the presence of additional scalar fields. Nevertheless, this matrix remains diagonal at large J. Since the fields in supersymmetric YM theories belong to the adjoint representation, the diagonal matrix elements are the same

$$\gamma_J^{(ab)} = 2\delta_{ab}\Gamma_{\rm cusp}(\alpha_s)\ln\, J + \mathcal{O}(\mathcal{J}') \tag{24}$$

with $a, b = (q, g, s)$ and $\Gamma_{\rm cusp}(\alpha_s)$ defined in the adjoint representation. As we will show in the next section, the fact that the anomalous dimension (12) turns out to be the Euler ψ function and, as a consequence, has a universal scaling behaviour (24), leads to a hidden integrability of the evolution equations in QCD [13–16].

4. INTEGRABILITY OF HIGH-ENERGY QCD

Scale dependence of the structure functions of deep inelastic scattering and hadronic light-cone wave functions can be studied in QCD using the Operator

Product Expansion. It can be reformulated as a problem of calculating the anomalous dimensions of the composite operators of a definite twist. The operators of the lowest twist have the following general form

$$\mathcal{O}^{(2)}_{N,k}(0) = (nD)^k \Phi_1(0)(nD)^{N-k}\Phi_2(0),$$
$$\mathcal{O}^{(3)}_{N,\mathbf{k}}(0) = (nD)^{k_1}\Phi_1(0)(nD)^{k_2}\Phi_2(0)(nD)^{N-k_1-k_2}\Phi_3(0), \tag{25}$$

where $\mathbf{k} \equiv (k_1, k_2)$ denotes the set of integer indices k_i; n_μ is a light-cone vector such that $n_\mu^2 = 0$; Φ_k denotes elementary fields (quarks, antiquarks, gluons), and $D_\mu = \partial_\mu - iA_\mu$ is a covariant derivative. The operators of a definite twist mix under renormalization with each other. In order to find their scaling dependence one has to diagonalize the corresponding matrix of the anomalous dimension and construct linear combination of such operators, the so-called conformal operators [17, 11]

$$\mathcal{O}^{\rm conf}_{N,q}(0) = \sum_{\mathbf{k}} C_{\mathbf{k},\mathbf{q}} \mathcal{O}_{N,\mathbf{k}}(0). \tag{26}$$

A unique feature of these operators is that they have an autonomous RG evolution

$$\Lambda^2 \frac{d}{d\Lambda^2}\, \mathcal{O}^{\rm conf}_{N,q}(0) = -\gamma_{N,q} \mathcal{O}^{\rm conf}_{N,q}(0). \tag{27}$$

Here Λ^2 is a UV cutoff and $\gamma_{N,q}$ is the corresponding anomalous dimension depending on some set of quantum numbers $\mathbf{q}$ to be specified below. It turns out that the problem of calculating the spectrum of the anomalous dimensions $\gamma_{N,q}$ to one-loop accuracy becomes equivalent to solving the Schrödinger equation for the $SL(2,\mathbb{R})$ Heisenberg spin magnet [13–16]. The number of sites in the magnet is equal to the number of fields entering into the operators under consideration.

To explain this correspondence it becomes convenient to introduce nonlocal light-cone (mesonic and baryonic) operators built from (two and three) quark fields of the same chirality

$$F(z_1, z_2) = \sum_{i=1}^{N_c} (\bar q_i^{\uparrow}\, \not{y})_\alpha (z_1 y)(\not{y} q_i^{\uparrow})_\beta (z_2 y),$$
$$F(z_1, z_2, z_3) = \sum_{i,j,k=1}^{N_c} \epsilon_{ijk} (\not{y} q_i^{\uparrow})_\alpha (z_1 y)(\not{y} q_j^{\uparrow})_\beta (z_2 y)(\not{y} q_k^{\uparrow})_\gamma (z_3 y) \tag{28}$$

with $(q_i^{\uparrow})_\alpha = (1+\gamma_5)(q_i)_\alpha/2$ is a quark field of the ith color and αth flavour. Here n_μ is a light-like vector ($n_\mu^2 = 0$) defining certain direction on the light-cone and the scalar variables z_i serve as coordinates of the fields along this direction. The quaks fields in (28) are transformed under the gauge transformations. It is tacitly assumed that the gauge invariance of the nonlocal operators $F(z_i)$ is restored by including the Wilson lines between the fields in the appropriate (fundamental or adjoint) representation. The conformal operators appear in the OPE expansion of the nonlocal operators (28) for small $z_1 - z_2$ and $z_2 - z_3$.

The field operators entering the definition of $F(z_i)$ are located on the light cone. This leads to the appearance of the additional light-cone singularities. They modify the renormalization properties of the nonlocal light-cone operators (28) and lead to nontrivial evolution equations which as we will show below are related to integrable chain models. We notice that there exists the following relation between the conformal three-particle operators (26) and the nonlocal operators (28)

$$\mathcal{O}^{\rm conf}_{N,q}(0) = \Psi_{N,q}(\partial_{z_1}, \partial_{z_2}, \partial_{z_3}) F(z_1, z_2, z_3)\bigg|_{z_i=0}, \tag{29}$$

where $\Psi_{N,q}(x_1, x_2, x_3)$ is a homogeneous polynomial in x_i of degree N

$$\Psi_{N,q}(x_1, x_2, x_3) = \sum_{\mathbf{k}} C_{\mathbf{k},\mathbf{q}} x_1^{k_1} x_2^{k_2} x_3^{N-k_1-k_2} \tag{30}$$

with the expansion coefficients $C_{\mathbf{k},\mathbf{q}}$ defined in (26). Similar relations hold for the twist-2 operators. The problem of defining the conformal operators is reduced to finding the polynomial coefficient functions $\Psi_{N,q}(x_i)$ and the corresponding anomalous dimensions $\gamma_{N,q}$.

Using the renormalization properties of the nonlocal light-cone operators (28) one can show [13–16] that to the one-loop accuracy the QCD evolution equation for the conformal operators (28) can be rewritten in the form of a Schrödinger equation

$$\mathcal{H} \Psi_{N,q}(x_i) = \gamma_{N,q} \Psi_{N,q}(x_i), \tag{31}$$

where the Hamiltonian $\mathcal{H}$ acts on the x_i variables which are conjugated to the derivatives ∂_{z_i} and, therefore, have the meaning of light-cone projection (np_i) of the momenta p_i carried by particles described by the quark fields in (28).

For mesonic operator $F(z_1, z_2)$, Eq. (28), the Hamiltonian $\mathcal{H}$ is given by [11]

$$\mathcal{H}^{(2)} = \frac{2\alpha_{\rm s}}{\pi} C_F \left[H_{qq}(J_{12}) + 1/4 \right], \qquad H_{qq}(J_{12}) = \psi(J_{12}) - \psi(2), \tag{32}$$

where $C_F = (N_c^2 - 1)/(2N_c)$ and the operator J_{12}, the so-called conformal spin, is defined as

$$J_{12}(J_{12} - 1) = -\left(\partial_{x_1} - \partial_{x_2}\right)^2 x_1 x_2. \tag{33}$$

The eigenfunctions for the Hamiltonian (32) take the form

$$\Psi^{(2)}_N(x_1, x_2) = (x_1 + x_2)^N C_N^{3/2}\left(\frac{x_1 - x_2}{x_1 + x_2}\right), \tag{34}$$

where $C_N^{3/2}$ are Gegenbauer polynomials. The corresponding eigenvalues define the anomalous dimensions of the twist-2 mesonic operators built from two quarks with the same helicity

$$\gamma_N^{(2)} = \frac{2\alpha_s}{\pi} C_F[\psi(N+2) - \psi(2) + 1/4] = \frac{2\alpha_s}{\pi} C_F \left[\sum_{k=1}^{N} \frac{1}{k+1} + \frac{1}{4}\right], \quad (35)$$

which coincides with (12). At large N this expression has well-known asymptotic behaviour $\gamma_N^{(2)} \sim (2\alpha_s C_F/\pi) \ln N$.

For baryonic operator built from three quark fields of the same chirality the evolution kernel is given by [13, 14]

$$\mathcal{H}^{(3)} = \frac{\alpha_s}{2\pi} \left\{(1 + 1/N_c) \left[H_{qq}(J_{12}) + H_{qq}(J_{23}) + H_{qq}(J_{31})\right] + 3C_F/2\right\} \quad (36)$$

with H_{qq} given by (32) and J_{ik} being the two-particle conformal spins. It is conformal symmetry of QCD Lagrangian which dictates that the two-particle Hamiltonian in (32) and (36) is a function of the conformal two-particle spin, but it does not fix this function. According to (32), the latter is given in QCD by the Euler ψ function and it leads to the following remarkable property of the evolution equations for anomalous dimensions of baryonic operators [13–16]. The Schrödinger equation (31) with the Hamiltonian defined in this way has a hidden integral of motion, $[\mathcal{H}^{(3)}, q] = 0$

$$q = i\,(\partial_{x_1} - \partial_{x_2})(\partial_{x_2} - \partial_{x_3})(\partial_{x_3} - \partial_{x_1})\, x_1 x_2 x_3, \quad (37)$$

and, as a consequence, the Schrödinger equation (31) is completely integrable. Moreover, one can identify (36) as the Hamiltonian of a quantum XXX Heisenberg magnet of $SL(2, \mathbb{R})$. This model is a generalization of the celebrated Heisenberg spin$-1/2$ magnet which has been studied in 1931 by Bethe as a model of one-dimensional metal and which has been solved by the method well known nowadays as the Bethe Ansatz. In application to QCD, the Bethe Ansatz technique allows one to solve the evolution equations and the detailed results can be found in [18]. The obtained solutions have direct applications to phenomenology of the baryon wave functions, scale dependence of twist-3 spin structure functions, etc.

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