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PATH-INTEGRAL APPROACH FOR SUPERINTEGRABLE POTENTIALS ON SPACES OF NONCONSTANT CURVATURE: II. DARBOUX SPACES $D_{\rm III}$ AND $D_{\rm IV}$

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This is the second paper on the path-integral approach of superintegrable systems on Darboux spaces, spaces of nonconstant curvature. We analyze in the spaces $D_{\rm III}$ and $D_{\rm IV}$ five and, respectively, four superintegrable potentials, which were first given by Kalnins et al. We are able to evaluate the path integral in most of the separating coordinate systems, leading to expressions for the Green functions, the discrete and continuous wave functions, and the discrete energy-spectra. In some cases, however, the discrete spectrum cannot be stated explicitly, because it is determined by a higher order polynomial equation. We also show that the free motion in Darboux space of type III can also contain bound states, provided the boundary conditions are appropriate. We can state the energy spectrum and the wave functions, respectively.

Это вторая статья, посвященная приближению интегралов по путям для суперинтегрируемых систем на пространствах Дарбу, пространствах переменной кривизны. На пространствах Дарбу $D_{\rm III}$ и $D_{\rm IV}$ проводится анализ пяти и, соответственно, четырех суперинтегрируемых потенциалов, которые впервые были представлены Калнинсом и др. Нам удалось вычислить интеграл по путям в наиболее разделяющихся системах координат, что приводит к выражениям для функций Грина, волновым функциям дискретного и непрерывного спектров и дискретному спектру энергий. Однако в некоторых случаях дискретный спектр установить не удается, так как он определяется полиномиальным уравнением более высокого порядка. Показано, что свободное движение в пространстве Дарбу III типа также может содержать связанные состояния при определенных граничных условиях. Соответственно, для них можно установить спектр энергий и волновые функции.

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1. INTRODUCTION

In the previous publication [21] we have started to study superintegrable systems on spaces of nonconstant curvature, i.e., Darboux spaces. These spaces were introduced by Kalnins et al. [26, 28]. In the first paper we have studied

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the Darboux spaces $D_{\rm I}$ and $D_{\rm II}$, and we continue our study by considering the two other Darboux spaces $D_{\rm III}$ and $D_{\rm IV}$ with five and, respectively, four superintegrable potentials as determined in [26].

We find a rich structure of the spectrum of these potentials yielding bound and continuous states. As it turns out, already the free motion on $D_{\rm III}$ can give a positive continuous and an infinite negative discrete spectrum. This situation is similar to that for the quantum motion on the $SU(1,1)$ manifold [2], respectively, on the $SU(2,2)$ [6] and $SO(2,2)$ manifold [30].

The notion of superintegrable systems was introduced by Winternitz and co-workers in [9, 47], Wojciechowski [48], and was developed further later on also by Evans [7]. Superintegrable potentials have the property of finding additional constants of motion. In two dimensions one has in total three functional independent constants of motion and in three dimensions one has four (minimal superintegrable) and five (maximal superintegrable) functional independent constants of motion. Well-known examples are the Coulomb potential with its Lenz–Runge vector and the harmonic oscillator with its quadrupole moment. Another property of superintegrable potentials is that usually the corresponding equations in classical and quantum mechanics separate in more than one coordinate system.

Similar studies of the quantum motion on spaces with and without curvature have been investigated in [17] for two- and three-dimensional flat space, in [18] for the two- and three-dimensional sphere, and in [19] and [20] for the two- and three-dimensional hyperboloid. In all these cases the path integral method [8, 22, 39, 45] was applied to find the bound and continuous states, i.e., wave functions and the explicit form of the spectrum. We have not considered complexified spaces as in [37] for the two-dimensional complex sphere or in [34–36] for the two-dimensional complex Euclidean space. In particular, in [34] coordinate systems on the two-dimensional complex sphere and corresponding superintegrable potentials, and in [36] coordinate systems on the two-dimensional complex plane and corresponding superintegrable potentials were discussed. The goal of [34, 36] was to extend the notion of superintegrable potentials of real spaces to the corresponding complexified spaces. The findings were that there are, in addition to the four coordinate systems on the real two-dimensional Euclidean plane, three more coordinate systems and also three more superintegrable potentials. Similarly, in addition to the two coordinate systems on the real two-dimensional sphere there are three more coordinate systems on the complex sphere and four more superintegrable potentials. This is not surprising because the complex plane contains not only the Euclidean plane but also the pseudo-Euclidean plane (10 coordinate systems [13, 23, 24]), and the complex sphere contains not only the real sphere but also the two-dimensional hyperboloid (9 coordinate systems [13, 24, 29, 43]).

However, a complexified space is an abstract object. In order to obtain the actual spectrum of a given potential formulated in a coordinate system one has to consider a real version of the complexified space, e.g., the complex sphere: One has to determine whether one considers the potential on the real sphere or on the real hyperboloid. The complexification serves only as a tool for a unified investigation.

Further studies on superintegrability in spaces with constant curvature are due to [31, 33] (hyperboloid with new potentials), [32] (sphere and Euclidean space), [37] and [38] with a general theory about the connection of separation in nonsubgroup coordinate systems of superintegrable systems and quasi-exactly-solvable problems [46].

An extension of the study of path integration on spaces of constant curvature is the investigation of path integral formulations in spaces of nonconstant curvature. Kalnins et al. [26, 28] denoted four types of two-dimensional spaces of nonconstant curvature, labeled by $D_{\rm I}$–$D_{\rm IV}$, which are called Darboux spaces [40]. In terms of the infinitesimal distance they are described by (the coordinates (u, v) will be called the (u, v) system; the (x, y) system in turn can be called light-cone coordinates):

$$\begin{aligned}
\text{(I)}\quad ds^2 &= (x+y)dxdy = \\
&= 2u(du^2+dv^2) \quad (x=u+iv, y=u-iv), \qquad (1.1)
\end{aligned}$$

$$\begin{aligned}
\text{(II)}\quad ds^2 &= \left(\frac{a}{(x-y)^2}+b\right)dxdy = \\
&= \frac{bu^2-a}{u^2}(du^2+dv^2) \quad \left(x=\frac{1}{2}(v+iu), y=\frac{1}{2}(v-iu)\right), \qquad (1.2)
\end{aligned}$$

$$\begin{aligned}
\text{(III)}\quad ds^2 &= \left(a\,\mathrm{e}^{-(x+y)/2}+b\,\mathrm{e}^{-x-y}\right)dxdy = \\
&= \mathrm{e}^{-2u}(b+a\,\mathrm{e}^{u})(du^2+dv^2) \quad (x=u-iv, y=u+iv), \qquad (1.3)
\end{aligned}$$

$$\begin{aligned}
\text{(IV)}\quad ds^2 &= -\frac{a\left(\mathrm{e}^{(x-y)/2}+\mathrm{e}^{(y-x)/2}\right)+b}{\left(\mathrm{e}^{(x-y)/2}-\mathrm{e}^{(y-x)/2}\right)^2}dxdy = \\
&= \left(\frac{a_+}{\sin^2 u}+\frac{a_-}{\cos^2 u}\right)(du^2+dv^2) \quad (x=u+iv, y=u-iv), \qquad (1.4)
\end{aligned}$$

where a and b are additional (real) parameters ($a_\pm = (a \pm 2b)/4$). These surfaces are also called surfaces of revolution [5, 25, 26]. Kalnins et al. [26, 28] studied not only the solution of the free motion, but also placed emphasis on the superintegrable systems in these spaces.

The Gaussian curvature in a space with metric $ds^2 = g(u,v)(du^2 + dv^2)$ is given by ($g = \det\, g(u,v)$)

$$G = -\frac{1}{2g}\left(\frac{\partial^2}{\partial u^2} + \frac{\partial^2}{\partial v^2}\right) \ln\, g. \tag{1.5}$$

Equation (1.5) will be used to discuss shortly the curvature properties of the Darboux spaces, including their limiting cases of constant curvature.

In the following sections we discuss superintegrable potentials in each of the two Darboux spaces $D_{\rm III}$ and $D_{\rm IV}$, respectively. We set up the classical Lagrangian and Hamiltonian, the quantum operator, and formulate and solve (if possible) the corresponding path integral. We also discuss some of the limiting cases of the Darboux spaces, i.e., where we obtain a space of constant (zero or negative) curvature. For the Darboux space $D_{\rm III}$ the zero-curvature case $\mathbb{R}^2$ emerges. In $D_{\rm IV}$ we find a hyperboloid.

In the last section we summarize our results, where we also include the findings of our previous paper which dealt with superintegrable potentials on $D_{\rm I}$ and $D_{\rm II}$.

In the first two appendices we add some additional material about the path integral evaluation of the free motion in $D_{\rm IV}$ in degenerate elliptic coordinates. In the third appendix we summarize briefly the path integral investigation of some remaining superintegrable potentials on the two-dimensional Euclidean plane. Finally, in the fourth appendix an example of a potential on the two-dimensional complex sphere will be given.

2. SUPERINTEGRABLE POTENTIALS ON DARBOUX SPACE $D_{\rm III}$

The coordinate systems to be considered in the Darboux space $D_{\rm III}$ are as follows:

$$((u,v)\text{ system})\quad x = v + iu, \quad y = v - iu, \tag{2.1}$$

$$(\text{Polar:})\quad \xi = \varrho\cos\varphi, \quad \eta = \varrho\sin\varphi \quad (\varrho > 0, \varphi \in [0, 2\pi]), \tag{2.2}$$

$$(\text{Parabolic:})\quad \xi = 2\,{\rm e}^{-u/2}\cos\,\frac{v}{2}, \quad \eta = 2\,{\rm e}^{-u/2}\sin\,\frac{v}{2},$$

$$u = \ln\frac{4}{\xi^2 + \eta^2},\ v = \arcsin\,\frac{2\xi\eta}{\xi^2 + \eta^2}\ (\xi \in \mathbb{R}, \eta > 0), \tag{2.3}$$

$$(\text{Elliptic:})\quad \xi = d\cosh\,\omega\cos\,\varphi,\ \eta = d\sinh\,\omega\sin\,\varphi\ (\omega > 0, \varphi \in [-\pi, \pi]), \tag{2.4}$$

$$(\text{Hyperbolic:})\quad \xi = \frac{\mu-\nu}{2\sqrt{\mu\nu}} + \sqrt{\mu\nu},\ \eta = i\left(\frac{\mu-\nu}{2\sqrt{\mu\nu}} - \sqrt{\mu\nu}\right)\ (\mu, \nu > 0). \tag{2.5}$$

For the line element we get (we also display where the metric is rescaled in such a way that we set $a = b = 1$ [26]):

$$ds^2 = \mathrm{e}^{-2u}(b + a\,\mathrm{e}^u)(du^2 + dv^2) = (\mathrm{e}^{-u} + \mathrm{e}^{-2u})(du^2 + dv^2), \quad (2.6)$$

$$\text{(Polar:)} = \left(a + \frac{b}{4}\varrho^2\right)(d\varrho^2 + \varrho^2 d\varphi^2) = \left(1 + \frac{1}{4}\varrho^2\right)(d\varrho^2 + \varrho^2 d\varphi^2), \quad (2.7)$$

$$\text{(Parabolic:)} = \left(a + \frac{b}{4}(\xi^2 + \eta^2)\right)(d\xi^2 + d\eta^2) = \\ = \left(1 + \frac{1}{4}(\xi^2 + \eta^2)\right)(d\xi^2 + d\eta^2), \quad (2.8)$$

$$\text{(Elliptic:)} = \left(a + \frac{b}{4}d^2(\sinh^2\omega + \cos^2\varphi)\right)d^2 \times \\ \times (\sinh^2\omega + \sin^2\varphi)(d\omega^2 + d\varphi^2), \quad (2.9)$$

$$\text{(Hyperbolic:)} = \left(a + \frac{b}{2}(\mu - \nu)\right)(\mu + \nu)\left(\frac{d\mu^2}{\mu^2} - \frac{d\nu^2}{\nu^2}\right). \quad (2.10)$$

For the Gaussian curvature we find

$$G = -\frac{ab\,\mathrm{e}^{-3u}}{(b\,\mathrm{e}^{-2u} + a\,\mathrm{e}^{-u})^4}. \quad (2.11)$$

For, e.g., $a = 1, b = 0$ we recover the two-dimensional flat space with the corresponding coordinate systems. To assure the positive definiteness of the metric (1.3), we can require $a, b > 0$. We introduce the following constants of motion on D_{III}:

$$X_1 = \frac{1}{4}\frac{\mathrm{e}^{2u}}{a + b\,\mathrm{e}^u}\cos v \cdot p_u^2 - \frac{1}{4}\frac{\mathrm{e}^u(e^u + 2)}{a + b\,\mathrm{e}^u}\cos v \cdot p_v^2 + \frac{1}{2}\mathrm{e}^u \sin v \cdot p_u p_v, \quad (2.12)$$

$$X_2 = \frac{1}{4}\frac{\mathrm{e}^{2u}}{a + b\,\mathrm{e}^u}\sin v \cdot p_u^2 - \frac{1}{4}\frac{\mathrm{e}^u(\mathrm{e}^u + 2)}{a + b\,\mathrm{e}^u}\sin v \cdot p_v^2 + \frac{1}{2}\mathrm{e}^u \cos v \cdot p_u p_v, \quad (2.13)$$

$$K = p_v. \quad (2.14)$$

These operators satisfy the Poisson relations

$$\{K, X_1\} = -X_2, \qquad \{K, X_2\} = X_1, \qquad \{X_1, X_2\} = K\tilde{\mathcal{H}}_0, \quad (2.15)$$

and the functional relation

$$X_1^2 + X_2^2 - \tilde{\mathcal{H}}_0^2 - \tilde{\mathcal{H}}_0 K^2 = 0. \quad (2.16)$$

Table 1. **Constants of motion and limiting cases of coordinate systems on $D_{\rm III}$**

Metric	Constant of motion	$D_{\rm III}$	E_2 $(a=1, b=0)$
$e^{-2u}(b+a\,e^u)(du^2+dv^2)$	K^2	(u,v) system	Cartesian
$\left(a+\frac{b}{4}\varrho^2\right)(d\varrho^2+\varrho^2 d\varphi^2$	X_2	Polar	Polar
$\left(a+\frac{b}{4}(\xi^2+\eta^2)\right)(d\xi^2+d\eta^2)$	X_1	Parabolic	Parabolic
$\left(a+\frac{b}{4}d^2(\sinh^2\omega+\cos^2\varphi)\right)d^2\times$ $\times(\sinh^2\omega+\sin^2\varphi)(d\omega^2+d\varphi^2)$	$d^2X_1+2K^2$	Elliptic	Elliptic

The operators K, X_1, X_2 can be used to characterize the separating coordinate systems on $D_{\rm III}$, as indicated in Table 1. The corresponding quantum operators are given by

$$X_1 = \frac{1}{4}\,e^u\left[\frac{e^u\cos v}{a+b\,e^u}\cdot\partial_u^2 - \frac{e^u+2}{a+b\,e^u}\cos v\cdot\partial_v^2 + \right.$$
$$\left. + \left(2\sin v\cdot\partial_u\partial_v + \cos v\cdot\partial_u + \sin v\cdot\partial_v\right)\right], \qquad (2.17)$$

$$X_2 = \frac{1}{4}\,e^u\left[\frac{e^u\sin v}{a+b\,e^u}\cdot\partial_u^2 - \frac{e^u+2}{a+b\,e^u}\sin v\cdot\partial_v^2 - \right.$$
$$\left. - \left(2\cos v\cdot\partial_u\partial_v - \sin v\cdot\partial_u + \cos v\cdot\partial_v\right)\right], \qquad (2.18)$$

$$K = \partial_v. \qquad (2.19)$$

These operators satisfy the commutation relations

$$[\widehat{K},\widehat{X}_1] = -\widehat{X}_2, \quad [\widehat{K},\widehat{X}_2] = \widehat{X}_1, \quad [\widehat{X}_1,\widehat{X}_2] = \widehat{K}\widehat{H}_0, \qquad (2.20)$$

and the relation

$$\widehat{X}_1^2 + \widehat{X}_2^2 - \widehat{H}_0^2 - \widehat{H}_0\widehat{K}^2 + \frac{1}{4}\widehat{H}_0 = 0. \qquad (2.21)$$

(Let us note that by $\tilde{\mathcal{H}}_0$ the classical Hamiltonian without the $1/2m$ factor is meant. Keeping this factor is no problem, however, in the present form the algebra is simpler.)

We now state the superintegrable potentials on $D_{\rm III}$:

$$V_1(u,v) = \frac{2k_1\,\mathrm{e}^{-u}\cos\,v/2 + 2k_2\,\mathrm{e}^{-u}\sin v/2 + k_3}{a + \dfrac{b}{4}\,\mathrm{e}^{-u}}, \tag{2.22}$$

$$V_2(u,v) = \frac{1}{a + b\,\mathrm{e}^{-u}}\left[-\alpha + \mathrm{e}^{u}\frac{\hbar^2}{8m}\left(\frac{k_1^2 - 1/4}{\cos^2 v/2} + \frac{k_1^2 - 1/4}{\cos^2 v/2}\right)\right], \tag{2.23}$$

$$V_3(u,v) = \frac{1}{a + b\,\mathrm{e}^{-u}}\left[-\alpha + \frac{\hbar^2}{2m}4\,\mathrm{e}^{u}\Big(c_1^2\,\mathrm{e}^{-iv} - 2c_2\,\mathrm{e}^{-2iv}\Big)\right], \tag{2.24}$$

Table 2. **Separation of variables for the superintegrable potentials on $D_{\rm III}$**

Potential	Constants of motion	Separating coordinate system
V_1	$R_1 = X_1 + \dfrac{2k_1\xi(2+\eta^2) - 2k_2\eta(2+\xi^2) + k_3(\eta^2-\xi^2)}{4a + b(\xi^2+\eta^2)}$ $R_2 = X_2 + \dfrac{k_1\eta(\eta^2-\xi^2+4) + k_2\xi(\xi^2-\eta^2+4) - 2xk_3\xi\eta}{4a + b(\xi^2+\eta^2)}$	Parabolic Translated Parabolic $(\xi,\eta\to\xi\eta\pm c)$
V_2	$R_1 = X_1 +$ $+\dfrac{\hbar^2/m((k_1^2-1/4)\eta^2(\eta^2+2) - (k_2^2-1/4)\xi^2(\xi^2+2)) - \alpha(\eta^2-\xi^2)}{4a + b(\xi^2+\eta^2)}$ $R_2 = K^2 + \dfrac{\hbar^2}{8m}((k_1^2-1/4)\dfrac{\eta^2}{\xi^2} + (k_2^2-1/4)\dfrac{\xi^2}{\eta^2})$	(u,v) system Polar Parabolic
V_3	$R_1 = X_1 + iX_2 - \dfrac{-\alpha\mu^2\nu^2 + c_1^2\mu\nu - 2c_2(1+\mu-\nu)}{(a + b/2(\mu-\nu))(\mu+\nu)}$ $R_2 = K^2 - c_1^2\dfrac{\mu-\nu}{\mu\nu} + c_2\dfrac{(\mu-\nu)^2}{\mu^2\nu^2}$	Polar Hyperbolic
V_4	$R_1 = X_1 + iX_2 - K^2 -$ $-\dfrac{\mu\nu(d_1(\nu-2) + d_2(\mu+2) + m\omega^2(\nu-\mu+\mu\nu))}{(a + b/2(\mu-\nu))(\mu+\nu)}$ $R_1 = X_1 - iX_2 -$ $-\dfrac{(\mu-\nu)((\mu-\nu)(d_1\mu + d_2\nu) - m\omega^2(\mu^2+\nu^2+\mu\nu(2+\mu-\nu)))}{4(a + b/2(\mu-\nu))(\mu+\nu)}$	Hyperbolic Elliptic
V_5	$R_1 = X_1 + \dfrac{\hbar^2 v_0^2}{8m}\dfrac{\eta^2-\xi^2}{a + b/4(\xi^2+\eta^2)}$ $R_2 = X_1 - \dfrac{n\hbar^2 v_0^2}{4m}\dfrac{\xi\eta}{a + b/4(\xi^2+\eta^2)}$ $R_3 = K = p_v$	(u,v) system Polar Parabolic Elliptic Hyperbolic

$$V_4(\mu,\nu) = \frac{1}{\left(a+\frac{b}{2}(\mu-\nu)\right)(\mu+\nu)}\left[d_1\mu + d_2\nu + \frac{m}{2}\omega^2(\mu^2-\nu^2)\right], \quad (2.25)$$

$$V_5(u,v) = \frac{1}{a+b\,\mathrm{e}^{-u}}\frac{\hbar^2 v_0^2}{2m}. \quad (2.26)$$

In Table 2 we list the properties of these potentials on $D_{\rm III}$, where the coordinate systems, where an explicit path integral solution is possible, are underlined. We see that V_5 is a special case, and it has three integrals of motion. We will threat this case in some more detail as in the other spaces, because on $D_{\rm III}$ the free quantum motion can give bound state solutions (provided the constants are chosen properly). This feature has not been discussed in [14].

2.1. The Superintegrable Potential V_1 on $D_{\rm III}$. We state the potential V_1 in the respective coordinate systems

$$V_1(u,v) = \frac{2k_1\mathrm{e}^{-u}\cos v/2 + 2k_2\,\mathrm{e}^{-u}\,\sin v/2 + k_3}{a+\frac{b}{4}\,\mathrm{e}^{-u}}, \quad (2.27)$$

$$= \frac{k_1\xi + k_2\eta + k_3}{a+\frac{b}{4}(\xi^2+\eta^2)}, \quad (2.28)$$

$$= \frac{k_1\xi + k_2\eta + (k_1c - k_2c + k_3)}{a+\frac{b}{4}((\xi+c)^2+(\eta-c)^2)}, \quad (2.29)$$

and V_1 is also separable in translated parabolic coordinates $\xi \to \xi + c, \eta \to \eta - c$. The translated parabolic coordinates just modify the solution of a shifted harmonic oscillator, and this case we do not discuss separately.

2.1.1. Separation of V_1 in Parabolic Coordinates. The classical Lagrangian and Hamiltonian in parabolic coordinates on $D_{\rm III}$ are given by

$$\mathcal{L}(\xi,\dot\xi,\eta,\dot\eta) = \frac{m}{2}\left(a+\frac{b}{4}\right)(\xi^2+\eta^2))(\dot\xi^2+\dot\eta^2) - V(\xi,\eta), \quad (2.30)$$

$$\mathcal{H}(\xi,p_\xi,\eta,p_\xi) = \frac{1}{2m}\frac{1}{a+\frac{b}{4}(\xi^2+\eta^2)}(p_\xi^2+p_\eta^2) + V(\xi,\eta). \quad (2.31)$$

The canonical momenta are given by

$$p_\xi = \frac{\hbar}{i}\left(\frac{\partial}{\partial\xi} + \frac{b\xi}{a+\frac{b}{4}(\xi^2+\eta^2)}\right),\ p_\eta = \frac{\hbar}{i}\left(\frac{\partial}{\partial\eta} + \frac{b\eta}{a+\frac{b}{4}(\xi^2+\eta^2)}\right), \quad (2.32)$$

and for the quantum Hamiltonian (product ordering) we find

$$H = -\frac{\hbar^2}{2m}\frac{1}{a+\frac{b}{4}(\xi^2+\eta^2)}\left(\frac{\partial^2}{\partial\xi}+\frac{\partial^2}{\partial\eta^2}\right)+V(\xi,\eta), \tag{2.33}$$

$$= \frac{1}{2m}\sqrt{\frac{1}{a+\frac{b}{4}(\xi^2+\eta^2)}}\,(p_\xi^2+p_\eta^2)\sqrt{\frac{1}{a+\frac{b}{4}(\xi^2+\eta^2)}}+V(\xi,\eta). \tag{2.34}$$

Therefore we obtain for the path integral formulation for V_1

$$K^{(V_1)}(\xi'',\xi',\eta'',\eta';T) = \int\limits_{\xi(t')=\xi'}^{\xi(t'')=\xi''}\mathcal{D}\xi(t)\int\limits_{\eta(t')=\eta'}^{\eta(t'')=\eta''}\mathcal{D}\eta(t)\left(a+\frac{b}{4}(\xi^2+\eta^2)\right)\times$$

$$\times\exp\left\{\frac{i}{\hbar}\int\limits_0^T\left[\left(a+\frac{b}{4}(\xi^2+\eta^2)\right)(\dot\xi^2+\dot\eta^2)-\frac{k_1\xi+k_2\eta+k_3}{\left(a+\frac{b}{4}(\xi^2+\eta^2)\right)}\right]dt\right\}=$$

$$=\int\limits_{-\infty}^{\infty}\frac{dE}{2\pi\hbar}\,\mathrm{e}^{-iET/\hbar}\int\limits_0^\infty ds''\exp\left[\frac{i}{\hbar}\left(aE-k_3-\frac{k_1^2+k_2^2}{2m\omega^2}\right)s''\right]\times$$

$$\times K^{(V_1)}(\xi'',\xi',\eta'',\eta';s''), \tag{2.35}$$

with the time-transformed path integral $K(s'')$ given by

$$K^{(V_1)}(\xi'',\xi',\eta'',\eta';s'') = \int\limits_{\xi(0)=\xi'}^{\xi(s'')=\xi''}\mathcal{D}\xi(s)\int\limits_{\eta(0)=\eta'}^{\eta(s'')=\eta''}\mathcal{D}\eta(s)\times$$

$$\times\exp\left\{\frac{i}{\hbar}\int\limits_0^{s''}\left[\frac{m}{2}\left((\dot\xi^2+\dot\eta^2)-\frac{m}{2}\omega^2(\tilde\xi^2+\tilde\eta^2)\right)\right]ds\right\}. \tag{2.36}$$

The transformed variables $\tilde\xi,\tilde\eta$ are given by $\tilde\xi=\xi+k_1/m\omega^2, \tilde\eta=\eta+k_2/m\omega^2$, and $\omega^2=-bE/2m$. Similarly as in [14] we can determine the Green function to

have the form

$$G^{(V_1)}(\xi'',\xi',\eta'',\eta';E)=$$
$$=\int d\mathcal{E}\,\frac{m}{\pi\hbar^2 b}\sqrt{-\frac{m}{2E}}\,\Gamma\biggl(\frac{1}{2}+\frac{\tilde{\mathcal{E}}}{b\hbar}\sqrt{-\frac{m}{2E}}\biggr)\Gamma\biggl(\frac{1}{2}+\frac{\mathcal{E}}{b\hbar}\sqrt{-\frac{m}{2E}}\biggr)\times$$
$$\times D_{-\frac{1}{2}+\frac{\tilde{\mathcal{E}}}{b\hbar}\sqrt{-\frac{m}{2E}}}\biggl(\sqrt[4]{-\frac{8mEb^2}{\hbar^2}}\,\tilde{\xi}_>\biggr)D_{-\frac{1}{2}+\frac{\tilde{\mathcal{E}}}{b\hbar}\sqrt{-\frac{m}{2E}}}\biggl(-\sqrt[4]{-\frac{8mEb^2}{\hbar^2}}\,\tilde{\xi}_<\biggr)\times$$
$$\times D_{-\frac{1}{2}+\frac{\mathcal{E}}{b\hbar}\sqrt{-\frac{m}{2E}}}\biggl(\sqrt[4]{-\frac{8mEb^2}{\hbar^2}}\,\tilde{\eta}_>\biggr)D_{-\frac{1}{2}+\frac{\mathcal{E}}{b\hbar}\sqrt{-\frac{m}{2E}}}\biggl(-\sqrt[4]{-\frac{8mEb^2}{\hbar^2}}\,\tilde{\eta}_<\biggr).\quad(2.37)$$

The $D_\nu(z)$ are parabolic cylinder-functions [10, p. 1064], and the $\tilde{\mathcal{E}}$ is defined by $\tilde{\mathcal{E}}=aE-k_3-(k_1^2+k_2^2)/bE-\mathcal{E}$. On the other hand, we can insert for the discrete part of the Green function the harmonic oscillator wave functions and obtain

$$G^{(V_1)}_{\rm disc}(\xi'',\xi',\eta'',\eta';E)=\sum_{n_\xi=0}^{\infty}\sum_{n_\eta=0}^{\infty}\frac{N^2_{n_\xi n_\eta}}{E_{n_\xi n_\eta}-E}\times$$
$$\times\Psi^{\rm (HO)}_{n_\xi}(\xi'')\Psi^{\rm (HO)}_{n_\xi}(\xi')\Psi^{\rm (HO)}_{n_\eta}(\eta'')\Psi^{\rm (HO)}_{n_\eta}(\eta').\quad(2.38)$$

The wave functions for the harmonic oscillator are given by the well-known form in terms of Hermite-polynomials [10]

$$\Psi^{\rm (HO)}_n(x)=\biggl(\frac{m\omega}{\pi\hbar}\biggr)^{1/4}\biggl(\frac{1}{2^n n!}\biggr)^{1/2}H_n\biggl(\sqrt{\frac{m\omega}{\hbar}}\,x\biggr)\exp\biggl(-\frac{m\omega}{2\hbar}x^2\biggr).\quad(2.39)$$

$E_{n_\xi n_\eta}$ is determined by the equation

$$aE-k_3-\frac{k_1^2+k_2^2}{2m\omega^2}-\hbar(n_\xi+n_\eta+1)\sqrt{-\frac{bE}{2m}}=0,\quad(2.40)$$

which is actually an equation of the fourth order in E

$$E^4_{n_\xi n_\eta}+\biggl(\frac{b\hbar^2}{2ma^2}(n_\xi+n_\eta+1)^2-\frac{2k_3}{a}\biggr)E^3_{n_\xi n_\eta}-$$
$$-\biggl(2\frac{k_1^2+k_2^2}{ab}-\frac{k_3^2}{a^2}\biggr)E^2_{n_\xi n_\eta}+2k_3\frac{k_1^2+k_2^2}{a^2b}E_{n_\xi n_\eta}-\frac{(k_1^2+k_2^2)^2}{a^2b^2}=0.\quad(2.41)$$

We dot not solve this equation. Note that for $k_1=k_2=k_3=0$ a discrete spectrum emerges for the free motion on $D_{\rm III}$, a feature which will be discussed

in more detail in the subsection for V_5. For the special case $k_1 = k_2 = 0$ we obtain the solution ($N = n_\xi + n_\eta + 1$)

$$E_{n_\xi n_\eta \pm} = -\frac{b\hbar^2 N^2}{4ma^2} + \frac{k_3}{a} \pm \frac{1}{a}\sqrt{\left(\frac{b\hbar^2 N^2}{4am}\right)^2 - \frac{bk_3\hbar^2 N^2}{2am} - k_3^2}. \tag{2.42}$$

Note that $\omega_{n_\xi n_\eta}$ must be taken on $\omega_{n_\xi n_\eta} = \sqrt{-bE_{n_\xi n_\eta}/2m}$. The normalization $N_{n_\xi n_\eta}$ is determined by the residuum in $G^{(V_1)}(E)$. If one fixes the parameters a and b and the specific surface of revolution, a more detailed investigation can be performed (special cases, limiting cases, which sign of the square-root gives a positive definite Hilbert space, etc.). Because we do not fix these parameters, we keep both signs of the square-root expression (recall that the free motion on $D_{\rm III}$ allows already a discrete spectrum reaching to $-\infty$).

Note that for the translated parabolic coordinates, the variables $\tilde{\xi}, \tilde{\eta}$ are translated by $\pm c$, respectively; and the quantity $\mathcal{E}$, by an additional $Ebc^2/2$.

2.2. The Superintegrable Potential V_2 on $D_{\rm III}$. We state the potential V_2 in the respective coordinate systems

$$V_2(u,v) = \frac{1}{a + b\,{\rm e}^{-u}}\left[-\alpha + {\rm e}^u\frac{\hbar^2}{8m}\left(\frac{k_1^2 - 1/4}{\cos^2 v/2} + \frac{k_1^2 - 1/4}{\cos^2 v/2}\right)\right], \tag{2.43}$$

$$= \frac{1}{a + \dfrac{b}{4}\varrho^2}\left[-\alpha + \frac{\hbar^2}{2m\varrho^2}\left(\frac{k_1^2 - 1/4}{\cos^2\varphi} + \frac{k_2^2 - 1/4}{\sin^2\varphi}\right)\right], \tag{2.44}$$

$$= \frac{1}{a + \dfrac{b}{4}(\xi^2 + \eta^2)}\left[-\alpha + \frac{\hbar^2}{2m}\left(\frac{k_1^2 - 1/4}{\xi^2} + \frac{k_2^2 - 1/4}{\eta^2}\right)\right], \tag{2.45}$$

$$= \frac{1}{a + b\,{\rm e}^{-u}}\left[-\alpha + \frac{\hbar^2}{2md^2}\left(\frac{k_1^2 - 1/4}{\cosh^2\omega\cos^2\varphi} + \frac{k_2^2 - 1/4}{\sinh^2\omega\sin^2\varphi}\right)\right]. \tag{2.46}$$

V_2 is obviously separable in elliptic coordinates, but the corresponding path integral is not solvable, so this case will be omitted.

2.2.1. Separation of V_2 in the (u, v) System. The classical Lagrangian and Hamiltonian are given by

$$\mathcal{L}(u, \dot{u}, v, \dot{v}) = \frac{m}{2}\frac{b + a\,{\rm e}^u}{{\rm e}^{2u}}(\dot{u}^2 + \dot{v}^2) - V(u,v), \tag{2.47}$$

$$\mathcal{H}(u, p_u, v, p_v) = \frac{1}{2m}\frac{{\rm e}^{2u}}{b + a\,{\rm e}^u}(p_u^2 + p_v^2) + V(u,v). \tag{2.48}$$

The canonical momenta are given by

$$p_u = \frac{\hbar}{i}\left(\frac{\partial}{\partial u} - \frac{1}{2}\frac{a\,\mathrm{e}^{-u} + 2b\,\mathrm{e}^{-2u}}{a\,\mathrm{e}^{-u} + b\,\mathrm{e}^{-2u}}\right), \quad p_v = \frac{\hbar}{i}\frac{\partial}{\partial v}, \tag{2.49}$$

and for the quantum Hamiltonian we find

$$H = -\frac{\hbar^2}{2m}\frac{1}{a\,\mathrm{e}^{-u} + b\,\mathrm{e}^{-2u}}\left(\frac{\partial^2}{\partial u^2} + \frac{\partial^2}{\partial v^2}\right) + V(u,v), \tag{2.50}$$

$$= \frac{1}{2m}\sqrt{\frac{1}{a\,\mathrm{e}^{-u} + b\,\mathrm{e}^{-2u}}}\left(p_u^2 + p_v^2\right)\sqrt{\frac{1}{a\,\mathrm{e}^{-u} + b\,\mathrm{e}^{-2u}}} + V(u,v). \tag{2.51}$$

Therefore we obtain for the path integral ($f(u) = (a\,\mathrm{e}^{-u} + b\,\mathrm{e}^{-2u})$)

$$\begin{aligned}
K^{(V_2)}(u'',u',v'',v';T) &= \int\limits_{u(t')=u'}^{u(t'')=u''}\mathcal{D}u(t)\int\limits_{v(t')=v'}^{v(t'')=v''}\mathcal{D}v(t)(a\,\mathrm{e}^{-u} + b\,\mathrm{e}^{-2u})\times \\
&\times\exp\left(\frac{i}{\hbar}\int\limits_0^T\left\{(a\,\mathrm{e}^{-u} + b\,\mathrm{e}^{-2u})(\dot{u}^2 + \dot{v}^2) -\right.\right. \\
&\left.\left.-\frac{1}{a + b\,\mathrm{e}^{-u}}\left[-\alpha + \mathrm{e}^u\frac{\hbar^2}{8m}\left(\frac{k_1^2 - 1/4}{\cos^2 v/2} + \frac{k_1^2 - 1/4}{\cos^2 v/2}\right)\right]\right\}dt\right) = \\
&= \frac{1}{[f(u')f(u'')]^{1/4}}\sum_{l=0}^{\infty}\Phi_l^{(k_2,k_1)}\left(\frac{v''}{2}\right)\Phi_l^{(k_2,k_1)}\left(\frac{v'}{2}\right)\times \\
&\times\int\limits_{u(t')=u'}^{u(t'')=u''}\mathcal{D}u(t)(a\,\mathrm{e}^{-u} + b\,\mathrm{e}^{-2u})^{1/2}\exp\left(\frac{i}{\hbar}\int\limits_0^T\left\{(a\,\mathrm{e}^{-u} + b\,\mathrm{e}^{-2u})\dot{u}^2 -\right.\right. \\
&\left.\left.-\frac{1}{a + b\,\mathrm{e}^{-u}}\left[-\alpha + \mathrm{e}^u\frac{\hbar^2}{8m}\big(2l + 1 + |k_1| + |k_2|\big)\right]\right\}dt\right) = \int\limits_{-\infty}^{\infty}\frac{dE}{2\pi\hbar}\,\mathrm{e}^{-iET/\hbar}\times \\
&\times\int\limits_0^{\infty}ds''\exp\left[-\frac{i}{\hbar}\frac{\hbar^2}{8m}\big(2l + 1 + |k_1| + |k_2|\big)^2 s''\right]K_l^{(V_2)}(u'',u';s''),
\end{aligned} \tag{2.52}$$

with the time-transformed path integral $K_l(s'')$ given by

$$K_l^{(V_2)}(u'',u';s'') = \\ = \int\limits_{u(0)=u'}^{u(s'')=u''} \mathcal{D}u(s)\exp\left[\frac{i}{\hbar}\int\limits_0^{s''}\left(\frac{m}{2}\dot{u}^2 + Eb\,\mathrm{e}^{-2u} + (aE-\alpha)\,\mathrm{e}^{-u}\right)ds\right]. \quad (2.53)$$

The $\Phi_n^{(k_1,k_2)}(\beta)$ are the wave functions of the Pöschl–Teller potential, which are given by

$$V(x) = \frac{\hbar^2}{2m}\left(\frac{\alpha^2-1/4}{\sin^2 x} + \frac{\beta^2-1/4}{\cos^2 x}\right), \quad (2.54)$$

$$\Phi_n^{(\alpha,\beta)}(x) = \left[2(\alpha+\beta+2l+1)\frac{l!\Gamma(\alpha+\beta+l+1)}{\Gamma(\alpha+l+1)\Gamma(\beta+l+1)}\right]^{1/2}\times \\ \times(\sin x)^{\alpha+1/2}(\cos x)^{\beta+1/2}P_n^{(\alpha,\beta)}(\cos 2x). \quad (2.55)$$

Equation (2.53) is a path integral for the Morse potential. Inserting the corresponding solution [22] we obtain

$$G^{(V_2)}(u'',u',v'',v';E) = \sum_{l=0}^{\infty}\Phi_l^{(k_2,k_1)}\left(\frac{v''}{2}\right)\Phi_l^{(k_2,k_1)}\left(\frac{v'}{2}\right)\times \\ \times\sqrt{-\frac{m}{2bE}}\,\frac{m\Gamma\left(\frac{1}{2}+\lambda+\frac{aE-\alpha}{\hbar}\sqrt{-\frac{m}{2bE}}\right)}{\hbar\Gamma(1+2\lambda)\,\mathrm{e}^{(u'+u'')/2}}\times \\ \times W_{\frac{aE-\alpha}{\hbar}\sqrt{-\frac{m}{2bE}},\lambda}\left(\frac{\sqrt{-8mbE}}{\hbar}\,\mathrm{e}^{-u_<}\right)M_{\frac{aE-\alpha}{\hbar}\sqrt{-\frac{m}{2bE}},\lambda}\left(\frac{\sqrt{-8mbE}}{\hbar}\,\mathrm{e}^{-u_>}\right). \quad (2.56)$$

Inserting the bound state wave functions for the Morse potential gives the bound state contribution of $G^{(V_2)}(E)$

$$G_{\text{disc}}^{(V_2)}(u'',u',v'',v';E) = \sum_{l=0}^{\infty}\Phi_l^{(k_2,k_1)}\left(\frac{v''}{2}\right)\Phi_l^{(k_2,k_1)}\left(\frac{v'}{2}\right)\times \\ \times\sum_{l=0}^{\infty}\frac{N_{nl}^2}{E_{nl}-E}\Psi_n^{(\mathrm{MP})}(u'')\Psi_n^{(\mathrm{MP})}(u'), \quad (2.57)$$

$$\Psi_n^{(\mathrm{MP})}(u) = N_{nl}\left[\left(-\frac{2mbE_{nl}}{\hbar^2}\right)^{\frac{aE_{nl}-\alpha}{\hbar}\sqrt{-\frac{m}{2bE_{nl}}}-n-1/2}\times\right.$$

$$\left.\times\frac{\left(\dfrac{aE_{nl}-\alpha}{\hbar}\sqrt{-\dfrac{m}{2bE_{nl}}}-2n-1\right)}{\Gamma\left(\dfrac{aE_{nl}-\alpha}{\hbar}\sqrt{-\dfrac{2m}{bE_{nl}}}-n\right)}\right]^{1/2}\times$$

$$\times\exp\left[(u'+u'')\left(\frac{aE_{nl}-\alpha}{\hbar}\sqrt{-\frac{m}{2bE_{nl}}}-n-\frac{1}{2}\right)-\frac{\sqrt{-2mbE_{nl}}}{\hbar}\,\mathrm{e}^u\right]\times$$

$$\times L_n^{\left(\frac{aE_{nl}-\alpha}{\hbar}\sqrt{-\frac{2m}{bE_{nl}}}-2n-1\right)}\left(\frac{-8mbE_{nl}}{\hbar}\,\mathrm{e}^u\right). \quad (2.58)$$

The $L_n^{(\alpha)}(z)$ are Laguerre polynomials [10]. Here, the spectrum E_{nl} is determined by

$$aE_{nl}-\alpha-\hbar\sqrt{-\frac{bE_{nl}}{2m}}\,(2n+2l+|k_1|+|k_2|+2), \quad (2.59)$$

which is a quadratic equation in E_{nl} with solution ($N=2n+2l+|k_1|+|k_2|+2$)

$$E_{nl\pm}=\frac{1}{2a^2}\left[-\left(\frac{b\hbar^2}{2m}N^2-2a\alpha\right)\pm\frac{b\hbar^2}{2m}N^2\sqrt{1-\frac{8a\alpha m}{b\hbar^2N^2}}\right], \quad (2.60)$$

and the normalization constants N_{nl} are determined by the residuum of (2.56). For large n, l we have

$$E_{nl-}\simeq-\frac{b\hbar^2}{m}(2n+2l+|k_1|+|k_2|+2)^2, \quad (2.61)$$

$$E_{nl+}\simeq-\frac{m\alpha^2}{2b\hbar^2(2n+2l+|k_1|+|k_2|+2)^2}, \quad (2.62)$$

with E_{nl+} showing a Coulomb-like behavior.

2.2.2. Separation of V_2 in Polar Coordinates. In the coordinates (ϱ,φ) the classical Lagrangian and Hamiltonian take on the form

$$\mathcal{L}(\varrho,\dot\varrho,\varphi,\dot\varphi)=\frac{m}{2}\left(a+\frac{b}{4}\varrho^2\right)(\dot\varrho^2+\varrho^2\dot\varphi^2)-V(\varrho,\varphi), \quad (2.63)$$

$$\mathcal{H}(\varrho,p_\varrho,\varphi,p_\varphi)=\frac{1}{2m}\frac{1}{a+\dfrac{b}{4}\varrho^2}\left(p_\varrho^2+\frac{1}{\varrho^2}p_\varphi^2\right)+V(\varrho,\varphi). \quad (2.64)$$

The canonical momenta are given by

$$p_\varrho = \frac{\hbar}{i}\left(\frac{\partial}{\partial\varrho} + \frac{b\varrho}{4a+b\varrho^2} + \frac{1}{2\varrho}\right), \quad p_\varphi = \frac{\hbar}{i}\frac{\partial}{\partial\varphi}. \tag{2.65}$$

Therefore the quantum Hamiltonian is given by

$$H = -\frac{\hbar^2}{2m}\frac{1}{a+\frac{b}{4}\varrho^2}\left(\frac{\partial^2}{\partial\varrho^2} + \frac{1}{\varrho}\frac{\partial}{\partial\varrho} + \frac{1}{\varrho^2}\frac{\partial^2}{\partial\varphi^2}\right) + V(\varrho,\varphi) = \tag{2.66}$$

$$= \frac{1}{2m}\sqrt{\frac{1}{a+\frac{b}{4}\varrho^2}}\left(p_\varrho^2 + \frac{1}{\varrho^2}p_\varphi^2\right)\sqrt{\frac{1}{a+\frac{b}{4}\varrho^2}} + V(\varrho,\varphi) - \left(a+\frac{b}{4}\varrho^2\right)^{-1}\frac{\hbar^2}{8m\varrho^2}, \tag{2.67}$$

and in this case we have an additional quantum potential $\propto \hbar^2$. This gives for the path integral $\left(f(\varrho) = a + \frac{b}{4}\varrho^2 = \sqrt{g}\right)$

$$\begin{aligned}
K^{(V_2)}(\varrho'',\varrho',\varphi'',\varphi';T) &= \int\limits_{\varrho(t')=\varrho'}^{\varrho(t'')=\varrho''}\mathcal{D}\varrho(t)\int\limits_{\varphi(t')=\varphi'}^{\varphi(t'')=\varphi''}\mathcal{D}\varphi(t) f(\varrho)\varrho\times\\
&\times\exp\left(\frac{i}{\hbar}\int_0^T\left\{\frac{m}{2}f(\varrho)(\dot\varrho^2+\varrho^2\dot\varphi^2) - \frac{1}{f(\varrho)}\times\right.\right.\\
&\times\left.\left.\left[-\alpha + \frac{\hbar^2}{2m\varrho^2}\left(\frac{k_1^2-1/4}{\cos^2\varphi} + \frac{k_2^2-1/4}{\sin^2\varphi} - \frac{1}{4}\right)\right]\right\}dt\right) =\\
&= \sum_{l=0}^\infty \Phi_l^{(k_2,k_1)}(\varphi'')\Phi_l^{(k_2,k_1)}(\varphi')\frac{1}{[(\varrho'\varrho'')^2 f(\varrho')f(\varrho'')]^{1/4}}\int\limits_{\varrho(t')=\varrho'}^{\varrho(t'')=\varrho''}\mathcal{D}\varrho(t) f^{1/2}(\varrho)\times\\
&\times\exp\left\{\frac{i}{\hbar}\int_0^T\left[\frac{m}{2}f(\varrho)\dot\varrho^2 - \frac{1}{f(\varrho)}\left(-\alpha + \frac{\hbar^2}{2m}\frac{\lambda^2-1/4}{\varrho^2}\right)\right]dt\right\} =\\
&= \frac{1}{\sqrt{\varrho'\varrho''}}\sum_{l=0}^\infty \Phi_l^{(k_2,k_1)}(\varphi'')\Phi_l^{(k_2,k_1)}(\varphi')\times\\
&\times\int_{-\infty}^\infty \frac{dE}{2\pi\hbar}\,\mathrm{e}^{-iET/\hbar}\int_0^\infty ds''\exp\left[\frac{i}{\hbar}(aE-\alpha)s''\right]K_l^{(V_2)}(\varrho'',\varrho';s''),
\end{aligned}\tag{2.68}$$

with the time-transformed path integral $K_l(s'')$ given by ($\lambda = 2l+|k_1|+|k_2|+1$))

$$K_l^{(V_2)}(\varrho'', \varrho'; s'') = \int\limits_{\varrho(0)=\varrho'}^{\varrho(s'')=\varrho''} \mathcal{D}\varrho(s) \times$$

$$\times \exp\left[\frac{i}{\hbar}\int\limits_0^{s''}\left(\frac{m}{2}\dot{\varrho}^2 + \frac{Eb}{4}\varrho^2 - \frac{\hbar^2}{2m}\frac{\lambda^2 - 1/4}{\varrho^2}\right)ds\right] =$$

$$= \frac{m\omega\sqrt{\varrho'\varrho''}}{i\hbar \sin\omega s''}\exp\left[-\frac{m\omega}{2i\hbar}({\varrho'}^2 + {\varrho''}^2)\cot\omega s''\right] I_\lambda\left(\frac{m\omega\varrho'\varrho''}{i\hbar \sin\omega s''}\right). \quad (2.69)$$

Performing the s'' integration yields the Green function

$$G^{(V_2)}(\varrho'', \varrho', \varphi'', \varphi'; E) = \sum_{l=0}^{\infty} \Phi_l^{(k_2,k_1)}(\varphi'')\Phi_l^{(k_2,k_1)}(\varphi') \times$$

$$\times \sqrt{-\frac{2m}{Eb}}\,\frac{\Gamma\left[\frac{1}{2}\left(1+\lambda - \frac{1}{\hbar}(aE-\alpha)\sqrt{-2m/bE}\right)\right]}{\Gamma(1+\lambda)\sqrt{\varrho'\varrho''}} M_{\frac{aE-\alpha}{2\hbar}\sqrt{-\frac{2m}{bE}},\frac{\lambda}{2}} \times$$

$$\times \left(\frac{m}{\hbar}\sqrt{-\frac{bE}{2m}}\,\varrho_<^2\right) M_{\frac{aE-\alpha}{2\hbar}\sqrt{-\frac{2m}{bE}},\frac{\lambda}{2}}\left(\frac{m}{\hbar}\sqrt{-\frac{bE}{2m}}\,\varrho_>^2\right). \quad (2.70)$$

Inserting the expansion into Laguerre polynomial yields the discrete contribution of the Green function

$$G_{\text{disc}}^{(V_2)}(\varrho'', \varrho', \varphi'', \varphi'; E) = \frac{1}{\sqrt{\varrho'\varrho''}}\sum_{l=0}^{\infty} \Phi_l^{(k_2,k_1)}(\varphi'')\Phi_l^{(k_2,k_1)}(\varphi') \times$$

$$\times \sum_{n=0}^{\infty} \frac{N_{nl}^2}{E_{nl} - E}\Psi_n^{(\text{RHO},\lambda)}(\varrho'')\Psi_n^{(\text{RHO},\lambda)}(\varrho'). \quad (2.71)$$

The wave functions for the radial harmonic oscillator $V(r) = \dfrac{m}{2}\omega^2 - \dfrac{\hbar^2}{2m}\dfrac{\lambda^2 - 1/4}{r^2}$ have the form [22, 44]

$$\Psi_n^{(\text{RHO},\lambda)}(r) =$$

$$= \sqrt{\frac{2m}{\hbar}\frac{n!}{\Gamma(n+\lambda+1)}}\, r\left(\frac{m\omega}{\hbar}r\right)^{\lambda/2}\exp\left(-\frac{m\omega}{2\hbar}r^2\right) L_n^{(\lambda)}\left(\frac{m\omega}{\hbar}r^2\right). \quad (2.72)$$

The spectrum E_{nl} is determined by

$$aE_{nl} - \alpha - \hbar\sqrt{-\frac{bE_{nl}}{2m}}\,(2n + 2l + |k_1| + |k_2| + 2), \tag{2.73}$$

which is the same as in (2.60). In the wave functions $\Psi_n^{(\mathrm{RHO},\lambda)}(\varrho)$ the quantity ω has to be taken on $\omega = \sqrt{-bE_{nl}/2m}$, and the normalization constants N_{nl} are determined by the residuum of (2.69).

2.2.3. Separation of V_2 in Parabolic Coordinates. We insert the potential V_2 into the path integral and obtain $\big(f = a + \frac{b}{4}(\xi^2 + \eta^2)\big)$

$$\begin{aligned}
K^{(V_2)}(\xi'', \xi', \eta'', \eta'; T) &= \int\limits_{\xi(t')=\xi'}^{\xi(t'')=\xi''} \mathcal{D}\xi(t) \int\limits_{\eta(t')=\eta'}^{\eta(t'')=\eta''} \mathcal{D}\eta(t) f(\xi,\eta)\times \\
&\times \exp\left(\frac{i}{\hbar}\int\limits_0^T \left\{ f(\xi,\eta)(\dot\xi^2 + \dot\eta^2) - \frac{1}{f(\xi,\eta)} \times \right.\right. \\
&\times \left.\left. \left[-\alpha + \frac{\hbar^2}{2m}\left(\frac{k_1^2 - 1/4}{\xi^2} + \frac{k_2^2 - 1/4}{\eta^2}\right)\right]\right\} dt\right) = \\
= \int\limits_{-\infty}^{\infty} \frac{dE}{2\pi\hbar}\,\mathrm{e}^{-iET/\hbar} &\int\limits_0^{\infty} ds'' \exp\left[\frac{i}{\hbar}\big(aE - \alpha\big)s''\right] K^{(V_2)}(\xi'', \xi', \eta'', \eta'; s''),
\end{aligned} \tag{2.74}$$

with the time-transformed path integral $K^{(V_2)}(s'')$ given by $(\omega^2 = -bE/2m)$

$$\begin{aligned}
K^{(V_2)}(\xi'', \xi', \eta'', \eta'; s'') &= \int\limits_{\xi(0)=\xi'}^{\xi(s'')=\xi''} \mathcal{D}\xi(s) \int\limits_{\eta(0)=\eta'}^{\eta(s'')=\eta''} \mathcal{D}\eta(s)\times \\
&\times \exp\left\{\frac{i}{\hbar}\int\limits_0^{s''} \left[\frac{m}{2}\left((\dot\xi^2 + \dot\eta^2) - \frac{m}{2}\omega^2(\xi^2 + \eta^2)\right) - \right.\right. \\
&\left.\left. - \frac{\hbar^2}{2m}\left(\frac{k_1^2 - 1/4}{\xi^2} + \frac{k_2^2 - 1/4}{\eta^2}\right)\right] ds\right\} = \frac{m\omega\sqrt{\xi'\xi''}}{i\hbar\sin\omega s''}\times \\
&\times \exp\left[-\frac{m\omega}{i\hbar\sin\omega s''}({\xi'}^2 + {\xi''}^2\cot\omega s'')\right] I_{k_2}\left(\frac{m\omega\xi'\xi''}{i\hbar\sin\omega s''}\right) \frac{m\omega\sqrt{\eta'\eta''}}{i\hbar\sin\omega s''}\times \\
&\times \exp\left[-\frac{m\omega}{i\hbar\sin\omega s''}({\eta'}^2 + {\eta''}^2\cot\omega s'')\right] I_{k_1}\left(\frac{m\omega\eta'\eta''}{i\hbar\sin\omega s''}\right).
\end{aligned} \tag{2.75}$$

Performing the s'' integration yields the Green function ($\tilde{\mathcal{E}} = aE - \alpha - \mathcal{E}$)

$$G^{(V_2)}(\xi'', \xi', \eta'', \eta'; E) = \int d\mathcal{E} \sqrt{-\frac{2m}{bE}} \frac{\Gamma\left[\frac{1}{2}(1 + |k_1| - \mathcal{E}\sqrt{-2m/bE}/\hbar)\right]}{\hbar\Gamma(1 + |k_1|)\sqrt{\xi'\xi''}} \times$$
$$\times W_{\mathcal{E}\sqrt{-2m/bE}/2\hbar, |k_1|/2}\left(\frac{m}{\hbar}\sqrt{-\frac{bE}{2m}}\,\xi_>^2\right) \times$$
$$\times M_{\mathcal{E}\sqrt{-2m/bE}/2\hbar, |k_1|/2}\left(\frac{m}{\hbar}\sqrt{-\frac{bE}{2m}}\,\xi_<^2\right) \times$$
$$\times \sqrt{-\frac{2m}{bE}} \frac{\Gamma\left[\frac{1}{2}(1 + |k_2| - \tilde{\mathcal{E}}\sqrt{-2m/bE}/\hbar)\right]}{\hbar\Gamma(1 + |k_2|)\sqrt{\eta'\eta''}} \times$$
$$\times W_{\tilde{\mathcal{E}}\sqrt{-2m/bE}/2\hbar, |k_2|/2}\left(\frac{m}{\hbar}\sqrt{-\frac{bE}{2m}}\,\eta_>^2\right) \times$$
$$\times M_{\tilde{\mathcal{E}}\sqrt{-2m/bE}/2\hbar, |k_2|/2}\left(\frac{m}{\hbar}\sqrt{-\frac{bE}{2m}}\,\eta_<^2\right). \quad (2.76)$$

On the other hand, we insert the expansion of the bound states of the radial harmonic oscillator and obtain for the discrete spectrum contribution of the Green function:

$$G^{(V_2)}(\xi'', \xi', \eta'', \eta'; E) = \sum_{n_\xi=0}^{\infty} \sum_{n_\eta=0}^{\infty} \frac{N_{n_\xi, n_\eta}^2}{E_{n_\xi, n_\eta} - E} \times$$
$$\times \Psi_{n_\xi}^{(\mathrm{RHO}, |k_1|)}(\xi'') \Psi_{n_\xi}^{(\mathrm{RHO}, |k_2|)}(\xi') \Psi_{n_\eta}^{(\mathrm{RHO}, |k_2|)}(\eta'') \Psi_{n_\eta}^{(\mathrm{RHO}, |k_1|)}(\eta'), \quad (2.77)$$

where the energy E_{n_ξ, n_η} is determined by the equation

$$2n_\xi + 2n_\eta + |k_1| + |k_2| + 2 = \frac{aE_{n_\xi, n_\eta} - \alpha}{\hbar} \sqrt{-\frac{2m}{bE_{n_\xi, n_\eta}}}, \quad (2.78)$$

which is equivalent with (2.60). The normalization constants $N_{n_\xi n_\eta}$ are determined by the residuum of (2.56), and ω in the $\Psi_{n_\xi}^{(\mathrm{RHO}, |k_2|)} \Psi_{n_\eta}^{(\mathrm{RHO}, |k_1|)}$ has to be taken on $\omega_{n_\xi, n_\eta} = \sqrt{-bE_{n_\xi, n_\eta}/2m}$.

2.3. The Superintegrable Potential V_3 on $D_{\rm III}$. First we state the potential V_3 in the respective coordinate systems

$$V_3(u,v)=\frac{1}{a+b\,\mathrm{e}^{-u}}\left[-\alpha+\frac{\hbar^2}{2m}4\,\mathrm{e}^{u}\Big(c_1^2\,\mathrm{e}^{-iv}-2c_2\,\mathrm{e}^{-2iv}\Big)\right], \tag{2.79}$$

$$=\frac{1}{a+\frac{b}{4}\varrho^2}\left[-\alpha+\frac{\hbar^2}{2m\varrho^2}4\Big(c_1^2\,\mathrm{e}^{-2i\varphi}-2c_2\,\mathrm{e}^{-4i\varphi}\Big)\right], \tag{2.80}$$

$$=\frac{-\alpha(\mu+\nu)+c_1^2\dfrac{\mu+\nu}{\mu\nu}-c_2\dfrac{\mu^2-\nu^2}{\mu^2\nu^2}}{\left(a+\dfrac{b}{2}(\mu-\nu)\right)(\mu+\nu)}. \tag{2.81}$$

In hyperbolic coordinates no closed solution can be obtained due to the involved mixture of linear, quadratic, inverse-linear and inverse-quadratic terms. In polar coordinates the path integral in ϱ turns out to be a path integral for the radial harmonic oscillator. Note that the (u,v) system is equivalent to polar coordinates.

2.3.1. Separation of V_3 in Polar Coordinates. We insert the potential V_3 into the path integral and get $\big(f(\varrho)=a+\frac{b}{4}\varrho^2=\sqrt{g}\big)$

$$K^{(V_3)}(\varrho'',\varrho',\varphi'',\varphi';T)=\int\limits_{\varrho(t')=\varrho'}^{\varrho(t'')=\varrho''}\mathcal{D}\varrho(t)\int\limits_{\varphi(t')=\varphi'}^{\varphi(t'')=\varphi''}\mathcal{D}\varphi(t)f(\varrho)\varrho\times$$

$$\times\exp\left(\frac{i}{\hbar}\int_0^T\left\{\frac{m}{2}f(\varrho)(\dot\varrho^2+\varrho^2\dot\varphi^2)-\right.\right.$$

$$\left.\left.-\frac{1}{f(\varrho)}\left[-\alpha+\frac{\hbar^2}{2m\varrho^2}4c_1^2\left(\mathrm{e}^{-4i\varphi}-2\frac{c_2}{c_1^2}\,\mathrm{e}^{-2i\varphi}-\frac{1}{4}\right)\right]\right\}dt\right)=$$

$$=\sum_{l=0}^{\infty}\Phi^{(c_1,c_2)}_{[\mathrm{cMP}],l}(\varphi'')\Phi^{(c_1,c_2)}_{[\mathrm{cMP}],l}(\varphi')\frac{1}{[(\varrho'\varrho'')^2f(\varrho')f(\varrho'')]^{1/4}}\times$$

$$\times \int\limits_{\varrho(t')=\varrho'}^{\varrho(t'')=\varrho''} \mathcal{D}\varrho(t) f^{1/2}(\varrho) \exp\left\{ \frac{i}{\hbar}\int\limits_0^T \left[\frac{m}{2} f(\varrho)\dot\varrho^2 - \right.\right.$$

$$\left.\left. - \frac{1}{f(\varrho)}\left(-\alpha + \frac{\hbar^2}{2m}\frac{\left(l + \frac{2c_2}{c_1} + \frac{1}{2}\right)^2 - \frac{1}{4}}{\varrho^2}\right)\right] dt \right\} =$$

$$= \frac{1}{\sqrt{\varrho'\varrho''}} \sum_{l=0}^{\infty} \Phi^{(c_1,c_2)}_{[\mathrm{cMP}],l}(\varphi'')\Phi^{(c_1,c_2)}_{[\mathrm{cMP}],l}(\varphi')\times$$

$$\times \int\limits_{-\infty}^{\infty} \frac{dE}{2\pi\hbar}\, \mathrm{e}^{-iET/\hbar} \int\limits_0^\infty ds'' \exp\left[\frac{i}{\hbar}(aE-\alpha)s''\right] K_l^{(V_3)}(\varrho'',\varrho';s''), \quad (2.82)$$

with the time-transformed path integral $K_l(s'')$ given by

$$K_l^{(V_3)}(\varrho'',\varrho';s'') =$$

$$= \int\limits_{\varrho(0)=\varrho'}^{\varrho(s'')=\varrho''} \mathcal{D}\varrho(s) \exp\left[\frac{i}{\hbar}\int\limits_0^{s''} \left(\frac{m}{2}\dot\varrho^2 + \frac{Eb}{4}\varrho^2 - \frac{\hbar^2}{2m}\frac{\left(l+\frac{2c_2}{c_1}+\frac{1}{2}\right)^2 - \frac{1}{4}}{\varrho^2}\right) ds\right] =$$

$$= \frac{m\omega\sqrt{\varrho'\varrho''}}{i\hbar\sin\omega s''} \exp\left[-\frac{m\omega}{2i\hbar}({\varrho'}^2 + {\varrho''}^2)\cot\omega s''\right] I_{l+\frac{2c_2}{c_1}+\frac{1}{2}}\left(\frac{m\omega\varrho'\varrho''}{i\hbar\sin\omega s''}\right). \quad (2.83)$$

By $\Phi^{(c_1,c_2)}_{[\mathrm{cMP}],l}(\varphi)$ we denote the wave functions of the complex periodic Morse potential in the variable φ with spectrum $E_l = \hbar^2\left(l + 2\frac{c_2}{c_1} + \frac{1}{2}\right)^2/2m$ [1,3,36, 42,50,51], c.f. Appendix C:

$$\Phi^{(c_1,c_2)}_{[\mathrm{cMP}],l}(\varphi) = \frac{\left(4\frac{c_2}{c_1} - 2n - 1\right) n!}{\Gamma\left(4\frac{c_2}{c_1} - 2n\right)} \left(4\frac{c_2}{c_1}\right)^{4\frac{c_2}{c_1}-2n-1} \times$$

$$\times \exp\left[-2i\left(2\frac{c_2}{c_1} - n - \frac{1}{2}\right)\varphi - 2c_1\,\mathrm{e}^{-2i\varphi}\right] L_n^{(4\frac{c_2}{c_1}-2n-1)}\left(4c_1\mathrm{e}^{-2i\varphi}\right). \quad (2.84)$$

Performing the s'' integration gives the Green function

$$G^{(V_3)}(\varrho'', \varrho', \varphi'', \varphi'; E) = \sum_{l=0}^{\infty} \Phi^{(c_1,c_2)}_{[\mathrm{cMP}],l}(\varphi'')\Phi^{(c_1,c_2)}_{[\mathrm{cMP}],l}(\varphi')\times$$
$$\times \sqrt{-\frac{2m}{Eb}} \frac{\Gamma\left[\frac{1}{2}\left(l + 2\frac{c_2}{c_1} + \frac{3}{2} - \frac{1}{\hbar}(aE-\alpha)\sqrt{-2m/bE}\right)\right]}{\Gamma\left(l + 2\frac{c_2}{c_1} + \frac{3}{2}\right)\sqrt{\varrho'\varrho''}}\times$$
$$\times M_{\frac{aE-\alpha}{2\hbar}\sqrt{-\frac{2m}{bE}}, \frac{1}{2}\left(l+2\frac{c_2}{c_1}+\frac{1}{2}\right)}\left(\frac{m}{\hbar}\sqrt{-\frac{bE}{2m}}\,\varrho_<^2\right)\times$$
$$\times M_{\frac{aE-\alpha}{2\hbar}\sqrt{-\frac{2m}{bE}}, \frac{1}{2}\left(l+2\frac{c_2}{c_1}+\frac{1}{2}\right)}\left(\frac{m}{\hbar}\sqrt{-\frac{bE}{2m}}\,\varrho_>^2\right). \quad (2.85)$$

Inserting the expansion into Laguerre polynomials yields the discrete contribution of the Green function $\left(\lambda = l + \frac{2c_2}{c_1} + \frac{1}{2}\right)$

$$G^{(V_3)}_{\mathrm{disc}}(\varrho'', \varrho', \varphi'', \varphi'; E) = \frac{1}{\sqrt{\varrho'\varrho''}} \sum_{l=0}^{\infty} \Phi^{(c_1,c_2)}_{[\mathrm{cMP}],l}(\varphi'')\Phi^{(c_1,c_2)}_{[\mathrm{cMP}],l}(\varphi')\times$$
$$\times \sum_{n=0}^{\infty} \frac{N_{nl}^2}{E_{nl}-E}\Psi_n^{(\mathrm{RHO},\lambda)}(\varrho'')\Psi_n^{(\mathrm{RHO},\lambda)}(\varrho'), \quad (2.86)$$

and the normalization constants N_{nl} are determined by the residuum of (2.85). Here, the spectrum E_{nl} is determined by

$$aE_{nl} - \alpha - \hbar\sqrt{-\frac{bE_{nl}}{2m}}\left(2n + 2l + \frac{c_2}{c_1} + 1\right), \quad (2.87)$$

which is quadratic equation in E_{nl} with solution $\left(N = 2n + 2l + \frac{c_2}{c_1} + 1\right)$

$$E_{nl\pm} = \frac{1}{2a^2}\left[-\left(\frac{b\hbar^2}{2m}N^2 - 2a\alpha\right) \pm \frac{b\hbar^2}{2m}N^2\sqrt{1 - \frac{8a\alpha m}{b\hbar^2 N^2}}\right]. \quad (2.88)$$

In the wave functions $\Psi_n^{(\mathrm{RHO},\lambda)}(\varrho)$ the quantity ω has to be taken on $\omega = \sqrt{-bE_{nl}/2m}$. For large n, l we have

$$E_{nl-} \simeq -\frac{b\hbar^2}{m}(2n+2l+1)^2, \tag{2.89}$$

$$E_{nl+} \simeq -\frac{m\alpha^2}{2b\hbar^2(2n+2l+1)^2}, \tag{2.90}$$

with E_{nl+} showing a Coulomb-like behavior.

2.4. The Superintegrable Potential V_4 on D_{III}.

$$V_4(\mu,\nu) = \frac{1}{\left(a+\frac{b}{2}(\mu-\nu)\right)(\mu+\nu)}\left[d_1\mu + d_2\nu + \frac{m}{2}\omega^2(\mu^2-\nu^2)\right], \tag{2.91}$$

$$= \frac{1}{a+b\,\mathrm{e}^{-u}}\big[2(d_1+d_2)(\cos 2\varphi - \cosh 2\omega) +$$

$$+ 2(d_1-d_2)(2i\sin 2\varphi + \sinh 2\omega) + 2d_3(2i\sin 2\varphi + \sinh 4\omega)\big]. \tag{2.92}$$

We can evaluate the path integral in hyperbolic coordinates (application of the Morse potential); in elliptic coordinates no closed solution can be found.

2.4.1. Separation of V_4 in Hyperbolic Coordinates. The classical Lagrangian and Hamiltonian have the form

$$\mathcal{L}(\mu,\dot\mu,\nu,\dot\nu) = \frac{m}{2}\left(a+\frac{b}{2}(\mu-\nu)\right)(\mu+\nu)\left(\frac{\dot\mu^2}{\mu^2}-\frac{\dot\nu^2}{\nu^2}\right) - V(\mu,\nu), \tag{2.93}$$

$$\mathcal{H}(\mu,p_\mu,\nu,p_\nu = \frac{1}{2m}\frac{\mu^2p_\mu^2-\nu^2p_\nu^2}{\left(a+\frac{b}{2}(\mu-\nu)\right)(\mu+\nu)} + V(\mu,\nu). \tag{2.94}$$

The canonical momentum operators are given by

$$p_\mu = \frac{\hbar}{i}\left[\frac{\partial}{\partial\mu} + \frac{1}{2}\left(+\frac{1}{\mu+\nu} + \frac{b}{a+\frac{b}{2}(\mu-\nu)} - \frac{1}{\mu}\right)\right], \tag{2.95}$$

$$p_\nu = \frac{\hbar}{i}\left[\frac{\partial}{\partial\mu} + \frac{1}{2}\left(+\frac{1}{\mu+\nu} - \frac{b}{a+\frac{b}{2}(\mu-\nu)} - \frac{1}{\nu}\right)\right], \tag{2.96}$$

and the quantum Hamiltonian has the form

$$H = -\frac{\hbar^2}{2m}\frac{1}{\left(a+\frac{b}{2}(\mu-\nu)\right)(\mu+\nu)}\times$$

$$\times\left[\mu^2\left(\frac{\partial^2}{\partial\mu^2}-\frac{1}{\mu}\frac{\partial}{\partial\mu}\right)-\nu^2\left(\frac{\partial^2}{\partial\nu^2}-\frac{1}{\nu}\frac{\partial}{\partial\nu}\right)\right]+V(\mu,\nu), \tag{2.97}$$

$$= \frac{1}{2m}\left[\frac{\mu}{\sqrt{\left(a+\frac{b}{2}(\mu-\nu)\right)(\mu+\nu)}}p_\mu^2\frac{\mu}{\sqrt{\left(a+\frac{b}{2}(\mu-\nu)\right)(\mu+\nu)}}-\right.$$

$$\left.-\frac{\nu}{\sqrt{\left(a+\frac{b}{2}(\mu-\nu)\right)(\mu+\nu)}}p_\nu^2\frac{\nu}{\sqrt{\left(a+\frac{b}{2}(\mu-\nu)\right)(\mu+\nu)}}\right]+V(\mu,\nu). \tag{2.98}$$

Note that from each coordinate there comes a quantum potential $\Delta V = \hbar^2/8m$, however they are canceling each other due to the minus-sign in the metric in ν.

We insert the potential V_4 into the path integral which has the form $\Big(f(\mu,\nu)= \left(a+\frac{b}{2}(\mu-\nu)\right)(\mu+\nu)\Big)$

$$K^{(V_4)}(\mu'',\mu',\nu'',\nu';T) = \int\limits_{\mu(t')=\mu'}^{\mu(t'')=\mu''}\mathcal{D}\mu(t)\int\limits_{\nu(t')=\nu'}^{\nu(t'')=\nu''}\mathcal{D}\nu(t)\frac{f(\mu,\nu)}{\mu\nu}\times$$

$$\times\exp\left\{\frac{i}{\hbar}\int_0^T\left[\frac{m}{2}f(\mu,\nu)\left(\frac{\dot\mu^2}{\mu^2}-\frac{\dot\nu^2}{\nu^2}\right)-\right.\right.$$

$$\left.\left.-\frac{1}{f(\mu,\nu)}\left(d_1\mu+d_2\nu+\frac{m}{2}\omega^2(\mu^2-\nu^2)\right)\right]dt\right\} =$$

$$= \int_{-\infty}^{\infty}\frac{dE}{2\pi\hbar}\,\mathrm{e}^{-iET/\hbar}\int_0^\infty ds''K^{(V_4)}(\mu'',\mu',\nu'',\nu';s''), \tag{2.99}$$

and the path integral $K^{(V_4)}(s'')$ is given by

$$K^{(V_4)}(\mu'',\mu',\nu'',\nu';s'') = \int\limits_{\mu(0)=\mu'}^{\mu(s'')=\mu''} \mathcal{D}\mu(s) \int\limits_{\nu(0)=\nu'}^{\nu(s'')=\nu''} \mathcal{D}\nu(s)\frac{1}{\mu\nu}\times$$
$$\times \exp\left\{\frac{i}{\hbar}\int\limits_0^{s''}\left[\frac{m}{2}\left(\frac{\dot\mu^2}{\mu^2}-\frac{\dot\nu^2}{\nu^2}\right)+aE(\mu+\nu)+\frac{1}{2}bE(\mu^2-\nu^2)-\right.\right.$$
$$\left.\left.-\left(d_1\mu+d_2\nu+\frac{m}{2}\omega^2(\mu^2-\nu^2)\right)\right]ds\right\}. \quad (2.100)$$

Each of the last path integrals has a similar form as the one discussed in [14]. One can perform the transformation $\mu = \mathrm{e}^x$, $\nu = \mathrm{e}^y$. Then the path integration in (μ,ν) gives a path integration in (x,y) of the following form:

$$K^{(V_4)}(x'',x',y'',y';s'') = \int\limits_{x(0)=x'}^{x(s'')=x''} \mathcal{D}x(s)\times$$
$$\times \exp\left\{\frac{i}{\hbar}\int\limits_0^{s''}\left[\frac{m}{2}\dot x^2-\frac{1}{2}(m\omega^2-bE)\,\mathrm{e}^{2x}-(d_1-aE)\,\mathrm{e}^{x}\right]ds\right\}\int\limits_{y(0)=y'}^{y(s'')=y''}\mathcal{D}y(s)\times$$
$$\times \exp\left\{-\frac{i}{\hbar}\int\limits_0^{s''}\left[\frac{m}{2}\dot y^2-\frac{1}{2}(m\omega^2-bE)\,\mathrm{e}^{2y}-(d_2+aE)\,\mathrm{e}^{y}\right]ds\right\}, \quad (2.101)$$

and we find the product of two path integrals for the Morse potential. This can be evaluated now as follows. We introduce the abbreviations

$$V_0^2 = \frac{m}{\hbar^2}(m\omega^2-bE), \quad \alpha_{x,y} = -\frac{d_{1,2}\mp aE}{m\omega^2-bE}. \quad (2.102)$$

We expand each path integral first into the discrete spectrum contribution by means of the known solution of the Morse potential in terms of Laguerre polynomials with the quantum numbers n and l, respectively, and the corresponding energy spectra. The s'' integration gives the energy spectrum

$$E_{n,l} = \frac{m\omega^2}{b} - \frac{m}{4b\hbar^2}\frac{(d_1+d_2)^2}{(n+l+1)^2}, \quad (2.103)$$

together with the wave functions ($N_{n,l}$ is determined by the corresponding residuum)

$$\Psi_{n,l}(x,y) = N_{n,l}\Psi_n^{(\mathrm{MP})}(x) \cdot \Psi_n^{(\mathrm{MP})}(y), \tag{2.104}$$

$$\Psi_k^{(\mathrm{MP})}(z) = \left(\frac{2\alpha_z V_0 - 2k - 1}{k!\Gamma(2\alpha_z V_0 - k)}\right)^{1/2} \times$$
$$\times (2V_0)^{\alpha_z V_0 - k - 1/2}\,\mathrm{e}^{(\alpha_z V_0 - k - 1/2)z - V_0\,\mathrm{e}^z} L_k^{(2\alpha_z V_0 - 2k - 1)}(2V_0\,\mathrm{e}^z), \tag{2.105}$$

for $z = x, y$ with $k = n, l$. The continuous spectrum is examined in an analogous way yielding

$$E = \frac{\hbar^2 p^2}{2m}, \tag{2.106}$$

with the wave functions

$$\Psi_{p,\lambda}(x,y) = \Psi_{p,\lambda}^{(\mathrm{MP})}(x) \cdot \Psi_{p,\lambda}^{(\mathrm{MP})}(y), \tag{2.107}$$

$$\Psi_{p,\lambda}^{(\mathrm{MP})}(z) = \left(\frac{p_\pm \sinh 2\pi p_\pm}{2\pi^2 V_0}\right)^{1/2} \times$$
$$\times \left|\Gamma\left(ip_\pm - \alpha_z + \frac{1}{2}\right)\right| \mathrm{e}^{-z} W_{\alpha_z V_0, ip_\pm}(2V_0\,\mathrm{e}^x), \tag{2.108}$$

with $p_\pm = p \pm \lambda$ for $z = x, y$. The entire Green function has the form

$$G(\mu'', \mu', \nu'', \nu'; E) = \sum_{n,l} \frac{\Psi_{n,l}(\mu'', \nu'')\Psi_{n,l}(\mu', \nu')}{E_{n,l} - E} +$$
$$+ \int dp \int d\lambda \frac{\Psi_{p,\lambda}(\mu'', \nu'')\Psi_{p,\lambda}^*(\mu', \nu')}{\frac{\hbar^2 p^2}{2m} - E}, \tag{2.109}$$

together with the replacement $\mu = \mathrm{e}^x$, $\nu = \mathrm{e}^y$. This concludes the discussion.

2.5. The Superintegrable Potential V_5 on $D_{\rm III}$. We display the potential V_5 in the respective coordinate systems

$$V_5(u,v) = \frac{1}{a+b\,\mathrm{e}^{-u}}\frac{\hbar^2 v_0^2}{2m}, \tag{2.110}$$

$$= \frac{1}{a+\dfrac{b}{4}\varrho^2}\frac{\hbar^2 v_0^2}{2m}, \tag{2.111}$$

$$= \frac{1}{a+\dfrac{b}{4}(\xi^2+\eta^2)}\frac{\hbar^2 v_0^2}{2m}, \tag{2.112}$$

$$= \frac{1}{a+\dfrac{b}{4}d^2(\sinh^2\omega+\cos^2\varphi)}\frac{\hbar^2 v_0^2}{2m}, \tag{2.113}$$

$$= \frac{1}{\Big(a+\dfrac{b}{2}(\mu-\nu)\Big)(\mu+\nu)}\frac{\hbar^2 v_0^2}{2m}. \tag{2.114}$$

We discuss the path integral solution of V_5 in some extend, where the case of elliptic coordinates is omitted due to intractability of this system in the path integral. Provided that $b>0$, there is in the case of the free motion a discrete spectrum

$$E_N = -\frac{\hbar^2}{2m}\frac{b}{a^2}(2N+1)^2, \tag{2.115}$$

with the principal quantum number $N \in \mathbb{N}$.

2.5.1. Separation of V_5 in the (u,v) System. We insert the potential V_5 into the path integral for the (u,v) system and obtain

$$K^{(V_5)}(u'',u',v'',v';T) = \int\limits_{u(t')=u'}^{u(t'')=u''}\mathcal{D}u(t)\int\limits_{v(t')=v'}^{v(t'')=v''}\mathcal{D}v(t)(a\,\mathrm{e}^{-u}+b\,\mathrm{e}^{-2u})\times$$

$$\times\exp\left\{\frac{i}{\hbar}\int\limits_0^T\left[\frac{m}{2}(a\,\mathrm{e}^{-u}+b\,\mathrm{e}^{-2u})(\dot u^2+\dot v^2)-\frac{1}{a+b\,\mathrm{e}^{-u}}\frac{\hbar^2 v_0^2}{2m}\right]dt\right\} =$$

$$= \int\limits_{-\infty}^{\infty}\frac{dE}{2\pi\hbar}\,\mathrm{e}^{-iET/\hbar}\int\limits_0^{\infty}ds''\mathrm{e}^{-i\hbar v_0^2 s''/2m}K^{(V_5)}(u'',u',v'',v';s''), \tag{2.116}$$

with the time-transformed path integral $K^{(V_5)}(s'')$ given by

$$K^{(V_5)}(u'',u',v'',v';s'') = \int\limits_{u(0)=u'}^{u(s'')=u''} \mathcal{D}u(s) \int\limits_{v(0)=v'}^{v(s'')=v''} \mathcal{D}v(s)\times$$

$$\times \exp\left(\frac{i}{\hbar}\int\limits_0^{s''}\left\{\frac{m}{2}(\dot u^2+\dot v^2)+Eb\left[\mathrm{e}^{-2u}+\left(\frac{aE-\hbar^2 v_0^2/2m}{Eb}\right)\mathrm{e}^{-u}\right]\right\}ds\right) =$$

$$= \sum_{l=0}^{\infty}\frac{\mathrm{e}^{il(v''-v')}}{2\pi}\,\mathrm{e}^{-i\hbar l^2 s''/2m}\int\limits_{u(0)=u'}^{u(s'')=u''}\mathcal{D}u(s)\times$$

$$\times \exp\left(\frac{i}{\hbar}\int\limits_0^{s''}\left\{\frac{m}{2}\dot u^2+Eb\left[\mathrm{e}^{-2u}+\left(\frac{aE-\hbar^2 v_0^2/2m}{Eb}\right)\mathrm{e}^{-u}\right]\right\}ds\right). \quad (2.117)$$

The path integral in u is a path integral for the Morse potential. Performing the s'' integration gives, c.f. [14], the Green function as follows ($\mathcal{E} = [Ea - (\hbar^2 v_0^2/2m)]\sqrt{-2m/bE}/2\hbar$):

$$G^{(V_5)}(u'',u',v'',v';E) = \sum_{l=-\infty}^{\infty}\frac{\mathrm{e}^{il(v''-v')}}{2\pi}\frac{m\Gamma\left(\frac{1}{2}+l-\mathcal{E}\right)}{\hbar\sqrt{-2mbE}\,\Gamma(1+2l)}\,\mathrm{e}^{(u'+u'')/2}\times$$

$$\times W_{\mathcal{E},l}\left(\frac{\sqrt{-8mbE}}{\hbar}\,\mathrm{e}^{-u_<}\right)M_{\mathcal{E},l}\left(\frac{\sqrt{-8mbE}}{\hbar}\,\mathrm{e}^{-u_>}\right). \quad (2.118)$$

The corresponding continuous part of the Green function is evaluated as [14]

$$G^{(V_5)}_{\mathrm{cont}}(u'',u',v'',v';E) = \sum_{l=-\infty}^{\infty}\frac{\mathrm{e}^{il(v''-v')}}{2\pi}\mathrm{e}^{(u'+u'')/2}\times$$

$$\times\int\limits_0^{\infty}\frac{\mathrm{e}^{\pi p/2}dp}{\dfrac{\hbar^2p^2}{2m}-E}\frac{\left|\Gamma\left(\frac{1}{2}+l+ip\right)\right|^2}{2\pi\Gamma^2(1+2l)}M_{ip/2,l}\Big(-2ip\,\mathrm{e}^{-u'}\Big)M_{-ip/2,l}\Big(2ip\,\mathrm{e}^{-u''}\Big). \quad (2.119)$$

In addition, we have a discrete spectrum. This is found by analyzing the poles of the Green function (2.118):

$$\frac{1}{2}+l-\frac{aE_{nl}-\dfrac{\hbar^2v_0^2}{2m}}{2\hbar}\sqrt{-\frac{2m}{bE_{nl}}} = -n. \quad (2.120)$$

In the case of $v_0 = 0$ this simplifies to

$$n + l + \frac{1}{2} - \frac{a}{2\hbar}\sqrt{-\frac{2m}{bE_{nl}}} = 0, \tag{2.121}$$

with the solution

$$E_{nl} = -\frac{\hbar^2}{2m}\frac{b}{a^2}(2n + 2l + 1)^2 \tag{2.122}$$

yielding for $b > 0$ an infinite number of bound states. For $v_0 \neq 0$, the equation for E_{nl} is a quadratic equation in E with solution

$$E_{nl\pm} = -\frac{\hbar^2}{2m}\frac{1}{2a^2}\times \\ \times \left[b(2n{+}2l{+}1)^2 - 2av_0^2 \pm b(2n{+}2l{+}1)^2\sqrt{1 - \frac{4av_0^2}{b(2n{+}2l{+}1)^2}}\right], \tag{2.123}$$

$$E_{nl+} \overset{(n,l)\to\infty}{=} -\frac{\hbar^2}{2m}\frac{b}{a^2}\left[(2n + 2l + 1)^2 - 2\frac{a}{b}v_0^2\right], \tag{2.124}$$

$$E_{nl-} \overset{(n,l)\to\infty}{=} -\frac{\hbar^2}{2bm}\frac{v_0^4}{(2n + 2l + 1)^2}. \tag{2.125}$$

For $v_0 = 0$, there is only E_{nl+}. For $(2n{+}2l{+}1)^2 < 4av_0^2/b$, there are semibound states located approximately around $E_0 = -\hbar^2 v_0^2/2ma$.

Therefore we have for the discrete spectrum contribution

$$G^{(V_5)}_{\text{disc}}(u'', u', v'', v'; E) = \\ = \sum_{l=-\infty}^{\infty}\frac{\mathrm{e}^{il(v''-v')}}{2\pi}\sum_{n=0}^{\infty}\frac{1}{E_{nl} - E}\Psi^{(V_5)}_{nl}(u'')\Psi^{(V_5)}_{nl}(u'), \tag{2.126}$$

with the functions $\Psi^{(V_5)}_{nl}(u)$ given by ($\mathcal{E}$ as in (2.118))

$$\Psi^{(V_5)}_{nl}(u) = N_{nl}\frac{(2\mathcal{E} - 2n - 1)n!}{\Gamma(2\mathcal{E} - n)}\left(\frac{\sqrt{-8mbE_{nl}}}{\hbar}\right)^{\mathcal{E}-n-1/2}\times \\ \times \exp\left[\left(\mathcal{E} - n - \frac{1}{2}\right)u - \sqrt{-\frac{8mbE_{nl}}{\hbar}}\,\mathrm{e}^u\right]\times \\ \times L_n^{(2\mathcal{E}-2n-1)}\left(\frac{\sqrt{-8mbE_{nl}}}{\hbar}\,\mathrm{e}^u\right). \tag{2.127}$$

The constant N_{nl} is determined by taking the Green function at the residuum E_{nl}. The wave functions vanish for $u \to \infty$ due to $\mathrm{e}^{-\sqrt{-8mbE_{nl}}\,\mathrm{e}^u/\hbar} = \mathrm{e}^{-2b\hbar(2n+2l+1)\mathrm{e}^u/a} \to 0$, provided $b/a > 0$ for all $n \in \mathbb{N}$, which shows that the discrete spectrum is indeed infinite. The feature that an homogeneous space with curvature has at the same time a discrete and a continuous spectrum is already known from the path integration on the $SU(1,1)$ group manifold [22]. Actually, this property allows the analysis of the modified Pöschl–Teller potential with its continuous and (finite) discrete spectrum.

2.5.2. Separation of V_5 in Polar Coordinates. We insert the potential V_5 into the path integral in polar coordinates and obtain

$$K^{(V_5)}(\varrho'',\varrho',\varphi'',\varphi';T) = \int\limits_{\varrho(t')=\varrho'}^{\varrho(t'')=\varrho''} \mathcal{D}\varrho(t) \int\limits_{\varphi(t')=\varphi'}^{\varphi(t'')=\varphi''} \mathcal{D}\varphi(t) \left(a + \frac{b}{4}\varrho^2\right)\varrho \times$$
$$\times \exp\left\{\frac{i}{\hbar}\int\limits_0^T \left[\frac{m}{2}\left(a+\frac{b}{4}\varrho^2\right)(\dot\varrho^2+\varrho^2\dot\varphi^2)+\left(a+\frac{b}{4}\varrho^2\right)^{-1}\frac{\hbar^2}{2m}\left(v_0^2+\frac{1}{4\varrho^2}\right)\right]dt\right\} =$$
$$= \int\limits_{-\infty}^{\infty}\frac{dE}{2\pi\hbar}\,\mathrm{e}^{-iET/\hbar}G^{(V_5)}(\varrho'',\varrho',\varphi'',\varphi';E), \quad (2.128)$$

and the Green function is evaluated to have the form [14] $\Big(\mathcal{E}=\dfrac{(aE-\hbar^2v_0^2/2m)}{\hbar\omega}$, $\omega^2 = -bE/2m\Big)$

$$G^{(V_5)}(\varrho'',\varrho',\varphi'',\varphi';E)\sum_{l=-\infty}^{\infty}\frac{\mathrm{e}^{il(\varphi''-\varphi')}}{2\pi}\frac{1}{\varrho'\varrho''}\sqrt{-\frac{2m}{2E}}\frac{\Gamma\left[\frac{1}{2}(1+l-\mathcal{E})\right]}{\Gamma(1+l)}\times$$
$$\times W_{\mathcal{E}/2,\frac{l}{2}}\left(\sqrt{-\frac{2mbE}{\hbar^2}}\,\varrho_>\right)M_{\mathcal{E}/2,\frac{l}{2}}\left(\sqrt{-\frac{2mbE}{\hbar^2}}\,\varrho_<\right). \quad (2.129)$$

The Green function has poles which are determined by

$$2n+l+1-\frac{1}{\hbar}\left(aE_{nl}-\frac{v_0^2\hbar^2}{2m}\right)\sqrt{-\frac{2m}{Eb_{nl}}} = 0. \quad (2.130)$$

In the case of $v_0 = 0$ this simplifies to

$$(2n+l+1)-\frac{a}{\hbar}\sqrt{-\frac{2m}{E_{nl}b}} = 0, \quad (2.131)$$

with the solution

$$E_{nl} = -\frac{\hbar^2}{2m}\frac{b}{a^2}(2n+l+1)^2 \qquad (2.132)$$

yielding for $b > 0$ an infinite number of bound states. For $v_0 \neq 0$ the equation for E_{nl} is a quadratic equation in E with solution

$$E_{nl\pm} = -\frac{\hbar^2}{2m}\frac{1}{2a^2}\times$$
$$\times\left[b(2n+l+1)^2 - 2av_0^2 \pm b(2n+l+1)^2\sqrt{1-\frac{4av_0^2}{b(2n+l+1)^2}}\right]. \qquad (2.133)$$

The limit of $N, l \to \infty$ yields

$$E_{nl+} \simeq -\frac{\hbar^2}{2m}\left[\frac{b}{a^2}(2n+l+1)^2 + \frac{v_0^2}{a}\right], \qquad (2.134)$$

$$E_{nl-} \simeq -\frac{\hbar^2}{2m}\frac{v_0^2}{4b(2n+l+1)^2}, \qquad (2.135)$$

and E_{nl+} corresponds in this limit to the spectrum of the free motion.

2.5.3. Separation of V_5 in Parabolic Coordinates. We insert the potential V_5 into the path integral in parabolic coordinates and obtain

$$K^{(V_5)}(\xi'',\xi',\eta'',\eta';T) = \int\limits_{\xi(t')=\xi'}^{\xi(t'')=\xi''}\mathcal{D}\xi(t)\int\limits_{\eta(t')=\eta'}^{\eta(t'')=\eta''}\mathcal{D}\eta(t)\left(a+\frac{b}{4}(\xi^2+\eta^2)\right)\times$$
$$\times\exp\left\{\frac{i}{\hbar}\int\limits_0^T\left[\frac{m}{2}\left(a+\frac{b}{4}(\xi^2+\eta^2)\right)(\dot\xi^2+\dot\eta^2) - \frac{1}{a+\frac{b}{4}(\xi^2+\eta^2)}\frac{\hbar^2 v_0^2}{2m}\right]dt\right\} =$$
$$= \int\limits_{-\infty}^{\infty}\frac{dE}{2\pi\hbar}\,\mathrm{e}^{-iET/\hbar}G^{(V_5)}(\xi'',\xi',\eta'',\eta';E), \qquad (2.136)$$

with the time-transformed path integral $K(s'')$ given by

$$K^{(V_5)}(\xi'',\xi',\eta'',\eta';s'') = \int\limits_{\xi(0)=\xi'}^{\xi(s'')=\xi''}\mathcal{D}\xi(s)\int\limits_{\eta(0)=\eta'}^{\eta(s'')=\eta''}\mathcal{D}\eta(s)\times$$
$$\times\exp\left\{\frac{i}{\hbar}\int\limits_0^{s''}\left[\frac{m}{2}(\dot\xi^2+\dot\eta^2)+E\frac{b}{4}(\xi^2+\eta^2)\right]ds + \frac{i}{\hbar}\left(aE - \frac{\hbar^2 v_0^2}{2m}\right)ds\right\}. \qquad (2.137)$$

The only difference in comparison with the result in [14] is the the additional $\dfrac{\hbar^2 v_0^2}{2m}$ term in the s'' integration. In order to find the discrete spectrum we insert the solution for the harmonic oscillator and get

$$G_{\rm disc}^{(V_5)}(\xi'',\xi',\eta'',\eta';E) =$$
$$= \sum_{n_\xi=0}^{\infty}\sum_{n_\eta=0}^{\infty} \frac{N_{n_\xi n_\eta}^2}{E_{n_\xi n_\eta}-E} \Psi_{n_\xi}^{(\rm HO)}(\xi'')\Psi_{n_\xi}^{(\rm HO)}(\xi'')\Psi_{n_\eta}^{(\rm HO)}(\eta'')\Psi_{n_\eta}^{(\rm HO)}(\eta''), \quad (2.138)$$

where $E_{n_\xi n_\eta}$ is determined by the equation

$$(n_\xi + n_\eta + 1) - \frac{1}{\hbar}\left(aE - \frac{\hbar^2 v_0^2}{2m}\right)\sqrt{-\frac{bE}{2m}} = 0, \quad (2.139)$$

which is (up to a different counting in the quantum numbers) identical with (2.131). The normalization $N_{n_\xi n_\eta}$ is determined by the residuum in $G^{(V_5)}(E)$. We do not state the continuous spectrum part, it can be derived from [14] by the replacement $aE \to aE - \hbar^2 v_0^2/2m$.

2.5.4. Separation of V_5 in Hyperbolic Coordinates. We insert the potential V_5 into the path integral in hyperbolic coordinates and obtain: The path integral has the form

$$K^{(V_5)}(\mu'',\mu',\nu'',\nu';T) =$$
$$= \int\limits_{\mu(t')=\mu'}^{\mu(t'')=\mu''} \mathcal{D}\mu(t) \int\limits_{\nu(t')=\nu'}^{\nu(t'')=\nu''} \mathcal{D}\nu(t) \frac{\left(a + \frac{b}{2}(\mu-\nu)\right)(\mu+\nu)}{\mu\nu} \times$$
$$\times \exp\left\{\frac{i}{\hbar}\int\limits_0^T \left[\left(a + \frac{b}{2}(\mu-\nu)\right)(\mu+\nu)\left(\frac{\dot\mu^2}{\mu^2} - \frac{\dot\nu^2}{\nu^2}\right) - \right.\right.$$
$$\left.\left. - \frac{1}{\left(a + \frac{b}{2}(\mu-\nu)\right)(\mu+\nu)} \frac{\hbar^2 v_0^2}{2m}\right] dt\right\} =$$
$$= \int\limits_{-\infty}^{\infty} \frac{dE}{2\pi\hbar}\, \mathrm{e}^{-iET/\hbar} \int\limits_0^{\infty} ds'' K^{(V_5)}(\mu'',\mu',\nu'',\nu';s''), \quad (2.140)$$

and the path integral $K^{(V_5)}(s'')$ is given by

$$K^{(V_5)}(\mu'',\mu',\nu'',\nu';s'') = \int\limits_{\mu(0)=\mu'}^{\mu(s'')=\mu''} \mathcal{D}\mu(s) \int\limits_{\nu(0)=\nu'}^{\nu(s'')=\nu''} \mathcal{D}\nu(s) \frac{1}{\mu\nu} \times$$
$$\times \exp\left\{ \frac{i}{\hbar} \int\limits_0^{s''} \left[\frac{m}{2}\left(\frac{\dot\mu^2}{\mu^2} - \frac{\dot\nu^2}{\nu^2} \right) + (\mu+\nu)\left(aE - \frac{\hbar^2 v_0^2}{2m} \right) + \right.\right.$$
$$\left.\left. + \frac{1}{2} bE(\mu^2 - \nu^2) \right] ds \right\}. \quad (2.141)$$

Each of the last path integrals has a similar form as the one discussed in [11]. One can perform the transformation $\mu = \mathrm{e}^x$, $\nu = \mathrm{e}^y$ yielding

$$K^{(V_5)}(x'',x',y'',y';s'') =$$
$$= \int\limits_{x(0)=x'}^{x(s'')=x''} \mathcal{D}x(s) \exp\left\{ \frac{i}{\hbar} \int\limits_0^{s''} \left[\frac{m}{2}\dot x^2 + \left(E\frac{b}{2}\,\mathrm{e}^{2x} + \left(aE - \frac{\hbar^2 v_0^2}{2m} \right)\mathrm{e}^{x} \right) \right] ds \right\} \times$$
$$\times \int\limits_{y(0)=y'}^{y(s'')=y''} \mathcal{D}y(s) \exp\left\{ -\frac{i}{\hbar} \int\limits_0^{s''} \left[\frac{m}{2}\dot y^2 + \left(E\frac{b}{2}\,\mathrm{e}^{2y} - \left(aE - \frac{\hbar^2 v_0^2}{2m} \right)\mathrm{e}^{y} \right) \right] ds \right\} \quad (2.142)$$

and we find the product of two path integrals for the Morse potential, however more complicated as in [14]. The continuous part of the spectrum can be analyzed similarly as in [14] yielding products of M-Whittaker functions. Analyzing the discrete spectrum contribution from the Morse potential we find the quantization condition

$$(n_\xi + n_\eta + 1) - \frac{1}{\hbar}\left(aE - \frac{\hbar^2 v_0^2}{2m} \right)\sqrt{-\frac{4m}{E_{nl} b}} = 0, \quad (2.143)$$

which is up to a different counting in the quantum numbers equivalent with (2.131). This concludes the discussion.

3. SUPERINTEGRABLE POTENTIALS ON DARBOUX SPACE $D_{\rm IV}$

Finally, we consider the Darboux space $D_{\rm IV}$. We have the coordinate systems:

$$((u,v) \text{ system:}) \quad x = v + iu, \quad y = v - iu \quad (u \in (0,\pi/2), v \in \mathbb{R}), \tag{3.1}$$

$$(\text{Equidistant:}) \quad u = \arctan(\mathrm{e}^{\alpha}), \quad v = \frac{\beta}{2} \quad (\alpha \in \mathbb{R}, \beta \in \mathbb{R}), \tag{3.2}$$

$$(\text{Horospherical:}) \quad x = \log\frac{\mu - i\nu}{2}, \quad y = \log\frac{\mu + i\nu}{2} \quad (\mu,\nu > 0), \tag{3.3}$$

$$\mu = 2\mathrm{e}^{v}\cos u, \quad \nu = -2\mathrm{e}^{v}\sin u, \tag{3.4}$$

$$(\text{Elliptic:}) \quad \mu = d\cosh\omega\cos\varphi, \quad \nu = d\sinh\omega\sin\varphi$$
$$(\omega > 0, \varphi \in (0,\pi/2)). \tag{3.5}$$

We obtain the following forms of the line-element ($a > 2b$, $a_\pm = (a \pm 2b)/4$):

$$\begin{aligned} ds^2 &= \frac{2b\cos u + a}{4\sin^2 u}(du^2 + dv^2) = \\ &= \left(\frac{a_+}{\sin^2 u} + \frac{a_-}{\cos^2 u}\right)(du^2 + dv^2) \\ &\quad (\text{rescaling } u/2 \to u:), \end{aligned} \tag{3.6}$$

$$(\text{Equidistant:}) = \frac{a - 2b\tanh\alpha}{4}(d\alpha^2 + \cosh^2\alpha\, d\beta^2), \tag{3.7}$$

$$(\text{Horospherical:}) = \left(\frac{a_+}{\nu^2} + \frac{a_-}{\mu^2}\right)(d\mu^2 + d\nu^2), \tag{3.8}$$

$$\begin{aligned} (\text{Elliptic:}) &= \left(\frac{a_-}{\cosh^2\omega\cos^2\varphi} + \frac{a_+}{\sinh^2\omega\sin^2\varphi}\right) \times \\ &\quad \times (\cosh^2\omega - \cos^2\varphi)(d\omega^2 + d\varphi^2), \\ &= \left(\frac{a_+}{\sin^2\varphi} + \frac{a_-}{\cos^2\varphi} + \frac{a_+}{\sinh^2\omega} - \frac{a_-}{\cosh^2\omega}\right) \times \\ &\quad \times (d\omega^2 + d\varphi^2), \end{aligned} \tag{3.9}$$

$$\begin{aligned} (\text{Degenerate elliptic I:}) &= \left[a_-\left(\frac{1}{\sinh^2\hat\omega} + \frac{1}{\sin^2\hat\varphi}\right) - a_+\left(\frac{1}{\cosh^2\hat\omega} - \frac{1}{\cos^2\hat\varphi}\right)\right] \times \\ &\quad \times (d\hat\omega^2 + d\hat\varphi^2) \quad (\gamma = 1), \end{aligned} \tag{3.10}$$

$$(\text{Degenerate elliptic II:}) = \frac{1}{4}\left(\frac{a_-}{\sinh^2\tilde\omega} + \frac{a_+}{\sin^2\tilde\varphi}\right)(d\tilde\omega^2 + d\tilde\varphi^2) \ (\gamma = 2). \tag{3.11}$$

We observe that the diagonal term in the metric corresponds in most cases to a combination of a Pöschl–Teller potential and a modified Pöschl–Teller, respectively. In particular, the (u,v) and the equidistant systems are the same, they

just differ in the parameterization. The limiting cases $a = 2b$ and $b = 0$ give particular cases for the metric on the two-dimensional hyperboloid. We have also displayed two versions of degenerate elliptic coordinates. They come from the observation that for the representatives

$$K^2, \quad X_2, \quad \gamma X_2 + K2, \quad X_1 + X_2 + \gamma K^2 \tag{3.12}$$

one can distinguish the cases $\gamma = 0$, $\gamma = 2$, and $\gamma \neq 0, 2$. For $\gamma \neq 0, 2$, one has coordinate systems which can be explicitly formulated in terms of the elliptic functions $\mathrm{sn}(\alpha, k), \mathrm{cn}(\beta, k)$, and only for a special choice of the parameter k they can be simplified in trigonometric and hyperbolic functions. Then the line element has the form

$$ds^2 = \frac{1}{4}[a_+ k^4 \mathrm{sn}^2(\alpha, k) - \mathrm{sn}^2(\beta, k) + k^2 a_-](d\alpha^2 + d\beta^2), \tag{3.13}$$

and separated equations are versions of Lame's equation, if we assume an Ansatz of the form $\Psi = A(\alpha)B(\beta)$ [28]:

$$\frac{\partial^2 A(\alpha)}{\partial \alpha^2} + \left(-\frac{1}{4}k^4 E a_+ \mathrm{sn}^2(\alpha, k) - \lambda_1\right) A(\alpha) = 0, \tag{3.14}$$

$$\frac{\partial^2 B(\beta)}{\partial \beta^2} + \left(-\frac{1}{4}k^4 E a_+ \mathrm{sn}^2(\beta, k) - \lambda_2\right) B(\beta) = 0, \tag{3.15}$$

where $\lambda_1 - \lambda_2 = -E a_- k^2/4$. k denotes the modulus of the elliptic functions.

In particular, for the potential V_2 one has the possibilities of taking $\gamma = 0$, and $\gamma = 2$. For $\gamma = 0$, the modulus k of the elliptic functions equals $k = -i$. We do not treat V_2 in these elliptic coordinates, but only the degenerate case of $\gamma = 2$.

For the potential V_3, however, the elliptic systems with $\gamma = 1$ can be explicitly worked out. We have stated the respective line elements for these two cases. Note that for $\gamma = 2$ the coordinate transformation can be put into

$$x = \ln\Big[\tan(\tilde{\varphi} - i\tilde{\omega})\Big], \quad y = \ln\Big[\tan(\tilde{\varphi} + i\tilde{\omega})\Big] \quad (\tilde{\omega} > 0, \tilde{\varphi} \in (0, \pi/4)). \tag{3.16}$$

We do not dwell into a discussion of elliptic systems any further, for details we refer to [26]. Let us finally note that the notion $elliptic$ is also used for the (ω, φ) system, and they must not be confused with the general elliptic coordinates just discussed.

Because we have not worked out the path integral for the free motion in these two further coordinate systems, this will be done in an appendix. For the

Gaussian curvature we obtain, e.g., in the (u, v) system

$$G = -\frac{\dfrac{a_+^2}{\sin^6 u} + \dfrac{a_-^2}{\cos^6 u} + \dfrac{a_- a_+}{\sin^4 u \cos^4 u}}{\left(\dfrac{a_+}{\sin^2 u} + \dfrac{a_-}{\cos^2 u}\right)^3}. \tag{3.17}$$

The case $a = 2b$ yields $a_- = 0$, and

$$G = -\frac{1}{b}, \tag{3.18}$$

and therefore again a space of constant curvature, the hyperboloid $\Lambda^{(2)}$ is given for $b > 0$. We have set the sign in the metric (1.4) in such a way that from $a = 2b > 0$ the hyperboloid $\Lambda^{(2)}$ emerges. We could also choose the metric (1.4) with the opposite sign, then $a = 2b < 0$ would give the same result. In the following it is understood that we make this restriction of positive definiteness of the metric and we do not dwell into the problem of continuation into nonpositive definiteness. Because the (u, v) coordinates and the equidistant system are the same, we do not evaluate the path integral in the equidistant system. In the following we assume $a_+ > 0$ and $a_+ > a_-$.

We introduce the following three constants of motion on D_{IV}:

$$X_1 = \mathrm{e}^{2v}(-\tilde{\mathcal{H}}_0 + \cos 2u \cdot p_u^2 + \sin 2u \cdot p_u p_v), \tag{3.19}$$

$$X_2 = \mathrm{e}^{2v}(-\tilde{\mathcal{H}}_0 + \cos 2u \cdot p_u^2 - \sin 2u \cdot p_u p_v), \tag{3.20}$$

$$K = p_v. \tag{3.21}$$

These integrals of motion satisfy the Poisson relations

$$\{K, X_1\} = 2X_1, \quad \{K, X_2\} = -2X_2, \quad \{X_1, X_2\} = -K^3 - 4aKH_0, \tag{3.22}$$

and satisfy the relation

$$X_1 X_2 - K^4 - aK^2 H_0 - H_0^2 = 0. \tag{3.23}$$

The corresponding quantum operators have the form

$$\widehat{H}_0 = \frac{\sin^2 2u}{2\cos 2u + a}(\partial_u^2 + \partial_v^2), \tag{3.24}$$

$$\widehat{X}_1 = \mathrm{e}^{2v}(-\widehat{H}_0 + \cos 2u \cdot (\partial_u^2 + \partial_v) + \sin 2u \cdot (\partial_u \partial_v + \partial_u), \tag{3.25}$$

$$\widehat{X}_2 = \mathrm{e}^{2v}(-\widehat{H}_0 + \cos 2u \cdot (\partial_u^2 - \partial_v) - \sin 2u \cdot (\partial_u \partial_v - \partial_u), \tag{3.26}$$

and the commutation relations read

$$[\widehat{K}, \widehat{X}_1] = 2\widehat{X}_1, \ [\widehat{K}, \widehat{X}_2] = -2\widehat{X}_2, \ [\widehat{X}_1, \widehat{X}_2] = -8\widehat{K}^3 - 4a\widehat{K}\widehat{H}_0 - 4\widehat{K} \tag{3.27}$$

and satisfy the operator relation

$$\frac{1}{2}\{\widehat{X}_1, \widehat{X}_2\} - \widehat{K}^4 - a\widehat{H}_0\widehat{K}^2 - 5\widehat{K}^2 - \widehat{H}_0^2 - a\widehat{H}_0 = 0. \tag{3.28}$$

In Table 3 we list the connection with these operators and the corresponding coordinate systems on D_{IV}.

Table 3. **Constants of motion and limiting cases of coordinate systems on D_{IV}**

Metric	Constants of motion	D_{IV}	$\Lambda^{(2)}$ $(a=2b)$	$\Lambda^{(2)}$ $(b=0)$
$\dfrac{2b\cos\, u+a}{4\sin^2 u}(du^2+dv^2)$	K^2	(u,v) system	Equidistant	Equidistant
$\left(\dfrac{a_+}{\nu^2}+\dfrac{a_-}{\mu^2}\right)(d\mu^2+d\nu^2)$	X_2	Horospherical	Horicyclic	Semicircular parabolic
$\left(\dfrac{a_-}{\cosh^2\omega\cos^2\varphi}+\dfrac{a_+}{\sinh^2\omega\sin^2\varphi}\right)\times$ $\times(\cosh^2\omega-\cos^2\varphi)(d\omega^2+d\varphi^2)$	$K^2+d^2X_2$	Elliptic	Elliptic-parabolic	Hyperbolic-parabolic
$\left[a_+k^2\left(\mathrm{sn}^2(\alpha,k)-\mathrm{sn}^2(\beta,k)\right)+a_-\right]\times$ $\times\dfrac{k^2}{4}(d^2\alpha+d^2\beta)$	$X_1+X_2+\gamma K^2$	Elliptic	Elliptic	Elliptic

We state the superintegrable potentials on D_{IV}:

$$V_1(u,v) = \left(\frac{a_+}{\sin^2 u} + \frac{a_-}{\cos^2 u}\right)^{-1} \times \\ \times \left[\frac{\hbar^2}{2m}\left(\frac{k^2-1/4}{\cos^2 u} + \frac{k^2-1/4}{\sin^2 u}\right) - 4\alpha \mathrm{e}^{2v} + 8m\omega^2\mathrm{e}^{4v}\right], \tag{3.29}$$

$$V_2(u,v) = \left(\frac{a_+}{\sin^2 u} + \frac{a_-}{\cos^2 u}\right)^{-1} \times \\ \times \left[\frac{\hbar^2}{2m}\left(\frac{k_1^2-1/4}{\sinh^2 v} - \frac{k_2^2-1/4}{\cosh^2 v}\right) - \frac{\alpha}{4}\left(\frac{1}{\sin^2 u} + \frac{1}{\cos^2 u}\right)\right], \tag{3.30}$$

$$V_3(\tilde{\omega},\tilde{\varphi}) = \frac{\hbar^2}{2m}\left(\frac{a_+}{\sinh^2\tilde{\omega}} - \frac{a_+}{\cosh^2\tilde{\omega}} + \frac{a_-}{\sin^2\tilde{\varphi}} + \frac{a_-}{\cos^2\tilde{\varphi}}\right)^{-1} \times \\ \times \left[\frac{c_3}{\sin^2\tilde{\varphi}} + \frac{c_2}{\cos^2\tilde{\varphi}} - \frac{c_3}{\sinh^2\tilde{\omega}} + \frac{c_2}{\cosh^2\tilde{\omega}}\right], \tag{3.31}$$

$$V_4(\mu,\nu) = \left(\frac{a_+}{\nu^2} + \frac{a_-}{\mu^2}\right)^{-1} \frac{\hbar^2}{2m}\left(k_0^2 - \frac{1}{4}\right)\left(\frac{1}{\mu^2} + \frac{1}{\nu^2}\right). \tag{3.32}$$

Table 4. **Separation of variables for the superintegrable potentials on $D_{\rm IV}$**

Potential	Constants of motion	Separating coordinate system
V_1	$R_1 = K^2 - \alpha(\mu^2+\nu^2) + \dfrac{m}{2}\omega^2(\mu^2+\nu^2)$ $R_2 = X_2 + \dfrac{-2\alpha(a_+\mu^2 - a_-\nu^2) + 8(k^2-1/4)\dfrac{\hbar^2}{m} + 2m\omega^2(a_+\mu^4 - a_-\nu^4)}{a_+\mu^2 + a_-\nu^2}$	(u,v) system Horospherical Elliptic
V_2	$R_1 = X_1 + X_2 + (2\cos u + a)^{-1}\dfrac{\hbar^2}{2m}\Bigg[\left(k_1^2+k_2^2-\dfrac{1}{2}\right) -$ $-2\left(k_3^2-\dfrac{1}{2}\right)\cosh 2v + (\cos 4u + 2a\cos 2u + 3)\left(\dfrac{k_1^2-1/4}{\sinh^2 v} - \dfrac{k_2^2-1/4}{\cosh^2 v}\right)\Bigg]$ $R_2 = K^2 + \dfrac{\hbar^2}{2m}\left(\dfrac{k_1^2-1/4}{\sinh^2 v} + \dfrac{k_2^2-1/4}{\cosh^2 v}\right)$	(u,v) system Degenerate elliptic I
V_3	$R_1 = X_1 + X_2 + 2K^2 + aH + \dfrac{\hbar^2}{2m}\left(\dfrac{a_+}{\sinh^2 2\tilde\omega} + \dfrac{a_-}{\sinh^2\tilde\omega}\right)^{-1}\times$ $\times\left[\dfrac{a_+}{\sinh^2 2\tilde\omega}\left(\dfrac{c_3}{\sin^2\tilde\varphi} + \dfrac{c_1}{\sin^2\tilde\varphi}\right) + \dfrac{a_-}{\sinh^2 2\tilde\varphi}\left(\dfrac{c_3}{\sinh^2\tilde\omega} - \dfrac{c_2}{\cos^2\tilde\omega}\right)\right]$ $R_2 = X_1 - X_2 + \dfrac{\hbar^2}{2m}\left(\dfrac{a_+}{\sinh^2 2\tilde\omega} + \dfrac{a_-}{\sinh^2\tilde\omega}\right)^{-1}\times$ $\times\Bigg[\dfrac{a_+}{\sinh^2 2\tilde\omega}\Bigg(c_1\cosh 2\tilde\omega\tan^2\tilde\varphi - c_2\cos 2\tilde\varphi -$ $-\dfrac{c_3(2\cos^2\tilde\varphi(\sinh^2\tilde\omega - \sin^2\tilde\varphi)+1)}{\sin^2\tilde\varphi}\Bigg) +$ $+\dfrac{a_-}{\sin^2 2\tilde\varphi}\Bigg(c_2\cos 2\tilde\varphi\tanh^2\tilde\omega + c_1\cosh 2\tilde\omega -$ $-\dfrac{c_3(2\cosh^2\tilde\omega(\sinh^2\tilde\omega - \sin^2\tilde\varphi)+1}{\sinh^2\tilde\omega}\Bigg)\Bigg]$	Degenerate elliptic I & II
V_4	$R_1 = X_1 + \dfrac{2\dfrac{\hbar^2}{m}(k_0^2-1/4)(\mu^2+\nu^2)}{a_+\mu^2 + a_-\nu^2}$ $R_2 = X_2 + \dfrac{32\dfrac{\hbar^2}{m}(k_0^2-1/4)}{a_+\mu^2 + a_-\nu^2}$ $R_3 = \mu p_\mu + \nu p_\nu$	(u,v) system Horospherical Elliptic

In Table 4 we list the properties of these potentials on $D_{\rm IV}$. We see that V_4 is a special case, and it has three integrals of motion. The variables $\tilde\omega, \tilde\varphi$ are defined by

$$x = \log\left[\tan\left(\tilde\varphi - i\tilde\omega\right)\right], \quad y = \log\left[\tan\left(\tilde\varphi + i\tilde\omega\right)\right]. \tag{3.33}$$

In terms of these coordinates the line element is given by

$$ds^2 = \frac{a+2b}{\sinh^2 2\tilde\omega} + \frac{a+2b}{\sin^2 2\tilde\varphi} = \frac{a_+}{\sinh^2\tilde\omega} - \frac{a_+}{\cosh^2\tilde\omega} - \frac{a_-}{\sin^2\tilde\varphi} + \frac{a_-}{\cos^2\tilde\varphi}. \tag{3.34}$$

3.1. The Superintegrable Potential V_1 on $D_{\rm IV}$. We start by stating the potential V_1 in the respective coordinate systems

$$V_1(u,v) = \left(\frac{a_+}{\sin^2 u} + \frac{a_-}{\cos^2 u}\right)^{-1} \times \times \left[\frac{\hbar^2}{2m}\left(\frac{k^2-1/4}{\cos^2 u} + \frac{k^2-1/4}{\sin^2 u}\right) - 4\alpha e^{2v} + 8m\omega^2 e^{4v}\right], \tag{3.35}$$

$$= \left(\frac{a_+}{\nu^2} + \frac{a_-}{\mu^2}\right)^{-1} \times \times \left[-\alpha + \frac{\hbar^2}{2m}\left(\frac{k^2-1/4}{\mu^2} + \frac{k^2-1/4}{\nu^2}\right) + \frac{m}{2}\omega^2(\mu^2+\nu^2)\right], \tag{3.36}$$

$$= d^2\left(\frac{a_+}{\sinh^2\omega\sin^2\varphi} + \frac{a_-}{\cosh^2\omega\cos^2\varphi}\right)^{-1} \times \times \left[-\alpha + \frac{\hbar^2}{2m}\left(\frac{k^2-1/4}{\sinh^2\omega\sin^2\varphi} + \frac{k^2-1/4}{\cosh^2\omega\cos^2\varphi}\right) + + \frac{m}{2}\omega^2 d^2(\cosh^2\omega - \sin^2\varphi)\right]. \tag{3.37}$$

The path integral for the potential V_1 can be solved in the (u,v) system and in horospherical coordinates. We also keep the parameters k_1 and k_2 different in comparison with Kalnins et al.

3.1.1. Separation of V_1 in the (u,v) System. The classical Lagrangian and Hamiltonian are given by

$$\mathcal{L}(u,\dot u,v,\dot v) = \frac{m}{2}\frac{2b\cos 2u + a}{\sin^2 2u}(\dot u^2 + \dot v^2) + V(u,v), \tag{3.38}$$

$$\mathcal{H}(u,p_u,v,p_v) = \frac{1}{2m}\frac{\sin^2 2u}{2b\cos 2u + a}(p_u^2 + p_v^2) + V(u,v). \tag{3.39}$$

The canonical momentum operators are given by

$$p_u = \frac{\hbar}{i}\left(\frac{\partial}{\partial u} + 2\cot\, 2u - \frac{2b\sin 2u}{2b\cos 2u + a}\right), \quad p_v = \frac{\hbar}{i}\frac{\partial}{\partial v}, \tag{3.40}$$

and the Hamiltonian operator has the form

$$H = -\frac{\hbar^2}{2m}\frac{\sin^2 2u}{2b\cos 2u + a}\left(\frac{\partial^2}{\partial u^2} + \frac{\partial^2}{\partial v^2}\right) + V(u,v) \tag{3.41}$$

$$= \frac{1}{2m}\frac{\sin 2u}{\sqrt{2b\cos 2u + a}}(p_u^2 + p_v^2)\frac{\sin 2u}{\sqrt{2b\cos 2u + a}} + V(u,v). \tag{3.42}$$

We insert V_1 into the path integral and obtain ($f = a_+/\sin^2 u + a_-/\cos^2 u$)

$$K^{(V_1)}(u'',u',v'',v';T) = \int\limits_{u(t')=u'}^{u(t'')=u''} \mathcal{D}u(t) \int\limits_{v(t')=v'}^{v(t'')=v''} \mathcal{D}v(t) f(u)\times$$

$$\times \exp\left(\frac{i}{\hbar}\int\limits_0^T \left\{\frac{m}{2}f(\dot u^2+\dot v^2) - \frac{1}{f}\left[\frac{\hbar^2}{2m}\left(\frac{k_1^2-1/4}{\cos^2 u} - \frac{k_2^2-1/4}{\sin^2 u}\right)+\right.\right.\right.$$

$$\left.\left.\left. + 8m\omega^2\left(\mathrm{e}^{4v} - \frac{\alpha}{2m\omega^2}\mathrm{e}^{2v}\right)\right]\right\}dt\right). \quad (3.43)$$

We see that the v dependence has the form of a Morse potential:

$$V^{(\mathrm{MP})}(x) = \frac{\hbar^2 V_0^2}{2M}\big(\mathrm{e}^{2x} - 2\tilde\alpha\,\mathrm{e}^{x}\big), \quad (3.44)$$

where the (finite) discrete energy spectrum is given by

$$E_l = -\frac{\hbar^2}{2M}\left(\tilde\alpha - l - \frac{1}{2}\right)^2. \quad (3.45)$$

Proceeding in the usual way we obtain for the time-transformed path integral

$$K^{(V_1)}(u'',u',v'',v';s'') = \int\limits_{u(0)=u'}^{u(s'')=u''} \mathcal{D}u(s) \int\limits_{v(0)=v'}^{v(s'')=v''} \mathcal{D}v(s)\times$$

$$\times \exp\left\{\frac{i}{\hbar}\int\limits_0^T \left[\frac{m}{2}(\dot u^2+\dot v^2) - \frac{\hbar^2}{2m}\left(\frac{\lambda_1^2-1/4}{\cos^2 u} - \frac{\lambda_2^2-1/4}{\sin^2 u}\right)-\right.\right.$$

$$\left.\left. - 8m\omega^2\left(\mathrm{e}^{4v} - \frac{\alpha}{2m\omega^2}\mathrm{e}^{2v}\right)\right]ds\right\} =$$

$$= \sum_n \Phi_n^{(\lambda_2,\lambda_1)}(u'')\Phi_n^{(\lambda_2,\lambda_1)}(u') \exp\left[-\frac{i}{\hbar}\frac{\hbar^2}{2m}(\lambda_1+\lambda_2+2n+1)2s''\right]\times$$

$$\times\left\{\int d\kappa \Psi_\kappa^{(\mathrm{MP})}(v'')\Phi_\kappa^{(MP)\,*}(u')\,\mathrm{e}^{-i\hbar\kappa^2 s''/2m}+\right.$$

$$\left. + \sum_l \Psi_l^{(\mathrm{MP})}(v'')\Phi_l^{(MP\,*)}(u')]\,\exp\left[\frac{i}{\hbar}\frac{\hbar^2}{2m}\left(\tilde\alpha - l - \frac{1}{2}\right)^2\right]\right\}. \quad (3.46)$$

Here, $\lambda_{1,2}^2 = k_{1,2}^2 - 2ma_{-,+}E/\hbar^2$, and in the variable v we have used the solution of the Morse potential and in the variable u the solution of the Pöschl–Teller potential, respectively. This form of the solution is convenient to obtain

the bound state solutions. The bound state energy levels are determined by

$$2(n+l+1)+\lambda_1+\lambda_2-\frac{\alpha}{\hbar\omega}=0. \tag{3.47}$$

By denoting

$$N_{n,l}=\left(2(n+l+1)-\frac{\alpha}{\hbar\omega}\right)^2-(k_1^2+k_2^2) \tag{3.48}$$

the quadratic equation in E can be solved to give (with the further abbreviation $K_a=4(a_+k_1^2+a_-k_2^2)$)

$$E_{n,l}= \\ =\frac{\hbar^2}{4mb^2}\left\{\pm\sqrt{(aN_{n,l}+K_a)^2-4b^2(N_{n,l}^2-4k_1^2k_2^2)}-(aN_{n,l}+K_a)\right\}. \tag{3.49}$$

We keep the $\pm$-sign to allow for different boundary conditions which may depend on the parameters a and b. For instance, for $a=2b$ we get the limiting case:

$$E_{n,l}=-\frac{\hbar^2}{2ma}\left[\left((2(n+l+1)+k_1^2-\frac{\alpha}{\hbar\omega}\right)^2-k_2^2\right]. \tag{3.50}$$

For $k_2=\pm 1/2$ it has the form of the usual zero-energy on the two-dimensional hyperboloid.

In order to obtain the continuous spectrum, the formulation in (u,v) coordinates is inconvenient. Following [12] we perform the coordinate transformation $\cos u=\tanh\tau$, and additionally we make a time-transformation with the time-transformation function $f=a_+/\sin^2 u+a_-/\cos^2 u$. Due to the coordinate transformation $\cos u=\tanh\tau$ additional quantum terms appear according to

$$\exp\left(\frac{im}{2\epsilon\hbar}\frac{\left(\Delta u^{(j)}\right)^2}{\cos u^{(j-1)}\cos u^{(j)}}\right)\doteq \\ \doteq\exp\left[\frac{im}{2\epsilon\hbar}\left(\Delta\tau^{(j)}\right)^2-i\frac{\hbar}{8m}\left(1+\frac{1}{\cosh^2\tau^{(j)}}\right)\right]. \tag{3.51}$$

We get for the path integral (3.43)

$$K^{(V_1)}(u'',u',v'',v';T)=\int_{-\infty}^{\infty}\frac{dE}{2\pi\hbar}\,\mathrm{e}^{-iET/\hbar}\times \\ \times\int_0^{\infty}ds''\exp\left[\frac{i}{\hbar}\left(a_+E-\frac{\hbar^2k_2^2}{2m}\right)\right]K^{(V_1)}(\tau'',\tau',v'',v';s''), \tag{3.52}$$

and the time-transformed path integral $K^{(V_1)}(s'')$ is given by

$$K^{(V_1)}(\tau'',\tau',v'',v';s'') = (\cosh\tau'\cosh\tau'')^{-1/2}\times \times\Bigg[\sum_l \Psi_l^{(\mathrm{MP})}(v')\Psi_l^{(\mathrm{MP})}(v'')K_l(\tau'',\tau';s'')+ + \int d\kappa \Psi_\kappa^{(\mathrm{MP})\,*}(v')\Psi_\kappa^{(\mathrm{MP})}(v'')K_\kappa(\tau'',\tau';s'')\Bigg], \quad (3.53)$$

$$K_{l,\kappa}^{(V_1)}(\tau'',\tau';s'') = \int\limits_{\tau(0)=\tau'}^{\tau(s'')=\tau''} \mathcal{D}\tau(s)\times \times\exp\left\{\frac{i}{\hbar}\int\limits_0^{s''}\left[\frac{m}{2}\dot\tau^2 - \frac{\hbar^2}{2m}\left(\frac{\lambda_1^2-1/4}{\sinh^2\tau} - \frac{\nu_{l,\kappa}^2-1/4}{\cosh^2\tau}\right)\right]ds\right\}. \quad (3.54)$$

The parameters $\lambda_{1,2}$ are the same as in the previous paragraph and ν is given by

$$\nu_l = \left|2l+1-\frac{\alpha}{\hbar\omega}\right| \quad \text{(discrete)}, \quad \nu_\kappa = i\kappa \quad \text{(continuous)}, \quad (3.55)$$

where discrete and continuous means the discrete and continuous contribution of the Morse potential. Of course, the analysis of the discrete spectrum gives the same result as before. The kernel $K_{l,\kappa}^{(V_1)}(s'')$ now allows us to write down the entire kernel $K^{(V_1)}(T)$ in terms of Morse wave functions and modified Pöschl–Teller wave functions in the following form:

$$K^{(V_1)}(u'',u',v'',v';T) = (\cosh\tau'\cosh\tau'')^{-1/2}\times \times\Bigg\{\sum_{ln} N_{ln}^2 \Psi_l^{(\mathrm{MP})\,*}(v')\Psi_l^{(\mathrm{MP})}(v'')\Psi_n^{(\lambda_1,\nu_l)\,*}(\tau')\Psi_n^{(\lambda_1,\nu_l)}(\tau'')\,\mathrm{e}^{-iE_{ln}T/\hbar}+ + \int dp \sum_l N_{lp}^2 \Psi_l^{(\mathrm{MP})\,*}(v')\Psi_l^{(\mathrm{MP})}(v'')\Psi_p^{(\lambda_1,\nu_l)\,*}(\tau')\Psi_p^{(\lambda_1,\nu_l)}(\tau'')\,\mathrm{e}^{-iE_pT/\hbar}+ + \int dp\int d\kappa N_{\kappa p}^2 \Psi_\kappa^{(\mathrm{MP})\,*}(v')\Psi_\kappa^{(\mathrm{MP})}(v'')\Psi_p^{(\lambda_1,i\kappa)\,*}(\tau')\Psi_p^{(\lambda_1,i\kappa)}(\tau'')\mathrm{e}^{-iE_pT/\hbar}\Bigg\}, \quad (3.56)$$

with the proper normalization constants $N_{ln}, N_{lp}, N_{\kappa p}$, where, e.g., N_{ln} is determined by the residuum corresponding to E_{ln} in the Green function, and with the continuous spectrum

$$E_p = \frac{\hbar^2}{2ma_+}(p^2+k_2^2). \quad (3.57)$$

Note that for $k_2 = 1/2$ we obtain the well-known zero energy on the two-dimensional hyperboloid, which appears here in a natural way after performing the coordinate transformation $\cos u = \tanh \tau$.

The $\Psi_p^{(\mu,\nu)}(\omega)$ are the modified Pöschl–Teller functions, which are given by

$$\Psi_n^{(\eta,\nu)}(r) = N_n^{(\eta,\nu)}(\sinh r)^{2k_2-\frac{1}{2}}(\cosh r)^{-2k_1+\frac{3}{2}}\times$$
$$\times\, {}_2F_1(-k_1+k_2+\kappa, -k_1+k_2-\kappa+1; 2k_2; -\sinh^2 r), \tag{3.58}$$

$$N_n^{(\eta,\nu)} = \frac{1}{\Gamma(2k_2)}\left[\frac{2(2\kappa-1)\Gamma(k_1+k_2-\kappa)\Gamma(k_1+k_2+\kappa-1)}{\Gamma(k_1-k_2+\kappa)\Gamma(k_1-k_2-\kappa+1)}\right]^{1/2}. \tag{3.59}$$

The scattering states are given by

$$V(r) = \frac{\hbar^2}{2m}\left(\frac{\eta^2-1/4}{\sinh^2 r} - \frac{\nu^2-1/4}{\cosh^2 r}\right),$$
$$\Psi_p^{(\eta,\nu)}(r) = N_p^{(\eta,\nu)}(\cosh r)^{2k_1-\frac{1}{2}}(\sinh r)^{2k_2-\frac{1}{2}}\times$$
$$\times\, {}_2F_1(k_1+k_2-\kappa, k_1+k_2+\kappa-1; 2k_2; -\sinh^2 r), \tag{3.60}$$

$$N_p^{(\eta,\nu)} = \frac{1}{\Gamma(2k_2)}\sqrt{\frac{p\sinh \pi p}{2\pi^2}}\Big[\Gamma(k_1+k_2-\kappa)\Gamma(-k_1+k_2+\kappa)\times$$
$$\times\,\Gamma(k_1+k_2+\kappa-1)\Gamma(-k_1+k_2-\kappa+1)\Big]^{1/2}, \tag{3.61}$$

k_1, k_2 defined by: $k_1 = \frac{1}{2}(1 \pm \nu)$, $k_2 = \frac{1}{2}(1 \pm \eta)$, where the correct sign depends on the boundary conditions for $r \to 0$ and $r \to \infty$, respectively. The number N_M denotes the maximal number of states with $0, 1, \ldots, N_M < k_1 - k_2 - \frac{1}{2}$, $\kappa = k_1 - k_2 - n$ for the bound states and $\kappa = \frac{1}{2}(1 + ip)$ for the scattering states; ${}_2F_1(a, b; c; z)$ is the hypergeometric function [10, p. 1057].

3.1.2. Separation of V_1 in Horospherical Coordinates. We evaluate the path integral for V_1 in horospherical coordinates. The classical Lagrangian and Hamiltonian are given by

$$\mathcal{L}(\mu, \dot\mu, \nu, \dot\nu) = \frac{m}{2}\left(\frac{a_+}{\nu^2} + \frac{a_-}{\mu^2}\right)(\dot\mu^2 + \dot\nu^2) - V(\mu, \nu), \tag{3.62}$$

$$\mathcal{H}(\mu, p_\mu, \nu, p_\nu) = \frac{1}{2m}\frac{\mu^2\nu^2(p_\mu^2 + p_\nu^2)}{a_+\mu^2 + a_-\nu^2} + V(\mu, \nu). \tag{3.63}$$

For the canonical momentum operators we have

$$p_\mu = \frac{\hbar}{i}\left(\frac{\partial}{\partial\mu} - \frac{\nu^2 a_-/\mu}{a_+\mu^2 + a_-\nu^2}\right), \tag{3.64}$$

$$p_\nu = \frac{\hbar}{i}\left(\frac{\partial}{\partial\nu} - \frac{\mu^2 a_+/\nu}{a_+\mu^2 + a_-\nu^2}\right), \tag{3.65}$$

and for the quantum Hamiltonian we get

$$H = -\frac{\hbar^2}{2m}\frac{\mu^2\nu^2}{a_+\mu^2 + a_-\nu^2}\left(\frac{\partial^2}{\partial\mu^2} + \frac{\partial^2}{\partial\nu^2}\right) + V(\mu,\nu), \tag{3.66}$$

$$= \frac{1}{2m}\sqrt{\frac{\mu^2\nu^2}{a_+\mu^2 + a_-\nu^2}}\,(p_\mu^2 + p_\nu^2)\sqrt{\frac{\mu^2\nu^2}{a_+\mu^2 + a_-\nu^2}} + V(\mu,\nu). \tag{3.67}$$

We insert V_1 into the path integral and obtain ($f = a_+/\nu^2 + a_-/\mu^2$ and keeping to constants $k_{1,2}$)

$$\begin{aligned} K^{(V_1)}(\mu'',\mu',\nu'',\nu';T) &= \int\limits_{\mu(t')=\mu'}^{\mu(t'')=\mu''}\mathcal{D}\mu(t)\int\limits_{\nu(t')=\nu'}^{\nu(t'')=\nu''}\mathcal{D}\nu(t)f(\mu,\nu)\times \\ &\times\exp\left\{\frac{i}{\hbar}\int\limits_0^T\left[\frac{m}{2}f(\mu,\nu)(\dot\mu^2+\dot\nu^2)-\right.\right. \\ &\left.\left.-\frac{1}{f(\mu,\nu)}\left(\frac{m}{2}\omega^2(\mu^2+\nu^2)-\alpha+\frac{\hbar^2}{2m}\left(\frac{k_1^2-1/4}{\mu^2}+\frac{k_2^2-1/4}{\nu^2}\right)\right)\right]dt\right\} = \\ &= \int\limits_{-\infty}^{\infty}\frac{dE}{2\pi\hbar}\,\mathrm{e}^{-iET/\hbar}\int\limits_0^\infty ds''\mathrm{e}^{i\alpha s''/\hbar}K^{(V_1)}(\mu'',\mu',\nu'',\nu';s''), \end{aligned} \tag{3.68}$$

and the time-transformed path integral $K^{(V_1)}(s'')$ is given by

$$\begin{aligned} K^{(V_1)}(\mu'',\mu',\nu'',\nu';s'') &= \int\limits_{\mu(0)=\mu'}^{\mu(s'')=\mu''}\mathcal{D}\mu(s)\times \\ &\times\exp\left\{\frac{i}{\hbar}\int\limits_0^{s''}\left[\frac{m}{2}(\dot\mu^2-\omega^2\mu^2)-\frac{\hbar^2}{2m}\frac{k_1^2-2ma_-E/\hbar^2-1/4}{\mu^2}\right]ds\right\}\times \\ &\times\int\limits_{\nu(0)=\nu'}^{\nu(s'')=\nu''}\mathcal{D}\nu(s)\exp\left\{\frac{i}{\hbar}\int\limits_0^{s''}\left[\frac{m}{2}(\dot\nu^2-\omega^2\nu^2)-\frac{\hbar^2}{2m}\frac{k_2^2-2ma_+E/\hbar^2-1/4}{\nu^2}\right]ds\right\} = \\ &= \frac{m^2\omega^2\sqrt{\mu'\mu''\nu'\nu''}}{i^2\hbar^2\sin^2\omega s''}\exp\left[-\frac{m\omega}{2i\hbar}(\mu'^2+\mu''^2+\nu'^2+\nu''^2)\cot\omega s''\right]\times \\ &\times I_{\lambda_1}\left(\frac{m\omega\mu'\mu''}{i\hbar\sin\omega s''}\right)I_{\lambda_2}\left(\frac{m\omega\nu'\nu''}{i\hbar\sin\omega s''}\right), \end{aligned} \tag{3.69}$$

where $\lambda_{1,2} = k_{1,2}^2 - 2ma_{\mp}E/\hbar^2$. We can extract the bound state wave functions for the bound state contribution of the Green function according to:

$$G^{(V_1)}(\mu'',\mu',\nu'',\nu';E) = \sum_{n_\mu=0}^{\infty}\sum_{n_\nu=0}^{\infty}\frac{N_{n_\mu n_\nu}^2}{E_{n_\mu n_\nu}-E}\times \\ \times\,\Psi_{n_\mu}^{(\mathrm{RHO},\lambda_1)}(\mu')\Psi_{n_\mu}^{(\mathrm{RHO},\lambda_1)}(\mu'')\Psi_{n_\nu}^{(\mathrm{RHO},\lambda_2)}(\nu')\Psi_{n_\nu}^{(\mathrm{RHO},\lambda_2)}(\nu''). \quad (3.70)$$

The bound states are determined by the equation

$$\frac{\alpha}{\hbar\omega} - 2(n_\mu+n_\nu+1) = \sqrt{k_1^2 - \frac{2ma_-E}{\hbar^2}} + \sqrt{k_2^2 - \frac{2ma_+E}{\hbar^2}}. \quad (3.71)$$

This quadratic equation in E is identical with (3.47).

3.2. The Superintegrable Potential V_2 on D_{IV}. We state the potential in the respective coordinate systems

$$V_2(u,v) = \left(\frac{a_+}{\sin^2 u}+\frac{a_-}{\cos^2 u}\right)^{-1}\frac{\hbar^2}{2m}\left[\frac{k_1^2-1/4}{\sinh^2 v}-\frac{k_2^2-1/4}{\cosh^2 v}+\right. \\ \left. +\left(k_3^2-\frac{1}{4}\right)\left(\frac{1}{\sin^2 u}+\frac{1}{\cos^2 u}\right)\right], \quad (3.72)$$

$$= 4\left(\frac{a_+}{\sinh^2 2\tilde\omega}+\frac{a_-}{\sin^2 2\tilde\varphi}\right)^{-1}\frac{\hbar^2}{2m}\left[\left(k_3^2-\frac{1}{4}\right)\left(\frac{1}{\sinh^2 2\tilde\omega}+\frac{1}{\sin^2 2\tilde\varphi}\right)+\right. \\ \left. +\left(\frac{k_2^2-1/4}{\cos^2 2\tilde\varphi}-\frac{k_1^2-1/4}{\cosh^2 2\tilde\omega}\right)\right]. \quad (3.73)$$

It is possible to evaluate the path integral for V_2 in the (u,v) and the degenerate elliptic system with $\gamma=2$. The elliptic system with $\gamma=0$ is not treated.

3.2.1. Separation of V_2 in the (u,v) System. We insert V_2 into the path integral and obtain ($f = a_+/\sin^2 u + a_-/\cos^2 u$)

$$K^{(V_2)}(u'',u',v'',v';T) = \int\limits_{u(t')=u'}^{u(t'')=u''}\mathcal{D}u(t)\int\limits_{v(t')=v'}^{v(t'')=v''}\mathcal{D}v(t)f(u)\times \\ \times\exp\left\{\frac{i}{\hbar}\int\limits_0^T\left[\frac{m}{2}f(\dot u^2+\dot v^2)-\frac{\hbar^2}{2mf}\left(\frac{k_1^2-1/4}{\sinh^2 v}-\frac{k_2^2-1/4}{\cosh^2 v}+\right.\right.\right. \\ \left.\left.\left. +\left(k_3^2-\frac{1}{4}\right)\left(\frac{1}{\sin^2 u}+\frac{1}{\cos^2 u}\right)\right)\right]dt\right\}. \quad (3.74)$$

This formulation in (u, v) coordinates is inconvenient. Following the procedure as for V_1 in the (u, v) system we perform the coordinate transformation $\cos u = \tanh\tau$, and get for the path integral (3.74)

$$K^{(V_2)}(u'', u', v'', v'; T) = \int_{-\infty}^{\infty} \frac{dE}{2\pi\hbar}\,\mathrm{e}^{-iET/\hbar}\times$$
$$\times \int_0^\infty ds''\,\exp\left[\frac{i}{\hbar}\left(a_+E - \frac{\hbar^2 k_3^2}{2m}\right)\right] K(\tau'', \tau', v'', v'; s''), \quad (3.75)$$

and the time-transformed path integral $K^{(V_2)}(s'')$ is given by

$$K^{(V_2)}(\tau'', \tau', v'', v'; s'') =$$
$$= (\cosh\,\tau'\cosh\,\tau'')^{-1/2}\sum_{n_v=0}^{N_{\max}} \Psi_{n_v}^{(k_1,k_2)}(v')\Psi_{n_v}^{(k_1,k_2)}(v'')\times$$
$$\times \int_{\tau(0)=\tau'}^{\tau(s'')=\tau''} \mathcal{D}\tau(s)\exp\left\{\frac{i}{\hbar}\int_0^{s''}\left[\frac{m}{2}\dot\tau^2 - \frac{\hbar^2}{2m}\left(\frac{\lambda_2^2-1/4}{\sinh^2\tau} - \frac{\lambda_1^2-1/4}{\cosh^2\tau}\right)\right]ds\right\} +$$
$$+ (\cosh\tau'\cosh\tau'')^{-1/2}\int dk_v\,\Psi_{k_v}^{(k_1,k_2)}(v')\Psi_{k_v}^{(k_1,k_2)}(v'')\times$$
$$\times \int_{\tau(0)=\tau'}^{\tau(s'')=\tau''} \mathcal{D}\tau(s)\exp\left\{\frac{i}{\hbar}\int_0^{s''}\left[\frac{m}{2}\dot\tau^2 - \frac{\hbar^2}{2m}\left(\frac{\lambda_2^2-1/4}{\sinh^2\tau} - \frac{-k_v^2-1/4}{\cosh^2\tau}\right)\right]ds\right\}.$$
$$(3.76)$$

$(\lambda_1^2 = (2n_v + |k_1| - |k_2| + 1)^2,\ \lambda_2^2 = k_3^2 - 2ma_-E/\hbar^2)$.

The v-path integration gives a discrete and continuous spectrum, thus two different parts for the τ-path integration. We therefore find for the Green function

$$G^{(V_2)}(\tau'', \tau', v'', v'; E) = (\cosh\tau'\cosh\tau'')^{-1/2}\times$$
$$\times \sum_{n_v=0}^{N_{\max}} \Psi_{n_v}^{(k_1,k_2)}(v')\Psi_{n_v}^{(k_1,k_2)}(v'')\frac{m}{\hbar^2}\frac{\Gamma(m_1 - L_{\lambda_1})\Gamma(L_{\lambda_1}+m_1+1)}{\Gamma(m_1+m_2+1)\Gamma(m_1-m_2+1)}\times$$
$$\times (\cosh\tau'\cosh\tau'')^{-(k_1-k_2)}(\tanh\tau'\tanh\tau'')^{m_1+m_2+1/2}\times$$
$$\times\,{}_2F_1\biggl(-L_{\lambda_1}+m_1, L_{\lambda_1}+m_1+1; m_1-m_2+1; \frac{1}{\cosh^2\tau_<}\biggr)\times$$

$$\times\, {}_2F_1\bigg(-L_{\lambda_1}+m_1, L_{\lambda_1}+m_1+1; m_1+m_2+1; \tanh^2\tau_>\bigg)+$$
$$+(\cosh\tau'\cosh\tau'')^{-1/2}\int dk_v \Psi_{k_v}^{(k_1,k_2)}(v')\Psi_{k_v}^{(k_1,k_2)}(v'')\times$$
$$\times\frac{m}{\hbar^2}\frac{\Gamma(m_1-L_{k_v})\Gamma(L_{k_v}+m_1+1)}{\Gamma(m_1+m_2+1)\Gamma(m_1-m_2+1)}\times$$
$$\times(\cosh\tau'\cosh\tau'')^{-(k_1-k_2)}(\tanh\tau'\tanh\tau'')^{m_1+m_2+1/2}\times$$
$$\times\, {}_2F_1\bigg(-L_{k_v}+m_1, L_{k_v}+m_1+1; m_1-m_2+1; \frac{1}{\cosh^2\tau_<}\bigg)\times$$
$$\times\, {}_2F_1\bigg(-L_{k_v}+m_1, L_{k_v}+m_1+1; m_1+m_2+1; \tanh^2\tau_>\bigg) \quad (3.77)$$

$\big(m_{1,2}=\frac{1}{2}(\lambda_2\pm\sqrt{2m\mathcal{E}}/\hbar)$, $L_{\lambda_1}=\frac{1}{2}(\lambda_1-1)$, $L_{k_v}=\frac{1}{2}(ik_v-1)$, $\mathcal{E}=a_+E-\hbar^2k_3^2/2m\big)$.

A discrete spectrum is only possible for the first summand in (3.76). First, we can analyze the discrete spectrum by looking at the poles in (3.77) which gives the equation

$$2(n_\tau+n_v)+\lambda_++\lambda_-+|k_2|-|k_1|=0 \quad (3.78)$$

($\lambda_\pm^2=k_3^2-2ma_\pm E/\hbar^2$). This gives a quadratic equation in E with solution ($N_k=2n_\tau-2n_v-|k_1|+|k_2|$)

$$E_{n_\tau n_v}=-\frac{a\hbar^2N_k^2}{4b^2}\left(1\mp\sqrt{1+\frac{4b^2}{a^2}\bigg(\frac{k_3^2}{N_k^2}-1\bigg)}\right). \quad (3.79)$$

The entire Green function in terms of the wave functions is given by

$$G^{(V_2)}(\tau'',\tau',v'',v';E)=(\cosh\tau'\cosh\tau'')^{-1/2}\int dp\frac{N_{pk_v}^2}{E_p-E}\int dk_v\times$$
$$\times\Psi_{k_v}^{(k_1,k_2)}(v')\Psi_{k_v}^{(k_1,k_2)}(v'')\Psi_p^{(\lambda_2,ik_v)}(\tau')\Psi_p^{(\lambda_2,ik_v)\,*}(\tau'')+$$
$$+(\cosh\tau'\cosh\tau'')^{-1/2}\sum_{n_\tau=0}^{\infty}\sum_{n_v=0}^{\infty}\Psi_{n_v}^{(k_1,k_2)}(v')\Psi_{n_v}^{(k_1,k_2)}(v'')\times$$
$$\times\left\{\sum_{n_\tau=0}^{N_{\max}}\frac{N_{n_\tau n_v}^2}{E_{n_\tau n_v}-E}\Psi_{n_\tau}^{(\lambda_2,\lambda_1)}(\tau')\Psi_{n_\tau}^{(\lambda_2,\lambda_1)}(\tau'')+\right.$$
$$\left.+\int dp\frac{N_{pn_v}^2}{E_p-E}\Psi_p^{(\lambda_2,\lambda_1)}(\tau')\Psi_p^{(\lambda_2,\lambda_1)\,*}(\tau'')\right\}, \quad (3.80)$$

where $N_{n_\tau n_v}, N_{k_\tau n_v}$ is determined by the residuum in (3.77). The continuous spectrum has the form

$$E_p = \frac{\hbar^2}{2ma_+}(p^2 + k_3^2). \tag{3.81}$$

For $k_3 = \pm 1/2$ we obtain the usual zero-point energy on the two-dimensional hyperboloid. Reinserting $\cos u = \tanh v$ gives the Green function in the (u, v) system.

3.2.2. Separation of V_2 in Degenerate Elliptic Coordinates. We insert the potential V_2 in degenerate elliptic coordinates into the path integral and obtain $(f(\tilde\omega, \tilde\varphi) = 4(a_+/\sinh^2 2\tilde\omega + a_-/\sin^2 2\tilde\varphi))$

$$K^{(V_2)}(\tilde\omega'', \tilde\omega', \tilde\varphi'', \tilde\varphi'; T) = \int\limits_{\tilde\omega(t')=\tilde\omega'}^{\tilde\omega(t'')=\tilde\omega''} \mathcal{D}\tilde\omega(t) \int\limits_{\tilde\varphi(t')=\tilde\varphi'}^{\tilde\varphi(t'')=\tilde\varphi''} \mathcal{D}\tilde\varphi(t) f(\tilde\omega, \tilde\varphi) \times$$
$$\times \exp\left\{\frac{i}{\hbar}\int\limits_0^T \left[\frac{m}{2} f(\tilde\omega, \tilde\varphi)(\dot{\tilde\omega}^2 + \dot{\tilde\varphi}^2) 4 \frac{\hbar^2}{2m f(\tilde\omega, \tilde\varphi)} \times\right.\right.$$
$$\left.\left. \times \left(\frac{k_1^2 - 1/4}{\sinh^2 2\tilde\omega} - \frac{k_2^2 - 1/4}{\cosh^2 2\tilde\omega} + \frac{k_3^2 - 1/4}{\sin^2 2\tilde\varphi} + \frac{k_2^2 - 1/4}{\cos^2 2\tilde\varphi}\right)\right] dt\right\}. \tag{3.82}$$

The calculation is similar as in the case of the (u, v) system: First, we rescale $2\tilde\omega \to \tilde\omega, 2\tilde\varphi \to \tilde\varphi$, then we perform the transformation $\cos\tilde\varphi = \tanh\tilde\tau$. Finally, we perform a time transformation in the path integral with the time transformation $f(\tilde\omega, \tilde\varphi) \to f(\tilde\omega, \tilde\tau)$ yielding

$$G^{(V_2)}(\tilde\tau'', \tilde\tau', \tilde\omega'', \tilde\omega'; E) =$$
$$= \int\limits_0^\infty ds'' \exp\left[\frac{i}{\hbar} s''\left(E a_- - \frac{\hbar^2 k_3^2}{2m}\right)\right] K^{(V_2)}(\tilde\tau'', \tilde\tau', \tilde\omega'', \tilde\omega'; s'') \tag{3.83}$$

with the transformed path integral $K^{(V_2)}(s'')$ given by

$$K^{(V_2)}(\tilde\tau'', \tilde\tau', \tilde\omega'', \tilde\omega'; s'') =$$
$$= \int\limits_{\tilde\tau(0)=\tilde\tau'}^{\tilde\tau(s'')=\tilde\tau''} \mathcal{D}\tilde\tau(s) \int\limits_{\tilde\omega(0)=\tilde\omega'}^{\tilde\omega(s'')=\tilde\omega''} \mathcal{D}\tilde\omega(s) \exp\left\{\frac{i}{\hbar}\int\limits_0^T \left[\frac{m}{2}(\dot{\tilde\tau}^2 + \cosh^2\tilde\tau\,\dot{\tilde\omega}^2) - \right.\right.$$
$$\left.\left. - \frac{\hbar^2}{2m}\left(\frac{k_1^2 - 1/4}{\sinh^2\tilde\tau} + \frac{1}{\cosh^2\tilde\tau}\left(\frac{\lambda_+^2 - 1/4}{\sinh^2\tilde\omega} - \frac{k_2^2 - 1/4}{\cosh^2\tilde\omega} + \frac{1}{4}\right)\right)\right] ds\right\}. \tag{3.84}$$

Again we evaluate this path integral by a successive $\tilde{\omega}$- and $\tilde{\tau}$-path integration. Performing finally the s'' integration we obtain

$$G^{(V_2)}(\tilde{\tau}'',\tilde{\tau}',\tilde{\omega}'',\tilde{\omega}';E) = (\cosh\tilde{\tau}'\cosh\tilde{\tau}'')^{-1/2}\times$$
$$\times\Bigg\{\int dp\frac{N^2_{k_{\tilde{\omega}}p}}{E_p-E}\int dk_{\tilde{\omega}}\,\Psi_p^{(k_1,ik_{\tilde{\omega}})}(\tilde{\tau}')\Psi_p^{(k_1,ik_{\tilde{\omega}})\,*}(\tilde{\tau}'')\Psi_{k_{\tilde{\omega}}}^{(\lambda_1,k_2)}(\tilde{\omega}')\Psi_{k_{\tilde{\omega}}}^{(\lambda_1,k_2)\,*}(\tilde{\omega}'')+$$
$$+\int dp\sum_{n_{\tilde{\omega}}=0}^{N_{\max}}\frac{N^2_{n_{\tilde{\omega}}p}}{E_p-E}\Psi_p^{(k_1,\epsilon_{n_{\tilde{\omega}}})}(\tilde{\tau}')\Psi_p^{(k_1,\epsilon_{n_{\tilde{\omega}}})\,*}(\tilde{\tau}'')\Psi_{n_{\tilde{\omega}}}^{(\lambda_1,k_2)}(\tilde{\omega}')\Psi_{n_{\tilde{\omega}}}^{(\lambda_1,k_2)}(\tilde{\omega}'')+$$
$$+\sum_{n_{\tilde{\tau}}=0}^{N_{\max}}\sum_{n_{\tilde{\omega}}=0}^{N_{\max}}\frac{N^2_{n_{\tilde{\tau}}n_{\tilde{\omega}}}}{E_{n_{\tilde{\tau}}n_{\tilde{\omega}}}-E}\Psi_{n_{\tilde{\tau}}}^{(k_1,\epsilon_{n_{\tilde{\omega}}})}(\tilde{\tau}')\Psi_{n_{\tilde{\tau}}}^{(k_1,\epsilon_{n_{\tilde{\omega}}})}(\tilde{\tau}'')\Psi_{n_{\tilde{\omega}}}^{(\lambda_1,k_2)}(\tilde{\omega}')\Psi_{n_{\tilde{\omega}}}^{(\lambda_1,k_2)}(\tilde{\omega}'')\Bigg\}. \tag{3.85}$$

The normalization constants $N_{k_{\tilde{\omega}}p}, N_{k_{\tilde{\omega}}p}, N_{n_{\tilde{\tau}}n_{\tilde{\omega}}}$ are determined by the respective residuum in $G^{(V_2)}(E)$ and the discrete spectrum is determined by the quadratic equation (3.78). The continuous spectrum has the form

$$E_p = \frac{\hbar^2}{2ma_-}(p^2+k_3^2). \tag{3.86}$$

The difference of E_p in comparison to the (u,v) system can be resolved by making in the (u,v) system the transformation $\sin u = \tanh\tau$ which changes the sign in the energy term. This concludes the discussion of V_2 on $D_{\rm IV}$.

3.3. The Superintegrable Potential V_3 on $D_{\rm IV}$. We state the potential in the respective coordinate systems

$$V_3(\tilde{\omega},\tilde{\varphi}) = \frac{\hbar^2}{2m}\left(\frac{4a_+}{\sinh^2 2\tilde{\omega}}+\frac{4a_-}{\sinh^2\tilde{\varphi}}\right)^{-1}\times$$
$$\times\left[\frac{c_1}{\cos^2\tilde{\varphi}}+\frac{c_2}{\cosh^2\tilde{\omega}}+c_3\left(\frac{1}{\sin^2\tilde{\varphi}}-\frac{1}{\sinh^2\tilde{\omega}}\right)\right], \tag{3.87}$$
$$= \frac{\hbar^2}{2m}\left[a_+\left(\frac{1}{\cosh^2\hat{\omega}}-\frac{1}{\cos^2\hat{\varphi}}\right)-a_-\left(\frac{1}{\sinh^2\hat{\omega}}+\frac{1}{\sin^2\hat{\varphi}}\right)\right]^{-1}\times$$
$$\times\left[\frac{c_3}{\sinh^2\hat{\omega}}+\frac{c_2}{\cosh^2\hat{\omega}}+c_3\left(\frac{1}{\sin^2\hat{\varphi}}-\frac{1}{\cos^2\hat{\varphi}}\right)\right]. \tag{3.88}$$

It is possible to evaluate the path integral for V_3 in both separating coordinate systems. However, due to the similarity in the evaluations, only the degenerate elliptic II case will be presented.

3.3.1. Separation of V_3 in Degenerate Elliptic Coordinates II. We insert the potential V_3 in the path integral formulation for degenerate elliptic coordinates on $D_{\rm IV}$ and obtain $f(\tilde\omega,\tilde\varphi)=4(a_+/\sinh^2 2\tilde\omega+a_-/\sin^2 2\tilde\varphi))$

$$K^{(V_3)}(\tilde\omega'',\tilde\omega',\varphi'',\varphi';T)=\int\limits_{\tilde\omega(t')=\tilde\omega'}^{\tilde\omega(t'')=\tilde\omega''}\mathcal{D}\tilde\omega(t)\int\limits_{\tilde\varphi(t')=\tilde\varphi'}^{\tilde\varphi(t'')=\tilde\varphi''}\mathcal{D}\tilde\varphi(t)f(\tilde\omega,\tilde\varphi)\times$$
$$\times\exp\left\{\frac{i}{\hbar}\int\limits_0^T\left[\frac{m}{2}f(\tilde\omega,\tilde\varphi)(\dot{\tilde\omega}^2+\dot{\tilde\varphi}^2)-\right.\right.$$
$$\left.\left.-\frac{\hbar^2}{2mf(\tilde\omega,\tilde\varphi)}\left(\frac{c_1}{\cos^2\tilde\varphi}+\frac{c_2}{\cosh^2\tilde\omega}+c_3\left(\frac{1}{\sin^2\tilde\varphi}-\frac{1}{\sinh^2\tilde\omega}\right)\right)\right]dt\right\}. \tag{3.89}$$

In order to obtain a convenient form to evaluate (3.89) we perform the coordinate transformation $\cos\tilde\varphi=\tanh\tilde\tau$ in the same way as for V_2. Performing also the corresponding time transformation gives

$$K^{(V_3)}(\tilde\omega'',\tilde\omega',\tilde\tau'',\tilde\tau';T)=\int\limits_{-\infty}^{\infty}\frac{dE}{2\pi\hbar}\,\mathrm{e}^{-iET/\hbar}\times$$
$$\times\int\limits_0^\infty ds''\exp\left[\frac{i}{\hbar}\left(\frac{\hbar^2}{2m}\lambda^2_{3^+_{a_-}}\right)\right]K^{(V_3)}(\tilde\omega'',\tilde\omega',\tilde\tau'',\tilde\tau';s''), \tag{3.90}$$

and the time-transformed path integral $K^{(V_3)}(s'')$ is given by

$$K^{(V_3)}(\tilde\omega'',\tilde\omega',\tilde\tau'',\tilde\tau';s'')=\int\limits_{\tilde\omega(0)=\tilde\omega'}^{\tilde\omega(s'')=\tilde\omega''}\mathcal{D}\tilde\omega(s)\int\limits_{\tilde\tau(0)=\tilde\tau'}^{\tilde\tau(s'')=\tilde\tau''}\mathcal{D}\tilde\tau(s)\cosh\tilde\tau\times$$
$$\times\exp\left\{\frac{i}{\hbar}\int\limits_0^{s''}\left[\frac{m}{2}(\dot{\tilde\tau}^2+\cosh^2\tilde\tau\dot{\tilde\omega}^2)-\frac{\hbar^2}{2m}\frac{\lambda^2_{1^+_{a_-}}-1/4}{\sinh^2\tilde\tau}-\right.\right.$$
$$\left.\left.-\frac{\hbar^2}{2m\cosh^2\tilde\tau}\left(\frac{\lambda^2_{3^-_{a+}}-1/4}{\sinh^2\tilde\omega}-\frac{\lambda^2_{2^-_{a+}}-1/4}{\cosh^2\tilde\omega}+\frac{1}{4}\right)\right]ds\right\} \tag{3.91}$$

$\big(\lambda^2_{i^\pm_{a_\pm}}=\dfrac{1}{4}\mp c_i-2ma_\pm E/\hbar^2,\ i=1,2,3\big)$. The latter path integral has the form of two successive modified Pöschl–Teller path integrations in $\tilde\omega$ and $\tilde\tau$. In the ω-path integration we get a contribution from the continuous and discrete spectrum. The continuous contribution gives in the $\tilde\tau$-path integration only a continuous part, whereas the other gives a discrete and continuous contribution in $\tilde\tau$. We denote the continuous parameter in $\tilde\omega$ by $p_{\tilde\omega}$, the discrete parameter in $\tilde\omega$ by $\epsilon_{n_{\tilde\omega}} = 2n_{\tilde\omega} + \lambda_{3^-_{a+}} - \lambda_{2^-_{a+}} - 1$, the continuous parameter in $\tilde\tau$ by p, the discrete parameter in $\tilde\tau$ by $\epsilon_{n_{\tilde\tau}} = 2n_{\tilde\tau} + \lambda_{1^+_{a-}} - \epsilon_{n_{\tilde\omega}} - 1$, therefore:

$$
\begin{aligned}
&K^{(V_3)}(\tilde\omega'',\tilde\omega',\tilde\tau'',\tilde\tau';s'') = \\
&= (\cosh\tilde\tau'\cosh\tilde\tau'')^{-1/2}\int_0^\infty dp_\omega \Psi_{p_{\tilde\omega}}^{(\lambda_{3^-_{a+}},\lambda_{2^-_{a+}})}(\tilde\omega')\Psi_{p_{\tilde\omega}}^{(\lambda_{3^-_{a+}},\lambda_{2^-_{a+}})\,*}(\tilde\omega'')\times \\
&\times \int_{\tilde\tau(0)=\tilde\tau'}^{\tilde\tau(s'')=\tilde\tau''} \mathcal{D}\tilde\tau(s)\,\exp\left\{\frac{i}{\hbar}\int_0^{s''}\left[\frac{m}{2}\dot{\tilde\tau}^2 - \frac{\hbar^2}{2m}\left(\frac{\lambda^2_{1^+_{a-}}-1/4}{\sinh^2\tilde\tau} + \frac{p^2_{\tilde\omega}+1/4}{\cosh^2\tilde\tau}\right)\right]ds\right\} + \\
&+ (\cosh\tilde\tau'\cosh\tilde\tau'')^{-1/2}\sum_{n_{\tilde\omega}=0}^{N_{\max}} \Psi_{n_{\tilde\omega}}^{(\lambda_{3^-_{a+}},\lambda_{2^-_{a+}})}(\tilde\omega')\Psi_{n_{\tilde\omega}}^{(\lambda_{3^-_{a+}},\lambda_{2^-_{a+}})}(\tilde\omega'')\times \\
&\times \int_{\tilde\tau(0)=\tilde\tau'}^{\tilde\tau(s'')=\tilde\tau''} \mathcal{D}\tilde\tau(s)\,\exp\left\{\frac{i}{\hbar}\int_0^{s''}\left[\frac{m}{2}\dot{\tilde\tau}^2 - \frac{\hbar^2}{2m}\left(\frac{\lambda^2_{1^+_{a-}}-1/4}{\sinh^2\tilde\tau} - \frac{\epsilon^2_{n_{\tilde\omega}}-1/4}{\cosh^2\tilde\tau}\right)\right]ds\right\} = \\
&= (\cosh\tilde\tau'\cosh\tilde\tau'')^{-1/2}\int_0^\infty dp_t\omega \Psi_{p_{\tilde\omega}}^{(\lambda_{3^-_{a+}},\lambda_{2^-_{a+}})}(\tilde\omega')\Psi_{p_{\tilde\omega}}^{(\lambda_{3^-_{a+}},\lambda_{2^-_{a+}})\,*}(\tilde\omega'')\times \\
&\times \int_0^\infty dp \Psi_p^{(\lambda_{1^+_{a-}},ip_{\tilde\omega})}(\tilde\omega')\Psi_p^{(\lambda_{1^+_{a-}},ip_{\tilde\omega})\,*}(\tilde\omega'')\,\mathrm{e}^{-is''\hbar p^2/2m} + \\
&+ (\cosh\tilde\tau'\cosh\tilde\tau'')^{-1/2}\sum_{n_{\tilde\omega}=0}^{N_{\max}} \Psi_{n_{\tilde\omega}}^{(\lambda_{3^-_{a+}},\lambda_{2^-_{a+}})}(\tilde\omega')\Psi_{n_{\tilde\omega}}^{(\lambda_{3^-_{a+}},\lambda_{2^-_{a+}})}(\tilde\omega'')\times \\
&\times \left\{\int_0^\infty dp \Psi_p^{(\lambda_{1^+_{a-}},\epsilon_{n_{\tilde\omega}})}(\tilde\omega')\Psi_p^{(\lambda_{1^+_{a-}},\epsilon_{n_{\tilde\omega}})\,*}(\tilde\omega'')\,\mathrm{e}^{-is''\hbar p^2/2m} + \right. \\
&\left. + \sum_{n_\omega=0}^{N_{\max}} \Psi_{n_\tau}^{(\lambda_{1^+_{a-}},\epsilon_{n_{\tilde\omega}})}(\tilde\tau')\Psi_{n_\tau}^{(\lambda_{1^+_{a-}},\epsilon_{n_{\tilde\omega}})}(\tilde\tau'')\,\mathrm{e}^{-i\hbar s''\epsilon^2_{n_{\tilde\tau}}/2m}\right\}. \qquad (3.92)
\end{aligned}
$$

Performing the s'' integration gives the spectrum. For the continuous spectrum we obtain

$$
E_p = \frac{\hbar^2}{2ma_-}\left(p^2 + \frac{1}{4} - c_3\right). \qquad (3.93)
$$

The discrete spectrum is determined by

$$2(n_{\tilde{\omega}}+n_{\tilde{\tau}})+\lambda_{1^+_{a_-}}+\lambda_{3^-_{a_+}}-\lambda_{2^-_{a_-}}-2=\lambda_{3^+_{a_-}}\,. \tag{3.94}$$

This is an equation in E in the eighth order which we will not solve.

3.4. The Superintegrable Potential V_4 on $D_{\rm IV}$. We state the potential in the respective coordinate systems

$$V_4(\mu,\nu)=\left(\frac{a_+}{\sin^2 u}+\frac{a_-}{\cos^2 u}\right)^{-1}\frac{\hbar^2}{2m}\left(k_0^2-\frac{1}{4}\right)\left(\frac{1}{\sin^2 u}+\frac{1}{\cos^2 u}\right), \tag{3.95}$$

$$=\left(\frac{a_+}{\nu^2}+\frac{a_-}{\mu^2}\right)^{-1}\frac{\hbar^2}{2m}\left(k_0^2-\frac{1}{4}\right)\left(\frac{1}{\nu^2}+\frac{1}{\mu^2}\right), \tag{3.96}$$

$$=\frac{\hbar^2}{2md^2}\left(\frac{a+2b}{\sinh^2 2\omega'}+\frac{a-2b}{\sin^2 2\varphi'}\right)^{-1}\left(k_0^2-\frac{1}{4}\right)\times$$
$$\times\left(\frac{1}{\cosh^2\omega\cos^2\varphi}+\frac{1}{\sinh^2\omega\sin^2\varphi}\right). \tag{3.97}$$

It is possible to evaluate the path integral for V_4 in all the separating coordinate systems. However, we evaluate the path integral for V_4 only in the (u,v) system because V_4 is trivial.

3.4.1. Separation of V_4 in the (u,v) System. We insert V_4 into the path integral and obtain ($f=a_+/\sin^2 u+a_-/\cos^2 u$)

$$K(u'',u',v'',v';T)=\int\limits_{u(t')=u'}^{u(t'')=u''}\mathcal{D}u(t)\int\limits_{v(t')=v'}^{v(t'')=v''}\mathcal{D}v(t)f(u)\times$$
$$\times\exp\left\{\frac{i}{\hbar}\int\limits_0^T\left[\frac{m}{2}f(u)(\dot{u}^2+\dot{v}^2)-\frac{\hbar^2}{2m}\frac{k_0^2-1/4}{f(u)}\left(\frac{1}{\sin^2 u}+\frac{1}{\cos^2 u}\right)\right]dt\right\}. \tag{3.98}$$

We proceed similarly as in [14]. Because the formulation in (u,v) coordinates is inconvenient, we perform following [12] the coordinate transformation $\cos u=\tanh\tau$. Further, we separate off the v-path integration, and additionally we make a time transformation with the time-transformation function $f=a_+/\sin^2 u+a_-/\cos^2 u$. Due to the coordinate transformation $\cos u=\tanh\tau$

additional quantum terms appear according to

$$\exp\left(\frac{im}{2\epsilon\hbar}\frac{\left(\Delta u^{(j)}\right)^2}{\cos u^{(j-1)}\cos u^{(j)}}\right)\doteq$$
$$\doteq\exp\left[\frac{im}{2\epsilon\hbar}\left(\Delta\tau^{(j)}\right)^2-i\frac{\hbar}{8m}\left(1+\frac{1}{\cosh^2\tau^{(j)}}\right)\right]. \quad (3.99)$$

We get for the path integral (3.98)

$$K(u'',u',v'',v';T)=\int_{-\infty}^{\infty}\frac{dE}{2\pi\hbar}\,\mathrm{e}^{-iET/\hbar}\times$$
$$\times\int_0^{\infty}ds''\,\exp\left[\frac{i}{\hbar}\left(a_+E-\frac{\hbar^2k_0^2}{2m}\right)\right]K(\tau'',\tau',v'',v';s''), \quad (3.100)$$

and the time-transformed path integral $K(s'')$ is given by

$$K(\tau'',\tau',v'',v';s'')=$$
$$=\int_{-\infty}^{\infty}dk_v\frac{\mathrm{e}^{ik_v(v''-v')}}{2\pi}(\cosh\tau'\cosh\tau'')^{-1/2}\int_{\tau(0)=\tau'}^{\tau(s'')=\tau''}\mathcal{D}\tau(s)\times$$
$$\times\exp\left\{\frac{i}{\hbar}\int_0^{s''}\left[\frac{m}{2}\dot\tau^2-\frac{\hbar^2}{2m}\left(\frac{\lambda_0^2-1/4}{\sinh^2\tau}-\frac{-k_v^2-1/4}{\cosh^2\tau}\right)\right]ds\right\}. \quad (3.101)$$

Inserting the solution for the modified Pöschl–Teller potential and evaluating the Green function on the cut yields for the path integral solution on $D_{\rm IV}$ as follows ($K(u'',u',v'',v';T)=K(\tau'',\tau',v'',v';T)$):

$$K(u'',u',v'',v';T)=$$
$$=\int_{-\infty}^{\infty}dk_v\int_0^{\infty}dp\,\mathrm{e}^{-iTE_p/\hbar}\Psi_{p,k_v}(\tau'',v'')\Psi^*_{p,k_v}(\tau',v'), \quad (3.102)$$

$$\Psi_{p,k_v}(\tau,v)=\frac{\mathrm{e}^{ik_v v}}{\sqrt{2\pi a_+\cosh\tau}}\,\Psi_p^{(\lambda_0,ik)}(\tau), \quad (3.103)$$

$$E_p=\frac{\hbar^2}{2ma_+}\left(p^2+k_0^2\right), \quad (3.104)$$

where $\lambda_0^2 = k_0^2 - 2ma_eE/\hbar^2$ and the wave functions for the modified Pöschl–Teller functions. Reinserting $\cos u = \tanh \tau$ gives the solution in terms of the variable u.

We also see from this example that the introduction of a third variable w, say, to a three-dimensional version of Darboux space D_{IV} allows separation of variables, where the additional quantum number k_0 corresponds to the motion in w.

4. SUMMARY AND DISCUSSION

In this paper we have finished the discussion of superintegrable potentials on spaces of nonconstant curvature. The results are very satisfactory. There are two potentials on D_{I}, four potentials on D_{II}, five potentials on D_{III}, and four potentials on D_{IV}, respectively. We could solve many of the emerging quantum mechanical problems. To give an overview, we summarize our results in Table 5. We list for each space the corresponding potentials including the general form of the solution (if explicitly possible). We omit the trivial potentials here, because they are separable in all corresponding coordinate systems.

In the first Darboux space D_{I} the superintegrable potentials were related to the Holt potential and a shifted isotropic harmonic oscillator in two-dimensional Euclidean space. Whereas the solution in the coordinate v can be expressed in terms of the wave functions for the radial harmonic oscillator (Laguerre polynomials) and the shifted harmonic oscillator (Hermite polynomials), the solution in the coordinate u was determined by a boundary condition for u. This gave wave functions in terms of parabolic cylinder functions and a transcendental equation for the bound state energy levels. The corresponding solution in the rotated (r, q) system was similar. An explicit solution in parabolic coordinates could not be found.

In the second Darboux space there were three nontrivial superintegrable potentials. The potentials were related to the Hold potential, the isotropic singular oscillator, and the Coulomb potential in two-dimensional Euclidean space. We found combinations of polynomial wave functions for the discrete states and combinations of polynomials and Whittaker functions for the scattering states. The discrete energy spectrum for the oscillator-related potentials was usually given by a quadratic equation in the energy. For the Coulomb-related potential we found an equation in eight order in the energy, which could be studied in a special case. Also, in the semiclassical limit, we found that the energy spectra indeed had the behavior of a harmonic oscillator and a Coulomb potential, respectively.

On D_{III} we had potentials related to a linear potential, a Coulomb potential, and a shifted oscillator in two-dimensional flat space. We found for the first po-

Table 5. **Solutions of the path integration for superintegrable potentials in Darboux spaces**

Space and potential	Solution in terms of the wave functions
$D_{\rm I}$	
V_1: (u,v) Parabolic	Hermite polynomials $\times$ Parabolic cylinder functions No explicit solution
V_2: (u,v) (r,q)	Hermite polynomials $\times$ Parabolic cylinder functions Hermite polynomials $\times$ Parabolic cylinder functions
$D_{\rm II}$	
V_1: (u,v) Parabolic	Hermite polynomial $\times$ Whittaker functions* No explicit solution
V_2: (u,v) Polar Elliptic	Laguerre polynomial $\times$ Whittaker functions* Gegenbauer polynomial $\times$ Whittaker functions* No explicit solution
V_3: Polar Parabolic Elliptic	Gegenbauer polynomials $\times$ Bessel functions Product of Whittaker functions* No explicit solution
$D_{\rm III}$	
V_1: Parabolic Translated parabolic	Product of Hermite polynomials/Parabolic cylinder functions Product of Hermite polynomials/Parabolic cylinder functions
V_2: (u,v) Polar Parabolic	Gegenbauer polynomials $\times$ Whittaker functions* Gegenbauer polynomials $\times$ Whittaker functions* Product of Whittaker functions*
V_3: Polar Hyperbolic	Gegenbauer polynomials $\times$ Whittaker functions* No explicit solution
V_4: Hyperbolic Elliptic	Product of Whittaker functions* No explicit solution
$D_{\rm IV}$	
V_1: (u,v) system Horospherical Elliptic	Product of hypergeometric functions Product of Whittaker functions* No explicit solution
V_2: (u,v) Degenerate elliptic	Hypergeometric functions Hypergeometric functions
V_3: Elliptic Degenerate elliptic	Hypergeometric functions Hypergeometric functions

*The notion Whittaker functions means for a disrete spectrum Laguerre polynomials and for a continuous spectrum Whittaker functions $W_{\mu,\nu}(z)$, respectively.

tential an equation in the fourth order in the energy E, and quadratic equations in the energy E for the second and third potentials. The Coulomb-related potential showed again in the semiclassical limit the behavior of a Coulomb potential. Of some special interest was the feature of the complex periodic Morse potential for the separation of V_3 in polar coordinates. Such complex potentials have attracted in the recent years some attention, because the involved $\mathcal{PT}$ symmetry in these potentials has the consequence that they, nevertheless, have a real spectrum, e.g., [3, 4, 42, 49–51]. Such kind of potentials also appear as subsystems in the list of superintegrable potentials on the complex Euclidean plane [36].

A special feature in $D_{\rm III}$ was that for the free motion there are already positive continuous and negative infinite discrete spectra. A similar feature also exists for the free quantum motion on the $SU(1,1)$ and $SO(2,2)$ hyperboloid.

In the fourth Darboux space we found potentials which were related to the Morse and Pöschl–Teller potential, and combined modified Pöschl–Teller potentials. The modified Pöschl–Teller potentials had, of course, solutions in terms of hypergeometric functions, respectively: Jacobi polynomials (discrete spectrum) and Jacobi functions (scattering states).

We were able to solve the various path integral representations, because we have now to our disposal not only the basic path integrals for the harmonic oscillator, the linear oscillator, the radial harmonic oscillator, and the (modified) Pöschl–Teller potential, but also path-integral identities derived from path integration on harmonic spaces like the elliptic and spheroidal path-integral representations with their more complicated special functions. This includes also numerous transformation techniques to find a particular solution based on one of the basic solutions. Various Green-function analysis techniques can be applied to find an expression not only for the Green function but also for the wave functions and the energy spectrum. Usually, we stated in all cases the solution for the discrete spectrum contribution, i.e., the energy spectrum and the bound-states wave functions. However, not in all cases we stated explictly the scattering states. In the cases where we omitted the explicit representation, this can be done in a straightforward way by inserting the corresponding solution by the potential problem in question and inserting the various coupling constants and scattering quantum numbers.

Let us also note that our solutions are often on a more or less formal level. Neither have we specified an embedding space, nor have we specified boundary conditions on our spaces. For instance, in $D_{\rm I}$ boundary conditions the signature of the ambient space is very important, because choosing a positive or negative signature of the ambient space changes the boundary conditions, and hence the quantization conditions [21]. The same line of reasoning is, of course, valid in the other three Darboux spaces. We have not discussed in detail special cases of the parameters (say a and b), including the limiting cases to flat spaces or spaces

with constant (negative) curvature. Such a discussion would go far beyond the scope of this paper.

Let us finally mention an important observation due to [26]. At the end of their paper Kalnins et al. gave a list of superintegrable potentials on the two-dimensional complex plane and complex sphere. As it turns out, all of the potentials on Darboux spaces can be generated by taking a two-dimensional line element and dividing this line element by a superintegrable potential belonging to a specific class [27]. Not every class generates a new potential on a Darboux space, some are simply related by a coordinate transformation, and some potentials can be generated from the Euclidean plane as well as the complex sphere. The appearance of the complex sphere is especially obvious in the general elliptic coordinate system on $D_{\rm IV}$. Some of the various different potentials coming from the complex plane and sphere are also related by the so-called «coupling constant metamorphosis». Coupling constant metamorphosis always comes into play if the energy E of the quantum system appears in the form of $E \cdot$ metric terms. This observation leads to the notion that every nondegenerate superintegrable system in two dimensions is «Stäckel equivalent» to a superintegrable system in a two-dimensional space of constant curvature [27].

In the language of path integrals coupling constant metamorphosis comes from «time-» or «space-time» transformations (also called Duru–Kleinert transformations [39]). Here the most important example is the Coulomb problem, where by means of a space-time transformation the Coulomb coupling α just becomes a constant and the emerging harmonic oscillator problem has the frequency $\omega^2 = -2E/m$, i.e., the negative energy of the Coulomb problem appears as a harmonic oscillator frequency. As we have seen, this kind of coupling constant metamorphosis or space-time transformation, respectively, had been indispensable tools in the path integral evaluations of the free motion and for the superintegrable potentials, and we can use both notions as synonymously.

We did not go into details of three-dimensional generalization of the Darboux spaces [15]. Of course, it is possible to extend the notion of superintegrability to three-dimensional Darboux spaces. In particular, in three dimensions there are more of such potentials. In total, there are five maximally superintegrable potentials [17], the first four of them are also superintegrable, including the singular harmonic oscillator, the Holt potential and the Coulomb potential. New features will arise due to the fact that on three-dimensional generalization of the more complicated Darboux spaces $D_{\rm III}$ and $D_{\rm IV}$, coordinate systems from the three-dimensional complex sphere come into play [30]. Studies along such lines will be performed in future investigations.

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Appendices

A. PATH INTEGRAL FOR THE FREE MOTION ON $D_{\rm IV}$ IN DEGENERATE ELLIPTIC COORDINATES ($\gamma = 1$)

We start by considering the metric in elliptic coordinates ($\gamma = 1$):

$$ds^2 = \left[a_-\left(\frac{1}{\sinh^2\hat\omega} + \frac{1}{\sin^2\hat\varphi}\right) - a_+\left(\frac{1}{\cosh^2\hat\omega} - \frac{1}{\cos^2\hat\varphi}\right)\right](d\hat\omega^2 + d\hat\varphi^2). \quad (A.1)$$

We formulate the path integral in the usual way. We perform the space-time transformation with the coordinate transformation $\cos\hat\varphi = \tanh\hat\tau$ yielding

$$K(\hat\omega'', \hat\omega', \hat\varphi'', \hat\varphi'; T) = \int\limits_{\hat\omega(t')=\hat\omega'}^{\hat\omega(t'')=\hat\omega''} \mathcal{D}\hat\omega(t)\times$$

$$\times \int\limits_{\hat\varphi(t')=\hat\varphi'}^{\hat\varphi(t'')=\hat\varphi''} \mathcal{D}\hat\varphi(t)\left[a_-\left(\frac{1}{\sinh^2\hat\omega} + \frac{1}{\sin^2\hat\varphi}\right) - a_+\left(\frac{1}{\cosh^2\hat\omega} - \frac{1}{\cos^2\hat\varphi}\right)\right]\times$$

$$\times \exp\left[\frac{im}{2\hbar}\int\limits_0^T\left(\frac{a_-}{\sinh^2\hat\omega} - \frac{a_+}{\cosh^2\hat\omega} + \frac{a_-}{\sin^2\hat\varphi} - \frac{a_+}{\cos^2\hat\varphi}\right)(\dot{\hat\omega}^2 + \dot{\hat\varphi}^2)dt\right] =$$

$$= \int\limits_{-\infty}^{\infty}\frac{dE}{2\pi\hbar}\,\mathrm{e}^{-iET/\hbar}\int\limits_0^\infty ds''\,\exp\left[\frac{i}{\hbar}\left(a_-E - \frac{\hbar^2}{8m}\right)s''\right]\times$$

$$\times K(\hat\omega'', \hat\omega', \hat\tau'', \hat\tau'; s'') \quad (A.2)$$

with the transformed path integral given by

$$K(\hat{\omega}'',\hat{\omega}',\hat{\tau}'',\hat{\tau}';s'') = \int\limits_{\hat{\tau}(0)=\hat{\tau}'}^{\hat{\tau}(s'')=\hat{\tau}''} \mathcal{D}\hat{\tau}(s) \int\limits_{\hat{\omega}(0)=\hat{\omega}'}^{\hat{\omega}(s'')=\hat{\omega}''} \mathcal{D}\hat{\omega}(s)\, \cosh\,\hat{\tau}\times$$

$$\times\exp\left(\frac{i}{\hbar}\int\limits_0^{s''}\left\{\frac{m}{2}(\dot{\hat{\tau}}^2+\cosh^2\hat{\tau}\dot{\hat{\omega}}^2)-\frac{\hbar^2}{2m}\left[\frac{1}{\cosh^2\hat{\tau}}\left(\frac{\lambda_-^2+1/4}{\sinh^2\hat{\omega}}-\frac{\lambda_+^2+1/4}{\cosh^2\hat{\omega}}+\frac{1}{4}\right)-\right.\right.\right.$$

$$\left.\left.\left.-\frac{\lambda_+^2+1/4}{\sinh^2\hat{\tau}}\right]\right\}ds\right), \quad \text{(A.3)}$$

where $\lambda_\pm^2 = \dfrac{1}{4} - 2ma_\pm E/\hbar^2$. The successive path integrations are of the modified Pöschl–Teller type. Therefore the solution can be written as follows:

$$K(\hat{\omega}'',\hat{\omega}',\hat{\varphi}'',\hat{\varphi}';T) = \int dk \int p\Psi_k^{(\lambda_-,\lambda_+)}(\hat{\omega}'')\Psi_k^{(\lambda_-,\lambda_+)\,*}(\hat{\omega}')\times$$

$$\times\,\Psi_p^{(\lambda_+,ik)}(\hat{\tau}'')\Psi_p^{(\lambda_+,ik)\,*}(\tau')\,\mathrm{e}^{-i\hbar Tp^2/2m} \quad \text{(A.4)}$$

with the energy spectrum

$$E_p = \frac{\hbar^2}{2ma_-}\left(p^2+\frac{1}{4}\right), \quad \text{(A.5)}$$

and we can reinsert $\tanh\hat{\tau} \to \cos\hat{\varphi}$. The difference of the energy spectra in degenerate elliptic and elliptic coordinates (interchanging of a_+ and a_-) can be removed by a shift of the coordinates $\tilde{\varphi}$ and $\hat{\varphi}$ by $\pi/2$, respectively.

B. PATH INTEGRAL FOR THE FREE MOTION ON D_{IV} IN DEGENERATE ELLIPTIC COORDINATES ($\gamma = 2$)

We start by considering the metric in degenerate elliptic coordinates ($\gamma = 2$):

$$ds^2 = \frac{1}{4}\left(\frac{a_+}{\sinh^2 2\tilde{\omega}}+\frac{a_-}{\sin^2 2\tilde{\varphi}}\right)(d\tilde{\omega}^2+d\tilde{\varphi}^2). \quad \text{(B.1)}$$

We formulate the path integral in the usual way. We scale both variables by the factor 2 and perform the space-time transformation with the coordinate transfor-

mation $\cos\tilde\varphi = \tanh\tilde\tau$ yielding $\big(\lambda^2 = \frac{1}{4} - 2ma_+E/\hbar^2\big)$:

$$K(\tilde\omega'',\tilde\omega',\tilde\varphi'',\tilde\varphi';T) = \frac{1}{2}\int\limits_{\tilde\omega(t')=\tilde\omega'}^{\tilde\omega(t'')=\tilde\omega''}\mathcal{D}\tilde\omega(t)\int\limits_{\tilde\varphi(t')=\tilde\varphi'}^{\tilde\varphi(t'')=\tilde\varphi''}\mathcal{D}\tilde\varphi(t)\bigg(\frac{a_+}{\sinh^2\tilde\omega}+\frac{a_-}{\sin^2\tilde\varphi}\bigg)\times$$
$$\times\exp\left[\frac{im}{2\hbar}\int\limits_0^T\left(\frac{a_+}{\sinh^2\tilde\omega}+\frac{a_-}{\sin^2\tilde\varphi}\right)(\dot{\tilde\omega}^2+\dot{\tilde\varphi}^2)dt\right] = \int\limits_{-\infty}^{\infty}\frac{dE}{2\pi\hbar}\,\mathrm{e}^{-iET/\hbar}\times$$
$$\times\int\limits_0^\infty ds''\,\exp\left[\frac{i}{\hbar}\left(a_-E-\frac{\hbar^2}{8m}\right)s''\right]K(\tilde\omega'',\tilde\omega',\tilde\tau'',\tilde\tau';s'') \quad \text{(B.2)}$$

with the transformed path integral given by

$$K(\tilde\omega'',\tilde\omega',\tilde\tau'',\tilde\tau';s'') = \int\limits_{\tilde\tau(0)=\tilde\tau'}^{\tilde\tau(s'')=\tilde\tau''}\mathcal{D}\tilde\tau(s)\int\limits_{\tilde\omega(0)=\tilde\omega'}^{\tilde\omega(s'')=\tilde\omega''}\mathcal{D}\tilde\omega(s)\cosh\tilde\tau\times$$
$$\times\exp\left\{\frac{i}{\hbar}\int\limits_0^{s''}\left[\frac{m}{2}(\dot{\tilde\tau}^2+\cosh^2\tilde\tau\,\dot{\tilde\omega}^2)-\frac{\hbar^2}{2m\cosh^2\tilde\tau}\left(\frac{\lambda^2+1/4}{\cosh^2\tilde\omega}+\frac{1}{4}\right)\right]ds\right\} =$$
$$= (\cosh\tilde\tau'\cosh\tilde\tau'')^{-1/2}\sum_\pm\int\limits_{\mathbb{R}}\frac{dk\,k\,\sinh\pi k}{\cosh^2\pi\lambda+\sinh^2\pi k}\times$$
$$\times P^{ik}_{i\lambda-1/2}(\pm\tanh\omega'')P^{-ik}_{i\lambda-1/2}(\pm\tanh\tilde\omega')\times$$
$$\times\sum_\pm\int\limits_{\mathbb{R}}\frac{dp\,p\,\sinh\pi p}{\cosh^2\pi k+\sinh^2\pi p}\times$$
$$\times P^{ip}_{ik-1/2}(\pm\tanh\tilde\tau'')P^{-ip}_{ik-1/2}(\pm\tanh\tilde\tau')\,\mathrm{e}^{-i\hbar Tp^2/2m}. \quad \text{(B.3)}$$

Therefore we obtain the wave functions and the energy spectrum, respectively,

$$\Psi_{k,p}(\tilde\tau,\tilde\omega) = \frac{1}{\sqrt{2\cosh\tilde\tau}}\left(\frac{k\,\sinh\pi k}{\cosh^2\pi\lambda+\sinh^2\pi k}\frac{p\,\sinh\pi p}{\cosh^2\pi k+\sinh^2\pi p}\right)^{1/2}\times$$
$$\times P^{ik}_{i\lambda-1/2}(\pm\tanh\omega)P^{ip}_{ik-1/2}(\pm\tanh\tilde\tau) \quad \text{(B.4)}$$

and $E_p = \dfrac{\hbar^2}{2ma_-}\left(p^2+\dfrac{1}{4}\right)$, and we can reinsert $\tanh\tilde\tau\to\cos\tilde\varphi$.

C. SUPERINTEGRABLE POTENTIALS ON $E(2,\mathbb{C})$

In this appendix we shortly discuss the path integral representation of superintegrable potentials on the two-dimensional complex Euclidean plane. A thorough path integral discussion on the real two-dimensional complex Euclidean plane has been done in [17], and therefore these solutions will not be repeated here, only some new due to the appearance of three more potentials V_5–V_7. In Table 6 we list the seven coordinate systems on the complex plane $E(2,\mathbb{C})$. As usual $P_1 = -i\hbar\partial_x$ and $P_2 = -i\hbar\partial_y$ denote the momentum operators, and $M = yP_1 - xP_2$ is the angular momentum. The potentials now read as follows [27, 34–36]:

$$\left.\begin{array}{ll}
V_5 = \dfrac{B}{2}(x-iy) & \underline{\text{Cartesian}} \\
 & \underline{\text{Semihyperbolic}} \\
 & \text{Light Cone} \\
\hline
V_6 = \dfrac{\alpha}{2\sqrt{x-iy}} & \underline{\text{Parabolic}} \\
 & \underline{\text{Semihyperbolic}} \\
 & \text{Light Cone} \\
\hline
V_7 = \dfrac{1}{2}\left[\alpha\dfrac{x^2+y^2}{(x+iy)^4} + \dfrac{\beta}{(x+iy)^2} + \gamma(x^2+y^2)\right] & \underline{\text{Polar}} \\
 & \text{Hyperbolic}
\end{array}\right\}. \tag{C.1}$$

In the <u>underlined cases</u> we give a (formal) path integral representation.

The Potential V_5. For the potential V_5 the corresponding Lagrangian has the form

$$\mathcal{L} = \frac{m}{2}(\dot{x}^2+\dot{y}^2) - \frac{B}{2}(x-iy). \tag{C.2}$$

Thus, we identify two linear potentials [13, 45]

$$\begin{aligned}
&K^{(V_5)}(x'',x',y'',y';T) = \\
&= \int\limits_{x(t')=x'}^{x(t'')=x''} \mathcal{D}x(t) \int\limits_{y(t')=y'}^{y(t'')=y''} \mathcal{D}y(t) \exp\left\{\frac{i}{\hbar}\int_{t'}^{t''}\left[\frac{m}{2}(\dot{x}^2+\dot{y}^2) - \frac{B}{2}(x-iy)\right]dt\right\} = \\
&= \left(\frac{m}{2\pi i\hbar T}\right) \exp\left[\frac{i}{\hbar}\left(\frac{m}{2}\frac{(x''-x')^2+(x''-x')^2}{T} - \right.\right. \\
&\qquad\qquad\qquad \left.\left. - \frac{BT}{4}(x'+x''-iy'-iy'')\right)\right],
\end{aligned} \tag{C.3}$$

Table 6. **Coordinate systems on the complex plane** $E(2,\mathbb{C})$

Coordinate system	Integrals of motion	Coordinates
1. Cartesian, $(x,y\in\mathbb{R})$	$I=p_1^2$	x,y
2. Polar $(\varrho>0,\varphi\in[0,\pi))$	$I=m^2$	$x=\varrho\cos\varphi$ $x=\varrho\sin\varphi$
3. Light cone $(x,y\in\mathbb{R})$	$I=(P_1+iP_2)^2$	$\hat{x}=x-iy$ $\hat{y}=x+iy$
4. Elliptic $(\omega>0,\alpha\in[0,2\pi))$	$I=M^2-a^2P_2^2$ $a\neq 0$	$x=\cosh\omega\cos\alpha$ $y=\sinh\omega\sin\alpha$
5. Parabolic $(\xi,\eta>0)$	$I=\{M,P_2\}$	$x=\frac{1}{2}(\xi^2-\eta^2)$ $y=\xi\eta$
6. Hyperbolic $(u,v>0)$	$I=M^2+(P_1+iP_2)^2$	$x=\frac{u^2+u^2v^2+v^2}{2uv}$ $y=i\frac{u^2-u^2v^2+v^2}{2uv}$
7. Semihyperbolic $(w,z\in\mathbb{R})$	$I=\{M,P_1+iP_2\}+(P_1-iP_2)^2$	$x=\frac{1}{2}(w-z)^2+\frac{1}{4}(w+z)$ $y=-\frac{1}{2}(w-z)^2-\frac{1}{4}(w+z)$

$$
= \left(\frac{4m}{\hbar^2 B}\right)^{4/3} \int\limits_{\mathbb{R}} dE\, \mathrm{e}^{-iET/\hbar} \int\limits_{\mathbb{R}} d\lambda \times
$$
$$
\times \operatorname{Ai}\left[\left(x' - \frac{2E+\lambda}{k}\right)\left(\frac{mB}{\hbar^2}\right)^{1/3}\right] \operatorname{Ai}\left[\left(x'' - \frac{2E+\lambda}{k}\right)\left(\frac{mB}{\hbar^2}\right)^{1/3}\right] \times
$$
$$
\times \operatorname{Ai}\left[i\left(y' - \frac{2E-\lambda}{k}\right)\left(\frac{mB}{\hbar^2}\right)^{1/3}\right] \operatorname{Ai}\left[i\left(y'' - \frac{2E-\lambda}{k}\right)\left(\frac{mB}{\hbar^2}\right)^{1/3}\right], \quad \text{(C.4)}
$$

with the continuous spectrum $E=\hbar^2p^2/2m$, and λ is the second separation constant.

For V_5 in the semihyperbolic coordinates we obtain for the corresponding Lagrangian ($\dot{w}=dw/dt$)

$$
\mathcal{L}_E = \frac{m}{2}(w-z)(\dot{w}^2-\dot{z}^2) - \frac{B}{2}(w+z) + E, \quad \text{(C.5)}
$$

which gives after a time transformation ($\dot{w} = dw/ds$, $\dot{z} = dz/ds$ and $dt = (w - z)ds$) a transformed Lagrangian

$$\tilde{\mathcal{L}}_E = \frac{m}{2}(\dot{w}^2 - \dot{z}^2) - \frac{B}{2}(w^2 - z^2) + E(w - z). \tag{C.6}$$

Therefore the potential v_5 has been transformed into the problem of a shifted harmonic oscillator, whose solution is well known. In order to determine the path integral solution we consider the Green function of the harmonic oscillator [22], use the convolution formula for the kernel in terms of a product of two Green functions

$$K^{(V_5)}(w'', w', z'', z'; T) = \int\limits_{-\infty}^{\infty} \frac{dE}{2\pi\hbar}\,\mathrm{e}^{-iET/\hbar} \times$$

$$\times \int\limits_0^{\infty} ds'' K_w(w'', w'; s'') \cdot K_z(z'', z'; s'') =$$

$$= \int\limits_{-\infty}^{\infty} \frac{dE}{2\pi\hbar}\,\mathrm{e}^{-iET/\hbar} \frac{\hbar}{2\pi i} \int d\mathcal{E} G_w(E; w'', w'; -\mathcal{E}) G_z(E; z'', z'; \mathcal{E}), \tag{C.7}$$

and obtain therefore

$$K^{(V_5)}(w'', w', z'', z'; T) = \int\limits_{w(t')=w'}^{w(t'')=w''} \mathcal{D}w(t) \times$$

$$\times \int\limits_{z(t')=z'}^{z(t'')=z''} \mathcal{D}z(t) \exp\left\{ \frac{i}{\hbar} \int\limits_{t'}^{t''} \left[\frac{m}{2}(w - z)(\dot{w}^2 - \dot{z}^2) - \frac{B}{2}(w + z) \right] dt \right\} =$$

$$= \frac{1}{4\pi^2} \int\limits_{-\infty}^{\infty} dE \int d\lambda \frac{m}{\pi\hbar^3} \sqrt{\frac{m}{B}}\,\Gamma^2\left(\frac{1}{2} - \frac{E + \lambda}{\hbar\omega} \right) \times$$

$$\times D_{-\frac{1}{2} + \frac{E+\lambda}{\hbar\omega}} \left[\sqrt{\frac{2}{\hbar}\sqrt{mB}} \left(w_> - \frac{E}{b} \right) \right] D_{-\frac{1}{2} + \frac{E+\lambda}{\hbar\omega}} \left[-\sqrt{\frac{2}{\hbar}\sqrt{mB}} \left(w_< - \frac{E}{b} \right) \right] \times$$

$$\times D_{-\frac{1}{2} + \frac{E+\lambda}{\hbar\omega}} \left[\sqrt{\frac{2}{\hbar}\sqrt{mB}} \left(z_> - \frac{E}{b} \right) \right] D_{-\frac{1}{2} + \frac{E+\lambda}{\hbar\omega}} \left[-\sqrt{\frac{2}{\hbar}\sqrt{mB}} \left(z_< - \frac{E}{b} \right) \right], \tag{C.8}$$

with the continuous spectrum $E = \hbar^2 p^2 / 2m$, and λ is the second separation constant. The Green function may be evaluated in terms of even and odd parabolic cylinder functions $E_\nu^{(0)}(z)$ and $E_\nu^{(1)}(z)$, e.g., [14,17,22,41], which is omitted here.

The Potential V_6. Let us consider the two Lagrangians of the potential V_6 expressed in parabolic and semihyperbolic coordinates, respectively,

$$\mathcal{L}_E = \frac{m}{2}(\xi^2+\eta^2)(\dot\xi^2+\dot\eta^2) + \sqrt{2}\,\alpha\frac{\xi - i\eta}{\xi^2+\eta^2} + E, \tag{C.9}$$

$$= \frac{m}{2}(w-z)(\dot w^2 - \dot z^2) + i\frac{\sqrt{2}\,\alpha}{w-z} + E, \tag{C.10}$$

which gives after a time transformation ($\dot\xi = d\xi/ds$, $\dot\eta = d\eta/ds$ and $dt = (\xi^2+\eta^2)ds$ in parabolic coordinates; $\dot w = dw/ds$, $\dot z = dz/ds$ and $dt = (w-z)ds$ in semihyperbolic coordinates) the transformed Lagrangians

$$\tilde{\mathcal{L}}_E = \frac{m}{2}(\dot\xi^2+\dot\eta^2) + \sqrt{2}\,\alpha(\xi - i\eta) + (\xi^2+\eta^2), \tag{C.11}$$

$$= \frac{m}{2}(\dot w^2 - \dot z^2) + i\sqrt{2}\,\alpha + E(w-z). \tag{C.12}$$

In parabolic coordinates we have a shifted harmonic oscillator and in semihyperbolic coordinates a linear potential plus a constant. The solution is consequently almost identical to the corresponding solutions for the potential V_5 with appropriate replacement of the coupling constants. See also [14,17,22,41] for more details.

The Potential V_7. Let us consider the last potential V_7. In polar coordinates we have the effective Lagrangian (note the additional $\hbar^2$-potential [22])

$$\mathcal{L} = \frac{m}{2}(\dot\varrho^2 + \varrho^2\dot\varphi^2 - \omega^2) - \frac{\hbar^2}{2mr^2}\bigg(\alpha\,\mathrm{e}^{-4i\varphi} - 2\beta\,\mathrm{e}^{-2i\varphi} - \frac{1}{4}\bigg). \tag{C.13}$$

In the variable φ we have a complex periodic Morse potential, the same kind of potentials we have encountered on $D_{\rm III}$ for V_3 in polar coordinates. We identify $\alpha = 4c_1^2$ and $\beta = c_2/c_1$. Furthermore we see that the remaining path integral in the variable ϱ is just a radial harmonic oscillator path integral. Putting everything together yields

$$K^{(V_7)}(\varrho'',\varrho',\varphi'',\varphi';T) = \int\limits_{\varrho(t')=\varrho'}^{\varrho(t'')=\varrho''}\mathcal{D}\varrho(t)\int\limits_{\varphi(t')=\varphi'}^{\varphi(t'')=\varphi''}\mathcal{D}\varphi(t)\times$$

$$\times\exp\left\{\frac{i}{\hbar}\int\limits_0^T\left[\frac{m}{2}(\dot\varrho^2+\varrho^2\dot\varphi^2-\omega^2\varrho^2) - \frac{\hbar^2}{2m\varrho^2}\left(\alpha\,\mathrm{e}^{-4i\varphi} - 2\beta\,\mathrm{e}^{-2i\varphi} - \frac{1}{4}\right)\right]dt\right\} =$$

$$= \sum_{l=0}^\infty \Phi^{(c_1,c_2)}_{[\mathrm{cMP}],l}(\varphi'')\Phi^{(c_1,c_2)}_{[\mathrm{cMP}],l}(\varphi')\frac{m\omega}{i\hbar\sin\omega T}\times$$

$$\times\exp\left[-\frac{m\omega}{2i\hbar}({\varrho'}^2+{\varrho''}^2)\cot\omega T\right] I_{l+2\frac{c_2}{c_1}+\frac{1}{2}}\left(\frac{m\omega\varrho'\varrho''}{i\hbar\sin\omega T}\right), \tag{C.14}$$

with the well-known expansion by means of the Hille–Hardy formula in terms of Laguerre polynomials for ϱ. We leave the result as it stands.

D. SUPERINTEGRABLE POTENTIALS ON $S(2,\mathbb{C})$

Let us shortly enumerate the superintegrable potentials on the complex sphere. On the real two-dimensional sphere there are two superintegrable potentials, a feature which has been already investigated, e.g., [18]. On the complex two-dimensional sphere there are four more potentials which are listed in (D.4) [27, 30, 34]. In the underlined cases we give a path integral representation. These representations remain, however, on a formal level, because the complex sphere is an abstract space and serves just as a tool to find the relevant potentials. Going to the corresponding real spaces, i.e., the sphere and the hyperboloid, respectively,

Table 7. **Coordinate systems on the complex sphere $S(2,\mathbb{C})$**

Coordinate system	Integrals of motion	Coordinates
1. Spherical $(\vartheta \in [0,\pi), \varphi \in [0,2\pi)$	$L = J_3^2$	$s_1 = \sin\vartheta\cos\varphi$ $s_2 = \sin\vartheta\sin\varphi$, $s_3 = \cos\vartheta$
2. Elliptic	$L = J - 1^2 + rJ_2^2$	$s_1^2 = \dfrac{(ru-1)(rv-1)}{1-r}$ $s_2^2 = \dfrac{r(u-1)(v-1)}{1-r}$, $z^2 = ruv$
3. Horospherical	$L = (J_1 + iJ_2)^2$	$s_1 = \dfrac{i}{2}\left(v + \dfrac{y^2-1}{v}\right)$ $s_2 = \dfrac{i}{2}\left(v + \dfrac{y^2-1}{v}\right)$, $s_3 = iy/v$
4. Degenerate Elliptic 1 $(\tau_{1,2} \in \mathbb{R})$	$L = (J_1 + ij_2)^2 - c^2 J_3^2$	$s_1 + is_3 = \dfrac{1}{\cosh\tau_1\cosh\tau_2}$ $s_2 - is_3 = \dfrac{\cosh\tau_2}{\cosh\tau_1} + \dfrac{\cosh\tau_1}{\cosh\tau_2} -$ $- \dfrac{1}{\cosh\tau_1\cosh\tau_2}$ $s_3 = \tanh\tau_1\tanh\tau_2$
5. Degenerate Elliptic 2 $(\xi, \eta > 0)$	$L = J_3(J_1 - iJ_2)^2$	$s_1 + is_2 = \dfrac{1}{\xi\eta}$ $s_1 + is_2 = -\dfrac{1}{4}\dfrac{(\xi^2-\eta^2)^2}{\xi\eta}$ $s_3 = \dfrac{1}{2}\dfrac{\xi^2+\eta^2}{\xi\eta}$

requires the real representation of the coordinate system in question, including the corresponding path integral representation.

In Table 7 we list the five coordinate systems on the complex sphere $S(2,\mathbb{C})$ according to [27, 30, 34]. Let us note that we can also use $v = i\,\mathrm{e}^{-ix}$ as a parameterization in the horospherical system $(x, y \in \mathbb{R})$. As usual, $J_1, J_2, J-3$ are the angular momentum operators in three dimensions.

The Potential V_3**.** Let us start superintegrable potential on the two-dimensional complex sphere. It has the form

$$V_3(\mathbf{s}) = \frac{\alpha}{s_3^2} + \frac{\beta}{(s_1 - is_2)^2} + \gamma\frac{(s_1 + is_2)}{(s_1 - is_2)^3}, \tag{D.1}$$

$$= \frac{\alpha}{\cos^2\vartheta^2} + \beta\frac{\mathrm{e}^{-2i\varphi}}{\sin^2\vartheta} - \gamma\frac{\mathrm{e}^{-4i\varphi}}{\sin^2\vartheta}, \tag{D.2}$$

$$= \mathrm{e}^{-2ix}\left(\gamma y^2 + \frac{\alpha}{y^2} + \beta\right) - \gamma\,\mathrm{e}^{-4ix}, \tag{D.3}$$

and we have inserted spherical and horospherical coordinates on the (complex) sphere, respectively,

$$\left.\begin{array}{ll}
V_3(\mathbf{s}) = \dfrac{\alpha}{s_3^2} + \dfrac{\beta}{(s_1 - is_2)^2} + \gamma\dfrac{s_1 + is_2}{(s_1 - is_2)^3} & \begin{array}{l}\underline{\text{Spherical}}\\ \underline{\text{Horospherical}}\\ \text{Degenerate}\\ \text{elliptic I}\end{array} \\
\hline
\begin{array}{l} V_4(\mathbf{s}) = \dfrac{\alpha}{(s_1 - is_2)^2} + \dfrac{\beta s_3}{\sqrt{s_1^2 + s_2^2}} + \\ \qquad + \dfrac{i\gamma}{\sqrt{(s_1 + is_2)(s_1 - is_2)^2}} \end{array} & \begin{array}{l}\text{Spherical}\\ \text{Degenerate}\\ \text{elliptic II}\end{array} \\
\hline
\begin{array}{l} V_5(\mathbf{s}) = \dfrac{\alpha z_+ + c^2 z_-}{\sqrt{(c^2 z_- - z_+)^2 - 4c^2 z_3}} + \\ \qquad + \dfrac{\beta(z_+ - c^2 z_-)(z_+ z_- + z_3^2)}{z_3^2\sqrt{(c^2 z_- - z_+)^2 - 4c^2 z_3}} + \gamma\dfrac{z_+ z_-}{z_3^2} \\ \left(z_\pm = s_1 \pm is_2,\ z_3 = \sqrt{1 - s_1^2 - s_2^2},\ c^2 = \dfrac{1+r}{1-r}\right) \end{array} & \begin{array}{l}\text{Elliptic}\\ \\ \text{Degenerate}\\ \text{elliptic I}\end{array} \\
\hline
V_6(\mathbf{s}) = \dfrac{\alpha}{(s_1 - is_2)^2} + \dfrac{\beta s_3}{(x - iy)^3} + \gamma\dfrac{1 - 4s_3^2}{(s_1 - is_2)^4} & \begin{array}{l}\underline{\text{Horospherical}}\\ \text{Degenerate}\\ \text{elliptic II}\end{array}
\end{array}\right\} . \tag{D.4}$$

This potential has now in spherical coordinates in the φ dependence the same structure as the potential V_7 on the complex plane, thus the solution is the same ($c_{1,2}$ in the complex Morse potential appropriately). In the ϑ dependence we obtain after the separation of φ a Pöschl–Teller potential. In comparison to V_7 with the complex plane, we must therefore replace the wave functions in ϱ in terms of Laguerre polynomials by the Pöschl–Teller wave functions $\Phi_n^{\left(\tilde{\alpha},l+2\frac{c_2}{c_1}+\frac{1}{2}\right)}(\vartheta)$ $\left(\tilde{\alpha}^2=2m\alpha/\hbar^2+\dfrac{1}{4}\right)$ and we have done. Summarizing we obtain

$$
\begin{aligned}
K^{(V_3)}(\vartheta'',\vartheta',\varphi'',\varphi';T) &= \int\limits_{\vartheta(t')=\vartheta'}^{\vartheta(t'')=\vartheta''} \mathcal{D}\vartheta(t) \int\limits_{\varphi(t')=\varphi'}^{\varphi(t'')=\varphi''} \mathcal{D}\varphi(t)\sin\vartheta\times \\
\times\exp\left\{\frac{i}{\hbar}\int\limits_{t'}^{t''}\left[\frac{m}{2}(\dot{\vartheta}^2+\sin^2\vartheta\dot{\varphi}^2)-\frac{\alpha}{\cos^2\vartheta}-\frac{1}{\sin^2\vartheta}\left(\beta\mathrm{e}^{-2i\varphi}-\gamma\mathrm{e}^{-4i\varphi}-\frac{1}{4}\right)\right]dt\right\} &= \\
=(\sin\vartheta'\sin\vartheta'')^{-1/2}\sum_{n=0}^{\infty}\sum_{l=0}^{\infty}\Phi^{(c_1,c_2)}_{[\mathrm{cMP}],l}(\varphi'')\Phi^{(c_1,c_2)}_{[\mathrm{cMP}],l}(\varphi')&\Phi_n^{\left(l+2\frac{c_2}{c_1}+\frac{1}{2},\tilde{\alpha}\right)}(\vartheta'')\times \\
\times\,\Phi_n^{\left(l+2\frac{c_2}{c_1}+\frac{1}{2},\tilde{\alpha}\right)}(\vartheta')\,\exp\left[-\frac{i}{\hbar}\frac{\hbar^2}{2m}\left(2n+l+2\frac{c_2}{c_1}+\frac{3}{2}\right)^2T\right]&. \qquad \text{(D.5)}
\end{aligned}
$$

In horospherical coordinates we have in the variable y a radial harmonic oscillator $\left(\text{set } \gamma=m\omega^2/2,\ \tilde{\alpha}^2=2m\alpha/\hbar^2+\dfrac{1}{4}\right)$ and in the same way ($c_{1,2}$ in the complex Morse potential appropriately)

$$
\begin{aligned}
K^{(V_3)}(x'',x',y'',y';T) &= \int\limits_{x(t')=x'}^{x(t'')=x''}\mathcal{D}x(t)\int\limits_{y(t')=y'}^{y(t'')=y''}\mathcal{D}y(t)\,\mathrm{e}^{2ix}\times \\
\times\exp\left\{\frac{i}{\hbar}\int\limits_{t'}^{t''}\left[\frac{m}{2}(\dot{x}+\mathrm{e}^{2ix}\dot{y}^2)-\mathrm{e}^{-2ix}\left(\gamma y^2+\frac{\alpha}{y^2}+\beta\right)-\gamma\mathrm{e}^{-4ix}\right]dt\right\} &= \\
=\mathrm{e}^{-i(x'+x'')}\sum_{n=0}^{\infty}\sum_{l=0}^{\infty}\Psi_l^{(\mathrm{RHO},\tilde{\alpha})}(y'')\Psi_l^{(\mathrm{RHO},\tilde{\alpha})}(y')&\Phi^{(c_1,c_2)}_{[\mathrm{cMP}],n}(\varphi'')\Phi^{(c_1,c_2)}_{[\mathrm{cMP}],n}(\varphi')\times \\
\times\exp\left[-\frac{i}{\hbar}\frac{\hbar^2}{2m}\left(n+2\frac{c_2}{c_1}+1\right)^2T\right]&, \qquad \text{(D.6)}
\end{aligned}
$$

and the $\Psi_l^{(\mathrm{RHO},\tilde{\alpha})}(y)$ are the wave functions of the radial harmonic oscillator [22].

The Potential V_6. As the last potential we consider V_6. We have (set $\gamma = -m\omega^2/8$)

$$V_6(\mathbf{s}) = \frac{\alpha}{(s_1 - is_2)^2} + \frac{\beta s_3}{(x - iy)^3} + \gamma \frac{1 - 4s_3^2}{(s_1 - is_2)^4} \tag{D.7}$$

$$= \mathrm{e}^{-2ix} \frac{m}{2}\omega^2 \left(y + \frac{i\beta}{m\omega^2}\right)^2 - \mathrm{e}^{-2ix}\left(\alpha + \frac{\beta^2}{2m\omega^2}\right) - \gamma\,\mathrm{e}^{-4ix}, \tag{D.8}$$

and we have inserted horospherical coordinates. This potential is, in the variable y, a shifted harmonic oscillator, however, the shift is a complex one. In the variable x we have the complex periodic Morse potential. Again, we encounter a complex potential, this time a $\mathcal{PT}$-symmetric harmonic oscillator with spectrum $E_l = \hbar\omega(l + 1/2)$, e.g., [49]. Consequently, we have in a similar way as before ($c_{1,2}$ in the complex Morse potential appropriately, set $\kappa = i\beta/m\omega^2$):

$$K^{(V_6)}(x'', x', y'', y'; T) = \int\limits_{x(t')=x'}^{x(t'')=x''} \mathcal{D}x(t) \int\limits_{y(t')=y'}^{y(t'')=y''} \mathcal{D}y(t)\,\mathrm{e}^{2ix} \times$$

$$\times \exp\left\{ \frac{i}{\hbar}\int_{t'}^{t''} \left[\frac{m}{2}(\dot{x}^2 + \mathrm{e}^{2ix}\dot{y}^2) - \left(\frac{m}{2}\omega^2 \left(y + \frac{i\beta}{m\omega^2}\right)^2 + \right.\right.\right.$$

$$\left.\left.\left. + \left(\alpha + \frac{\beta^2}{2m\omega^2}\right)\right)\mathrm{e}^{-2ix} - \gamma\,\mathrm{e}^{-4ix} \right] dt \right\} =$$

$$= \mathrm{e}^{-i(x'+x'')} \sum_{l=0}^{\infty} \Psi_l^{(\mathrm{HO})}(y'')\Psi_l^{(\mathrm{HO})}(y') \sum_{n=0}^{\infty} \Phi_{[\mathrm{cMP}],n}^{(c_1,c_2)}(\varphi'')\Phi_{[\mathrm{cMP}],n}^{(c_1,c_2)}(\varphi') \times$$

$$\times \exp\left[-\frac{i}{\hbar}\frac{\hbar^2}{2m}\left(n + 2\frac{c_2}{c_1} + 1\right)^2 T \right], \tag{D.9}$$

and the $\Psi_l^{(\mathrm{HO},\kappa)}(y)$ are the wave functions of the shifted harmonic oscillator [22]. The representations of the potentials V_4 and V_5 in the separating coordinate systems lead to intractable powers in the various coordinates, respectively, powers of $\cosh\tau_{1,2}$, i.e., highly anharmonic terms which cannot be treated. The same holds for V_3 and V_6 in the remaining separating coordinate systems. This concludes the discussion.

REFERENCES

1. *Bender C. M. et al.* The $\mathcal{C}$ Operator in $\mathcal{PT}$-Symmetric Quantum Theories // J. Phys. A: Math. Gen. 2004. V. 37. P. 10139–10165.
2. *Böhm M., Junker G.* Path Integration over Compact and Noncompact Rotation Groups // J. Math. Phys. 1987. V. 28. P. 1978–1994.

3. *Bagchi B., Quesne C., Znojil M.* Generalized Continuity Equation and Modified Normalization in PT-Symmetric Quantum Mechanics // Mod. Phys. Lett. A. 2001. V. 16. P. 2047–2057.

4. *Cannata F., Junker J., Trost J.* Schrödinger Operators with Complex Potential but Real Spectrum // Phys. Lett. A. 1998. V. 246. P. 219–226.

5. *Daskaloyannis C., Ypsilantis K.* Unified Treatment and Classification of Superintegrable Systems with Integrals Quadratic in Momenta on a Two-Dimensional Manifold // J. Math. Phys. 2006. V. 45. P. 042904.

6. *Del Olmo M. A., Rodríguez M. A., Winternitz P.* The Conformal Group $SU(2,2)$ and Integrable Systems on a Lorentzian Hyperboloid // Fortschr. Phys. 1996. Bd. 44. S. 199–233; Integrable Systems Based on $SU(p,q)$ Homogeneous Manifolds // J. Math. Phys. 1993. V. 34. P. 5118–5139.

7. *Evans N. W.* Superintegrability in Classical Mechanics // Phys. Rev. A. 1990. V. 41. P. 5666–5676; Group Theory of the Smorodinsky–Winternitz System // J. Math. Phys. 1991. V. 32. P. 3369–3375; Superintegrability of the Winternitz System // Phys. Lett. A. 1990. V. 147. P. 483–486.

8. *Feynman R. P., Hibbs A.* Quantum Mechanics and Path Integrals. N. Y.: McGraw Hill, 1965.

9. *Friš J. et al.* On Higher Symmetries in Quantum Mechanics // Phys. Lett. 1965. V. 16. P. 354–356;
Friš J. et al. Symmetry Groups in Classical and Quantum Mechanics // Sov. J. Nucl. Phys. 1967. V. 4. P. 444–450.

10. *Gradshteyn I. S., Ryzhik I. M.* Table of Integrals, Series, and Products. N. Y.: Acad. Press, 1980.

11. *Grosche C.* The Path Integral on the Poincaré Upper Half-Plane with a Magnetic Field and for the Morse Potential // Ann. Phys. (N. Y.) 1988. V. 187. P. 110–134.

12. *Grosche C.* The Path Integral on the Poincaré Disc, the Poincaré Upper Half-Plane and on the Hyperbolic Strip // Fortschr. Phys. 1990. Bd. 38. S. 531–569.

13. *Grosche C.* Path Integrals, Hyperbolic Spaces, and Selberg Trace Formulae. Singapore: World Sci., 1996.

14. *Grosche C.* Path Integration on Darboux Spaces // Part. Nucl. 2006. V. 37. P. 368–389.

15. *Grosche C.* Path Integral Approach for Spaces of Nonconstant Curvature in Three Dimensions. DESY Report, DESY 05-221. 2005; quant-ph/0511135; Proc. of the II Intern. Workshop on Superintegrable Systems in Classical and Quantum Mechanics, Dubna, June 27 – July 1, 2005 // Phys. At. Nucl. 2006. V. 69.

16. *Grosche C. et al.* Quantum Motion on the Three-Dimensional Sphere: The Ellipso-Cylindrical Bases // J. Phys. A: Math. Gen. 1997. V. 30. P. 1629–1657.

17. *Grosche C., Pogosyan G. S., Sissakian A. N.* Path Integral Discussion for Smorodinsky–Winternitz Potentials: I. Two- and Three-Dimensional Euclidean Space // Fortschr. Phys. 1995. Bd. 43. S. 453–521.

18. *Grosche C., Pogosyan G. S., Sissakian A. N.* Path Integral Discussion for Smorodinsky–Winternitz Potentials: II. The Two- and Three-Dimensional Sphere // Ibid. S. 523–563.

19. *Grosche C., Pogosyan G. S., Sissakian A. N.* Path Integral Approach to Superintegrable Potentials on the Two-Dimensional Hyperboloid // Part. Nucl. 1996. V. 27. P. 244–278.

20. *Grosche C., Pogosyan G. S., Sissakian A. N.* Path Integral Approach for Superintegrable Potentials on the Three-Dimensional Hyperboloid // Part. Nucl. 1997. V. 28. P. 486–519.

21. *Grosche C., Pogosyan G. S., Sissakian A. N.* Path Integral Approach for Superintegrable Potentials on Spaces of Nonconstant Curvature: I. Darboux Spaces $D_{\rm I}$ and $D_{\rm II}$. DESY preprint, DESY 06-113. 2006; Part. Nucl. 2007. V. 38, No. 3. P. 587.

22. *Grosche C., Steiner F.* Handbook of Feynman Path Integrals. Springer Tracts in Modern Physics. Berlin; Heidelberg, 1998. V. 145.

23. *Kalnins E. G.* On the Separation of Variables for the Laplace Equation $\Delta\psi + K^2\psi = 0$ in Two- and Three-Dimensional Minkowski Space // SIAM J. Math. Anal. 1975. V. 6. P. 340–374.

24. *Kalnins E. G.* Separation of Variables for Riemannian Spaces of Constant Curvature. Essex: Longman Sci. & Technical, 1986.

25. *Kalnins E. G., Kress J. M., Miller W. (Jr.).* Jacobi, Ellipsoidal Coordinates, and Superintegrable Systems // J. Nonlin. Math. Phys. 2005. V. 12. P. 209–229.

26. *Kalnins E. G. et al.* Superintegrable Systems in Darboux Spaces // J. Math. Phys. 2003. V. 44. P. 5811–5848.

27. *Kalnins E. G. et al.* Completeness of Superintegrability in Two-Dimensional Constant-Curvature Spaces // J. Phys. A: Math. Gen. 2001. V. 34. P. 4705–4720.

28. *Kalnins E. G., Kress J. M., Winternitz P.* Superintegrability in a Two-Dimensional Space of Nonconstant Curvature // J. Math. Phys. 2002. V. 43. P. 970–983.

29. *Kalnins E. G., Miller W. (Jr.).* Lie Theory and Separation of Variables. 4. The Groups $SO(2,1)$ and $SO(3)$ // J. Math. Phys. 1974. V. 15. P. 1263–1274.

30. *Kalnins E. G., Miller W. (Jr.).* The Wave Equation, $\underset{\sim}{O}(2,2)$, and Separation of Variables on Hyperboloids // Proc. Roy. Soc. A (Edinburgh). 1977. V. 79. P. 227–256;
Kalnins E. G., Miller W. (Jr.). Lie Theory and the Wave Equation in Space-Time. 2. The Group $SO(4,\mathbb{C})$ // SIAM J. Math. Anal. 1978. V. 9. P. 12–33.

31. *Kalnins E. G. et al.* Superintegrability on the Two-Dimensional Hyperboloid II // J. Math. Phys. 1999. V. 40. P. 2291–2306.

32. *Kalnins E. G., Miller W. (Jr.), Pogosyan G. S.* Superintegrability and Associated Polynomial Solutions. Euclidean Space and the Sphere in Two Dimensions // J. Math. Phys. 1996. V. 37. P. 6439–6467.

33. *Kalnins E. G., Miller W. (Jr.), Pogosyan G. S.* Superintegrability on the Two-Dimensional Hyperboloid // J. Math. Phys. 1997. V. 38. P. 5416–5433.

34. *Kalnins E. G., Miller W. (Jr.), Pogosyan G. S.* Completeness of Multiseparable Superintegrability on the Complex 2-Sphere // J. Phys. A: Math. Gen. 2000. V. 33. P. 6791–6806.

35. *Kalnins E. G., Miller W. (Jr.), Pogosyan G. S.* Completeness of Multiseparable Superintegrability in $E_{2,C}$ // Ibid. P. 4105–4120.

36. *Kalnins E. G., Miller W. (Jr.), Pogosyan G. S.* Superintegrability on Two-Dimensional Complex Euclidean Space. Algebraic Methods in Physics: A Symp. for the 60th Birthdays of Jiří Patera and Pavel Winternitz // CRM Series in Math. Phys. Berlin; Heidelberg, 2001. P. 95–103.

37. *Kalnins E. G., Miller W. (Jr.), Pogosyan G. S.* Completeness of Multiseparable Superintegrability in Two Dimensions // Phys. At. Nucl. 2002. V. 65. P. 1033–1035.

38. *Kalnins E. G., Miller W. (Jr.), Pogosyan G. S.* Exact and Quasi-Exact Solvability of Two-Dimensional Superintegrable Quantum Systems. I. Euclidean Space. math-ph/0412035.

39. *Kleinert H.* Path Integrals in Quantum Mechanics, Statistics and Polymer Physics. Singapore: World Sci., 1990.

40. *Koenigs G.* Sur les géodésiques a intégrales quadratiques // *Darboux G.* Lecons sur la théorie générale des surface. Chelsea Publ., 1972. V. 4. P. 368–404.

41. *Lutsenko I. V. et al.* Hydrogen Atom as Indicator of Hidden Symmetry of a Ring-Shaped Potential // Theor. Math. Phys. 1990. V. 83. P. 633–639.

42. *Mostafazadeh A., Batal A.* Physical Aspects of Pseudo-Hermitian and PT-Symmetric Quantum Mechanics // J. Phys. A: Math. Gen. A. 2004. V. 37. P. 11645–11680.

43. *Olevskiĭ M. N.* Triorthogonal Systems in Spaces of Constant Curvature in which the Equation $\Delta_2 u + \lambda u = 0$ Allows the Complete Separation of Variables // Mat. Sb. 1950. V. 27. P. 379–426 (in Russian).

44. *Peak D., Inomata A.* Summation over Feynman Histories in Polar Coordinates // J. Math. Phys. 1969. V. 10. P. 1422–1428.

45. *Schulman L. S.* Techniques and Applications of Path Integration. N. Y.: John Wiley & Sons, 1981.

46. *Ushveridze A.* Quasi-Exactly Solvable Models in Quantum Mechanics. Bristol: Inst. of Phys. Publ., 1994.

47. *Winternitz P. et al.* Symmetry Groups in Classical and Quantum Mechanics // Sov. J. Nucl. Phys. 1967. V. 4. P. 444–450.

48. *Wojciechowski S.* Superintegrability of the Calogero–Moser System // Phys. Lett. A. 1983. V. 95. P. 279–281.

49. *Znojil M.* $\mathcal{PT}$-Symmetric Harmonic Oscillators // Phys. Lett. A. 1999. V. 259. P. 220–223.

50. *Znojil M.* Exact Solution for Morse Oscillator in PT-Symmetric Quantum Mechanics // Phys. Lett. A. 1999. V. 264. P. 108–111.

51. *Znojil M.* Workshop on Pseudo-Hermitian Hamiltonians in Quantum Physics. I–III // Czech. J. Phys. 2004. V. 54. P. 1–156; 1005–1148; V. 55. P. 1045–1192;
http://gemma.ujf.cas.cz/%7Eznojil/conf/index.html

ФИЗИКА ЭЛЕМЕНТАРНЫХ ЧАСТИЦ И АТОМНОГО ЯДРА
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PATH-INTEGRAL APPROACH FOR SUPERINTEGRABLE POTENTIALS ON SPACES OF NONCONSTANT CURVATURE: II. DARBOUX SPACES $D_{\rm III}$ AND $D_{\rm IV}$

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This is the second paper on the path-integral approach of superintegrable systems on Darboux spaces, spaces of nonconstant curvature. We analyze in the spaces $D_{\rm III}$ and $D_{\rm IV}$ five and, respectively, four superintegrable potentials, which were first given by Kalnins et al. We are able to evaluate the path integral in most of the separating coordinate systems, leading to expressions for the Green functions, the discrete and continuous wave functions, and the discrete energy-spectra. In some cases, however, the discrete spectrum cannot be stated explicitly, because it is determined by a higher order polynomial equation. We also show that the free motion in Darboux space of type III can also contain bound states, provided the boundary conditions are appropriate. We can state the energy spectrum and the wave functions, respectively.

Это вторая статья, посвященная приближению интегралов по путям для суперинтегрируемых систем на пространствах Дарбу, пространствах переменной кривизны. На пространствах Дарбу $D_{\rm III}$ и $D_{\rm IV}$ проводится анализ пяти и, соответственно, четырех суперинтегрируемых потенциалов, которые впервые были представлены Калнинсом и др. Нам удалось вычислить интеграл по путям в наиболее разделяющихся системах координат, что приводит к выражениям для функций Грина, волновым функциям дискретного и непрерывного спектров и дискретному спектру энергий. Однако в некоторых случаях дискретный спектр установить не удается, так как он определяется полиномиальным уравнением более высокого порядка. Показано, что свободное движение в пространстве Дарбу III типа также может содержать связанные состояния при определенных граничных условиях. Соответственно, для них можно установить спектр энергий и волновые функции.

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1. INTRODUCTION

In the previous publication [21] we have started to study superintegrable systems on spaces of nonconstant curvature, i.e., Darboux spaces. These spaces were introduced by Kalnins et al. [26, 28]. In the first paper we have studied

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the Darboux spaces $D_{\rm I}$ and $D_{\rm II}$, and we continue our study by considering the two other Darboux spaces $D_{\rm III}$ and $D_{\rm IV}$ with five and, respectively, four superintegrable potentials as determined in [26].

We find a rich structure of the spectrum of these potentials yielding bound and continuous states. As it turns out, already the free motion on $D_{\rm III}$ can give a positive continuous and an infinite negative discrete spectrum. This situation is similar to that for the quantum motion on the $SU(1,1)$ manifold [2], respectively, on the $SU(2,2)$ [6] and $SO(2,2)$ manifold [30].

The notion of superintegrable systems was introduced by Winternitz and co-workers in [9, 47], Wojciechowski [48], and was developed further later on also by Evans [7]. Superintegrable potentials have the property of finding additional constants of motion. In two dimensions one has in total three functional independent constants of motion and in three dimensions one has four (minimal superintegrable) and five (maximal superintegrable) functional independent constants of motion. Well-known examples are the Coulomb potential with its Lenz–Runge vector and the harmonic oscillator with its quadrupole moment. Another property of superintegrable potentials is that usually the corresponding equations in classical and quantum mechanics separate in more than one coordinate system.

Similar studies of the quantum motion on spaces with and without curvature have been investigated in [17] for two- and three-dimensional flat space, in [18] for the two- and three-dimensional sphere, and in [19] and [20] for the two- and three-dimensional hyperboloid. In all these cases the path integral method [8, 22, 39, 45] was applied to find the bound and continuous states, i.e., wave functions and the explicit form of the spectrum. We have not considered complexified spaces as in [37] for the two-dimensional complex sphere or in [34–36] for the two-dimensional complex Euclidean space. In particular, in [34] coordinate systems on the two-dimensional complex sphere and corresponding superintegrable potentials, and in [36] coordinate systems on the two-dimensional complex plane and corresponding superintegrable potentials were discussed. The goal of [34, 36] was to extend the notion of superintegrable potentials of real spaces to the corresponding complexified spaces. The findings were that there are, in addition to the four coordinate systems on the real two-dimensional Euclidean plane, three more coordinate systems and also three more superintegrable potentials. Similarly, in addition to the two coordinate systems on the real two-dimensional sphere there are three more coordinate systems on the complex sphere and four more superintegrable potentials. This is not surprising because the complex plane contains not only the Euclidean plane but also the pseudo-Euclidean plane (10 coordinate systems [13, 23, 24]), and the complex sphere contains not only the real sphere but also the two-dimensional hyperboloid (9 coordinate systems [13, 24, 29, 43]).

However, a complexified space is an abstract object. In order to obtain the actual spectrum of a given potential formulated in a coordinate system one has to consider a real version of the complexified space, e.g., the complex sphere: One has to determine whether one considers the potential on the real sphere or on the real hyperboloid. The complexification serves only as a tool for a unified investigation.

Further studies on superintegrability in spaces with constant curvature are due to [31, 33] (hyperboloid with new potentials), [32] (sphere and Euclidean space), [37] and [38] with a general theory about the connection of separation in nonsubgroup coordinate systems of superintegrable systems and quasi-exactly-solvable problems [46].

An extension of the study of path integration on spaces of constant curvature is the investigation of path integral formulations in spaces of nonconstant curvature. Kalnins et al. [26, 28] denoted four types of two-dimensional spaces of nonconstant curvature, labeled by D_{I}–D_{IV}, which are called Darboux spaces [40]. In terms of the infinitesimal distance they are described by (the coordinates (u,v) will be called the (u,v) system; the (x,y) system in turn can be called light-cone coordinates):

$$\begin{aligned}(\mathrm{I})\quad ds^2 &= (x+y)dxdy = \\ &= 2u(du^2+dv^2) \quad (x=u+iv, y=u-iv),\end{aligned} \tag{1.1}$$

$$\begin{aligned}(\mathrm{II})\quad ds^2 &= \left(\frac{a}{(x-y)^2}+b\right)dxdy = \\ &= \frac{bu^2-a}{u^2}(du^2+dv^2) \quad \left(x=\frac{1}{2}(v+iu), y=\frac{1}{2}(v-iu)\right),\end{aligned} \tag{1.2}$$

$$\begin{aligned}(\mathrm{III})\quad ds^2 &= \left(a\,\mathrm{e}^{-(x+y)/2}+b\,\mathrm{e}^{-x-y}\right)dxdy = \\ &= \mathrm{e}^{-2u}(b+a\,\mathrm{e}^{u})(du^2+dv^2) \quad (x=u-iv, y=u+iv),\end{aligned} \tag{1.3}$$

$$\begin{aligned}(\mathrm{IV})\quad ds^2 &= -\frac{a\left(\mathrm{e}^{(x-y)/2}+\mathrm{e}^{(y-x)/2}\right)+b}{\left(\mathrm{e}^{(x-y)/2}-\mathrm{e}^{(y-x)/2}\right)^2}dxdy = \\ &= \left(\frac{a_+}{\sin^2 u}+\frac{a_-}{\cos^2 u}\right)(du^2+dv^2) \quad (x=u+iv, y=u-iv),\end{aligned} \tag{1.4}$$

where a and b are additional (real) parameters ($a_\pm = (a \pm 2b)/4$). These surfaces are also called surfaces of revolution [5, 25, 26]. Kalnins et al. [26, 28] studied not only the solution of the free motion, but also placed emphasis on the superintegrable systems in these spaces.

The Gaussian curvature in a space with metric $ds^2 = g(u,v)(du^2 + dv^2)$ is given by ($g = \det\, g(u,v)$)

$$G = -\frac{1}{2g}\left(\frac{\partial^2}{\partial u^2} + \frac{\partial^2}{\partial v^2}\right)\ln\, g. \tag{1.5}$$

Equation (1.5) will be used to discuss shortly the curvature properties of the Darboux spaces, including their limiting cases of constant curvature.

In the following sections we discuss superintegrable potentials in each of the two Darboux spaces $D_{\rm III}$ and $D_{\rm IV}$, respectively. We set up the classical Lagrangian and Hamiltonian, the quantum operator, and formulate and solve (if possible) the corresponding path integral. We also discuss some of the limiting cases of the Darboux spaces, i.e., where we obtain a space of constant (zero or negative) curvature. For the Darboux space $D_{\rm III}$ the zero-curvature case $\mathbb{R}^2$ emerges. In $D_{\rm IV}$ we find a hyperboloid.

In the last section we summarize our results, where we also include the findings of our previous paper which dealt with superintegrable potentials on $D_{\rm I}$ and $D_{\rm II}$.

In the first two appendices we add some additional material about the path integral evaluation of the free motion in $D_{\rm IV}$ in degenerate elliptic coordinates. In the third appendix we summarize briefly the path integral investigation of some remaining superintegrable potentials on the two-dimensional Euclidean plane. Finally, in the fourth appendix an example of a potential on the two-dimensional complex sphere will be given.

2. SUPERINTEGRABLE POTENTIALS ON DARBOUX SPACE $D_{\rm III}$

The coordinate systems to be considered in the Darboux space $D_{\rm III}$ are as follows:

$$((u,v)\text{ system})\quad x = v + iu, \quad y = v - iu, \tag{2.1}$$

$$(\text{Polar:})\quad \xi = \varrho\cos\varphi, \quad \eta = \varrho\sin\varphi \quad (\varrho > 0, \varphi \in [0, 2\pi]), \tag{2.2}$$

$$(\text{Parabolic:})\quad \xi = 2\,\mathrm{e}^{-u/2}\cos\frac{v}{2}, \quad \eta = 2\,\mathrm{e}^{-u/2}\sin\frac{v}{2},$$

$$u = \ln\frac{4}{\xi^2+\eta^2},\; v = \arcsin\frac{2\xi\eta}{\xi^2+\eta^2}\; (\xi \in \mathbb{R}, \eta > 0), \tag{2.3}$$

$$(\text{Elliptic:})\quad \xi = d\cosh\,\omega\cos\,\varphi,\; \eta = d\sinh\,\omega\sin\,\varphi\; (\omega > 0, \varphi\in[-\pi,\pi]), \tag{2.4}$$

$$(\text{Hyperbolic:})\quad \xi = \frac{\mu-\nu}{2\sqrt{\mu\nu}} + \sqrt{\mu\nu},\; \eta = i\left(\frac{\mu-\nu}{2\sqrt{\mu\nu}} - \sqrt{\mu\nu}\right)\; (\mu,\nu > 0). \tag{2.5}$$

For the line element we get (we also display where the metric is rescaled in such a way that we set $a = b = 1$ [26]):

$$ds^2 = \mathrm{e}^{-2u}(b + a\,\mathrm{e}^u)(du^2 + dv^2) = (\mathrm{e}^{-u} + \mathrm{e}^{-2u})(du^2 + dv^2), \tag{2.6}$$

$$\text{(Polar:)} = \left(a + \frac{b}{4}\varrho^2\right)(d\varrho^2 + \varrho^2 d\varphi^2) = \left(1 + \frac{1}{4}\varrho^2\right)(d\varrho^2 + \varrho^2 d\varphi^2), \tag{2.7}$$

$$\text{(Parabolic:)} = \left(a + \frac{b}{4}(\xi^2 + \eta^2)\right)(d\xi^2 + d\eta^2) = \\ = \left(1 + \frac{1}{4}(\xi^2 + \eta^2)\right)(d\xi^2 + d\eta^2), \tag{2.8}$$

$$\text{(Elliptic:)} = \left(a + \frac{b}{4}d^2(\sinh^2\omega + \cos^2\varphi)\right)d^2 \times \\ \times (\sinh^2\omega + \sin^2\varphi)(d\omega^2 + d\varphi^2), \tag{2.9}$$

$$\text{(Hyperbolic:)} = \left(a + \frac{b}{2}(\mu - \nu)\right)(\mu + \nu)\left(\frac{d\mu^2}{\mu^2} - \frac{d\nu^2}{\nu^2}\right). \tag{2.10}$$

For the Gaussian curvature we find

$$G = -\frac{ab\,\mathrm{e}^{-3u}}{(b\,\mathrm{e}^{-2u} + a\,\mathrm{e}^{-u})^4}. \tag{2.11}$$

For, e.g., $a = 1, b = 0$ we recover the two-dimensional flat space with the corresponding coordinate systems. To assure the positive definiteness of the metric (1.3), we can require $a, b > 0$. We introduce the following constants of motion on D_{III}:

$$X_1 = \frac{1}{4}\frac{\mathrm{e}^{2u}}{a + b\,\mathrm{e}^u}\cos v \cdot p_u^2 - \frac{1}{4}\frac{\mathrm{e}^u(e^u + 2)}{a + b\,\mathrm{e}^u}\cos v \cdot p_v^2 + \frac{1}{2}\mathrm{e}^u \sin v \cdot p_u p_v, \tag{2.12}$$

$$X_2 = \frac{1}{4}\frac{\mathrm{e}^{2u}}{a + b\,\mathrm{e}^u}\sin v \cdot p_u^2 - \frac{1}{4}\frac{\mathrm{e}^u(\mathrm{e}^u + 2)}{a + b\,\mathrm{e}^u}\sin v \cdot p_v^2 + \frac{1}{2}\mathrm{e}^u \cos v \cdot p_u p_v, \tag{2.13}$$

$$K = p_v. \tag{2.14}$$

These operators satisfy the Poisson relations

$$\{K, X_1\} = -X_2, \qquad \{K, X_2\} = X_1, \qquad \{X_1, X_2\} = K\tilde{\mathcal{H}}_0, \tag{2.15}$$

and the functional relation

$$X_1^2 + X_2^2 - \tilde{\mathcal{H}}_0^2 - \tilde{\mathcal{H}}_0 K^2 = 0. \tag{2.16}$$

Table 1. **Constants of motion and limiting cases of coordinate systems on $D_{\rm III}$**

Metric	Constant of motion	$D_{\rm III}$	E_2 $(a=1, b=0)$
$e^{-2u}(b+a\,e^u)(du^2+dv^2)$	K^2	(u,v) system	Cartesian
$\left(a+\frac{b}{4}\varrho^2\right)(d\varrho^2+\varrho^2 d\varphi^2$	X_2	Polar	Polar
$\left(a+\frac{b}{4}(\xi^2+\eta^2)\right)(d\xi^2+d\eta^2)$	X_1	Parabolic	Parabolic
$\left(a+\frac{b}{4}d^2(\sinh^2\omega+\cos^2\varphi)\right)d^2\times$ $\times(\sinh^2\omega+\sin^2\varphi)(d\omega^2+d\varphi^2)$	$d^2X_1+2K^2$	Elliptic	Elliptic

The operators K, X_1, X_2 can be used to characterize the separating coordinate systems on $D_{\rm III}$, as indicated in Table 1. The corresponding quantum operators are given by

$$X_1 = \frac{1}{4}\,e^u\left[\frac{e^u\cos v}{a+b\,e^u}\cdot\partial_u^2 - \frac{e^u+2}{a+b\,e^u}\cos v\cdot\partial_v^2 + \right.$$
$$\left. + \left(2\sin v\cdot\partial_u\partial_v + \cos v\cdot\partial_u + \sin v\cdot\partial_v\right)\right], \qquad (2.17)$$

$$X_2 = \frac{1}{4}\,e^u\left[\frac{e^u\sin v}{a+b\,e^u}\cdot\partial_u^2 - \frac{e^u+2}{a+b\,e^u}\sin v\cdot\partial_v^2 - \right.$$
$$\left. - \left(2\cos v\cdot\partial_u\partial_v - \sin v\cdot\partial_u + \cos v\cdot\partial_v\right)\right], \qquad (2.18)$$

$$K = \partial_v. \qquad (2.19)$$

These operators satisfy the commutation relations

$$[\widehat{K},\widehat{X}_1] = -\widehat{X}_2, \quad [\widehat{K},\widehat{X}_2] = \widehat{X}_1, \quad [\widehat{X}_1,\widehat{X}_2] = \widehat{K}\widehat{H}_0, \qquad (2.20)$$

and the relation

$$\widehat{X}_1^2 + \widehat{X}_2^2 - \widehat{H}_0^2 - \widehat{H}_0\widehat{K}^2 + \frac{1}{4}\widehat{H}_0 = 0. \qquad (2.21)$$

(Let us note that by $\tilde{\mathcal{H}}_0$ the classical Hamiltonian without the $1/2m$ factor is meant. Keeping this factor is no problem, however, in the present form the algebra is simpler.)

We now state the superintegrable potentials on D_{III}:

$$V_1(u,v) = \frac{2k_1\,\mathrm{e}^{-u}\cos\,v/2 + 2k_2\,\mathrm{e}^{-u}\sin v/2 + k_3}{a + \dfrac{b}{4}\,\mathrm{e}^{-u}}, \tag{2.22}$$

$$V_2(u,v) = \frac{1}{a + b\,\mathrm{e}^{-u}}\left[-\alpha + \mathrm{e}^{u}\frac{\hbar^2}{8m}\left(\frac{k_1^2 - 1/4}{\cos^2 v/2} + \frac{k_1^2 - 1/4}{\cos^2 v/2}\right)\right], \tag{2.23}$$

$$V_3(u,v) = \frac{1}{a + b\,\mathrm{e}^{-u}}\left[-\alpha + \frac{\hbar^2}{2m}4\,\mathrm{e}^{u}\left(c_1^2\,\mathrm{e}^{-iv} - 2c_2\,\mathrm{e}^{-2iv}\right)\right], \tag{2.24}$$

Table 2. **Separation of variables for the superintegrable potentials on D_{III}**

Potential	Constants of motion	Separating coordinate system
V_1	$R_1 = X_1 + \dfrac{2k_1\xi(2+\eta^2) - 2k_2\eta(2+\xi^2) + k_3(\eta^2-\xi^2)}{4a + b(\xi^2+\eta^2)}$ $R_2 = X_2 + \dfrac{k_1\eta(\eta^2-\xi^2+4) + k_2\xi(\xi^2-\eta^2+4) - 2xk_3\xi\eta}{4a+b(\xi^2+\eta^2)}$	Parabolic Translated Parabolic $(\xi,\eta \to \xi\eta \pm c)$
V_2	$R_1 = X_1 +$ $+\dfrac{\hbar^2/m((k_1^2-1/4)\eta^2(\eta^2+2) - (k_2^2-1/4)\xi^2(\xi^2+2)) - \alpha(\eta^2-\xi^2)}{4a+b(\xi^2+\eta^2)}$ $R_2 = K^2 + \dfrac{\hbar^2}{8m}((k_1^2-1/4)\dfrac{\eta^2}{\xi^2} + (k_2^2-1/4)\dfrac{\xi^2}{\eta^2})$	(u,v) system Polar Parabolic
V_3	$R_1 = X_1 + iX_2 - \dfrac{-\alpha\mu^2\nu^2 + c_1^2\mu\nu - 2c_2(1+\mu-\nu)}{(a+b/2(\mu-\nu))(\mu+\nu)}$ $R_2 = K^2 - c_1^2\dfrac{\mu-\nu}{\mu\nu} + c_2\dfrac{(\mu-\nu)^2}{\mu^2\nu^2}$	Polar Hyperbolic
V_4	$R_1 = X_1 + iX_2 - K^2 -$ $-\dfrac{\mu\nu(d_1(\nu-2) + d_2(\mu+2) + m\omega^2(\nu-\mu+\mu\nu))}{(a+b/2(\mu-\nu))(\mu+\nu)}$ $R_1 = X_1 - iX_2 -$ $-\dfrac{(\mu-\nu)((\mu-\nu)(d_1\mu+d_2\nu) - m\omega^2(\mu^2+\nu^2+\mu\nu(2+\mu-\nu)))}{4(a+b/2(\mu-\nu))(\mu+\nu)}$	Hyperbolic Elliptic
V_5	$R_1 = X_1 + \dfrac{\hbar^2 v_0^2}{8m}\dfrac{\eta^2-\xi^2}{a+b/4(\xi^2+\eta^2)}$ $R_2 = X_1 - \dfrac{n\hbar^2 v_0^2}{4m}\dfrac{\xi\eta}{a+b/4(\xi^2+\eta^2)}$ $R_3 = K = p$	(u,v) system Polar Parabolic Elliptic Hyperbolic

$$V_4(\mu,\nu)=\frac{1}{\left(a+\dfrac{b}{2}(\mu-\nu)\right)(\mu+\nu)}\left[d_1\mu+d_2\nu+\frac{m}{2}\omega^2(\mu^2-\nu^2)\right], \qquad (2.25)$$

$$V_5(u,v)=\frac{1}{a+b\,\mathrm{e}^{-u}}\frac{\hbar^2 v_0^2}{2m}. \qquad (2.26)$$

In Table 2 we list the properties of these potentials on D_{III}, where the coordinate systems, where an explicit path integral solution is possible, are underlined. We see that V_5 is a special case, and it has three integrals of motion. We will threat this case in some more detail as in the other spaces, because on D_{III} the free quantum motion can give bound state solutions (provided the constants are chosen properly). This feature has not been discussed in [14].

2.1. The Superintegrable Potential V_1 on D_{III}. We state the potential V_1 in the respective coordinate systems

$$V_1(u,v)=\frac{2k_1\mathrm{e}^{-u}\cos v/2+2k_2\,\mathrm{e}^{-u}\,\sin v/2+k_3}{a+\dfrac{b}{4}\,\mathrm{e}^{-u}}, \qquad (2.27)$$

$$=\frac{k_1\xi+k_2\eta+k_3}{a+\dfrac{b}{4}(\xi^2+\eta^2)}, \qquad (2.28)$$

$$=\frac{k_1\xi+k_2\eta+(k_1c-k_2c+k_3)}{a+\dfrac{b}{4}((\xi+c)^2+(\eta-c)^2)}, \qquad (2.29)$$

and V_1 is also separable in translated parabolic coordinates $\xi\to\xi+c, \eta\to\eta-c$. The translated parabolic coordinates just modify the solution of a shifted harmonic oscillator, and this case we do not discuss separately.

2.1.1. Separation of V_1 in Parabolic Coordinates. The classical Lagrangian and Hamiltonian in parabolic coordinates on D_{III} are given by

$$\mathcal{L}(\xi,\dot\xi,\eta,\dot\eta)=\frac{m}{2}\left(a+\frac{b}{4}\right)(\xi^2+\eta^2))(\dot\xi^2+\dot\eta^2)-V(\xi,\eta), \qquad (2.30)$$

$$\mathcal{H}(\xi,p_\xi,\eta,p_\xi)=\frac{1}{2m}\frac{1}{a+\dfrac{b}{4}(\xi^2+\eta^2)}(p_\xi^2+p_\eta^2)+V(\xi,\eta). \qquad (2.31)$$

The canonical momenta are given by

$$p_\xi=\frac{\hbar}{i}\left(\frac{\partial}{\partial\xi}+\frac{b\xi}{a+\dfrac{b}{4}(\xi^2+\eta^2)}\right),\quad p_\eta=\frac{\hbar}{i}\left(\frac{\partial}{\partial\eta}+\frac{b\eta}{a+\dfrac{b}{4}(\xi^2+\eta^2)}\right), \qquad (2.32)$$

and for the quantum Hamiltonian (product ordering) we find

$$H = -\frac{\hbar^2}{2m}\frac{1}{a+\frac{b}{4}(\xi^2+\eta^2)}\left(\frac{\partial^2}{\partial\xi}+\frac{\partial^2}{\partial\eta^2}\right)+V(\xi,\eta), \tag{2.33}$$

$$= \frac{1}{2m}\sqrt{\frac{1}{a+\frac{b}{4}(\xi^2+\eta^2)}}\,(p_\xi^2+p_\eta^2)\sqrt{\frac{1}{a+\frac{b}{4}(\xi^2+\eta^2)}}+V(\xi,\eta). \tag{2.34}$$

Therefore we obtain for the path integral formulation for V_1

$$K^{(V_1)}(\xi'',\xi',\eta'',\eta';T) = \int\limits_{\xi(t')=\xi'}^{\xi(t'')=\xi''}\mathcal{D}\xi(t)\int\limits_{\eta(t')=\eta'}^{\eta(t'')=\eta''}\mathcal{D}\eta(t)\left(a+\frac{b}{4}(\xi^2+\eta^2)\right)\times$$

$$\times\exp\left\{\frac{i}{\hbar}\int\limits_0^T\left[\left(a+\frac{b}{4}(\xi^2+\eta^2)\right)(\dot\xi^2+\dot\eta^2)-\frac{k_1\xi+k_2\eta+k_3}{\left(a+\frac{b}{4}(\xi^2+\eta^2)\right)}\right]dt\right\}=$$

$$=\int\limits_{-\infty}^{\infty}\frac{dE}{2\pi\hbar}\,\mathrm{e}^{-iET/\hbar}\int\limits_0^\infty ds''\exp\left[\frac{i}{\hbar}\left(aE-k_3-\frac{k_1^2+k_2^2}{2m\omega^2}\right)s''\right]\times$$

$$\times K^{(V_1)}(\xi'',\xi',\eta'',\eta';s''), \tag{2.35}$$

with the time-transformed path integral $K(s'')$ given by

$$K^{(V_1)}(\xi'',\xi',\eta'',\eta';s'') = \int\limits_{\xi(0)=\xi'}^{\xi(s'')=\xi''}\mathcal{D}\xi(s)\int\limits_{\eta(0)=\eta'}^{\eta(s'')=\eta''}\mathcal{D}\eta(s)\times$$

$$\times\exp\left\{\frac{i}{\hbar}\int\limits_0^{s''}\left[\frac{m}{2}\left((\dot\xi^2+\dot\eta^2)-\frac{m}{2}\omega^2(\tilde\xi^2+\tilde\eta^2)\right)\right]ds\right\}. \tag{2.36}$$

The transformed variables $\tilde\xi,\tilde\eta$ are given by $\tilde\xi=\xi+k_1/m\omega^2, \tilde\eta=\eta+k_2/m\omega^2$, and $\omega^2=-bE/2m$. Similarly as in [14] we can determine the Green function to

have the form

$$G^{(V_1)}(\xi'',\xi',\eta'',\eta';E) =$$

$$= \int d\mathcal{E}\, \frac{m}{\pi\hbar^2 b}\sqrt{-\frac{m}{2E}}\,\Gamma\biggl(\frac{1}{2}+\frac{\tilde{\mathcal{E}}}{b\hbar}\sqrt{-\frac{m}{2E}}\biggr)\Gamma\biggl(\frac{1}{2}+\frac{\mathcal{E}}{b\hbar}\sqrt{-\frac{m}{2E}}\biggr)\times$$

$$\times\, D_{-\frac{1}{2}+\frac{\tilde{\mathcal{E}}}{\hbar}\sqrt{-\frac{}{2}}}\biggl(\sqrt[4]{-\frac{8mEb^2}{\hbar^2}}\,\tilde{\xi}_>\biggr) D_{-\frac{1}{2}+\frac{\tilde{\mathcal{E}}}{\hbar}\sqrt{-\frac{}{2}}}\biggl(-\sqrt[4]{-\frac{8mEb^2}{\hbar^2}}\,\tilde{\xi}_<\biggr)\times$$

$$\times\, D_{-\frac{1}{2}+\frac{\mathcal{E}}{\hbar}\sqrt{-\frac{}{2}}}\biggl(\sqrt[4]{-\frac{8mEb^2}{\hbar^2}}\,\tilde{\eta}_>\biggr) D_{-\frac{1}{2}+\frac{\mathcal{E}}{\hbar}\sqrt{-\frac{}{2}}}\biggl(-\sqrt[4]{-\frac{8mEb^2}{\hbar^2}}\,\tilde{\eta}_<\biggr). \quad (2.37)$$

The $D_\nu(z)$ are parabolic cylinder-functions [10, p. 1064], and the $\tilde{\mathcal{E}}$ is defined by $\tilde{\mathcal{E}} = aE - k_3 - (k_1^2+k_2^2)/bE - \mathcal{E}$. On the other hand, we can insert for the discrete part of the Green function the harmonic oscillator wave functions and obtain

$$G^{(V_1)}_{\rm disc}(\xi'',\xi',\eta'',\eta';E) = \sum_{n\ =0}^{\infty}\sum_{n\ =0}^{\infty} \frac{N^2_{n\ n}}{E_{n\ n}\ - E}\times$$

$$\times\, \Psi_n^{(\rm HO)}(\xi'')\Psi_n^{(\rm HO)}(\xi')\Psi_n^{(\rm HO)}(\eta'')\Psi_n^{(\rm HO)}(\eta'). \quad (2.38)$$

The wave functions for the harmonic oscillator are given by the well-known form in terms of Hermite-polynomials [10]

$$\Psi_n^{(\rm HO)}(x) = \biggl(\frac{m\omega}{\pi\hbar}\biggr)^{1/4}\biggl(\frac{1}{2^n n!}\biggr)^{1/2} H_n\biggl(\sqrt{\frac{m\omega}{\hbar}}\,x\biggr)\exp\biggl(-\frac{m\omega}{2\hbar}x^2\biggr). \quad (2.39)$$

$E_{n\ n}$ is determined by the equation

$$aE - k_3 - \frac{k_1^2+k_2^2}{2m\omega^2} - \hbar(n_\xi+n_\eta+1)\sqrt{-\frac{bE}{2m}} = 0, \quad (2.40)$$

which is actually an equation of the fourth order in E

$$E^4_{n\ n} + \biggl(\frac{b\hbar^2}{2ma^2}(n_\xi+n_\eta+1)^2 - \frac{2k_3}{a}\biggr)E^3_{n\ n} -$$

$$-\biggl(2\frac{k_1^2+k_2^2}{ab} - \frac{k_3^2}{a^2}\biggr)E^2_{n\ n} + 2k_3\frac{k_1^2+k_2^2}{a^2 b}E_{n\ n} - \frac{(k_1^2+k_2^2)^2}{a^2b^2} = 0. \quad (2.41)$$

We dot not solve this equation. Note that for $k_1 = k_2 = k_3 = 0$ a discrete spectrum emerges for the free motion on $D_{\rm III}$, a feature which will be discussed

in more detail in the subsection for V_5. For the special case $k_1 = k_2 = 0$ we obtain the solution ($N = n_\xi + n_\eta + 1$)

$$E_{n\ n\ \pm} = -\frac{b\hbar^2 N^2}{4ma^2} + \frac{k_3}{a} \pm \frac{1}{a}\sqrt{\left(\frac{b\hbar^2 N^2}{4am}\right)^2 - \frac{bk_3\hbar^2 N^2}{2am} - k_3^2}. \tag{2.42}$$

Note that $\omega_{n\ n}$ must be taken on $\omega_{n\ n} = \sqrt{-bE_{n\ n}/2m}$. The normalization $N_{n\ n}$ is determined by the residuum in $G^{(V_1)}(E)$. If one fixes the parameters a and b and the specific surface of revolution, a more detailed investigation can be performed (special cases, limiting cases, which sign of the square-root gives a positive definite Hilbert space, etc.). Because we do not fix these parameters, we keep both signs of the square-root expression (recall that the free motion on D_{III} allows already a discrete spectrum reaching to $-\infty$).

Note that for the translated parabolic coordinates, the variables $\tilde{\xi}, \tilde{\eta}$ are translated by $\pm c$, respectively; and the quantity $\mathcal{E}$, by an additional $Ebc^2/2$.

2.2. The Superintegrable Potential V_2 on D_{III}. We state the potential V_2 in the respective coordinate systems

$$V_2(u,v) = \frac{1}{a + b\,\mathrm{e}^{-u}}\left[-\alpha + \mathrm{e}^u \frac{\hbar^2}{8m}\left(\frac{k_1^2 - 1/4}{\cos^2 v/2} + \frac{k_1^2 - 1/4}{\cos^2 v/2}\right)\right], \tag{2.43}$$

$$= \frac{1}{a + \frac{b}{4}\varrho^2}\left[-\alpha + \frac{\hbar^2}{2m\varrho^2}\left(\frac{k_1^2 - 1/4}{\cos^2\varphi} + \frac{k_2^2 - 1/4}{\sin^2\varphi}\right)\right], \tag{2.44}$$

$$= \frac{1}{a + \frac{b}{4}(\xi^2 + \eta^2)}\left[-\alpha + \frac{\hbar^2}{2m}\left(\frac{k_1^2 - 1/4}{\xi^2} + \frac{k_2^2 - 1/4}{\eta^2}\right)\right], \tag{2.45}$$

$$= \frac{1}{a + b\,\mathrm{e}^{-u}}\left[-\alpha + \frac{\hbar^2}{2md^2}\left(\frac{k_1^2 - 1/4}{\cosh^2\omega\cos^2\varphi} + \frac{k_2^2 - 1/4}{\sinh^2\omega\sin^2\varphi}\right)\right]. \tag{2.46}$$

V_2 is obviously separable in elliptic coordinates, but the corresponding path integral is not solvable, so this case will be omitted.

2.2.1. Separation of V_2 in the (u, v) System. The classical Lagrangian and Hamiltonian are given by

$$\mathcal{L}(u,\dot{u},v,\dot{v}) = \frac{m}{2}\frac{b + a\,\mathrm{e}^u}{\mathrm{e}^{2u}}(\dot{u}^2 + \dot{v}^2) - V(u,v), \tag{2.47}$$

$$\mathcal{H}(u,p_u,v,p_v) = \frac{1}{2m}\frac{\mathrm{e}^{2u}}{b + a\,\mathrm{e}^u}(p_u^2 + p_v^2) + V(u,v). \tag{2.48}$$

The canonical momenta are given by

$$p_u = \frac{\hbar}{i}\left(\frac{\partial}{\partial u} - \frac{1}{2}\frac{a\,\mathrm{e}^{-u} + 2b\,\mathrm{e}^{-2u}}{a\,\mathrm{e}^{-u} + b\,\mathrm{e}^{-2u}}\right), \quad p_v = \frac{\hbar}{i}\frac{\partial}{\partial v}, \tag{2.49}$$

and for the quantum Hamiltonian we find

$$H = -\frac{\hbar^2}{2m}\frac{1}{a\,\mathrm{e}^{-u} + b\,\mathrm{e}^{-2u}}\left(\frac{\partial^2}{\partial u^2} + \frac{\partial^2}{\partial v^2}\right) + V(u,v), \tag{2.50}$$

$$= \frac{1}{2m}\sqrt{\frac{1}{a\,\mathrm{e}^{-u} + b\,\mathrm{e}^{-2u}}}\left(p_u^2 + p_v^2\right)\sqrt{\frac{1}{a\,\mathrm{e}^{-u} + b\,\mathrm{e}^{-2u}}} + V(u,v). \tag{2.51}$$

Therefore we obtain for the path integral ($f(u) = (a\,\mathrm{e}^{-u} + b\,\mathrm{e}^{-2u})$)

$$K^{(V_2)}(u'',u',v'',v';T) = \int\limits_{u(t')=u'}^{u(t'')=u''}\mathcal{D}u(t)\int\limits_{v(t')=v'}^{v(t'')=v''}\mathcal{D}v(t)(a\,\mathrm{e}^{-u} + b\,\mathrm{e}^{-2u})\times$$

$$\times\exp\left(\frac{i}{\hbar}\int\limits_0^T\left\{(a\,\mathrm{e}^{-u} + b\,\mathrm{e}^{-2u})(\dot{u}^2 + \dot{v}^2) - \right.\right.$$

$$\left.\left. - \frac{1}{a + b\,\mathrm{e}^{-u}}\left[-\alpha + \mathrm{e}^u\frac{\hbar^2}{8m}\left(\frac{k_1^2 - 1/4}{\cos^2 v/2} + \frac{k_1^2 - 1/4}{\cos^2 v/2}\right)\right]\right\}dt\right) =$$

$$= \frac{1}{[f(u')f(u'')]^{1/4}}\sum_{l=0}^{\infty}\Phi_l^{(k_2,k_1)}\left(\frac{v''}{2}\right)\Phi_l^{(k_2,k_1)}\left(\frac{v'}{2}\right)\times$$

$$\times\int\limits_{u(t')=u'}^{u(t'')=u''}\mathcal{D}u(t)(a\,\mathrm{e}^{-u} + b\,\mathrm{e}^{-2u})^{1/2}\exp\left(\frac{i}{\hbar}\int\limits_0^T\left\{(a\,\mathrm{e}^{-u} + b\,\mathrm{e}^{-2u})\dot{u}^2 - \right.\right.$$

$$\left.\left. - \frac{1}{a + b\,\mathrm{e}^{-u}}\left[-\alpha + \mathrm{e}^u\frac{\hbar^2}{8m}\big(2l + 1 + |k_1| + |k_2|\big)\right]\right\}dt\right) = \int\limits_{-\infty}^{\infty}\frac{dE}{2\pi\hbar}\,\mathrm{e}^{-iET/\hbar}\times$$

$$\times\int\limits_0^{\infty}ds''\exp\left[-\frac{i}{\hbar}\frac{\hbar^2}{8m}\big(2l + 1 + |k_1| + |k_2|\big)^2 s''\right]K_l^{(V_2)}(u'',u';s''), \tag{2.52}$$

with the time-transformed path integral $K_l(s'')$ given by

$$K_l^{(V_2)}(u'',u';s'') =$$
$$= \int\limits_{u(0)=u'}^{u(s'')=u''} \mathcal{D}u(s) \exp\left[\frac{i}{\hbar}\int\limits_0^{s''}\left(\frac{m}{2}\dot{u}^2 + Eb\,\mathrm{e}^{-2u} + (aE-\alpha)\,\mathrm{e}^{-u}\right)ds\right]. \quad (2.53)$$

The $\Phi_n^{(k_1,k_2)}(\beta)$ are the wave functions of the Pöschl–Teller potential, which are given by

$$V(x) = \frac{\hbar^2}{2m}\left(\frac{\alpha^2-1/4}{\sin^2 x} + \frac{\beta^2-1/4}{\cos^2 x}\right), \quad (2.54)$$

$$\Phi_n^{(\alpha,\beta)}(x) = \left[2(\alpha+\beta+2l+1)\frac{l!\Gamma(\alpha+\beta+l+1)}{\Gamma(\alpha+l+1)\Gamma(\beta+l+1)}\right]^{1/2} \times$$
$$\times\,(\sin x)^{\alpha+1/2}(\cos x)^{\beta+1/2}P_n^{(\alpha,\beta)}(\cos 2x). \quad (2.55)$$

Equation (2.53) is a path integral for the Morse potential. Inserting the corresponding solution [22] we obtain

$$G^{(V_2)}(u'',u',v'',v';E) = \sum_{l=0}^{\infty}\Phi_l^{(k_2,k_1)}\left(\frac{v''}{2}\right)\Phi_l^{(k_2,k_1)}\left(\frac{v'}{2}\right)\times$$
$$\times\sqrt{-\frac{m}{2bE}}\,\frac{m\Gamma\left(\frac{1}{2}+\lambda+\frac{aE-\alpha}{\hbar}\sqrt{-\frac{m}{2bE}}\right)}{\hbar\Gamma(1+2\lambda)\,\mathrm{e}^{(u'+u'')/2}}\times$$
$$\times\,W_{\frac{-}{\hbar}\sqrt{-\frac{}{2}},\lambda}\left(\frac{\sqrt{-8mbE}}{\hbar}\,\mathrm{e}^{-u}\right)M_{\frac{-}{\hbar}\sqrt{-\frac{}{2}},\lambda}\left(\frac{\sqrt{-8mbE}}{\hbar}\,\mathrm{e}^{-u}\right). \quad (2.56)$$

Inserting the bound state wave functions for the Morse potential gives the bound state contribution of $G^{(V_2)}(E)$

$$G_{\mathrm{disc}}^{(V_2)}(u'',u',v'',v';E) = \sum_{l=0}^{\infty}\Phi_l^{(k_2,k_1)}\left(\frac{v''}{2}\right)\Phi_l^{(k_2,k_1)}\left(\frac{v'}{2}\right)\times$$
$$\times\sum_{l=0}^{\infty}\frac{N_{nl}^2}{E_{nl}-E}\Psi_n^{(\mathrm{MP})}(u'')\Psi_n^{(\mathrm{MP})}(u'), \quad (2.57)$$

$$\Psi_n^{(\mathrm{MP})}(u) = N_{nl}\left[\left(-\frac{2mbE_{nl}}{\hbar^2}\right)^{\frac{\ \ -\ \ }{\hbar}\sqrt{-\frac{\ }{2}}-n-1/2}\times\right.$$

$$\left.\times\frac{\left(\dfrac{aE_{nl}-\alpha}{\hbar}\sqrt{-\dfrac{m}{2bE_{nl}}}-2n-1\right)}{\Gamma\left(\dfrac{aE_{nl}-\alpha}{\hbar}\sqrt{-\dfrac{2m}{bE_{nl}}}-n\right)}\right]^{1/2}\times$$

$$\times\exp\left[(u'+u'')\left(\frac{aE_{nl}-\alpha}{\hbar}\sqrt{-\frac{m}{2bE_{nl}}}-n-\frac{1}{2}\right)-\frac{\sqrt{-2mbE_{nl}}}{\hbar}\,\mathrm{e}^u\right]\times$$

$$\times L_n^{\left(\frac{\ \ -\ \ }{\hbar}\sqrt{-\frac{2}{\ }}-2n-1\right)}\left(\frac{-8mbE_{nl}}{\hbar}\,\mathrm{e}^u\right). \quad (2.58)$$

The $L_n^{(\alpha)}(z)$ are Laguerre polynomials [10]. Here, the spectrum E_{nl} is determined by

$$aE_{nl}-\alpha-\hbar\sqrt{-\frac{bE_{nl}}{2m}}\,(2n+2l+|k_1|+|k_2|+2), \quad (2.59)$$

which is a quadratic equation in E_{nl} with solution ($N = 2n+2l+|k_1|+|k_2|+2$)

$$E_{nl\pm} = \frac{1}{2a^2}\left[-\left(\frac{b\hbar^2}{2m}N^2-2a\alpha\right)\pm\frac{b\hbar^2}{2m}N^2\sqrt{1-\frac{8a\alpha m}{b\hbar^2N^2}}\right], \quad (2.60)$$

and the normalization constants N_{nl} are determined by the residuum of (2.56). For large n, l we have

$$E_{nl-} \simeq -\frac{b\hbar^2}{m}(2n+2l+|k_1|+|k_2|+2)^2, \quad (2.61)$$

$$E_{nl+} \simeq -\frac{m\alpha^2}{2b\hbar^2(2n+2l+|k_1|+|k_2|+2)^2}, \quad (2.62)$$

with E_{nl+} showing a Coulomb-like behavior.

2.2.2. Separation of V_2 in Polar Coordinates. In the coordinates (ϱ, φ) the classical Lagrangian and Hamiltonian take on the form

$$\mathcal{L}(\varrho,\dot\varrho,\varphi,\dot\varphi) = \frac{m}{2}\left(a+\frac{b}{4}\varrho^2\right)(\dot\varrho^2+\varrho^2\dot\varphi^2)-V(\varrho,\varphi), \quad (2.63)$$

$$\mathcal{H}(\varrho,p_\varrho,\varphi,p_\varphi) = \frac{1}{2m}\frac{1}{a+\dfrac{b}{4}\varrho^2}\left(p_\varrho^2+\frac{1}{\varrho^2}p_\varphi^2\right)+V(\varrho,\varphi). \quad (2.64)$$

The canonical momenta are given by

$$p_\varrho = \frac{\hbar}{i}\left(\frac{\partial}{\partial\varrho} + \frac{b\varrho}{4a+b\varrho^2} + \frac{1}{2\varrho}\right), \quad p_\varphi = \frac{\hbar}{i}\frac{\partial}{\partial\varphi}. \tag{2.65}$$

Therefore the quantum Hamiltonian is given by

$$H = -\frac{\hbar^2}{2m}\frac{1}{a+\frac{b}{4}\varrho^2}\left(\frac{\partial^2}{\partial\varrho^2} + \frac{1}{\varrho}\frac{\partial}{\partial\varrho} + \frac{1}{\varrho^2}\frac{\partial^2}{\partial\varphi^2}\right) + V(\varrho,\varphi) = \tag{2.66}$$

$$= \frac{1}{2m}\sqrt{\frac{1}{a+\frac{b}{4}\varrho^2}}\left(p_\varrho^2 + \frac{1}{\varrho^2}p_\varphi^2\right)\sqrt{\frac{1}{a+\frac{b}{4}\varrho^2}} + \\ + V(\varrho,\varphi) - \left(a+\frac{b}{4}\varrho^2\right)^{-1}\frac{\hbar^2}{8m\varrho^2}, \tag{2.67}$$

and in this case we have an additional quantum potential $\propto \hbar^2$. This gives for the path integral $\Big(f(\varrho) = a + \frac{b}{4}\varrho^2 = \sqrt{g}\Big)$

$$\begin{aligned} K^{(V_2)}(\varrho'',\varrho',\varphi'',\varphi';T) &= \int\limits_{\varrho(t')=\varrho'}^{\varrho(t'')=\varrho''}\mathcal{D}\varrho(t)\int\limits_{\varphi(t')=\varphi'}^{\varphi(t'')=\varphi''}\mathcal{D}\varphi(t) f(\varrho)\varrho\times \\ &\times\exp\left(\frac{i}{\hbar}\int\limits_0^T\left\{\frac{m}{2}f(\varrho)(\dot\varrho^2+\varrho^2\dot\varphi^2) - \frac{1}{f(\varrho)}\times\right.\right. \\ &\times\left.\left.\left[-\alpha + \frac{\hbar^2}{2m\varrho^2}\left(\frac{k_1^2-1/4}{\cos^2\varphi} + \frac{k_2^2-1/4}{\sin^2\varphi} - \frac{1}{4}\right)\right]\right\}dt\right) = \\ &= \sum_{l=0}^\infty \Phi_l^{(k_2,k_1)}(\varphi'')\Phi_l^{(k_2,k_1)}(\varphi')\frac{1}{[(\varrho'\varrho'')^2 f(\varrho')f(\varrho'')]^{1/4}}\int\limits_{\varrho(t')=\varrho'}^{\varrho(t'')=\varrho''}\mathcal{D}\varrho(t) f^{1/2}(\varrho)\times \\ &\times\exp\left\{\frac{i}{\hbar}\int\limits_0^T\left[\frac{m}{2}f(\varrho)\dot\varrho^2 - \frac{1}{f(\varrho)}\left(-\alpha + \frac{\hbar^2}{2m}\frac{\lambda^2-1/4}{\varrho^2}\right)\right]dt\right\} = \\ &= \frac{1}{\sqrt{\varrho'\varrho''}}\sum_{l=0}^\infty \Phi_l^{(k_2,k_1)}(\varphi'')\Phi_l^{(k_2,k_1)}(\varphi')\times \\ &\times\int\limits_{-\infty}^\infty \frac{dE}{2\pi\hbar}\,\mathrm{e}^{-iET/\hbar}\int\limits_0^\infty ds''\exp\left[\frac{i}{\hbar}(aE-\alpha)s''\right]K_l^{(V_2)}(\varrho'',\varrho';s''), \end{aligned} \tag{2.68}$$

with the time-transformed path integral $K_l(s'')$ given by ($\lambda = 2l + |k_1| + |k_2| + 1$))

$$K_l^{(V_2)}(\varrho'', \varrho'; s'') = \int\limits_{\varrho(0)=\varrho'}^{\varrho(s'')=\varrho''} \mathcal{D}\varrho(s) \times$$

$$\times \exp\left[\frac{i}{\hbar}\int\limits_0^{s''}\left(\frac{m}{2}\dot{\varrho}^2 + \frac{Eb}{4}\varrho^2 - \frac{\hbar^2}{2m}\frac{\lambda^2 - 1/4}{\varrho^2}\right)ds\right] =$$

$$= \frac{m\omega\sqrt{\varrho'\varrho''}}{i\hbar\sin\omega s''}\exp\left[-\frac{m\omega}{2i\hbar}(\varrho'^2 + \varrho''^2)\cot\omega s''\right] I_\lambda\left(\frac{m\omega\varrho'\varrho''}{i\hbar\sin\omega s''}\right). \quad (2.69)$$

Performing the s'' integration yields the Green function

$$G^{(V_2)}(\varrho'', \varrho', \varphi'', \varphi'; E) = \sum_{l=0}^{\infty}\Phi_l^{(k_2,k_1)}(\varphi'')\Phi_l^{(k_2,k_1)}(\varphi') \times$$

$$\times \sqrt{-\frac{2m}{Eb}}\,\frac{\Gamma\left[\frac{1}{2}\left(1 + \lambda - \frac{1}{\hbar}(aE - \alpha)\sqrt{-2m/bE}\right)\right]}{\Gamma(1+\lambda)\sqrt{\varrho'\varrho''}} M_{\frac{-}{2\hbar}\sqrt{-\frac{2}{\ }},\frac{\ }{2}} \times$$

$$\times \left(\frac{m}{\hbar}\sqrt{-\frac{bE}{2m}}\,\varrho_<^2\right) M_{\frac{-}{2\hbar}\sqrt{-\frac{2}{\ }},\frac{\ }{2}}\left(\frac{m}{\hbar}\sqrt{-\frac{bE}{2m}}\,\varrho_>^2\right). \quad (2.70)$$

Inserting the expansion into Laguerre polynomial yields the discrete contribution of the Green function

$$G_{\rm disc}^{(V_2)}(\varrho'', \varrho', \varphi'', \varphi'; E) = \frac{1}{\sqrt{\varrho'\varrho''}}\sum_{l=0}^{\infty}\Phi_l^{(k_2,k_1)}(\varphi'')\Phi_l^{(k_2,k_1)}(\varphi') \times$$

$$\times \sum_{n=0}^{\infty}\frac{N_{nl}^2}{E_{nl} - E}\Psi_n^{({\rm RHO},\lambda)}(\varrho'')\Psi_n^{({\rm RHO},\lambda)}(\varrho'). \quad (2.71)$$

The wave functions for the radial harmonic oscillator $V(r) = \frac{m}{2}\omega^2 - \frac{\hbar^2}{2m}\frac{\lambda^2 - 1/4}{r^2}$ have the form [22, 44]

$$\Psi_n^{({\rm RHO},\lambda)}(r) =$$

$$= \sqrt{\frac{2m}{\hbar}\frac{n!}{\Gamma(n+\lambda+1)}}\, r\left(\frac{m\omega}{\hbar}r\right)^{\lambda/2}\exp\left(-\frac{m\omega}{2\hbar}r^2\right)L_n^{(\lambda)}\left(\frac{m\omega}{\hbar}r^2\right). \quad (2.72)$$

The spectrum E_{nl} is determined by

$$aE_{nl} - \alpha - \hbar\sqrt{-\frac{bE_{nl}}{2m}}\,(2n + 2l + |k_1| + |k_2| + 2), \tag{2.73}$$

which is the same as in (2.60). In the wave functions $\Psi_n^{(\mathrm{RHO},\lambda)}(\varrho)$ the quantity ω has to be taken on $\omega = \sqrt{-bE_{nl}/2m}$, and the normalization constants N_{nl} are determined by the residuum of (2.69).

2.2.3. Separation of V_2 in Parabolic Coordinates. We insert the potential V_2 into the path integral and obtain $\big(f = a + \frac{b}{4}(\xi^2 + \eta^2)\big)$

$$K^{(V_2)}(\xi'', \xi', \eta'', \eta'; T) = \int\limits_{\xi(t')=\xi'}^{\xi(t'')=\xi''} \mathcal{D}\xi(t) \int\limits_{\eta(t')=\eta'}^{\eta(t'')=\eta''} \mathcal{D}\eta(t) f(\xi, \eta) \times$$

$$\times \exp\left(\frac{i}{\hbar}\int\limits_0^T \left\{ f(\xi, \eta)(\dot\xi^2 + \dot\eta^2) - \frac{1}{f(\xi, \eta)} \times\right.\right.$$

$$\left.\left.\times \left[-\alpha + \frac{\hbar^2}{2m}\left(\frac{k_1^2 - 1/4}{\xi^2} + \frac{k_2^2 - 1/4}{\eta^2}\right)\right]\right\} dt\right) =$$

$$= \int\limits_{-\infty}^{\infty} \frac{dE}{2\pi\hbar}\, \mathrm{e}^{-iET/\hbar} \int\limits_0^{\infty} ds'' \exp\left[\frac{i}{\hbar}\big(aE - \alpha\big)s''\right] K^{(V_2)}(\xi'', \xi', \eta'', \eta'; s''), \tag{2.74}$$

with the time-transformed path integral $K^{(V_2)}(s'')$ given by $(\omega^2 = -bE/2m)$

$$K^{(V_2)}(\xi'', \xi', \eta'', \eta'; s'') = \int\limits_{\xi(0)=\xi'}^{\xi(s'')=\xi''} \mathcal{D}\xi(s) \int\limits_{\eta(0)=\eta'}^{\eta(s'')=\eta''} \mathcal{D}\eta(s) \times$$

$$\times \exp\left\{\frac{i}{\hbar}\int\limits_0^{s''} \left[\frac{m}{2}\left((\dot\xi^2 + \dot\eta^2) - \frac{m}{2}\omega^2(\xi^2 + \eta^2)\right) - \right.\right.$$

$$\left.\left. - \frac{\hbar^2}{2m}\left(\frac{k_1^2 - 1/4}{\xi^2} + \frac{k_2^2 - 1/4}{\eta^2}\right)\right] ds\right\} = \frac{m\omega\sqrt{\xi'\xi''}}{i\hbar\,\sin\,\omega s''} \times$$

$$\times \exp\left[-\frac{m\omega}{i\hbar\,\sin\,\omega s''}({\xi'}^2 + {\xi''}^2 \cot\,\omega s'')\right] I_{k_2}\left(\frac{m\omega\xi'\xi''}{i\hbar\,\sin\,\omega s''}\right) \frac{m\omega\sqrt{\eta'\eta''}}{i\hbar\,\sin\,\omega s''} \times$$

$$\times \exp\left[-\frac{m\omega}{i\hbar\,\sin\,\omega s''}({\eta'}^2 + {\eta''}^2 \cot\,\omega s'')\right] I_{k_1}\left(\frac{m\omega\eta'\eta''}{i\hbar\,\sin\,\omega s''}\right). \tag{2.75}$$

Performing the s'' integration yields the Green function ($\tilde{\mathcal{E}} = aE - \alpha - \mathcal{E}$)

$$
\begin{aligned}
G^{(V_2)}(\xi'', \xi', \eta'', \eta'; E) = \int d\mathcal{E} \sqrt{-\frac{2m}{bE}} \frac{\Gamma\left[\frac{1}{2}(1 + |k_1| - \mathcal{E}\sqrt{-2m/bE}/\hbar)\right]}{\hbar\Gamma(1 + |k_1|)\sqrt{\xi'\xi''}} \times \\
\times W_{\mathcal{E}\sqrt{-2m/bE}/2\hbar, |k_1|/2}\left(\frac{m}{\hbar}\sqrt{-\frac{bE}{2m}}\,\xi_>^2\right) \times \\
\times M_{\mathcal{E}\sqrt{-2m/bE}/2\hbar, |k_1|/2}\left(\frac{m}{\hbar}\sqrt{-\frac{bE}{2m}}\,\xi_<^2\right) \times \\
\times \sqrt{-\frac{2m}{bE}} \frac{\Gamma\left[\frac{1}{2}(1 + |k_2| - \tilde{\mathcal{E}}\sqrt{-2m/bE}/\hbar)\right]}{\hbar\Gamma(1 + |k_2|)\sqrt{\eta'\eta''}} \times \\
\times W_{\tilde{\mathcal{E}}\sqrt{-2m/bE}/2\hbar, |k_2|/2}\left(\frac{m}{\hbar}\sqrt{-\frac{bE}{2m}}\,\eta_>^2\right) \times \\
\times M_{\tilde{\mathcal{E}}\sqrt{-2m/bE}/2\hbar, |k_2|/2}\left(\frac{m}{\hbar}\sqrt{-\frac{bE}{2m}}\,\eta_<^2\right).
\end{aligned}
\tag{2.76}
$$

On the other hand, we insert the expansion of the bound states of the radial harmonic oscillator and obtain for the discrete spectrum contribution of the Green function:

$$
\begin{aligned}
G^{(V_2)}(\xi'', \xi', \eta'', \eta'; E) = \sum_{n\ =0}^{\infty} \sum_{n\ =0}^{\infty} \frac{N_{n\ ,n}^2}{E_{n\ ,n}\ - E} \times \\
\times \Psi_n^{(\mathrm{RHO},|k_1|)}(\xi'')\Psi_n^{(\mathrm{RHO},|k_2|)}(\xi')\Psi_n^{(\mathrm{RHO},|k_2|)}(\eta'')\Psi_n^{(\mathrm{RHO},|k_1|)}(\eta'),
\end{aligned}
\tag{2.77}
$$

where the energy $E_{n\ ,n}$ is determined by the equation

$$
2n_\xi + 2n_\eta + |k_1| + |k_2| + 2 = \frac{aE_{n\ ,n}\ - \alpha}{\hbar}\sqrt{-\frac{2m}{bE_{n\ ,n}}},
\tag{2.78}
$$

which is equivalent with (2.60). The normalization constants $N_{n\ n}$ are determined by the residuum of (2.56), and ω in the $\Psi_n^{(\mathrm{RHO},|k_2|)}\Psi_n^{(\mathrm{RHO},|k_1|)}$ has to be taken on $\omega_{n\ ,n}\ = \sqrt{-bE_{n\ ,n}\ /2m}$.

2.3. The Superintegrable Potential V_3 on D_{III}. First we state the potential V_3 in the respective coordinate systems

$$V_3(u,v) = \frac{1}{a + b\,\mathrm{e}^{-u}}\left[-\alpha + \frac{\hbar^2}{2m} 4\,\mathrm{e}^{u}\Big(c_1^2\,\mathrm{e}^{-iv} - 2c_2\,\mathrm{e}^{-2iv}\Big)\right], \tag{2.79}$$

$$= \frac{1}{a + \frac{b}{4}\varrho^2}\left[-\alpha + \frac{\hbar^2}{2m\varrho^2} 4\Big(c_1^2\,\mathrm{e}^{-2i\varphi} - 2c_2\,\mathrm{e}^{-4i\varphi}\Big)\right], \tag{2.80}$$

$$= \frac{-\alpha(\mu+\nu) + c_1^2\dfrac{\mu+\nu}{\mu\nu} - c_2\dfrac{\mu^2-\nu^2}{\mu^2\nu^2}}{\left(a + \dfrac{b}{2}(\mu-\nu)\right)(\mu+\nu)}. \tag{2.81}$$

In hyperbolic coordinates no closed solution can be obtained due to the involved mixture of linear, quadratic, inverse-linear and inverse-quadratic terms. In polar coordinates the path integral in ϱ turns out to be a path integral for the radial harmonic oscillator. Note that the (u,v) system is equivalent to polar coordinates.

2.3.1. Separation of V_3 in Polar Coordinates. We insert the potential V_3 into the path integral and get $\big(f(\varrho) = a + \frac{b}{4}\varrho^2 = \sqrt{g}\big)$

$$K^{(V_3)}(\varrho'',\varrho',\varphi'',\varphi';T) = \int\limits_{\varrho(t')=\varrho'}^{\varrho(t'')=\varrho''} \mathcal{D}\varrho(t) \int\limits_{\varphi(t')=\varphi'}^{\varphi(t'')=\varphi''} \mathcal{D}\varphi(t) f(\varrho)\varrho \times$$

$$\times \exp\left(\frac{i}{\hbar}\int_0^T \left\{\frac{m}{2} f(\varrho)(\dot\varrho^2 + \varrho^2\dot\varphi^2) - \right.\right.$$

$$\left.\left. - \frac{1}{f(\varrho)}\left[-\alpha + \frac{\hbar^2}{2m\varrho^2} 4c_1^2\left(\mathrm{e}^{-4i\varphi} - 2\frac{c_2}{c_1^2}\,\mathrm{e}^{-2i\varphi} - \frac{1}{4}\right)\right]\right\} dt\right) =$$

$$= \sum_{l=0}^{\infty} \Phi^{(c_1,c_2)}_{[\mathrm{cMP}],l}(\varphi'')\Phi^{(c_1,c_2)}_{[\mathrm{cMP}],l}(\varphi') \frac{1}{[(\varrho'\varrho'')^2 f(\varrho') f(\varrho'')]^{1/4}} \times$$

$$\times \int\limits_{\varrho(t')=\varrho'}^{\varrho(t'')=\varrho''} \mathcal{D}\varrho(t) f^{1/2}(\varrho) \exp\left\{\frac{i}{\hbar}\int\limits_0^T \left[\frac{m}{2} f(\varrho)\dot{\varrho}^2 - \right.\right.$$

$$\left.\left. - \frac{1}{f(\varrho)}\left(-\alpha + \frac{\hbar^2}{2m}\frac{\left(l + \dfrac{2c_2}{c_1} + \dfrac{1}{2}\right)^2 - \dfrac{1}{4}}{\varrho^2}\right)\right] dt\right\} =$$

$$= \frac{1}{\sqrt{\varrho'\varrho''}} \sum_{l=0}^{\infty} \Phi_{[\mathrm{cMP}],l}^{(c_1,c_2)}(\varphi'') \Phi_{[\mathrm{cMP}],l}^{(c_1,c_2)}(\varphi') \times$$

$$\times \int\limits_{-\infty}^{\infty} \frac{dE}{2\pi\hbar} \mathrm{e}^{-iET/\hbar} \int\limits_0^{\infty} ds'' \exp\left[\frac{i}{\hbar}(aE - \alpha)s''\right] K_l^{(V_3)}(\varrho'', \varrho'; s''), \quad (2.82)$$

with the time-transformed path integral $K_l(s'')$ given by

$$K_l^{(V_3)}(\varrho'', \varrho'; s'') =$$

$$= \int\limits_{\varrho(0)=\varrho'}^{\varrho(s'')=\varrho''} \mathcal{D}\varrho(s) \exp\left[\frac{i}{\hbar}\int\limits_0^{s''}\left(\frac{m}{2}\dot{\varrho}^2 + \frac{Eb}{4}\varrho^2 - \frac{\hbar^2}{2m}\frac{\left(l + \dfrac{2c_2}{c_1} + \dfrac{1}{2}\right)^2 - \dfrac{1}{4}}{\varrho^2}\right) ds\right] =$$

$$= \frac{m\omega\sqrt{\varrho'\varrho''}}{i\hbar \sin \omega s''} \exp\left[-\frac{m\omega}{2i\hbar}({\varrho'}^2 + {\varrho''}^2) \cot \omega s''\right] I_{l+\frac{2\ 2}{1}+\frac{1}{2}}\left(\frac{m\omega\varrho'\varrho''}{i\hbar \sin \omega s''}\right). \quad (2.83)$$

By $\Phi_{[\mathrm{cMP}],l}^{(c_1,c_2)}(\varphi)$ we denote the wave functions of the complex periodic Morse potential in the variable φ with spectrum $E_l = \hbar^2 \left(l + 2\dfrac{c_2}{c_1} + \dfrac{1}{2}\right)^2 / 2m$ [1,3,36, 42,50,51], c.f. Appendix C:

$$\Phi_{[\mathrm{cMP}],l}^{(c_1,c_2)}(\varphi) = \frac{\left(4\dfrac{c_2}{c_1} - 2n - 1\right) n!}{\Gamma\left(4\dfrac{c_2}{c_1} - 2n\right)} \left(4\frac{c_2}{c_1}\right)^{4\frac{2}{1}-2n-1} \times$$

$$\times \exp\left[-2i\left(2\frac{c_2}{c_1} - n - \frac{1}{2}\right)\varphi - 2c_1\,\mathrm{e}^{-2i\varphi}\right] L_n^{(4\frac{2}{1}-2n-1)}\left(4c_1\mathrm{e}^{-2i\varphi}\right). \quad (2.84)$$

Performing the s'' integration gives the Green function

$$
\begin{aligned}
G^{(V_3)}(\varrho'',\varrho',\varphi'',\varphi';E) &= \sum_{l=0}^{\infty}\Phi^{(c_1,c_2)}_{[\mathrm{cMP}],l}(\varphi'')\Phi^{(c_1,c_2)}_{[\mathrm{cMP}],l}(\varphi')\times \\
&\times\sqrt{-\frac{2m}{Eb}}\,\frac{\Gamma\left[\frac{1}{2}\left(l+2\frac{c_2}{c_1}+\frac{3}{2}-\frac{1}{\hbar}(aE-\alpha)\sqrt{-2m/bE}\right)\right]}{\Gamma\left(l+2\frac{c_2}{c_1}+\frac{3}{2}\right)\sqrt{\varrho'\varrho''}}\times \\
&\times M_{\frac{-}{2\hbar}\sqrt{-\frac{2}{\ }},\frac{1}{2}\left(l+2\frac{2}{1}+\frac{1}{2}\right)}\left(\frac{m}{\hbar}\sqrt{-\frac{bE}{2m}}\,\varrho^2_<\right)\times \\
&\times M_{\frac{-}{2\hbar}\sqrt{-\frac{2}{\ }},\frac{1}{2}\left(l+2\frac{2}{1}+\frac{1}{2}\right)}\left(\frac{m}{\hbar}\sqrt{-\frac{bE}{2m}}\,\varrho^2_>\right). \qquad (2.85)
\end{aligned}
$$

Inserting the expansion into Laguerre polynomials yields the discrete contribution of the Green function $\left(\lambda = l+\frac{2c_2}{c_1}+\frac{1}{2}\right)$

$$
\begin{aligned}
G^{(V_3)}_{\mathrm{disc}}(\varrho'',\varrho',\varphi'',\varphi';E) &= \frac{1}{\sqrt{\varrho'\varrho''}}\sum_{l=0}^{\infty}\Phi^{(c_1,c_2)}_{[\mathrm{cMP}],l}(\varphi'')\Phi^{(c_1,c_2)}_{[\mathrm{cMP}],l}(\varphi')\times \\
&\times\sum_{n=0}^{\infty}\frac{N^2_{nl}}{E_{nl}-E}\Psi^{(\mathrm{RHO},\lambda)}_n(\varrho'')\Psi^{(\mathrm{RHO},\lambda)}_n(\varrho'), \qquad (2.86)
\end{aligned}
$$

and the normalization constants N_{nl} are determined by the residuum of (2.85). Here, the spectrum E_{nl} is determined by

$$
aE_{nl}-\alpha-\hbar\sqrt{-\frac{bE_{nl}}{2m}}\left(2n+2l+\frac{c_2}{c_1}+1\right), \qquad (2.87)
$$

which is quadratic equation in E_{nl} with solution $\left(N = 2n+2l+\frac{c_2}{c_1}+1\right)$

$$
E_{nl\pm} = \frac{1}{2a^2}\left[-\left(\frac{b\hbar^2}{2m}N^2-2a\alpha\right)\pm\frac{b\hbar^2}{2m}N^2\sqrt{1-\frac{8a\alpha m}{b\hbar^2N^2}}\right]. \qquad (2.88)
$$

In the wave functions $\Psi_n^{(\mathrm{RHO},\lambda)}(\varrho)$ the quantity ω has to be taken on $\omega = \sqrt{-bE_{nl}/2m}$. For large n, l we have

$$E_{nl-} \simeq -\frac{b\hbar^2}{m}(2n+2l+1)^2, \tag{2.89}$$

$$E_{nl+} \simeq -\frac{m\alpha^2}{2b\hbar^2(2n+2l+1)^2}, \tag{2.90}$$

with E_{nl+} showing a Coulomb-like behavior.

2.4. The Superintegrable Potential V_4 on D_{III}.

$$V_4(\mu,\nu) = \frac{1}{\left(a+\frac{b}{2}(\mu-\nu)\right)(\mu+\nu)}\left[d_1\mu + d_2\nu + \frac{m}{2}\omega^2(\mu^2-\nu^2)\right], \tag{2.91}$$

$$= \frac{1}{a+b\,\mathrm{e}^{-u}}\big[2(d_1+d_2)(\cos 2\varphi - \cosh 2\omega) +$$

$$+ 2(d_1-d_2)(2i\,\sin 2\varphi + \sinh 2\omega) + 2d_3(2i\,\sin 2\varphi + \sinh 4\omega)\big]. \tag{2.92}$$

We can evaluate the path integral in hyperbolic coordinates (application of the Morse potential); in elliptic coordinates no closed solution can be found.

2.4.1. Separation of V_4 in Hyperbolic Coordinates. The classical Lagrangian and Hamiltonian have the form

$$\mathcal{L}(\mu,\dot\mu,\nu,\dot\nu) = \frac{m}{2}\left(a+\frac{b}{2}(\mu-\nu)\right)(\mu+\nu)\left(\frac{\dot\mu^2}{\mu^2}-\frac{\dot\nu^2}{\nu^2}\right) - V(\mu,\nu), \tag{2.93}$$

$$\mathcal{H}(\mu,p_\mu,\nu,p_\nu = \frac{1}{2m}\frac{\mu^2p_\mu^2-\nu^2p_\nu^2}{\left(a+\frac{b}{2}(\mu-\nu)\right)(\mu+\nu)} + V(\mu,\nu). \tag{2.94}$$

The canonical momentum operators are given by

$$p_\mu = \frac{\hbar}{i}\left[\frac{\partial}{\partial\mu} + \frac{1}{2}\left(+\frac{1}{\mu+\nu} + \frac{b}{a+\frac{b}{2}(\mu-\nu)} - \frac{1}{\mu}\right)\right], \tag{2.95}$$

$$p_\nu = \frac{\hbar}{i}\left[\frac{\partial}{\partial\mu} + \frac{1}{2}\left(+\frac{1}{\mu+\nu} - \frac{b}{a+\frac{b}{2}(\mu-\nu)} - \frac{1}{\nu}\right)\right], \tag{2.96}$$

and the quantum Hamiltonian has the form

$$H = -\frac{\hbar^2}{2m}\frac{1}{\left(a+\frac{b}{2}(\mu-\nu)\right)(\mu+\nu)}\times$$

$$\times\left[\mu^2\left(\frac{\partial^2}{\partial\mu^2}-\frac{1}{\mu}\frac{\partial}{\partial\mu}\right)-\nu^2\left(\frac{\partial^2}{\partial\nu^2}-\frac{1}{\nu}\frac{\partial}{\partial\nu}\right)\right]+V(\mu,\nu), \tag{2.97}$$

$$=\frac{1}{2m}\left[\frac{\mu}{\sqrt{\left(a+\frac{b}{2}(\mu-\nu)\right)(\mu+\nu)}}p_\mu^2\frac{\mu}{\sqrt{\left(a+\frac{b}{2}(\mu-\nu)\right)(\mu+\nu)}}-\right.$$

$$\left.-\frac{\nu}{\sqrt{\left(a+\frac{b}{2}(\mu-\nu)\right)(\mu+\nu)}}p_\nu^2\frac{\nu}{\sqrt{\left(a+\frac{b}{2}(\mu-\nu)\right)(\mu+\nu)}}\right]+V(\mu,\nu). \tag{2.98}$$

Note that from each coordinate there comes a quantum potential $\Delta V=\hbar^2/8m$, however they are canceling each other due to the minus-sign in the metric in ν.

We insert the potential V_4 into the path integral which has the form $\Big(f(\mu,\nu)=\left(a+\frac{b}{2}(\mu-\nu)\right)(\mu+\nu)\Big)$

$$K^{(V_4)}(\mu'',\mu',\nu'',\nu';T)=\int\limits_{\mu(t')=\mu'}^{\mu(t'')=\mu''}\mathcal{D}\mu(t)\int\limits_{\nu(t')=\nu'}^{\nu(t'')=\nu''}\mathcal{D}\nu(t)\frac{f(\mu,\nu)}{\mu\nu}\times$$

$$\times\exp\left\{\frac{i}{\hbar}\int_0^T\left[\frac{m}{2}f(\mu,\nu)\left(\frac{\dot\mu^2}{\mu^2}-\frac{\dot\nu^2}{\nu^2}\right)-\right.\right.$$

$$\left.\left.-\frac{1}{f(\mu,\nu)}\left(d_1\mu+d_2\nu+\frac{m}{2}\omega^2(\mu^2-\nu^2)\right)\right]dt\right\}=$$

$$=\int_{-\infty}^{\infty}\frac{dE}{2\pi\hbar}\,\mathrm{e}^{-iET/\hbar}\int_0^\infty ds''K^{(V_4)}(\mu'',\mu',\nu'',\nu';s''), \tag{2.99}$$

and the path integral $K^{(V_4)}(s'')$ is given by

$$K^{(V_4)}(\mu'',\mu',\nu'',\nu';s'') = \int\limits_{\mu(0)=\mu'}^{\mu(s'')=\mu''} \mathcal{D}\mu(s) \int\limits_{\nu(0)=\nu'}^{\nu(s'')=\nu''} \mathcal{D}\nu(s)\frac{1}{\mu\nu}\times$$

$$\times \exp\left\{\frac{i}{\hbar}\int\limits_0^{s''}\left[\frac{m}{2}\left(\frac{\dot{\mu}^2}{\mu^2}-\frac{\dot{\nu}^2}{\nu^2}\right)+aE(\mu+\nu)+\frac{1}{2}bE(\mu^2-\nu^2)-\right.\right.$$

$$\left.\left.-\left(d_1\mu+d_2\nu+\frac{m}{2}\omega^2(\mu^2-\nu^2)\right)\right]ds\right\}. \quad (2.100)$$

Each of the last path integrals has a similar form as the one discussed in [14]. One can perform the transformation $\mu = \mathrm{e}^x$, $\nu = \mathrm{e}^y$. Then the path integration in (μ,ν) gives a path integration in (x,y) of the following form:

$$K^{(V_4)}(x'',x',y'',y';s'') = \int\limits_{x(0)=x'}^{x(s'')=x''} \mathcal{D}x(s)\times$$

$$\times \exp\left\{\frac{i}{\hbar}\int\limits_0^{s''}\left[\frac{m}{2}\dot{x}^2-\frac{1}{2}(m\omega^2-bE)\,\mathrm{e}^{2x}-(d_1-aE)\,\mathrm{e}^{x}\right]ds\right\}\int\limits_{y(0)=y'}^{y(s'')=y''}\mathcal{D}y(s)\times$$

$$\times \exp\left\{-\frac{i}{\hbar}\int\limits_0^{s''}\left[\frac{m}{2}\dot{y}^2-\frac{1}{2}(m\omega^2-bE)\,\mathrm{e}^{2y}-(d_2+aE)\,\mathrm{e}^{y}\right]ds\right\}, \quad (2.101)$$

and we find the product of two path integrals for the Morse potential. This can be evaluated now as follows. We introduce the abbreviations

$$V_0^2 = \frac{m}{\hbar^2}(m\omega^2-bE), \quad \alpha_{x,y} = -\frac{d_{1,2}\mp aE}{m\omega^2-bE}. \quad (2.102)$$

We expand each path integral first into the discrete spectrum contribution by means of the known solution of the Morse potential in terms of Laguerre polynomials with the quantum numbers n and l, respectively, and the corresponding energy spectra. The s'' integration gives the energy spectrum

$$E_{n,l} = \frac{m\omega^2}{b} - \frac{m}{4b\hbar^2}\frac{(d_1+d_2)^2}{(n+l+1)^2}, \quad (2.103)$$

together with the wave functions ($N_{n,l}$ is determined by the corresponding residuum)

$$\Psi_{n,l}(x,y) = N_{n,l}\Psi_n^{(\mathrm{MP})}(x)\cdot\Psi_n^{(\mathrm{MP})}(y), \tag{2.104}$$

$$\begin{aligned}\Psi_k^{(\mathrm{MP})}(z) &= \left(\frac{2\alpha_z V_0 - 2k - 1}{k!\Gamma(2\alpha_z V_0 - k)}\right)^{1/2}\times \\ &\times (2V_0)^{\alpha\ \ V_0-k-1/2}\,\mathrm{e}^{(\alpha\ \ V_0-k-1/2)z - V_0\,\mathrm{e}}\ L_k^{(2\alpha\ \ V_0-2k-1)}(2V_0\,\mathrm{e}^z),\end{aligned} \tag{2.105}$$

for $z = x, y$ with $k = n, l$. The continuous spectrum is examined in an analogous way yielding

$$E = \frac{\hbar^2 p^2}{2m}, \tag{2.106}$$

with the wave functions

$$\Psi_{p,\lambda}(x,y) = \Psi_{p,\lambda}^{(\mathrm{MP})}(x)\cdot\Psi_{p,\lambda}^{(\mathrm{MP})}(y), \tag{2.107}$$

$$\begin{aligned}\Psi_{p,\lambda}^{(\mathrm{MP})}(z) &= \left(\frac{p_\pm \sinh 2\pi p_\pm}{2\pi^2 V_0}\right)^{1/2}\times \\ &\times \left|\Gamma\left(ip_\pm - \alpha_z + \frac{1}{2}\right)\right|\,\mathrm{e}^{-z}W_{\alpha\ \ V_0, ip_\pm}(2V_0\,\mathrm{e}^x),\end{aligned} \tag{2.108}$$

with $p_\pm = p \pm \lambda$ for $z = x, y$. The entire Green function has the form

$$\begin{aligned}G(\mu'',\mu',\nu'',\nu';E) &= \sum_{n,l}\frac{\Psi_{n,l}(\mu'',\nu'')\Psi_{n,l}(\mu',\nu')}{E_{n,l}-E} + \\ &+ \int dp\int d\lambda \frac{\Psi_{p,\lambda}(\mu'',\nu'')\Psi_{p,\lambda}^*(\mu',\nu')}{\dfrac{\hbar^2p^2}{2m} - E},\end{aligned} \tag{2.109}$$

together with the replacement $\mu = \mathrm{e}^x$, $\nu = \mathrm{e}^y$. This concludes the discussion.

2.5. The Superintegrable Potential V_5 on $D_{\rm III}$. We display the potential V_5 in the respective coordinate systems

$$V_5(u,v) = \frac{1}{a+b\,\mathrm{e}^{-u}}\frac{\hbar^2 v_0^2}{2m}, \tag{2.110}$$

$$= \frac{1}{a+\dfrac{b}{4}\varrho^2}\frac{\hbar^2 v_0^2}{2m}, \tag{2.111}$$

$$= \frac{1}{a+\dfrac{b}{4}(\xi^2+\eta^2)}\frac{\hbar^2 v_0^2}{2m}, \tag{2.112}$$

$$= \frac{1}{a+\dfrac{b}{4}d^2(\sinh^2\omega+\cos^2\varphi)}\frac{\hbar^2 v_0^2}{2m}, \tag{2.113}$$

$$= \frac{1}{\Big(a+\dfrac{b}{2}(\mu-\nu)\Big)(\mu+\nu)}\frac{\hbar^2 v_0^2}{2m}. \tag{2.114}$$

We discuss the path integral solution of V_5 in some extend, where the case of elliptic coordinates is omitted due to intractability of this system in the path integral. Provided that $b > 0$, there is in the case of the free motion a discrete spectrum

$$E_N = -\frac{\hbar^2}{2m}\frac{b}{a^2}(2N+1)^2, \tag{2.115}$$

with the principal quantum number $N \in \mathbb{N}$.

2.5.1. Separation of V_5 in the (u,v) System. We insert the potential V_5 into the path integral for the (u,v) system and obtain

$$K^{(V_5)}(u'',u',v'',v';T) = \int\limits_{u(t')=u'}^{u(t'')=u''}\mathcal{D}u(t)\int\limits_{v(t')=v'}^{v(t'')=v''}\mathcal{D}v(t)(a\,\mathrm{e}^{-u}+b\,\mathrm{e}^{-2u})\times$$

$$\times\exp\left\{\frac{i}{\hbar}\int\limits_0^T\left[\frac{m}{2}(a\,\mathrm{e}^{-u}+b\,\mathrm{e}^{-2u})(\dot u^2+\dot v^2) - \frac{1}{a+b\,\mathrm{e}^{-u}}\frac{\hbar^2 v_0^2}{2m}\right]dt\right\} =$$

$$= \int\limits_{-\infty}^{\infty}\frac{dE}{2\pi\hbar}\,\mathrm{e}^{-iET/\hbar}\int\limits_0^{\infty}ds''\mathrm{e}^{-i\hbar v_0^2 s''/2m}K^{(V_5)}(u'',u',v'',v';s''), \tag{2.116}$$

with the time-transformed path integral $K^{(V_5)}(s'')$ given by

$$K^{(V_5)}(u'',u',v'',v';s'') = \int\limits_{u(0)=u'}^{u(s'')=u''} \mathcal{D}u(s) \int\limits_{v(0)=v'}^{v(s'')=v''} \mathcal{D}v(s)\times$$

$$\times \exp\left(\frac{i}{\hbar}\int\limits_0^{s''}\left\{\frac{m}{2}(\dot{u}^2+\dot{v}^2)+Eb\left[\mathrm{e}^{-2u}+\left(\frac{aE-\hbar^2v_0^2/2m}{Eb}\right)\mathrm{e}^{-u}\right]\right\}ds\right) =$$

$$= \sum_{l=0}^{\infty}\frac{\mathrm{e}^{il(v''-v')}}{2\pi}\,\mathrm{e}^{-i\hbar l^2 s''/2m}\int\limits_{u(0)=u'}^{u(s'')=u''}\mathcal{D}u(s)\times$$

$$\times \exp\left(\frac{i}{\hbar}\int\limits_0^{s''}\left\{\frac{m}{2}\dot{u}^2+Eb\left[\mathrm{e}^{-2u}+\left(\frac{aE-\hbar^2v_0^2/2m}{Eb}\right)\mathrm{e}^{-u}\right]\right\}ds\right). \quad (2.117)$$

The path integral in u is a path integral for the Morse potential. Performing the s'' integration gives, c.f. [14], the Green function as follows ($\mathcal{E} = [Ea - (\hbar^2 v_0^2/2m)]\sqrt{-2m/bE}/2\hbar$):

$$G^{(V_5)}(u'',u',v'',v';E) = \sum_{l=-\infty}^{\infty}\frac{\mathrm{e}^{il(v''-v')}}{2\pi}\,\frac{m\Gamma\left(\frac{1}{2}+l-\mathcal{E}\right)}{\hbar\sqrt{-2mbE}\,\Gamma(1+2l)}\,\mathrm{e}^{(u'+u'')/2}\times$$

$$\times W_{\mathcal{E},l}\left(\frac{\sqrt{-8mbE}}{\hbar}\,\mathrm{e}^{-u}\right)M_{\mathcal{E},l}\left(\frac{\sqrt{-8mbE}}{\hbar}\,\mathrm{e}^{-u}\right). \quad (2.118)$$

The corresponding continuous part of the Green function is evaluated as [14]

$$G^{(V_5)}_{\mathrm{cont}}(u'',u',v'',v';E) = \sum_{l=-\infty}^{\infty}\frac{\mathrm{e}^{il(v''-v')}}{2\pi}\mathrm{e}^{(u'+u'')/2}\times$$

$$\times\int\limits_0^{\infty}\frac{\mathrm{e}^{\pi p/2}dp}{\dfrac{\hbar^2p^2}{2m}-E}\,\frac{\left|\Gamma\left(\frac{1}{2}+l+ip\right)\right|^2}{2\pi\Gamma^2(1+2l)}M_{ip/2,l}\Big(-2ip\,\mathrm{e}^{-u'}\Big)M_{-ip/2,l}\Big(2ip\,\mathrm{e}^{-u''}\Big). \quad (2.119)$$

In addition, we have a discrete spectrum. This is found by analyzing the poles of the Green function (2.118):

$$\frac{1}{2}+l-\frac{aE_{nl}-\dfrac{\hbar^2v_0^2}{2m}}{2\hbar}\sqrt{-\frac{2m}{bE_{nl}}} = -n. \quad (2.120)$$

In the case of $v_0 = 0$ this simplifies to

$$n + l + \frac{1}{2} - \frac{a}{2\hbar}\sqrt{-\frac{2m}{bE_{nl}}} = 0, \tag{2.121}$$

with the solution

$$E_{nl} = -\frac{\hbar^2}{2m}\frac{b}{a^2}(2n + 2l + 1)^2 \tag{2.122}$$

yielding for $b > 0$ an infinite number of bound states. For $v_0 \neq 0$, the equation for E_{nl} is a quadratic equation in E with solution

$$E_{nl\pm} = -\frac{\hbar^2}{2m}\frac{1}{2a^2}\times \\ \times\left[b(2n+2l+1)^2 - 2av_0^2 \pm b(2n+2l+1)^2\sqrt{1 - \frac{4av_0^2}{b(2n+2l+1)^2}}\right], \tag{2.123}$$

$$E_{nl+} \overset{(n,l)\to\infty}{=} -\frac{\hbar^2}{2m}\frac{b}{a^2}\left[(2n + 2l + 1)^2 - 2\frac{a}{b}v_0^2\right], \tag{2.124}$$

$$E_{nl-} \overset{(n,l)\to\infty}{=} -\frac{\hbar^2}{2bm}\frac{v_0^4}{(2n + 2l + 1)^2}. \tag{2.125}$$

For $v_0 = 0$, there is only E_{nl+}. For $(2n+2l+1)^2 < 4av_0^2/b$, there are semibound states located approximately around $E_0 = -\hbar^2 v_0^2/2ma$.

Therefore we have for the discrete spectrum contribution

$$G^{(V_5)}_{\text{disc}}(u'', u', v'', v'; E) = \\ = \sum_{l=-\infty}^{\infty}\frac{e^{il(v''-v')}}{2\pi}\sum_{n=0}^{\infty}\frac{1}{E_{nl} - E}\Psi^{(V_5)}_{nl}(u'')\Psi^{(V_5)}_{nl}(u'), \tag{2.126}$$

with the functions $\Psi^{(V_5)}_{nl}(u)$ given by ($\mathcal{E}$ as in (2.118))

$$\Psi^{(V_5)}_{nl}(u) = N_{nl}\frac{(2\mathcal{E} - 2n - 1)n!}{\Gamma(2\mathcal{E} - n)}\left(\frac{\sqrt{-8mbE_{nl}}}{\hbar}\right)^{\mathcal{E}-n-1/2}\times \\ \times\exp\left[\left(\mathcal{E} - n - \frac{1}{2}\right)u - \sqrt{-\frac{8mbE_{nl}}{\hbar}}\,e^u\right]\times \\ \times L_n^{(2\mathcal{E}-2n-1)}\left(\frac{\sqrt{-8mbE_{nl}}}{\hbar}\,e^u\right). \tag{2.127}$$

The constant N_{nl} is determined by taking the Green function at the residuum E_{nl}. The wave functions vanish for $u \to \infty$ due to $\mathrm{e}^{-\sqrt{-8mbE}\ \mathrm{e}\ /\hbar} = \mathrm{e}^{-2b\hbar(2n+2l+1)\mathrm{e}\ /a} \to 0$, provided $b/a > 0$ for all $n \in \mathbb{N}$, which shows that the discrete spectrum is indeed infinite. The feature that an homogeneous space with curvature has at the same time a discrete and a continuous spectrum is already known from the path integration on the $SU(1,1)$ group manifold [22]. Actually, this property allows the analysis of the modified Pöschl–Teller potential with its continuous and (finite) discrete spectrum.

2.5.2. Separation of V_5 in Polar Coordinates. We insert the potential V_5 into the path integral in polar coordinates and obtain

$$K^{(V_5)}(\varrho'',\varrho',\varphi'',\varphi';T) = \int\limits_{\varrho(t')=\varrho'}^{\varrho(t'')=\varrho''} \mathcal{D}\varrho(t) \int\limits_{\varphi(t')=\varphi'}^{\varphi(t'')=\varphi''} \mathcal{D}\varphi(t)\left(a+\frac{b}{4}\varrho^2\right)\varrho\times$$
$$\times\exp\left\{\frac{i}{\hbar}\int\limits_0^T\left[\frac{m}{2}\left(a+\frac{b}{4}\varrho^2\right)(\dot\varrho^2+\varrho^2\dot\varphi^2)+\left(a+\frac{b}{4}\varrho^2\right)^{-1}\frac{\hbar^2}{2m}\left(v_0^2+\frac{1}{4\varrho^2}\right)\right]dt\right\}=$$
$$=\int\limits_{-\infty}^{\infty}\frac{dE}{2\pi\hbar}\,\mathrm{e}^{-iET/\hbar}G^{(V_5)}(\varrho'',\varrho',\varphi'',\varphi';E), \quad (2.128)$$

and the Green function is evaluated to have the form [14] $\left(\mathcal{E}=\frac{(aE-\hbar^2v_0^2/2m)}{\hbar\omega},\ \omega^2=-bE/2m\right)$

$$G^{(V_5)}(\varrho'',\varrho',\varphi'',\varphi';E)\sum_{l=-\infty}^{\infty}\frac{\mathrm{e}^{il(\varphi''-\varphi')}}{2\pi}\frac{1}{\varrho'\varrho''}\sqrt{-\frac{2m}{2E}}\,\frac{\Gamma\left[\frac{1}{2}(1+l-\mathcal{E})\right]}{\Gamma(1+l)}\times$$
$$\times W_{\mathcal{E}/2,\frac{}{2}}\left(\sqrt{-\frac{2mbE}{\hbar^2}}\,\varrho_>\right)M_{\mathcal{E}/2,\frac{}{2}}\left(\sqrt{-\frac{2mbE}{\hbar^2}}\,\varrho_<\right). \quad (2.129)$$

The Green function has poles which are determined by

$$2n+l+1-\frac{1}{\hbar}\left(aE_{nl}-\frac{v_0^2\hbar^2}{2m}\right)\sqrt{-\frac{2m}{Eb_{nl}}}=0. \quad (2.130)$$

In the case of $v_0=0$ this simplifies to

$$(2n+l+1)-\frac{a}{\hbar}\sqrt{-\frac{2m}{E_{nl}b}}=0, \quad (2.131)$$

with the solution

$$E_{nl} = -\frac{\hbar^2}{2m}\frac{b}{a^2}(2n+l+1)^2 \tag{2.132}$$

yielding for $b > 0$ an infinite number of bound states. For $v_0 \neq 0$ the equation for E_{nl} is a quadratic equation in E with solution

$$E_{nl\pm} = -\frac{\hbar^2}{2m}\frac{1}{2a^2}\times$$
$$\times\left[b(2n+l+1)^2 - 2av_0^2 \pm b(2n+l+1)^2\sqrt{1-\frac{4av_0^2}{b(2n+l+1)^2}}\right]. \tag{2.133}$$

The limit of $N, l \to \infty$ yields

$$E_{nl+} \simeq -\frac{\hbar^2}{2m}\left[\frac{b}{a^2}(2n+l+1)^2 + \frac{v_0^2}{a}\right], \tag{2.134}$$

$$E_{nl-} \simeq -\frac{\hbar^2}{2m}\frac{v_0^2}{4b(2n+l+1)^2}, \tag{2.135}$$

and E_{nl+} corresponds in this limit to the spectrum of the free motion.

2.5.3. Separation of V_5 in Parabolic Coordinates. We insert the potential V_5 into the path integral in parabolic coordinates and obtain

$$K^{(V_5)}(\xi'',\xi',\eta'',\eta';T) = \int\limits_{\xi(t')=\xi'}^{\xi(t'')=\xi''}\mathcal{D}\xi(t)\int\limits_{\eta(t')=\eta'}^{\eta(t'')=\eta''}\mathcal{D}\eta(t)\left(a+\frac{b}{4}(\xi^2+\eta^2)\right)\times$$
$$\times\exp\left\{\frac{i}{\hbar}\int\limits_0^T\left[\frac{m}{2}\left(a+\frac{b}{4}(\xi^2+\eta^2)\right)(\dot\xi^2+\dot\eta^2) - \frac{1}{a+\frac{b}{4}(\xi^2+\eta^2)}\frac{\hbar^2v_0^2}{2m}\right]dt\right\} =$$
$$= \int\limits_{-\infty}^{\infty}\frac{dE}{2\pi\hbar}\,\mathrm{e}^{-iET/\hbar}G^{(V_5)}(\xi'',\xi',\eta'',\eta';E), \tag{2.136}$$

with the time-transformed path integral $K(s'')$ given by

$$K^{(V_5)}(\xi'',\xi',\eta'',\eta';s'') = \int\limits_{\xi(0)=\xi'}^{\xi(s'')=\xi''}\mathcal{D}\xi(s)\int\limits_{\eta(0)=\eta'}^{\eta(s'')=\eta''}\mathcal{D}\eta(s)\times$$
$$\times\exp\left\{\frac{i}{\hbar}\int\limits_0^{s''}\left[\frac{m}{2}(\dot\xi^2+\dot\eta^2)+E\frac{b}{4}(\xi^2+\eta^2)\right]ds+\frac{i}{\hbar}\left(aE-\frac{\hbar^2v_0^2}{2m}\right)ds\right\}. \tag{2.137}$$

The only difference in comparison with the result in [14] is the the additional $\dfrac{\hbar^2 v_0^2}{2m}$ term in the s'' integration. In order to find the discrete spectrum we insert the solution for the harmonic oscillator and get

$$G^{(V_5)}_{\rm disc}(\xi'',\xi',\eta'',\eta';E) =$$
$$= \sum_{n\ =0}^{\infty}\sum_{n\ =0}^{\infty} \frac{N^2_{n\ n}}{E_{n\ n}\ -E}\Psi_n^{(\rm HO)}(\xi'')\Psi_n^{(\rm HO)}(\xi'')\Psi_n^{(\rm HO)}(\eta'')\Psi_n^{(\rm HO)}(\eta''), \quad (2.138)$$

where $E_{n\ n}$ is determined by the equation

$$(n_\xi + n_\eta + 1) - \frac{1}{\hbar}\left(aE - \frac{\hbar^2 v_0^2}{2m}\right)\sqrt{-\frac{bE}{2m}} = 0, \quad (2.139)$$

which is (up to a different counting in the quantum numbers) identical with (2.131). The normalization $N_{n\ n}$ is determined by the residuum in $G^{(V_5)}(E)$. We do not state the continuous spectrum part, it can be derived from [14] by the replacement $aE \to aE - \hbar^2 v_0^2/2m$.

2.5.4. Separation of V_5 in Hyperbolic Coordinates. We insert the potential V_5 into the path integral in hyperbolic coordinates and obtain: The path integral has the form

$$K^{(V_5)}(\mu'',\mu',\nu'',\nu';T) =$$
$$= \int\limits_{\mu(t')=\mu'}^{\mu(t'')=\mu''}\mathcal{D}\mu(t)\int\limits_{\nu(t')=\nu'}^{\nu(t'')=\nu''}\mathcal{D}\nu(t)\frac{\left(a+\dfrac{b}{2}(\mu-\nu)\right)(\mu+\nu)}{\mu\nu}\times$$
$$\times \exp\left\{\frac{i}{\hbar}\int\limits_0^T\left[\left(a+\frac{b}{2}(\mu-\nu)\right)(\mu+\nu)\left(\frac{\dot\mu^2}{\mu^2}-\frac{\dot\nu^2}{\nu^2}\right)-\right.\right.$$
$$\left.\left.-\frac{1}{\left(a+\dfrac{b}{2}(\mu-\nu)\right)(\mu+\nu)}\frac{\hbar^2 v_0^2}{2m}\right]dt\right\} =$$
$$= \int\limits_{-\infty}^{\infty}\frac{dE}{2\pi\hbar}\,\mathrm{e}^{-iET/\hbar}\int\limits_0^{\infty}ds''K^{(V_5)}(\mu'',\mu',\nu'',\nu';s''), \quad (2.140)$$

and the path integral $K^{(V_5)}(s'')$ is given by

$$K^{(V_5)}(\mu'',\mu',\nu'',\nu';s'') = \int\limits_{\mu(0)=\mu'}^{\mu(s'')=\mu''} \mathcal{D}\mu(s) \int\limits_{\nu(0)=\nu'}^{\nu(s'')=\nu''} \mathcal{D}\nu(s)\frac{1}{\mu\nu}\times$$
$$\times \exp\left\{\frac{i}{\hbar}\int\limits_0^{s''}\left[\frac{m}{2}\left(\frac{\dot{\mu}^2}{\mu^2}-\frac{\dot{\nu}^2}{\nu^2}\right)+(\mu+\nu)\left(aE-\frac{\hbar^2 v_0^2}{2m}\right)+\right.\right.$$
$$\left.\left.+\frac{1}{2}bE(\mu^2-\nu^2)\right]ds\right\}. \quad (2.141)$$

Each of the last path integrals has a similar form as the one discussed in [11]. One can perform the transformation $\mu = \mathrm{e}^x$, $\nu = \mathrm{e}^y$ yielding

$$K^{(V_5)}(x'',x',y'',y';s'') =$$
$$= \int\limits_{x(0)=x'}^{x(s'')=x''} \mathcal{D}x(s) \exp\left\{\frac{i}{\hbar}\int\limits_0^{s''}\left[\frac{m}{2}\dot{x}^2+\left(E\frac{b}{2}\,\mathrm{e}^{2x}+\left(aE-\frac{\hbar^2 v_0^2}{2m}\right)\mathrm{e}^x\right)\right]ds\right\}\times$$
$$\times \int\limits_{y(0)=y'}^{y(s'')=y''} \mathcal{D}y(s) \exp\left\{-\frac{i}{\hbar}\int\limits_0^{s''}\left[\frac{m}{2}\dot{y}^2+\left(E\frac{b}{2}\,\mathrm{e}^{2y}-\left(aE-\frac{\hbar^2 v_0^2}{2m}\right)\mathrm{e}^y\right)\right]ds\right\}$$
$$(2.142)$$

and we find the product of two path integrals for the Morse potential, however more complicated as in [14]. The continuous part of the spectrum can be analyzed similarly as in [14] yielding products of M-Whittaker functions. Analyzing the discrete spectrum contribution from the Morse potential we find the quantization condition

$$(n_\xi+n_\eta+1)-\frac{1}{\hbar}\left(aE-\frac{\hbar^2 v_0^2}{2m}\right)\sqrt{-\frac{4m}{E_{nl}b}}=0, \qquad (2.143)$$

which is up to a different counting in the quantum numbers equivalent with (2.131). This concludes the discussion.

3. SUPERINTEGRABLE POTENTIALS ON DARBOUX SPACE D_{IV}

Finally, we consider the Darboux space D_{IV}. We have the coordinate systems:

$$\text{((u,v) system:)}\quad x = v + iu, \quad y = v - iu \quad (u \in (0,\pi/2), v \in \mathbb{R}), \tag{3.1}$$

$$\text{(Equidistant:)}\quad u = \arctan(\mathrm{e}^{\alpha}), \quad v = \frac{\beta}{2} \quad (\alpha \in \mathbb{R}, \beta \in \mathbb{R}), \tag{3.2}$$

$$\text{(Horospherical:)}\quad x = \log\frac{\mu - i\nu}{2}, \quad y = \log\frac{\mu + i\nu}{2} \quad (\mu, \nu > 0), \tag{3.3}$$

$$\mu = 2\mathrm{e}^{v}\cos u, \quad \nu = -2\mathrm{e}^{v}\sin u, \tag{3.4}$$

$$\text{(Elliptic:)}\quad \mu = d\cosh\omega\cos\varphi, \quad \nu = d\sinh\omega\sin\varphi$$
$$(\omega > 0, \varphi \in (0,\pi/2)). \tag{3.5}$$

We obtain the following forms of the line-element ($a > 2b$, $a_\pm = (a \pm 2b)/4$):

$$\begin{aligned} ds^2 &= \frac{2b\cos u + a}{4\sin^2 u}(du^2 + dv^2) = \\ &= \left(\frac{a_+}{\sin^2 u} + \frac{a_-}{\cos^2 u}\right)(du^2 + dv^2) \\ &\quad \text{(rescaling } u/2 \to u:), \end{aligned} \tag{3.6}$$

$$\text{(Equidistant:)} = \frac{a - 2b\tanh\alpha}{4}(d\alpha^2 + \cosh^2\alpha\, d\beta^2), \tag{3.7}$$

$$\text{(Horospherical:)} = \left(\frac{a_+}{\nu^2} + \frac{a_-}{\mu^2}\right)(d\mu^2 + d\nu^2), \tag{3.8}$$

$$\begin{aligned} \text{(Elliptic:)} &= \left(\frac{a_-}{\cosh^2\omega\cos^2\varphi} + \frac{a_+}{\sinh^2\omega\sin^2\varphi}\right)\times \\ &\quad \times(\cosh^2\omega - \cos^2\varphi)(d\omega^2 + d\varphi^2), \\ &= \left(\frac{a_+}{\sin^2\varphi} + \frac{a_-}{\cos^2\varphi} + \frac{a_+}{\sinh^2\omega} - \frac{a_-}{\cosh^2\omega}\right)\times \\ &\quad \times(d\omega^2 + d\varphi^2), \end{aligned} \tag{3.9}$$

$$\begin{aligned} \text{(Degenerate elliptic I:)} &= \left[a_-\left(\frac{1}{\sinh^2\hat\omega} + \frac{1}{\sin^2\hat\varphi}\right) - a_+\left(\frac{1}{\cosh^2\hat\omega} - \frac{1}{\cos^2\hat\varphi}\right)\right]\times \\ &\quad \times(d\hat\omega^2 + d\hat\varphi^2) \quad (\gamma = 1), \end{aligned} \tag{3.10}$$

$$\text{(Degenerate elliptic II:)} = \frac{1}{4}\left(\frac{a_-}{\sinh^2\tilde\omega} + \frac{a_+}{\sin^2\tilde\varphi}\right)(d\tilde\omega^2 + d\tilde\varphi^2)\ (\gamma = 2). \tag{3.11}$$

We observe that the diagonal term in the metric corresponds in most cases to a combination of a Pöschl–Teller potential and a modified Pöschl–Teller, respectively. In particular, the (u, v) and the equidistant systems are the same, they

just differ in the parameterization. The limiting cases $a = 2b$ and $b = 0$ give particular cases for the metric on the two-dimensional hyperboloid. We have also displayed two versions of degenerate elliptic coordinates. They come from the observation that for the representatives

$$K^2, \quad X_2, \quad \gamma X_2 + K2, \quad X_1 + X_2 + \gamma K^2 \tag{3.12}$$

one can distinguish the cases $\gamma = 0$, $\gamma = 2$, and $\gamma \neq 0, 2$. For $\gamma \neq 0, 2$, one has coordinate systems which can be explicitly formulated in terms of the elliptic functions $\mathrm{sn}(\alpha, k), \mathrm{cn}(\beta, k)$, and only for a special choice of the parameter k they can be simplified in trigonometric and hyperbolic functions. Then the line element has the form

$$ds^2 = \frac{1}{4}[a_+ k^4 \mathrm{sn}^2(\alpha, k) - \mathrm{sn}^2(\beta, k) + k^2 a_-](d\alpha^2 + d\beta^2), \tag{3.13}$$

and separated equations are versions of Lame's equation, if we assume an Ansatz of the form $\Psi = A(\alpha)B(\beta)$ [28]:

$$\frac{\partial^2 A(\alpha)}{\partial \alpha^2} + \left(-\frac{1}{4}k^4 E a_+ \mathrm{sn}^2(\alpha, k) - \lambda_1\right) A(\alpha) = 0, \tag{3.14}$$

$$\frac{\partial^2 B(\beta)}{\partial \beta^2} + \left(-\frac{1}{4}k^4 E a_+ \mathrm{sn}^2(\beta, k) - \lambda_2\right) B(\beta) = 0, \tag{3.15}$$

where $\lambda_1 - \lambda_2 = -Ea_- k^2/4$. k denotes the modulus of the elliptic functions.

In particular, for the potential V_2 one has the possibilities of taking $\gamma = 0$, and $\gamma = 2$. For $\gamma = 0$, the modulus k of the elliptic functions equals $k = -i$. We do not treat V_2 in these elliptic coordinates, but only the degenerate case of $\gamma = 2$.

For the potential V_3, however, the elliptic systems with $\gamma = 1$ can be explicitly worked out. We have stated the respective line elements for these two cases. Note that for $\gamma = 2$ the coordinate transformation can be put into

$$x = \ln\Big[\tan(\tilde\varphi - i\tilde\omega)\Big], \quad y = \ln\Big[\tan(\tilde\varphi + i\tilde\omega)\Big] \quad (\tilde\omega > 0, \tilde\varphi \in (0, \pi/4)). \tag{3.16}$$

We do not dwell into a discussion of elliptic systems any further, for details we refer to [26]. Let us finally note that the notion *elliptic* is also used for the (ω, φ) system, and they must not be confused with the general elliptic coordinates just discussed.

Because we have not worked out the path integral for the free motion in these two further coordinate systems, this will be done in an appendix. For the

Gaussian curvature we obtain, e.g., in the (u, v) system

$$G = -\frac{\dfrac{a_+^2}{\sin^6 u} + \dfrac{a_-^2}{\cos^6 u} + \dfrac{a_- a_+}{\sin^4 u \cos^4 u}}{\left(\dfrac{a_+}{\sin^2 u} + \dfrac{a_-}{\cos^2 u}\right)^3}. \tag{3.17}$$

The case $a = 2b$ yields $a_- = 0$, and

$$G = -\frac{1}{b}, \tag{3.18}$$

and therefore again a space of constant curvature, the hyperboloid $\Lambda^{(2)}$ is given for $b > 0$. We have set the sign in the metric (1.4) in such a way that from $a = 2b > 0$ the hyperboloid $\Lambda^{(2)}$ emerges. We could also choose the metric (1.4) with the opposite sign, then $a = 2b < 0$ would give the same result. In the following it is understood that we make this restriction of positive definiteness of the metric and we do not dwell into the problem of continuation into nonpositive definiteness. Because the (u, v) coordinates and the equidistant system are the same, we do not evaluate the path integral in the equidistant system. In the following we assume $a_+ > 0$ and $a_+ > a_-$.

We introduce the following three constants of motion on D_{IV}:

$$X_1 = \mathrm{e}^{2v}(-\tilde{\mathcal{H}}_0 + \cos 2u \cdot p_u^2 + \sin 2u \cdot p_u p_v), \tag{3.19}$$

$$X_2 = \mathrm{e}^{2v}(-\tilde{\mathcal{H}}_0 + \cos 2u \cdot p_u^2 - \sin 2u \cdot p_u p_v), \tag{3.20}$$

$$K = p_v. \tag{3.21}$$

These integrals of motion satisfy the Poisson relations

$$\{K, X_1\} = 2X_1, \quad \{K, X_2\} = -2X_2, \quad \{X_1, X_2\} = -K^3 - 4aKH_0, \tag{3.22}$$

and satisfy the relation

$$X_1 X_2 - K^4 - aK^2 H_0 - H_0^2 = 0. \tag{3.23}$$

The corresponding quantum operators have the form

$$\widehat{H}_0 = \frac{\sin^2 2u}{2\cos 2u + a}(\partial_u^2 + \partial_v^2), \tag{3.24}$$

$$\widehat{X}_1 = \mathrm{e}^{2v}(-\widehat{H}_0 + \cos 2u \cdot (\partial_u^2 + \partial_v) + \sin 2u \cdot (\partial_u \partial_v + \partial_u), \tag{3.25}$$

$$\widehat{X}_2 = \mathrm{e}^{2v}(-\widehat{H}_0 + \cos 2u \cdot (\partial_u^2 - \partial_v) - \sin 2u \cdot (\partial_u \partial_v - \partial_u), \tag{3.26}$$

and the commutation relations read

$$[\widehat{K}, \widehat{X}_1] = 2\widehat{X}_1,\ [\widehat{K}, \widehat{X}_2] = -2\widehat{X}_2,\ [\widehat{X}_1, \widehat{X}_2] = -8\widehat{K}^3 - 4a\widehat{K}\widehat{H}_0 - 4\widehat{K} \tag{3.27}$$

and satisfy the operator relation

$$\frac{1}{2}\{\widehat{X}_1, \widehat{X}_2\} - \widehat{K}^4 - a\widehat{H}_0\widehat{K}^2 - 5\widehat{K}^2 - \widehat{H}_0^2 - a\widehat{H}_0 = 0. \qquad (3.28)$$

In Table 3 we list the connection with these operators and the corresponding coordinate systems on D_{IV}.

Table 3. **Constants of motion and limiting cases of coordinate systems on D_{IV}**

Metric	Constants of motion	D_{IV}	$\Lambda^{(2)}$ $(a = 2b)$	$\Lambda^{(2)}$ $(b = 0)$
$\frac{2b\cos\,u + a}{4\sin^2 u}(du^2 + dv^2)$	K^2	(u, v) system	Equidistant	Equidistant
$\left(\frac{a_+}{\nu^2} + \frac{a_-}{\mu^2}\right)(d\mu^2 + d\nu^2)$	X_2	Horospherical	Horicyclic	Semicircular parabolic
$\left(\frac{a_-}{\cosh^2\omega\cos^2\varphi} + \frac{a_+}{\sinh^2\omega\sin^2\varphi}\right)\times$ $\times(\cosh^2\omega - \cos^2\varphi)(d\omega^2 + d\varphi^2)$	$K^2 + d^2 X_2$	Elliptic	Elliptic-parabolic	Hyperbolic-parabolic
$\Big[a_+ k^2\Big(\mathrm{sn}^2(\alpha, k) - \mathrm{sn}^2(\beta, k)\Big) + a_-\Big]\times$ $\times\frac{k^2}{4}(d^2\alpha + d^2\beta)$	$X_1 + X_2 + \gamma K^2$	Elliptic	Elliptic	Elliptic

We state the superintegrable potentials on D_{IV}:

$$V_1(u, v) = \left(\frac{a_+}{\sin^2 u} + \frac{a_-}{\cos^2 u}\right)^{-1}\times \\ \times\left[\frac{\hbar^2}{2m}\left(\frac{k^2 - 1/4}{\cos^2 u} + \frac{k^2 - 1/4}{\sin^2 u}\right) - 4\alpha \mathrm{e}^{2v} + 8m\omega^2\mathrm{e}^{4v}\right], \qquad (3.29)$$

$$V_2(u, v) = \left(\frac{a_+}{\sin^2 u} + \frac{a_-}{\cos^2 u}\right)^{-1}\times \\ \times\left[\frac{\hbar^2}{2m}\left(\frac{k_1^2 - 1/4}{\sinh^2 v} - \frac{k_2^2 - 1/4}{\cosh^2 v}\right) - \frac{\alpha}{4}\left(\frac{1}{\sin^2 u} + \frac{1}{\cos^2 u}\right)\right], \qquad (3.30)$$

$$V_3(\tilde{\omega}, \tilde{\varphi}) = \frac{\hbar^2}{2m}\left(\frac{a_+}{\sinh^2\tilde{\omega}} - \frac{a_+}{\cosh^2\tilde{\omega}} + \frac{a_-}{\sin^2\tilde{\varphi}} + \frac{a_-}{\cos^2\tilde{\varphi}}\right)^{-1}\times \\ \times\left[\frac{c_3}{\sin^2\tilde{\varphi}} + \frac{c_2}{\cos^2\tilde{\varphi}} - \frac{c_3}{\sinh^2\tilde{\omega}} + \frac{c_2}{\cosh^2\tilde{\omega}}\right], \qquad (3.31)$$

$$V_4(\mu, \nu) = \left(\frac{a_+}{\nu^2} + \frac{a_-}{\mu^2}\right)^{-1}\frac{\hbar^2}{2m}\left(k_0^2 - \frac{1}{4}\right)\left(\frac{1}{\mu^2} + \frac{1}{\nu^2}\right). \qquad (3.32)$$

Table 4. **Separation of variables for the superintegrable potentials on D_{IV}**

Potential	Constants of motion	Separating coordinate system
V_1	$R_1 = K^2 - \alpha(\mu^2+\nu^2) + \frac{m}{2}\omega^2(\mu^2+\nu^2)$ $R_2 = X_2 + \dfrac{-2\alpha(a_+\mu^2 - a_-\nu^2) + 8(k^2-1/4)\frac{\hbar^2}{m} + 2m\omega^2(a_+\mu^4 - a_-\nu^4)}{a_+\mu^2 + a_-\nu^2}$	(u,v) system Horospherical Elliptic
V_2	$R_1 = X_1 + X_2 + (2\cos u + a)^{-1}\frac{\hbar^2}{2m}\Big[\Big(k_1^2 + k_2^2 - \frac{1}{2}\Big) -$ $-2\Big(k_3^2 - \frac{1}{2}\Big)\cosh 2v + (\cos 4u + 2a\cos 2u + 3)\Big(\frac{k_1^2 - 1/4}{\sinh^2 v} - \frac{k_2^2 - 1/4}{\cosh^2 v}\Big)\Big]$ $R_2 = K^2 + \frac{\hbar^2}{2m}\Big(\frac{k_1^2-1/4}{\sinh^2 v} + \frac{k_2^2-1/4}{\cosh^2 v}\Big)$	(u,v) system Degenerate elliptic I
V_3	$R_1 = X_1 + X_2 + 2K^2 + aH + \frac{\hbar^2}{2m}\Big(\frac{a_+}{\sinh^2 2\tilde\omega} + \frac{a_-}{\sinh^2\tilde\omega}\Big)^{-1}\times$ $\times\Big[\frac{a_+}{\sinh^2 2\tilde\omega}\Big(\frac{c_3}{\sin^2\tilde\varphi} + \frac{c_1}{\sin^2\tilde\varphi}\Big) + \frac{a_-}{\sinh^2 2\tilde\varphi}\Big(\frac{c_3}{\sinh^2\tilde\omega} - \frac{c_2}{\cos^2\tilde\omega}\Big)\Big]$ $R_2 = X_1 - X_2 + \frac{\hbar^2}{2m}\Big(\frac{a_+}{\sinh^2 2\tilde\omega} + \frac{a_-}{\sinh^2\tilde\omega}\Big)^{-1}\times$ $\times\Big[\frac{a_+}{\sinh^2 2\tilde\omega}\Big(c_1\cosh 2\tilde\omega\tan^2\tilde\varphi - c_2\cos 2\tilde\varphi -$ $-\frac{c_3(2\cos^2\tilde\varphi(\sinh^2\tilde\omega - \sin^2\tilde\varphi)+1)}{\sin^2\tilde\varphi}\Big) +$ $+\frac{a_-}{\sin^2 2\tilde\varphi}\Big(c_2\cos 2\tilde\varphi\tanh^2\tilde\omega + c_1\cosh 2\tilde\omega -$ $-\frac{c_3(2\cosh^2\tilde\omega(\sinh^2\tilde\omega - \sin^2\tilde\varphi)+1}{\sinh^2\tilde\omega}\Big)\Big]$	Degenerate elliptic I & II
V_4	$R_1 = X_1 + \dfrac{2\frac{\hbar^2}{m}(k_0^2 - 1/4)(\mu^2+\nu^2)}{a_+\mu^2 + a_-\nu^2}$ $R_2 = X_2 + \dfrac{32\frac{\hbar^2}{m}(k_0^2 - 1/4)}{a_+\mu^2 + a_-\nu^2}$ $R_3 = \mu p \quad + \nu p$	(u,v) system Horospherical Elliptic

In Table 4 we list the properties of these potentials on D_{IV}. We see that V_4 is a special case, and it has three integrals of motion. The variables $\tilde\omega, \tilde\varphi$ are defined by

$$x = \log\,[\tan\,(\tilde\varphi - i\tilde\omega)], \quad y = \log\,[\tan\,(\tilde\varphi + i\tilde\omega)]. \tag{3.33}$$

In terms of these coordinates the line element is given by

$$ds^2 = \frac{a+2b}{\sinh^2 2\tilde\omega} + \frac{a+2b}{\sin^2 2\tilde\varphi} = \frac{a_+}{\sinh^2\tilde\omega} - \frac{a_+}{\cosh^2\tilde\omega} - \frac{a_-}{\sin^2\tilde\varphi} + \frac{a_-}{\cos^2\tilde\varphi}. \tag{3.34}$$

3.1. The Superintegrable Potential V_1 on $D_{\rm IV}$. We start by stating the potential V_1 in the respective coordinate systems

$$V_1(u,v) = \left(\frac{a_+}{\sin^2 u} + \frac{a_-}{\cos^2 u}\right)^{-1} \times \times \left[\frac{\hbar^2}{2m}\left(\frac{k^2-1/4}{\cos^2 u} + \frac{k^2-1/4}{\sin^2 u}\right) - 4\alpha e^{2v} + 8m\omega^2 e^{4v}\right], \qquad (3.35)$$

$$= \left(\frac{a_+}{\nu^2} + \frac{a_-}{\mu^2}\right)^{-1} \times \times \left[-\alpha + \frac{\hbar^2}{2m}\left(\frac{k^2-1/4}{\mu^2} + \frac{k^2-1/4}{\nu^2}\right) + \frac{m}{2}\omega^2(\mu^2+\nu^2)\right], \qquad (3.36)$$

$$= d^2\left(\frac{a_+}{\sinh^2\omega\sin^2\varphi} + \frac{a_-}{\cosh^2\omega\cos^2\varphi}\right)^{-1} \times \times \left[-\alpha + \frac{\hbar^2}{2m}\left(\frac{k^2-1/4}{\sinh^2\omega\sin^2\varphi} + \frac{k^2-1/4}{\cosh^2\omega\cos^2\varphi}\right) + + \frac{m}{2}\omega^2 d^2(\cosh^2\omega - \sin^2\varphi)\right]. \qquad (3.37)$$

The path integral for the potential V_1 can be solved in the (u,v) system and in horospherical coordinates. We also keep the parameters k_1 and k_2 different in comparison with Kalnins et al.

3.1.1. Separation of V_1 in the (u,v) System. The classical Lagrangian and Hamiltonian are given by

$$\mathcal{L}(u,\dot u,v,\dot v) = \frac{m}{2}\frac{2b\cos 2u + a}{\sin^2 2u}(\dot u^2 + \dot v^2) + V(u,v), \qquad (3.38)$$

$$\mathcal{H}(u,p_u,v,p_v) = \frac{1}{2m}\frac{\sin^2 2u}{2b\cos 2u + a}(p_u^2 + p_v^2) + V(u,v). \qquad (3.39)$$

The canonical momentum operators are given by

$$p_u = \frac{\hbar}{i}\left(\frac{\partial}{\partial u} + 2\cot\, 2u - \frac{2b\sin 2u}{2b\cos 2u + a}\right), \quad p_v = \frac{\hbar}{i}\frac{\partial}{\partial v}, \qquad (3.40)$$

and the Hamiltonian operator has the form

$$H = -\frac{\hbar^2}{2m}\frac{\sin^2 2u}{2b\cos 2u + a}\left(\frac{\partial^2}{\partial u^2} + \frac{\partial^2}{\partial v^2}\right) + V(u,v) \qquad (3.41)$$

$$= \frac{1}{2m}\frac{\sin 2u}{\sqrt{2b\cos 2u + a}}(p_u^2 + p_v^2)\frac{\sin 2u}{\sqrt{2b\cos 2u + a}} + V(u,v). \qquad (3.42)$$

We insert V_1 into the path integral and obtain ($f = a_+/\sin^2 u + a_-/\cos^2 u$)

$$K^{(V_1)}(u'', u', v'', v'; T) = \int\limits_{u(t')=u'}^{u(t'')=u''} \mathcal{D}u(t) \int\limits_{v(t')=v'}^{v(t'')=v''} \mathcal{D}v(t) f(u) \times$$

$$\times \exp\left(\frac{i}{\hbar}\int\limits_0^T \left\{\frac{m}{2} f(\dot{u}^2 + \dot{v}^2) - \frac{1}{f}\left[\frac{\hbar^2}{2m}\left(\frac{k_1^2 - 1/4}{\cos^2 u} - \frac{k_2^2 - 1/4}{\sin^2 u}\right) + \right.\right.\right.$$

$$\left.\left.\left. + 8m\omega^2\left(\mathrm{e}^{4v} - \frac{\alpha}{2m\omega^2}\mathrm{e}^{2v}\right)\right]\right\} dt\right). \quad (3.43)$$

We see that the v dependence has the form of a Morse potential:

$$V^{(\mathrm{MP})}(x) = \frac{\hbar^2 V_0^2}{2M}\left(\mathrm{e}^{2x} - 2\tilde{\alpha}\,\mathrm{e}^{x}\right), \quad (3.44)$$

where the (finite) discrete energy spectrum is given by

$$E_l = -\frac{\hbar^2}{2M}\left(\tilde{\alpha} - l - \frac{1}{2}\right)^2. \quad (3.45)$$

Proceeding in the usual way we obtain for the time-transformed path integral

$$K^{(V_1)}(u'', u', v'', v'; s'') = \int\limits_{u(0)=u'}^{u(s'')=u''} \mathcal{D}u(s) \int\limits_{v(0)=v'}^{v(s'')=v''} \mathcal{D}v(s) \times$$

$$\times \exp\left\{\frac{i}{\hbar}\int\limits_0^T \left[\frac{m}{2}(\dot{u}^2 + \dot{v}^2) - \frac{\hbar^2}{2m}\left(\frac{\lambda_1^2 - 1/4}{\cos^2 u} - \frac{\lambda_2^2 - 1/4}{\sin^2 u}\right) - \right.\right.$$

$$\left.\left. - 8m\omega^2\left(\mathrm{e}^{4v} - \frac{\alpha}{2m\omega^2}\mathrm{e}^{2v}\right)\right] ds\right\} =$$

$$= \sum_n \Phi_n^{(\lambda_2,\lambda_1)}(u'')\Phi_n^{(\lambda_2,\lambda_1)}(u') \exp\left[-\frac{i}{\hbar}\frac{\hbar^2}{2m}(\lambda_1 + \lambda_2 + 2n + 1)2s''\right] \times$$

$$\times \left\{\int d\kappa \Psi_\kappa^{(\mathrm{MP})}(v'')\Phi_\kappa^{(MP)\,*}(u')\,\mathrm{e}^{-i\hbar\kappa^2 s''/2m} + \right.$$

$$\left. + \sum_l \Psi_l^{(\mathrm{MP})}(v'')\Phi_l^{(MP\,*)}(u')]\ \exp\left[\frac{i}{\hbar}\frac{\hbar^2}{2m}\left(\tilde{\alpha} - l - \frac{1}{2}\right)^2\right]\right\}. \quad (3.46)$$

Here, $\lambda_{1,2}^2 = k_{1,2}^2 - 2ma_{-,+}E/\hbar^2$, and in the variable v we have used the solution of the Morse potential and in the variable u the solution of the Pöschl–Teller potential, respectively. This form of the solution is convenient to obtain

the bound state solutions. The bound state energy levels are determined by

$$2(n+l+1)+\lambda_1+\lambda_2-\frac{\alpha}{\hbar\omega}=0. \tag{3.47}$$

By denoting

$$N_{n,l}=\left(2(n+l+1)-\frac{\alpha}{\hbar\omega}\right)^2-(k_1^2+k_2^2) \tag{3.48}$$

the quadratic equation in E can be solved to give (with the further abbreviation $K_a=4(a_+k_1^2+a_-k_2^2)$)

$$E_{n,l}= \\ =\frac{\hbar^2}{4mb^2}\left\{\pm\sqrt{(aN_{n,l}+K_a)^2-4b^2(N_{n,l}^2-4k_1^2k_2^2)}-(aN_{n,l}+K_a)\right\}. \tag{3.49}$$

We keep the $\pm$-sign to allow for different boundary conditions which may depend on the parameters a and b. For instance, for $a=2b$ we get the limiting case:

$$E_{n,l}=-\frac{\hbar^2}{2ma}\left[\left((2(n+l+1)+k_1^2-\frac{\alpha}{\hbar\omega}\right)^2-k_2^2\right]. \tag{3.50}$$

For $k_2=\pm 1/2$ it has the form of the usual zero-energy on the two-dimensional hyperboloid.

In order to obtain the continuous spectrum, the formulation in (u,v) coordinates is inconvenient. Following [12] we perform the coordinate transformation $\cos u=\tanh\tau$, and additionally we make a time-transformation with the time-transformation function $f=a_+/\sin^2 u+a_-/\cos^2 u$. Due to the coordinate transformation $\cos u=\tanh\tau$ additional quantum terms appear according to

$$\exp\left(\frac{im}{2\epsilon\hbar}\frac{\left(\Delta u^{(j)}\right)^2}{\cos u^{(j-1)}\cos u^{(j)}}\right)\doteq \\ \doteq\exp\left[\frac{im}{2\epsilon\hbar}\left(\Delta\tau^{(j)}\right)^2-i\frac{\hbar}{8m}\left(1+\frac{1}{\cosh^2\tau^{(j)}}\right)\right]. \tag{3.51}$$

We get for the path integral (3.43)

$$K^{(V_1)}(u'',u',v'',v';T)=\int_{-\infty}^{\infty}\frac{dE}{2\pi\hbar}\,\mathrm{e}^{-iET/\hbar}\times \\ \times\int_0^\infty ds''\exp\left[\frac{i}{\hbar}\left(a_+E-\frac{\hbar^2k_2^2}{2m}\right)\right]K^{(V_1)}(\tau'',\tau',v'',v';s''), \tag{3.52}$$

and the time-transformed path integral $K^{(V_1)}(s'')$ is given by

$$K^{(V_1)}(\tau'',\tau',v'',v';s'') = (\cosh\tau'\cosh\tau'')^{-1/2}\times \\ \times\left[\sum_l \Psi_l^{(\mathrm{MP})}(v')\Psi_l^{(\mathrm{MP})}(v'')K_l(\tau'',\tau';s'')+ \\ +\int d\kappa \Psi_\kappa^{(\mathrm{MP})\,*}(v')\Psi_\kappa^{(\mathrm{MP})}(v'')K_\kappa(\tau'',\tau';s'')\right], \quad (3.53)$$

$$K_{l,\kappa}^{(V_1)}(\tau'',\tau';s'') = \int\limits_{\tau(0)=\tau'}^{\tau(s'')=\tau''}\mathcal{D}\tau(s)\times \\ \times\exp\left\{\frac{i}{\hbar}\int\limits_0^{s''}\left[\frac{m}{2}\dot\tau^2-\frac{\hbar^2}{2m}\left(\frac{\lambda_1^2-1/4}{\sinh^2\tau}-\frac{\nu_{l,\kappa}^2-1/4}{\cosh^2\tau}\right)\right]ds\right\}. \quad (3.54)$$

The parameters $\lambda_{1,2}$ are the same as in the previous paragraph and ν is given by

$$\nu_l = \left|2l+1-\frac{\alpha}{\hbar\omega}\right| \quad \text{(discrete)}, \quad \nu_\kappa = i\kappa \quad \text{(continuous)}, \quad (3.55)$$

where discrete and continuous means the discrete and continuous contribution of the Morse potential. Of course, the analysis of the discrete spectrum gives the same result as before. The kernel $K_{l,\kappa}^{(V_1)}(s'')$ now allows us to write down the entire kernel $K^{(V_1)}(T)$ in terms of Morse wave functions and modified Pöschl–Teller wave functions in the following form:

$$K^{(V_1)}(u'',u',v'',v';T) = (\cosh\tau'\cosh\tau'')^{-1/2}\times \\ \times\left\{\sum_{ln} N_{ln}^2\Psi_l^{(\mathrm{MP})\,*}(v')\Psi_l^{(\mathrm{MP})}(v'')\Psi_n^{(\lambda_1,\nu\ )\,*}(\tau')\Psi_n^{(\lambda_1,\nu\ )}(\tau'')\,\mathrm{e}^{-iE\ \ T/\hbar}+ \right.\\ +\int dp\sum_l N_{lp}^2\Psi_l^{(\mathrm{MP})\,*}(v')\Psi_l^{(\mathrm{MP})}(v'')\Psi_p^{(\lambda_1,\nu\ )\,*}(\tau')\Psi_p^{(\lambda_1,\nu\ )}(\tau'')\,\mathrm{e}^{-iE\ \ T/\hbar}+ \\ \left.+\int dp\int d\kappa N_{\kappa p}^2\Psi_\kappa^{(\mathrm{MP})\,*}(v')\Psi_\kappa^{(\mathrm{MP})}(v'')\Psi_p^{(\lambda_1,i\kappa)\,*}(\tau')\Psi_p^{(\lambda_1,i\kappa)}(\tau'')\mathrm{e}^{-iE\ \ T/\hbar}\right\}, \quad (3.56)$$

with the proper normalization constants $N_{ln}, N_{lp}, N_{\kappa p}$, where, e.g., N_{ln} is determined by the residuum corresponding to E_{ln} in the Green function, and with the continuous spectrum

$$E_p = \frac{\hbar^2}{2ma_+}(p^2+k_2^2). \quad (3.57)$$

Note that for $k_2 = 1/2$ we obtain the well-known zero energy on the two-dimensional hyperboloid, which appears here in a natural way after performing the coordinate transformation $\cos u = \tanh \tau$.

The $\Psi_p^{(\mu,\nu)}(\omega)$ are the modified Pöschl–Teller functions, which are given by

$$\Psi_n^{(\eta,\nu)}(r) = N_n^{(\eta,\nu)} (\sinh r)^{2k_2 - \frac{1}{2}} (\cosh r)^{-2k_1 + \frac{3}{2}} \times \\ \times {}_2F_1(-k_1 + k_2 + \kappa, -k_1 + k_2 - \kappa + 1; 2k_2; -\sinh^2 r), \quad (3.58)$$

$$N_n^{(\eta,\nu)} = \frac{1}{\Gamma(2k_2)} \left[\frac{2(2\kappa - 1)\Gamma(k_1 + k_2 - \kappa)\Gamma(k_1 + k_2 + \kappa - 1)}{\Gamma(k_1 - k_2 + \kappa)\Gamma(k_1 - k_2 - \kappa + 1)} \right]^{1/2}. \quad (3.59)$$

The scattering states are given by

$$V(r) = \frac{\hbar^2}{2m} \left(\frac{\eta^2 - 1/4}{\sinh^2 r} - \frac{\nu^2 - 1/4}{\cosh^2 r} \right),$$

$$\Psi_p^{(\eta,\nu)}(r) = N_p^{(\eta,\nu)} (\cosh r)^{2k_1 - \frac{1}{2}} (\sinh r)^{2k_2 - \frac{1}{2}} \times \\ \times {}_2F_1(k_1 + k_2 - \kappa, k_1 + k_2 + \kappa - 1; 2k_2; -\sinh^2 r), \quad (3.60)$$

$$N_p^{(\eta,\nu)} = \frac{1}{\Gamma(2k_2)} \sqrt{\frac{p \sinh \pi p}{2\pi^2}} \Big[\Gamma(k_1 + k_2 - \kappa)\Gamma(-k_1 + k_2 + \kappa) \times \\ \times \Gamma(k_1 + k_2 + \kappa - 1)\Gamma(-k_1 + k_2 - \kappa + 1) \Big]^{1/2}, \quad (3.61)$$

k_1, k_2 defined by: $k_1 = \frac{1}{2}(1 \pm \nu)$, $k_2 = \frac{1}{2}(1 \pm \eta)$, where the correct sign depends on the boundary conditions for $r \to 0$ and $r \to \infty$, respectively. The number N_M denotes the maximal number of states with $0, 1, \ldots, N_M < k_1 - k_2 - \frac{1}{2}$, $\kappa = k_1 - k_2 - n$ for the bound states and $\kappa = \frac{1}{2}(1 + ip)$ for the scattering states; ${}_2F_1(a, b; c; z)$ is the hypergeometric function [10, p. 1057].

3.1.2. Separation of V_1 in Horospherical Coordinates. We evaluate the path integral for V_1 in horospherical coordinates. The classical Lagrangian and Hamiltonian are given by

$$\mathcal{L}(\mu, \dot{\mu}, \nu, \dot{\nu}) = \frac{m}{2} \left(\frac{a_+}{\nu^2} + \frac{a_-}{\mu^2} \right) (\dot{\mu}^2 + \dot{\nu}^2) - V(\mu, \nu), \quad (3.62)$$

$$\mathcal{H}(\mu, p_\mu, \nu, p_\nu) = \frac{1}{2m} \frac{\mu^2 \nu^2 (p_\mu^2 + p_\nu^2)}{a_+ \mu^2 + a_- \nu^2} + V(\mu, \nu). \quad (3.63)$$

For the canonical momentum operators we have

$$p_\mu = \frac{\hbar}{i} \left(\frac{\partial}{\partial \mu} - \frac{\nu^2 a_- / \mu}{a_+ \mu^2 + a_- \nu^2} \right), \quad (3.64)$$

$$p_\nu = \frac{\hbar}{i} \left(\frac{\partial}{\partial \nu} - \frac{\mu^2 a_+ / \nu}{a_+ \mu^2 + a_- \nu^2} \right), \quad (3.65)$$

and for the quantum Hamiltonian we get

$$H = -\frac{\hbar^2}{2m}\frac{\mu^2\nu^2}{a_+\mu^2 + a_-\nu^2}\left(\frac{\partial^2}{\partial\mu^2} + \frac{\partial^2}{\partial\nu^2}\right) + V(\mu,\nu), \tag{3.66}$$

$$= \frac{1}{2m}\sqrt{\frac{\mu^2\nu^2}{a_+\mu^2 + a_-\nu^2}}\,(p_\mu^2 + p_\nu^2)\sqrt{\frac{\mu^2\nu^2}{a_+\mu^2 + a_-\nu^2}} + V(\mu,\nu). \tag{3.67}$$

We insert V_1 into the path integral and obtain ($f = a_+/\nu^2 + a_-/\mu^2$ and keeping to constants $k_{1,2}$)

$$\begin{aligned} K^{(V_1)}(\mu'',\mu',\nu'',\nu';T) &= \int\limits_{\mu(t')=\mu'}^{\mu(t'')=\mu''}\mathcal{D}\mu(t)\int\limits_{\nu(t')=\nu'}^{\nu(t'')=\nu''}\mathcal{D}\nu(t) f(\mu,\nu)\times \\ &\times \exp\left\{\frac{i}{\hbar}\int\limits_0^T\left[\frac{m}{2}f(\mu,\nu)(\dot\mu^2+\dot\nu^2) - \right.\right. \\ &\left.\left. -\frac{1}{f(\mu,\nu)}\left(\frac{m}{2}\omega^2(\mu^2+\nu^2) - \alpha + \frac{\hbar^2}{2m}\left(\frac{k_1^2-1/4}{\mu^2} + \frac{k_2^2-1/4}{\nu^2}\right)\right)\right]dt\right\} = \\ &= \int\limits_{-\infty}^{\infty}\frac{dE}{2\pi\hbar}\,\mathrm{e}^{-iET/\hbar}\int\limits_0^\infty ds''\mathrm{e}^{i\alpha s''/\hbar}K^{(V_1)}(\mu'',\mu',\nu'',\nu';s''), \end{aligned} \tag{3.68}$$

and the time-transformed path integral $K^{(V_1)}(s'')$ is given by

$$\begin{aligned} K^{(V_1)}(\mu'',\mu',\nu'',\nu';s'') &= \int\limits_{\mu(0)=\mu'}^{\mu(s'')=\mu''}\mathcal{D}\mu(s)\times \\ &\times \exp\left\{\frac{i}{\hbar}\int\limits_0^{s''}\left[\frac{m}{2}(\dot\mu^2 - \omega^2\mu^2) - \frac{\hbar^2}{2m}\frac{k_1^2 - 2ma_-E/\hbar^2 - 1/4}{\mu^2}\right]ds\right\}\times \\ &\times \int\limits_{\nu(0)=\nu'}^{\nu(s'')=\nu''}\mathcal{D}\nu(s)\exp\left\{\frac{i}{\hbar}\int\limits_0^{s''}\left[\frac{m}{2}(\dot\nu^2-\omega^2\nu^2) - \frac{\hbar^2}{2m}\frac{k_2^2-2ma_+E/\hbar^2-1/4}{\nu^2}\right]ds\right\} = \\ &= \frac{m^2\omega^2\sqrt{\mu'\mu''\nu'\nu''}}{i^2\hbar^2\sin^2\omega s''}\exp\left[-\frac{m\omega}{2i\hbar}({\mu'}^2 + {\mu''}^2 + {\nu'}^2 + {\nu''}^2)\cot\omega s''\right]\times \\ &\times I_{\lambda_1}\left(\frac{m\omega\mu'\mu''}{i\hbar\sin\omega s''}\right)I_{\lambda_2}\left(\frac{m\omega\nu'\nu''}{i\hbar\sin\omega s''}\right), \end{aligned} \tag{3.69}$$

where $\lambda_{1,2} = k_{1,2}^2 - 2ma_{\mp}E/\hbar^2$. We can extract the bound state wave functions for the bound state contribution of the Green function according to:

$$G^{(V_1)}(\mu'', \mu', \nu'', \nu'; E) = \sum_{n\ =0}^{\infty} \sum_{n\ =0}^{\infty} \frac{N_{n\ n}^2}{E_{n\ n} - E} \times$$
$$\times \Psi_n^{(\mathrm{RHO},\lambda_1)}(\mu') \Psi_n^{(\mathrm{RHO},\lambda_1)}(\mu'') \Psi_n^{(\mathrm{RHO},\lambda_2)}(\nu') \Psi_n^{(\mathrm{RHO},\lambda_2)}(\nu''). \quad (3.70)$$

The bound states are determined by the equation

$$\frac{\alpha}{\hbar\omega} - 2(n_\mu + n_\nu + 1) = \sqrt{k_1^2 - \frac{2ma_- E}{\hbar^2}} + \sqrt{k_2^2 - \frac{2ma_+ E}{\hbar^2}}. \quad (3.71)$$

This quadratic equation in E is identical with (3.47).

3.2. The Superintegrable Potential V_2 on D_{IV}. We state the potential in the respective coordinate systems

$$V_2(u, v) = \left(\frac{a_+}{\sin^2 u} + \frac{a_-}{\cos^2 u}\right)^{-1} \frac{\hbar^2}{2m} \left[\frac{k_1^2 - 1/4}{\sinh^2 v} - \frac{k_2^2 - 1/4}{\cosh^2 v} + \right.$$
$$\left. + \left(k_3^2 - \frac{1}{4}\right)\left(\frac{1}{\sin^2 u} + \frac{1}{\cos^2 u}\right)\right], \quad (3.72)$$
$$= 4\left(\frac{a_+}{\sinh^2 2\tilde{\omega}} + \frac{a_-}{\sin^2 2\tilde{\varphi}}\right)^{-1} \frac{\hbar^2}{2m} \left[\left(k_3^2 - \frac{1}{4}\right)\left(\frac{1}{\sinh^2 2\tilde{\omega}} + \frac{1}{\sin^2 2\tilde{\varphi}}\right) + \right.$$
$$\left. + \left(\frac{k_2^2 - 1/4}{\cos^2 2\tilde{\varphi}} - \frac{k_1^2 - 1/4}{\cosh^2 2\tilde{\omega}}\right)\right]. \quad (3.73)$$

It is possible to evaluate the path integral for V_2 in the (u, v) and the degenerate elliptic system with $\gamma = 2$. The elliptic system with $\gamma = 0$ is not treated.

3.2.1. Separation of V_2 in the (u, v) System. We insert V_2 into the path integral and obtain ($f = a_+/\sin^2 u + a_-/\cos^2 u$)

$$K^{(V_2)}(u'', u', v'', v'; T) = \int\limits_{u(t')=u'}^{u(t'')=u''} \mathcal{D}u(t) \int\limits_{v(t')=v'}^{v(t'')=v''} \mathcal{D}v(t) f(u) \times$$
$$\times \exp\left\{\frac{i}{\hbar}\int_0^T \left[\frac{m}{2} f(\dot{u}^2 + \dot{v}^2) - \frac{\hbar^2}{2mf}\left(\frac{k_1^2 - 1/4}{\sinh^2 v} - \frac{k_2^2 - 1/4}{\cosh^2 v} + \right.\right.\right.$$
$$\left.\left.\left. + \left(k_3^2 - \frac{1}{4}\right)\left(\frac{1}{\sin^2 u} + \frac{1}{\cos^2 u}\right)\right)\right] dt\right\}. \quad (3.74)$$

This formulation in (u, v) coordinates is inconvenient. Following the procedure as for V_1 in the (u, v) system we perform the coordinate transformation $\cos u = \tanh\tau$, and get for the path integral (3.74)

$$K^{(V_2)}(u'', u', v'', v'; T) = \int_{-\infty}^{\infty} \frac{dE}{2\pi\hbar}\, e^{-iET/\hbar} \times \\ \times \int_0^\infty ds'' \exp\left[\frac{i}{\hbar}\left(a_+ E - \frac{\hbar^2 k_3^2}{2m}\right)\right] K(\tau'', \tau', v'', v'; s''), \quad (3.75)$$

and the time-transformed path integral $K^{(V_2)}(s'')$ is given by

$$K^{(V_2)}(\tau'', \tau', v'', v'; s'') = \\ = (\cosh\tau' \cosh\tau'')^{-1/2} \sum_{n=0}^{N_{\max}} \Psi_n^{(k_1,k_2)}(v')\Psi_n^{(k_1,k_2)}(v'') \times \\ \times \int_{\tau(0)=\tau'}^{\tau(s'')=\tau''} \mathcal{D}\tau(s) \exp\left\{\frac{i}{\hbar}\int_0^{s''}\left[\frac{m}{2}\dot{\tau}^2 - \frac{\hbar^2}{2m}\left(\frac{\lambda_2^2 - 1/4}{\sinh^2\tau} - \frac{\lambda_1^2 - 1/4}{\cosh^2\tau}\right)\right] ds\right\} + \\ + (\cosh\tau' \cosh\tau'')^{-1/2} \int dk_v \Psi_k^{(k_1,k_2)}(v')\Psi_k^{(k_1,k_2)}(v'') \times \\ \times \int_{\tau(0)=\tau'}^{\tau(s'')=\tau''} \mathcal{D}\tau(s) \exp\left\{\frac{i}{\hbar}\int_0^{s''}\left[\frac{m}{2}\dot{\tau}^2 - \frac{\hbar^2}{2m}\left(\frac{\lambda_2^2 - 1/4}{\sinh^2\tau} - \frac{-k_v^2 - 1/4}{\cosh^2\tau}\right)\right] ds\right\}. \quad (3.76)$$

($\lambda_1^2 = (2n_v + |k_1| - |k_2| + 1)^2$, $\lambda_2^2 = k_3^2 - 2ma_- E/\hbar^2$).

The v-path integration gives a discrete and continuous spectrum, thus two different parts for the τ-path integration. We therefore find for the Green function

$$G^{(V_2)}(\tau'', \tau', v'', v'; E) = (\cosh\tau' \cosh\tau'')^{-1/2} \times \\ \times \sum_{n=0}^{N_{\max}} \Psi_n^{(k_1,k_2)}(v')\Psi_n^{(k_1,k_2)}(v'') \frac{m}{\hbar^2} \frac{\Gamma(m_1 - L_{\lambda_1})\Gamma(L_{\lambda_1} + m_1 + 1)}{\Gamma(m_1 + m_2 + 1)\Gamma(m_1 - m_2 + 1)} \times \\ \times (\cosh\tau' \cosh\tau'')^{-(k_1 - k_2)} (\tanh\tau' \tanh\tau'')^{m_1 + m_2 + 1/2} \times \\ \times {}_2F_1\left(-L_{\lambda_1} + m_1, L_{\lambda_1} + m_1 + 1; m_1 - m_2 + 1; \frac{1}{\cosh^2\tau_<}\right) \times$$

$$\times\,{}_2F_1\biggl(-L_{\lambda_1}+m_1, L_{\lambda_1}+m_1+1; m_1+m_2+1; \tanh^2\tau_>\biggr)+$$
$$+(\cosh\tau'\cosh\tau'')^{-1/2}\int dk_v\,\Psi_k^{(k_1,k_2)}(v')\Psi_k^{(k_1,k_2)}(v'')\times$$
$$\times\frac{m}{\hbar^2}\frac{\Gamma(m_1-L_k\ )\Gamma(L_k\ +m_1+1)}{\Gamma(m_1+m_2+1)\Gamma(m_1-m_2+1)}\times$$
$$\times(\cosh\tau'\cosh\tau'')^{-(k_1-k_2)}(\tanh\tau'\tanh\tau'')^{m_1+m_2+1/2}\times$$
$$\times\,{}_2F_1\biggl(-L_k\ +m_1, L_k\ +m_1+1; m_1-m_2+1; \frac{1}{\cosh^2\tau_<}\biggr)\times$$
$$\times\,{}_2F_1\biggl(-L_k\ +m_1, L_k\ +m_1+1; m_1+m_2+1; \tanh^2\tau_>\biggr) \tag{3.77}$$

$\bigl(m_{1,2}=\frac{1}{2}(\lambda_2\pm\sqrt{2m\mathcal{E}}/\hbar)$, $L_{\lambda_1}=\frac{1}{2}(\lambda_1-1)$, $L_k\ =\frac{1}{2}(ik_v-1)$, $\mathcal{E}=a_+E-\hbar^2k_3^2/2m\bigr)$.

A discrete spectrum is only possible for the first summand in (3.76). First, we can analyze the discrete spectrum by looking at the poles in (3.77) which gives the equation

$$2(n_\tau+n_v)+\lambda_++\lambda_-+|k_2|-|k_1|=0 \tag{3.78}$$

($\lambda_\pm^2=k_3^2-2ma_\pm E/\hbar^2$). This gives a quadratic equation in E with solution ($N_k=2n_\tau-2n_v-|k_1|+|k_2|$)

$$E_{n\ n}\ =-\frac{a\hbar^2N_k^2}{4b^2}\left(1\mp\sqrt{1+\frac{4b^2}{a^2}\biggl(\frac{k_3^2}{N_k^2}-1\biggr)}\right). \tag{3.79}$$

The entire Green function in terms of the wave functions is given by

$$G^{(V_2)}(\tau'',\tau',v'',v';E)=(\cosh\tau'\cosh\tau'')^{-1/2}\int dp\frac{N_{pk}^2}{E_p-E}\int dk_v\times$$
$$\times\,\Psi_k^{(k_1,k_2)}(v')\Psi_k^{(k_1,k_2)}(v'')\Psi_p^{(\lambda_2,ik\ )}(\tau')\Psi_p^{(\lambda_2,ik\ )\,*}(\tau'')+$$
$$+(\cosh\tau'\cosh\tau'')^{-1/2}\sum_{n\ =0}^{\infty}\sum_{n\ =0}^{\infty}\Psi_n^{(k_1,k_2)}(v')\Psi_n^{(k_1,k_2)}(v'')\times$$
$$\times\left\{\sum_{n\ =0}^{N_{\max}}\frac{N_{n\ n}^2}{E_{n\ n}\ -E}\Psi_n^{(\lambda_2,\lambda_1)}(\tau')\Psi_n^{(\lambda_2,\lambda_1)}(\tau'')+\right.$$
$$\left.+\int dp\frac{N_{pn}^2}{E_p-E}\Psi_p^{(\lambda_2,\lambda_1)}(\tau')\Psi_p^{(\lambda_2,\lambda_1)\,*}(\tau'')\right\}, \tag{3.80}$$

where $N_{n\ n}, N_{k\ n}$ is determined by the residuum in (3.77). The continuous spectrum has the form

$$E_p = \frac{\hbar^2}{2ma_+}(p^2 + k_3^2). \tag{3.81}$$

For $k_3 = \pm 1/2$ we obtain the usual zero-point energy on the two-dimensional hyperboloid. Reinserting $\cos u = \tanh v$ gives the Green function in the (u, v) system.

3.2.2. Separation of V_2 in Degenerate Elliptic Coordinates. We insert the potential V_2 in degenerate elliptic coordinates into the path integral and obtain $(f(\tilde\omega, \tilde\varphi) = 4(a_+/\sinh^2 2\tilde\omega + a_-/\sin^2 2\tilde\varphi))$

$$\begin{aligned} K^{(V_2)}(\tilde\omega'', \tilde\omega', \tilde\varphi'', \tilde\varphi'; T) &= \int\limits_{\tilde\omega(t')=\tilde\omega'}^{\tilde\omega(t'')=\tilde\omega''} \mathcal{D}\tilde\omega(t) \int\limits_{\tilde\varphi(t')=\tilde\varphi'}^{\tilde\varphi(t'')=\tilde\varphi''} \mathcal{D}\tilde\varphi(t) f(\tilde\omega, \tilde\varphi) \times \\ &\times \exp\left\{\frac{i}{\hbar}\int_0^T \left[\frac{m}{2} f(\tilde\omega, \tilde\varphi)(\dot{\tilde\omega}^2 + \dot{\tilde\varphi}^2) 4 \frac{\hbar^2}{2mf(\tilde\omega, \tilde\varphi)} \times \right.\right. \\ &\left.\left. \times \left(\frac{k_1^2 - 1/4}{\sinh^2 2\tilde\omega} - \frac{k_2^2 - 1/4}{\cosh^2 2\tilde\omega} + \frac{k_3^2 - 1/4}{\sin^2 2\tilde\varphi} + \frac{k_2^2 - 1/4}{\cos^2 2\tilde\varphi}\right)\right] dt\right\}. \end{aligned} \tag{3.82}$$

The calculation is similar as in the case of the (u, v) system: First, we rescale $2\tilde\omega \to \tilde\omega, 2\tilde\varphi \to \tilde\varphi$, then we perform the transformation $\cos\tilde\varphi = \tanh\tilde\tau$. Finally, we perform a time transformation in the path integral with the time transformation $f(\tilde\omega, \tilde\varphi) \to f(\tilde\omega, \tilde\tau)$ yielding

$$\begin{aligned} &G^{(V_2)}(\tilde\tau'', \tilde\tau', \tilde\omega'', \tilde\omega'; E) = \\ &\quad = \int_0^\infty ds'' \exp\left[\frac{i}{\hbar} s''\left(Ea_- - \frac{\hbar^2 k_3^2}{2m}\right)\right] K^{(V_2)}(\tilde\tau'', \tilde\tau', \tilde\omega'', \tilde\omega'; s'') \end{aligned} \tag{3.83}$$

with the transformed path integral $K^{(V_2)}(s'')$ given by

$$\begin{aligned} &K^{(V_2)}(\tilde\tau'', \tilde\tau', \tilde\omega'', \tilde\omega'; s'') = \\ &\quad = \int\limits_{\tilde\tau(0)=\tilde\tau'}^{\tilde\tau(s'')=\tilde\tau''} \mathcal{D}\tilde\tau(s) \int\limits_{\tilde\omega(0)=\tilde\omega'}^{\tilde\omega(s'')=\tilde\omega''} \mathcal{D}\tilde\omega(s) \exp\left\{\frac{i}{\hbar}\int_0^T \left[\frac{m}{2}(\dot{\tilde\tau}^2 + \cosh^2\tilde\tau\,\dot{\tilde\omega}^2) - \right.\right. \\ &\quad \left.\left. - \frac{\hbar^2}{2m}\left(\frac{k_1^2 - 1/4}{\sinh^2\tilde\tau} + \frac{1}{\cosh^2\tilde\tau}\left(\frac{\lambda_+^2 - 1/4}{\sinh^2\tilde\omega} - \frac{k_2^2 - 1/4}{\cosh^2\tilde\omega} + \frac{1}{4}\right)\right)\right] ds\right\}. \end{aligned} \tag{3.84}$$

Again we evaluate this path integral by a successive $\tilde{\omega}$- and $\tilde{\tau}$-path integration. Performing finally the s'' integration we obtain

$$
\begin{aligned}
&G^{(V_2)}(\tilde{\tau}'',\tilde{\tau}',\tilde{\omega}'',\tilde{\omega}';E)=(\cosh\tilde{\tau}'\cosh\tilde{\tau}'')^{-1/2}\times\\
&\times\Bigg\{\int dp\frac{N^2_{k_{\tilde{\omega}}p}}{E_p-E}\int dk_{\tilde{\omega}}\,\Psi_p^{(k_1,ik_{\tilde{\omega}})}(\tilde{\tau}')\Psi_p^{(k_1,ik_{\tilde{\omega}})\,*}(\tilde{\tau}'')\Psi_{k_{\tilde{\omega}}}^{(\lambda_1,k_2)}(\tilde{\omega}')\Psi_{k_{\tilde{\omega}}}^{(\lambda_1,k_2)\,*}(\tilde{\omega}'')+\\
&+\int dp\sum_{n_{\tilde{\omega}}=0}^{N_{\max}}\frac{N^2_{n_{\tilde{\omega}}p}}{E_p-E}\Psi_p^{(k_1,\epsilon\quad\tilde{\omega})}(\tilde{\tau}')\Psi_p^{(k_1,\epsilon\quad\tilde{\omega})\,*}(\tilde{\tau}'')\Psi_{n_{\tilde{\omega}}}^{(\lambda_1,k_2)}(\tilde{\omega}')\Psi_{n_{\tilde{\omega}}}^{(\lambda_1,k_2)}(\tilde{\omega}'')+\\
&+\sum_{n_{\tilde{\ }}=0}^{N_{\max}}\sum_{n_{\tilde{\omega}}=0}^{N_{\max}}\frac{N^2_{n_{\tilde{\ }}n_{\tilde{\omega}}}}{E_{n_{\tilde{\ }}n_{\tilde{\omega}}}-E}\Psi_{n_{\tilde{\ }}}^{(k_1,\epsilon\quad\tilde{\omega})}(\tilde{\tau}')\Psi_{n_{\tilde{\ }}}^{(k_1,\epsilon\quad\tilde{\omega})}(\tilde{\tau}'')\Psi_{n_{\tilde{\omega}}}^{(\lambda_1,k_2)}(\tilde{\omega}')\Psi_{n_{\tilde{\omega}}}^{(\lambda_1,k_2)}(\tilde{\omega}'')\Bigg\}.
\end{aligned}
\tag{3.85}
$$

The normalization constants $N_{k_{\tilde{\omega}}p}, N_{k_{\tilde{\omega}}p}, N_{n_{\tilde{\ }}n_{\tilde{\omega}}}$ are determined by the respective residuum in $G^{(V_2)}(E)$ and the discrete spectrum is determined by the quadratic equation (3.78). The continuous spectrum has the form

$$
E_p=\frac{\hbar^2}{2ma_-}(p^2+k_3^2). \tag{3.86}
$$

The difference of E_p in comparison to the (u,v) system can be resolved by making in the (u,v) system the transformation $\sin u=\tanh\tau$ which changes the sign in the energy term. This concludes the discussion of V_2 on $D_{\rm IV}$.

3.3. The Superintegrable Potential V_3 on $D_{\rm IV}$. We state the potential in the respective coordinate systems

$$
\begin{aligned}
V_3(\tilde{\omega},\tilde{\varphi})&=\frac{\hbar^2}{2m}\left(\frac{4a_+}{\sinh^2 2\tilde{\omega}}+\frac{4a_-}{\sinh^2\tilde{\varphi}}\right)^{-1}\times\\
&\qquad\times\left[\frac{c_1}{\cos^2\tilde{\varphi}}+\frac{c_2}{\cosh^2\tilde{\omega}}+c_3\left(\frac{1}{\sin^2\tilde{\varphi}}-\frac{1}{\sinh^2\tilde{\omega}}\right)\right], \qquad (3.87)\\
&=\frac{\hbar^2}{2m}\left[a_+\left(\frac{1}{\cosh^2\hat{\omega}}-\frac{1}{\cos^2\hat{\varphi}}\right)-a_-\left(\frac{1}{\sinh^2\hat{\omega}}+\frac{1}{\sin^2\hat{\varphi}}\right)\right]^{-1}\times\\
&\qquad\times\left[\frac{c_3}{\sinh^2\hat{\omega}}+\frac{c_2}{\cosh^2\hat{\omega}}+c_3\left(\frac{1}{\sin^2\hat{\varphi}}-\frac{1}{\cos^2\hat{\varphi}}\right)\right]. \qquad (3.88)
\end{aligned}
$$

It is possible to evaluate the path integral for V_3 in both separating coordinate systems. However, due to the similarity in the evaluations, only the degenerate elliptic II case will be presented.

3.3.1. Separation of V_3 in Degenerate Elliptic Coordinates II. We insert the potential V_3 in the path integral formulation for degenerate elliptic coordinates on D_{IV} and obtain $f(\tilde{\omega},\tilde{\varphi}) = 4(a_+/\sinh^2 2\tilde{\omega} + a_-/\sin^2 2\tilde{\varphi}))$

$$K^{(V_3)}(\tilde{\omega}'',\tilde{\omega}',\varphi'',\varphi';T) = \int\limits_{\tilde{\omega}(t')=\tilde{\omega}'}^{\tilde{\omega}(t'')=\tilde{\omega}''} \mathcal{D}\tilde{\omega}(t) \int\limits_{\tilde{\varphi}(t')=\tilde{\varphi}'}^{\tilde{\varphi}(t'')=\tilde{\varphi}''} \mathcal{D}\tilde{\varphi}(t) f(\tilde{\omega},\tilde{\varphi})\times$$
$$\times \exp\left\{\frac{i}{\hbar}\int\limits_0^T \left[\frac{m}{2} f(\tilde{\omega},\tilde{\varphi})(\dot{\tilde{\omega}}^2 + \dot{\tilde{\varphi}}^2) - \right.\right.$$
$$\left.\left. - \frac{\hbar^2}{2mf(\tilde{\omega},\tilde{\varphi})}\left(\frac{c_1}{\cos^2\tilde{\varphi}} + \frac{c_2}{\cosh^2\tilde{\omega}} + c_3\left(\frac{1}{\sin^2\tilde{\varphi}} - \frac{1}{\sinh^2\tilde{\omega}}\right)\right)\right] dt\right\}. \quad (3.89)$$

In order to obtain a convenient form to evaluate (3.89) we perform the coordinate transformation $\cos\tilde{\varphi} = \tanh\tilde{\tau}$ in the same way as for V_2. Performing also the corresponding time transformation gives

$$K^{(V_3)}(\tilde{\omega}'',\tilde{\omega}',\tilde{\tau}'',\tilde{\tau}';T) = \int\limits_{-\infty}^{\infty} \frac{dE}{2\pi\hbar}\, \mathrm{e}^{-iET/\hbar}\times$$
$$\times \int\limits_0^{\infty} ds'' \exp\left[\frac{i}{\hbar}\left(\frac{\hbar^2}{2m}\lambda^2_{3^+_-}\right)\right] K^{(V_3)}(\tilde{\omega}'',\tilde{\omega}',\tilde{\tau}'',\tilde{\tau}';s''), \quad (3.90)$$

and the time-transformed path integral $K^{(V_3)}(s'')$ is given by

$$K^{(V_3)}(\tilde{\omega}'',\tilde{\omega}',\tilde{\tau}'',\tilde{\tau}';s'') = \int\limits_{\tilde{\omega}(0)=\tilde{\omega}'}^{\tilde{\omega}(s'')=\tilde{\omega}''} \mathcal{D}\tilde{\omega}(s) \int\limits_{\tilde{\tau}(0)=\tilde{\tau}'}^{\tilde{\tau}(s'')=\tilde{\tau}''} \mathcal{D}\tilde{\tau}(s)\cosh\tilde{\tau}\times$$
$$\times \exp\left\{\frac{i}{\hbar}\int\limits_0^{s''}\left[\frac{m}{2}(\dot{\tilde{\tau}}^2 + \cosh^2\tilde{\tau}\dot{\tilde{\omega}}^2) - \frac{\hbar^2}{2m}\frac{\lambda^2_{1^+_-} - 1/4}{\sinh^2\tilde{\tau}} - \right.\right.$$
$$\left.\left. - \frac{\hbar^2}{2m\cosh^2\tilde{\tau}}\left(\frac{\lambda^2_{3^-_+} - 1/4}{\sinh^2\tilde{\omega}} - \frac{\lambda^2_{2^-_+} - 1/4}{\cosh^2\tilde{\omega}} + \frac{1}{4}\right)\right] ds\right\} \quad (3.91)$$

$\big(\lambda^2_{i^\pm_\pm} = \frac{1}{4} \mp c_i - 2ma_\pm E/\hbar^2,\ i = 1,2,3\big)$. The latter path integral has the form of two successive modified Pöschl–Teller path integrations in $\tilde{\omega}$ and $\tilde{\tau}$. In the ω-path integration we get a contribution from the continuous and discrete spectrum. The continuous contribution gives in the $\tilde{\tau}$-path integration only a continuous part, whereas the other gives a discrete and continuous contribution in $\tilde{\tau}$. We denote the continuous parameter in $\tilde{\omega}$ by $p_{\tilde{\omega}}$, the discrete parameter in $\tilde{\omega}$ by $\epsilon_{n_{\tilde{\omega}}} = 2n_{\tilde{\omega}} + \lambda_{3^-_+} - \lambda_{2^-_+} - 1$, the continuous parameter in $\tilde{\tau}$ by p, the discrete parameter in $\tilde{\tau}$ by $\epsilon_{n_{\tilde{\tau}}} = 2n_{\tilde{\tau}} + \lambda_{1^+_-} - \epsilon_{n_{\tilde{\omega}}} - 1$, therefore:

$$\begin{aligned}
&K^{(V_3)}(\tilde{\omega}'',\tilde{\omega}',\tilde{\tau}'',\tilde{\tau}';s'') = \\
&= (\cosh\tilde{\tau}'\cosh\tilde{\tau}'')^{-1/2}\int_0^\infty dp_\omega \Psi_{p_{\tilde{\omega}}}^{(\lambda_{3^-_+},\lambda_{2^-_+})}(\tilde{\omega}')\Psi_{p_{\tilde{\omega}}}^{(\lambda_{3^-_+},\lambda_{2^-_+})*}(\tilde{\omega}'')\times \\
&\times \int_{\tilde{\tau}(0)=\tilde{\tau}'}^{\tilde{\tau}(s'')=\tilde{\tau}''}\mathcal{D}\tilde{\tau}(s)\exp\left\{\frac{i}{\hbar}\int_0^{s''}\left[\frac{m}{2}\dot{\tilde{\tau}}^2 - \frac{\hbar^2}{2m}\left(\frac{\lambda^2_{1^+_-}-1/4}{\sinh^2\tilde{\tau}} + \frac{p^2_{\tilde{\omega}}+1/4}{\cosh^2\tilde{\tau}}\right)\right]ds\right\} + \\
&+ (\cosh\tilde{\tau}'\cosh\tilde{\tau}'')^{-1/2}\sum_{n_{\tilde{\omega}}=0}^{N_{\max}}\Psi_{n_{\tilde{\omega}}}^{(\lambda_{3^-_+},\lambda_{2^-_+})}(\tilde{\omega}')\Psi_{n_{\tilde{\omega}}}^{(\lambda_{3^-_+},\lambda_{2^-_+})}(\tilde{\omega}'')\times \\
&\times \int_{\tilde{\tau}(0)=\tilde{\tau}'}^{\tilde{\tau}(s'')=\tilde{\tau}''}\mathcal{D}\tilde{\tau}(s)\exp\left\{\frac{i}{\hbar}\int_0^{s''}\left[\frac{m}{2}\dot{\tilde{\tau}}^2 - \frac{\hbar^2}{2m}\left(\frac{\lambda^2_{1^+_-}-1/4}{\sinh^2\tilde{\tau}} - \frac{\epsilon^2_{n_{\tilde{\omega}}}-1/4}{\cosh^2\tilde{\tau}}\right)\right]ds\right\} = \\
&= (\cosh\tilde{\tau}'\cosh\tilde{\tau}'')^{-1/2}\int_0^\infty dp_t\omega \Psi_{p_{\tilde{\omega}}}^{(\lambda_{3^-_+},\lambda_{2^-_+})}(\tilde{\omega}')\Psi_{p_{\tilde{\omega}}}^{(\lambda_{3^-_+},\lambda_{2^-_+})*}(\tilde{\omega}'')\times \\
&\times \int_0^\infty dp \Psi_p^{(\lambda_{1^+_-},ip_{\tilde{\omega}})}(\tilde{\omega}')\Psi_p^{(\lambda_{1^+_-},ip_{\tilde{\omega}})*}(\tilde{\omega}'')\,\mathrm{e}^{-is''\hbar p^2/2m} + \\
&+ (\cosh\tilde{\tau}'\cosh\tilde{\tau}'')^{-1/2}\sum_{n_{\tilde{\omega}}=0}^{N_{\max}}\Psi_{n_{\tilde{\omega}}}^{(\lambda_{3^-_+},\lambda_{2^-_+})}(\tilde{\omega}')\Psi_{n_{\tilde{\omega}}}^{(\lambda_{3^-_+},\lambda_{2^-_+})}(\tilde{\omega}'')\times \\
&\times \left\{\int_0^\infty dp \Psi_p^{(\lambda_{1^+_-},\epsilon_{\tilde{\omega}})}(\tilde{\omega}')\Psi_p^{(\lambda_{1^+_-},\epsilon_{\tilde{\omega}})*}(\tilde{\omega}'')\,\mathrm{e}^{-is''\hbar p^2/2m} + \right. \\
&\left. + \sum_{n_\omega=0}^{N_{\max}}\Psi_n^{(\lambda_{1^+_-},\epsilon_{\tilde{\omega}})}(\tilde{\tau}')\Psi_n^{(\lambda_{1^+_-},\epsilon_{\tilde{\omega}})}(\tilde{\tau}'')\,\mathrm{e}^{-i\hbar s''\epsilon^2_{-}/2m}\right\}. \qquad (3.92)
\end{aligned}$$

Performing the s'' integration gives the spectrum. For the continuous spectrum we obtain

$$E_p = \frac{\hbar^2}{2ma_-}\left(p^2 + \frac{1}{4} - c_3\right). \qquad (3.93)$$

The discrete spectrum is determined by

$$2(n_{\tilde{\omega}}+n_{\tilde{\tau}})+\lambda_{1\substack{+\\-}}+\lambda_{3\substack{-\\+}}-\lambda_{2\substack{-\\-}}-2=\lambda_{3\substack{+\\-}}\,. \tag{3.94}$$

This is an equation in E in the eighth order which we will not solve.

3.4. The Superintegrable Potential V_4 on $D_{\rm IV}$. We state the potential in the respective coordinate systems

$$V_4(\mu,\nu)=\left(\frac{a_+}{\sin^2 u}+\frac{a_-}{\cos^2 u}\right)^{-1}\frac{\hbar^2}{2m}\left(k_0^2-\frac{1}{4}\right)\left(\frac{1}{\sin^2 u}+\frac{1}{\cos^2 u}\right), \tag{3.95}$$

$$=\left(\frac{a_+}{\nu^2}+\frac{a_-}{\mu^2}\right)^{-1}\frac{\hbar^2}{2m}\left(k_0^2-\frac{1}{4}\right)\left(\frac{1}{\nu^2}+\frac{1}{\mu^2}\right), \tag{3.96}$$

$$=\frac{\hbar^2}{2md^2}\left(\frac{a+2b}{\sinh^2 2\omega'}+\frac{a-2b}{\sin^2 2\varphi'}\right)^{-1}\left(k_0^2-\frac{1}{4}\right)\times$$
$$\times\left(\frac{1}{\cosh^2\omega\cos^2\varphi}+\frac{1}{\sinh^2\omega\sin^2\varphi}\right). \tag{3.97}$$

It is possible to evaluate the path integral for V_4 in all the separating coordinate systems. However, we evaluate the path integral for V_4 only in the (u,v) system because V_4 is trivial.

3.4.1. Separation of V_4 in the (u,v) System. We insert V_4 into the path integral and obtain ($f=a_+/\sin^2 u+a_-/\cos^2 u$)

$$K(u'',u',v'',v';T)=\int\limits_{u(t')=u'}^{u(t'')=u''}\mathcal{D}u(t)\int\limits_{v(t')=v'}^{v(t'')=v''}\mathcal{D}v(t)f(u)\times$$
$$\times\exp\left\{\frac{i}{\hbar}\int\limits_0^T\left[\frac{m}{2}f(u)(\dot{u}^2+\dot{v}^2)-\frac{\hbar^2}{2m}\frac{k_0^2-1/4}{f(u)}\left(\frac{1}{\sin^2 u}+\frac{1}{\cos^2 u}\right)\right]dt\right\}. \tag{3.98}$$

We proceed similarly as in [14]. Because the formulation in (u,v) coordinates is inconvenient, we perform following [12] the coordinate transformation $\cos u=\tanh\tau$. Further, we separate off the v-path integration, and additionally we make a time transformation with the time-transformation function $f=a_+/\sin^2 u+a_-/\cos^2 u$. Due to the coordinate transformation $\cos u=\tanh\tau$

additional quantum terms appear according to

$$\exp\left(\frac{im}{2\epsilon\hbar}\frac{\left(\Delta u^{(j)}\right)^2}{\cos u^{(j-1)}\cos u^{(j)}}\right) \doteq$$

$$\doteq \exp\left[\frac{im}{2\epsilon\hbar}\left(\Delta\tau^{(j)}\right)^2 - i\frac{\hbar}{8m}\left(1+\frac{1}{\cosh^2\tau^{(j)}}\right)\right]. \quad (3.99)$$

We get for the path integral (3.98)

$$K(u'',u',v'',v';T) = \int_{-\infty}^{\infty}\frac{dE}{2\pi\hbar}\,\mathrm{e}^{-iET/\hbar}\times$$

$$\times\int_0^{\infty} ds''\,\exp\left[\frac{i}{\hbar}\left(a_+E - \frac{\hbar^2 k_0^2}{2m}\right)\right]K(\tau'',\tau',v'',v';s''), \quad (3.100)$$

and the time-transformed path integral $K(s'')$ is given by

$$K(\tau'',\tau',v'',v';s'') =$$

$$= \int_{-\infty}^{\infty} dk_v \frac{\mathrm{e}^{ik\ \ (v''-v')}}{2\pi}(\cosh\tau'\cosh\tau'')^{-1/2}\int_{\tau(0)=\tau'}^{\tau(s'')=\tau''}\mathcal{D}\tau(s)\times$$

$$\times\exp\left\{\frac{i}{\hbar}\int_0^{s''}\left[\frac{m}{2}\dot{\tau}^2 - \frac{\hbar^2}{2m}\left(\frac{\lambda_0^2-1/4}{\sinh^2\tau} - \frac{-k_v^2-1/4}{\cosh^2\tau}\right)\right]ds\right\}. \quad (3.101)$$

Inserting the solution for the modified Pöschl–Teller potential and evaluating the Green function on the cut yields for the path integral solution on D_{IV} as follows ($K(u'',u',v'',v';T) = K(\tau'',\tau',v'',v';T)$):

$$K(u'',u',v'',v';T) =$$

$$= \int_{-\infty}^{\infty} dk_v \int_0^{\infty} dp\,\mathrm{e}^{-iTE\ \ /\hbar}\Psi_{p,k\ \ }(\tau'',v'')\Psi^*_{p,k\ \ }(\tau',v'), \quad (3.102)$$

$$\Psi_{p,k\ \ }(\tau,v) = \frac{\mathrm{e}^{ik\ \ v}}{\sqrt{2\pi a_+\cosh\tau}}\,\Psi_p^{(\lambda_0,ik)}(\tau), \quad (3.103)$$

$$E_p = \frac{\hbar^2}{2ma_+}\left(p^2+k_0^2\right), \quad (3.104)$$

where $\lambda_0^2 = k_0^2 - 2ma_e E/\hbar^2$ and the wave functions for the modified Pöschl–Teller functions. Reinserting $\cos u = \tanh \tau$ gives the solution in terms of the variable u.

We also see from this example that the introduction of a third variable w, say, to a three-dimensional version of Darboux space $D_{\rm IV}$ allows separation of variables, where the additional quantum number k_0 corresponds to the motion in w.

4. SUMMARY AND DISCUSSION

In this paper we have finished the discussion of superintegrable potentials on spaces of nonconstant curvature. The results are very satisfactory. There are two potentials on $D_{\rm I}$, four potentials on $D_{\rm II}$, five potentials on $D_{\rm III}$, and four potentials on $D_{\rm IV}$, respectively. We could solve many of the emerging quantum mechanical problems. To give an overview, we summarize our results in Table 5. We list for each space the corresponding potentials including the general form of the solution (if explicitly possible). We omit the trivial potentials here, because they are separable in all corresponding coordinate systems.

In the first Darboux space $D_{\rm I}$ the superintegrable potentials were related to the Holt potential and a shifted isotropic harmonic oscillator in two-dimensional Euclidean space. Whereas the solution in the coordinate v can be expressed in terms of the wave functions for the radial harmonic oscillator (Laguerre polynomials) and the shifted harmonic oscillator (Hermite polynomials), the solution in the coordinate u was determined by a boundary condition for u. This gave wave functions in terms of parabolic cylinder functions and a transcendental equation for the bound state energy levels. The corresponding solution in the rotated (r, q) system was similar. An explicit solution in parabolic coordinates could not be found.

In the second Darboux space there were three nontrivial superintegrable potentials. The potentials were related to the Hold potential, the isotropic singular oscillator, and the Coulomb potential in two-dimensional Euclidean space. We found combinations of polynomial wave functions for the discrete states and combinations of polynomials and Whittaker functions for the scattering states. The discrete energy spectrum for the oscillator-related potentials was usually given by a quadratic equation in the energy. For the Coulomb-related potential we found an equation in eight order in the energy, which could be studied in a special case. Also, in the semiclassical limit, we found that the energy spectra indeed had the behavior of a harmonic oscillator and a Coulomb potential, respectively.

On $D_{\rm III}$ we had potentials related to a linear potential, a Coulomb potential, and a shifted oscillator in two-dimensional flat space. We found for the first po-

Table 5. **Solutions of the path integration for superintegrable potentials in Darboux spaces**

Space and potential	Solution in terms of the wave functions
$D_{\rm I}$	
V_1: (u,v)	Hermite polynomials × Parabolic cylinder functions
Parabolic	No explicit solution
V_2: (u,v)	Hermite polynomials × Parabolic cylinder functions
(r,q)	Hermite polynomials × Parabolic cylinder functions
$D_{\rm II}$	
V_1: (u,v)	Hermite polynomial × Whittaker functions*
Parabolic	No explicit solution
V_2: (u,v)	Laguerre polynomial × Whittaker functions*
Polar	Gegenbauer polynomial × Whittaker functions*
Elliptic	No explicit solution
V_3: Polar	Gegenbauer polynomials × Bessel functions
Parabolic	Product of Whittaker functions*
Elliptic	No explicit solution
$D_{\rm III}$	
V_1: Parabolic	Product of Hermite polynomials/Parabolic cylinder functions
Translated parabolic	Product of Hermite polynomials/Parabolic cylinder functions
V_2: (u,v)	Gegenbauer polynomials × Whittaker functions*
Polar	Gegenbauer polynomials × Whittaker functions*
Parabolic	Product of Whittaker functions*
V_3: Polar	Gegenbauer polynomials × Whittaker functions*
Hyperbolic	No explicit solution
V_4: Hyperbolic	Product of Whittaker functions*
Elliptic	No explicit solution
$D_{\rm IV}$	
V_1: (u,v) system	Product of hypergeometric functions
Horospherical	Product of Whittaker functions*
Elliptic	No explicit solution
V_2: (u,v)	Hypergeometric functions
Degenerate elliptic	Hypergeometric functions
V_3: Elliptic	Hypergeometric functions
Degenerate elliptic	Hypergeometric functions

*The notion Whittaker functions means for a disrete spectrum Laguerre polynomials and for a continuous spectrum Whittaker functions $W_{\mu,\nu}(z)$, respectively.

tential an equation in the fourth order in the energy E, and quadratic equations in the energy E for the second and third potentials. The Coulomb-related potential showed again in the semiclassical limit the behavior of a Coulomb potential. Of some special interest was the feature of the complex periodic Morse potential for the separation of V_3 in polar coordinates. Such complex potentials have attracted in the recent years some attention, because the involved $\mathcal{PT}$ symmetry in these potentials has the consequence that they, nevertheless, have a real spectrum, e.g., [3, 4, 42, 49–51]. Such kind of potentials also appear as subsystems in the list of superintegrable potentials on the complex Euclidean plane [36].

A special feature in $D_{\rm III}$ was that for the free motion there are already positive continuous and negative infinite discrete spectra. A similar feature also exists for the free quantum motion on the $SU(1,1)$ and $SO(2,2)$ hyperboloid.

In the fourth Darboux space we found potentials which were related to the Morse and Pöschl–Teller potential, and combined modified Pöschl–Teller potentials. The modified Pöschl–Teller potentials had, of course, solutions in terms of hypergeometric functions, respectively: Jacobi polynomials (discrete spectrum) and Jacobi functions (scattering states).

We were able to solve the various path integral representations, because we have now to our disposal not only the basic path integrals for the harmonic oscillator, the linear oscillator, the radial harmonic oscillator, and the (modified) Pöschl–Teller potential, but also path-integral identities derived from path integration on harmonic spaces like the elliptic and spheroidal path-integral representations with their more complicated special functions. This includes also numerous transformation techniques to find a particular solution based on one of the basic solutions. Various Green-function analysis techniques can be applied to find an expression not only for the Green function but also for the wave functions and the energy spectrum. Usually, we stated in all cases the solution for the discrete spectrum contribution, i.e., the energy spectrum and the bound-states wave functions. However, not in all cases we stated explictly the scattering states. In the cases where we omitted the explicit representation, this can be done in a straightforward way by inserting the corresponding solution by the potential problem in question and inserting the various coupling constants and scattering quantum numbers.

Let us also note that our solutions are often on a more or less formal level. Neither have we specified an embedding space, nor have we specified boundary conditions on our spaces. For instance, in $D_{\rm I}$ boundary conditions the signature of the ambient space is very important, because choosing a positive or negative signature of the ambient space changes the boundary conditions, and hence the quantization conditions [21]. The same line of reasoning is, of course, valid in the other three Darboux spaces. We have not discussed in detail special cases of the parameters (say a and b), including the limiting cases to flat spaces or spaces

with constant (negative) curvature. Such a discussion would go far beyond the scope of this paper.

Let us finally mention an important observation due to [26]. At the end of their paper Kalnins et al. gave a list of superintegrable potentials on the two-dimensional complex plane and complex sphere. As it turns out, all of the potentials on Darboux spaces can be generated by taking a two-dimensional line element and dividing this line element by a superintegrable potential belonging to a specific class [27]. Not every class generates a new potential on a Darboux space, some are simply related by a coordinate transformation, and some potentials can be generated from the Euclidean plane as well as the complex sphere. The appearance of the complex sphere is especially obvious in the general elliptic coordinate system on $D_{\rm IV}$. Some of the various different potentials coming from the complex plane and sphere are also related by the so-called «coupling constant metamorphosis». Coupling constant metamorphosis always comes into play if the energy E of the quantum system appears in the form of $E\cdot$ metric terms. This observation leads to the notion that every nondegenerate superintegrable system in two dimensions is «Stäckel equivalent» to a superintegrable system in a two-dimensional space of constant curvature [27].

In the language of path integrals coupling constant metamorphosis comes from «time-» or «space-time» transformations (also called Duru–Kleinert transformations [39]). Here the most important example is the Coulomb problem, where by means of a space-time transformation the Coulomb coupling α just becomes a constant and the emerging harmonic oscillator problem has the frequency $\omega^2 = -2E/m$, i.e., the negative energy of the Coulomb problem appears as a harmonic oscillator frequency. As we have seen, this kind of coupling constant metamorphosis or space-time transformation, respectively, had been indispensable tools in the path integral evaluations of the free motion and for the superintegrable potentials, and we can use both notions as synonymously.

We did not go into details of three-dimensional generalization of the Darboux spaces [15]. Of course, it is possible to extend the notion of superintegrability to three-dimensional Darboux spaces. In particular, in three dimensions there are more of such potentials. In total, there are five maximally superintegrable potentials [17], the first four of them are also superintegrable, including the singular harmonic oscillator, the Holt potential and the Coulomb potential. New features will arise due to the fact that on three-dimensional generalization of the more complicated Darboux spaces $D_{\rm III}$ and $D_{\rm IV}$, coordinate systems from the three-dimensional complex sphere come into play [30]. Studies along such lines will be performed in future investigations.

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Appendices

A. PATH INTEGRAL FOR THE FREE MOTION ON $D_{\rm IV}$ IN DEGENERATE ELLIPTIC COORDINATES ($\gamma = 1$)

We start by considering the metric in elliptic coordinates ($\gamma = 1$):

$$ds^2 = \left[a_-\left(\frac{1}{\sinh^2\hat\omega}+\frac{1}{\sin^2\hat\varphi}\right)-a_+\left(\frac{1}{\cosh^2\hat\omega}-\frac{1}{\cos^2\hat\varphi}\right)\right](d\hat\omega^2+d\hat\varphi^2). \quad \text{(A.1)}$$

We formulate the path integral in the usual way. We perform the space-time transformation with the coordinate transformation $\cos\hat\varphi = \tanh\hat\tau$ yielding

$$K(\hat\omega'',\hat\omega',\hat\varphi'',\hat\varphi';T) = \int\limits_{\hat\omega(t')=\hat\omega'}^{\hat\omega(t'')=\hat\omega''} \mathcal{D}\hat\omega(t)\times$$

$$\times \int\limits_{\hat\varphi(t')=\hat\varphi'}^{\hat\varphi(t'')=\hat\varphi''} \mathcal{D}\hat\varphi(t)\left[a_-\left(\frac{1}{\sinh^2\hat\omega}+\frac{1}{\sin^2\hat\varphi}\right)-a_+\left(\frac{1}{\cosh^2\hat\omega}-\frac{1}{\cos^2\hat\varphi}\right)\right]\times$$

$$\times\exp\left[\frac{im}{2\hbar}\int\limits_0^T\left(\frac{a_-}{\sinh^2\hat\omega}-\frac{a_+}{\cosh^2\hat\omega}+\frac{a_-}{\sin^2\hat\varphi}-\frac{a_+}{\cos^2\hat\varphi}\right)(\dot{\hat\omega}^2+\dot{\hat\varphi}^2)dt\right] =$$

$$= \int\limits_{-\infty}^{\infty}\frac{dE}{2\pi\hbar}\,\mathrm{e}^{-iET/\hbar}\int\limits_0^\infty ds''\,\exp\left[\frac{i}{\hbar}\left(a_-E-\frac{\hbar^2}{8m}\right)s''\right]\times$$

$$\times K(\hat\omega'',\hat\omega',\hat\tau'',\hat\tau';s'') \quad \text{(A.2)}$$

with the transformed path integral given by

$$K(\hat{\omega}'', \hat{\omega}', \hat{\tau}'', \hat{\tau}'; s'') = \int\limits_{\hat{\tau}(0)=\hat{\tau}'}^{\hat{\tau}(s'')=\hat{\tau}''} \mathcal{D}\hat{\tau}(s) \int\limits_{\hat{\omega}(0)=\hat{\omega}'}^{\hat{\omega}(s'')=\hat{\omega}''} \mathcal{D}\hat{\omega}(s)\, \cosh\hat{\tau}\times$$

$$\times\exp\left(\frac{i}{\hbar}\int\limits_0^{s''}\left\{\frac{m}{2}(\dot{\hat{\tau}}^2+\cosh^2\hat{\tau}\dot{\hat{\omega}}^2)-\frac{\hbar^2}{2m}\left[\frac{1}{\cosh^2\hat{\tau}}\left(\frac{\lambda_-^2+1/4}{\sinh^2\hat{\omega}}-\frac{\lambda_+^2+1/4}{\cosh^2\hat{\omega}}+\frac{1}{4}\right)-\right.\right.\right.$$
$$\left.\left.\left.-\frac{\lambda_+^2+1/4}{\sinh^2\hat{\tau}}\right]\right\}ds\right), \quad \text{(A.3)}$$

where $\lambda_\pm^2 = \frac{1}{4} - 2ma_\pm E/\hbar^2$. The successive path integrations are of the modified Pöschl–Teller type. Therefore the solution can be written as follows:

$$K(\hat{\omega}'', \hat{\omega}', \hat{\varphi}'', \hat{\varphi}'; T) = \int dk \int p\Psi_k^{(\lambda_-,\lambda_+)}(\hat{\omega}'')\Psi_k^{(\lambda_-,\lambda_+)\,*}(\hat{\omega}')\times$$
$$\times\,\Psi_p^{(\lambda_+,ik)}(\hat{\tau}'')\Psi_p^{(\lambda_+,ik)\,*}(\tau')\,\mathrm{e}^{-i\hbar T p^2/2m} \quad \text{(A.4)}$$

with the energy spectrum

$$E_p = \frac{\hbar^2}{2ma_-}\left(p^2 + \frac{1}{4}\right), \quad \text{(A.5)}$$

and we can reinsert $\tanh\hat{\tau} \to \cos\hat{\varphi}$. The difference of the energy spectra in degenerate elliptic and elliptic coordinates (interchanging of a_+ and a_-) can be removed by a shift of the coordinates $\tilde{\varphi}$ and $\hat{\varphi}$ by $\pi/2$, respectively.

B. PATH INTEGRAL FOR THE FREE MOTION ON D_{IV} IN DEGENERATE ELLIPTIC COORDINATES ($\gamma = 2$)

We start by considering the metric in degenerate elliptic coordinates ($\gamma = 2$):

$$ds^2 = \frac{1}{4}\left(\frac{a_+}{\sinh^2 2\tilde{\omega}} + \frac{a_-}{\sin^2 2\tilde{\varphi}}\right)(d\tilde{\omega}^2 + d\tilde{\varphi}^2). \quad \text{(B.1)}$$

We formulate the path integral in the usual way. We scale both variables by the factor 2 and perform the space-time transformation with the coordinate transfor-

mation $\cos\tilde\varphi = \tanh\tilde\tau$ yielding $\big(\lambda^2 = \frac{1}{4} - 2ma_+E/\hbar^2\big)$:

$$K(\tilde\omega'',\tilde\omega',\tilde\varphi'',\tilde\varphi';T) = \frac{1}{2}\int\limits_{\tilde\omega(t')=\tilde\omega'}^{\tilde\omega(t'')=\tilde\omega''}\mathcal{D}\tilde\omega(t)\int\limits_{\tilde\varphi(t')=\tilde\varphi'}^{\tilde\varphi(t'')=\tilde\varphi''}\mathcal{D}\tilde\varphi(t)\bigg(\frac{a_+}{\sinh^2\tilde\omega}+\frac{a_-}{\sin^2\tilde\varphi}\bigg)\times$$
$$\times\exp\left[\frac{im}{2\hbar}\int\limits_0^T\left(\frac{a_+}{\sinh^2\tilde\omega}+\frac{a_-}{\sin^2\tilde\varphi}\right)(\dot{\tilde\omega}^2+\dot{\tilde\varphi}^2)dt\right] = \int\limits_{-\infty}^{\infty}\frac{dE}{2\pi\hbar}\,\mathrm{e}^{-iET/\hbar}\times$$
$$\times\int\limits_0^\infty ds''\,\exp\left[\frac{i}{\hbar}\left(a_-E-\frac{\hbar^2}{8m}\right)s''\right]K(\tilde\omega'',\tilde\omega',\tilde\tau'',\tilde\tau';s'') \quad \text{(B.2)}$$

with the transformed path integral given by

$$K(\tilde\omega'',\tilde\omega',\tilde\tau'',\tilde\tau';s'') = \int\limits_{\tilde\tau(0)=\tilde\tau'}^{\tilde\tau(s'')=\tilde\tau''}\mathcal{D}\tilde\tau(s)\int\limits_{\tilde\omega(0)=\tilde\omega'}^{\tilde\omega(s'')=\tilde\omega''}\mathcal{D}\tilde\omega(s)\cosh\tilde\tau\times$$
$$\times\exp\left\{\frac{i}{\hbar}\int\limits_0^{s''}\left[\frac{m}{2}(\dot{\tilde\tau}^2+\cosh^2\tilde\tau\dot{\tilde\omega}^2)-\frac{\hbar^2}{2m\cosh^2\tilde\tau}\left(\frac{\lambda^2+1/4}{\cosh^2\tilde\omega}+\frac{1}{4}\right)\right]ds\right\} =$$
$$= (\cosh\tilde\tau'\cosh\tilde\tau'')^{-1/2}\sum_\pm\int\limits_{\mathbb{R}}\frac{dk\,k\,\sinh\pi k}{\cosh^2\pi\lambda+\sinh^2\pi k}\times$$
$$\times P^{ik}_{i\lambda-1/2}(\pm\tanh\omega'')P^{-ik}_{i\lambda-1/2}(\pm\tanh\tilde\omega')\times$$
$$\times\sum_\pm\int\limits_{\mathbb{R}}\frac{dp\,p\,\sinh\pi p}{\cosh^2\pi k+\sinh^2\pi p}\times$$
$$\times P^{ip}_{ik-1/2}(\pm\tanh\tilde\tau'')P^{-ip}_{ik-1/2}(\pm\tanh\tilde\tau')\,\mathrm{e}^{-i\hbar Tp^2/2m}. \quad \text{(B.3)}$$

Therefore we obtain the wave functions and the energy spectrum, respectively,

$$\Psi_{k,p}(\tilde\tau,\tilde\omega) = \frac{1}{\sqrt{2\cosh\tilde\tau}}\left(\frac{k\,\sinh\pi k}{\cosh^2\pi\lambda+\sinh^2\pi k}\frac{p\,\sinh\pi p}{\cosh^2\pi k+\sinh^2\pi p}\right)^{1/2}\times$$
$$\times P^{ik}_{i\lambda-1/2}(\pm\tanh\omega)P^{ip}_{ik-1/2}(\pm\tanh\tilde\tau) \quad \text{(B.4)}$$

and $E_p = \dfrac{\hbar^2}{2ma_-}\left(p^2+\dfrac{1}{4}\right)$, and we can reinsert $\tanh\tilde\tau\to\cos\tilde\varphi$.

C. SUPERINTEGRABLE POTENTIALS ON $E(2,\mathbb{C})$

In this appendix we shortly discuss the path integral representation of superintegrable potentials on the two-dimensional complex Euclidean plane. A thorough path integral discussion on the real two-dimensional complex Euclidean plane has been done in [17], and therefore these solutions will not be repeated here, only some new due to the appearance of three more potentials V_5–V_7. In Table 6 we list the seven coordinate systems on the complex plane $E(2,\mathbb{C})$. As usual $P_1=-i\hbar\partial_x$ and $P_2=-i\hbar\partial_y$ denote the momentum operators, and $M=yP_1-xP_2$ is the angular momentum. The potentials now read as follows [27, 34–36]:

Potential	Coordinate systems
$V_5=\dfrac{B}{2}(x-iy)$	Cartesian, Semihyperbolic, Light Cone
$V_6=\dfrac{\alpha}{2\sqrt{x-iy}}$	Parabolic, Semihyperbolic, Light Cone
$V_7=\dfrac{1}{2}\left[\alpha\dfrac{x^2+y^2}{(x+iy)^4}+\dfrac{\beta}{(x+iy)^2}+\gamma(x^2+y^2)\right]$	Polar, Hyperbolic

(C.1)

In the underlined cases we give a (formal) path integral representation.

The Potential V_5. For the potential V_5 the corresponding Lagrangian has the form

$$\mathcal{L}=\frac{m}{2}(\dot{x}^2+\dot{y}^2)-\frac{B}{2}(x-iy). \tag{C.2}$$

Thus, we identify two linear potentials [13, 45]

$$K^{(V_5)}(x'',x',y'',y';T)=$$
$$=\int\limits_{x(t')=x'}^{x(t'')=x''}\mathcal{D}x(t)\int\limits_{y(t')=y'}^{y(t'')=y''}\mathcal{D}y(t)\exp\left\{\frac{i}{\hbar}\int\limits_{t'}^{t''}\left[\frac{m}{2}(\dot{x}^2+\dot{y}^2)-\frac{B}{2}(x-iy)\right]dt\right\}=$$
$$=\left(\frac{m}{2\pi i\hbar T}\right)\exp\left[\frac{i}{\hbar}\left(\frac{m}{2}\frac{(x''-x')^2+(x''-x')^2}{T}-\right.\right.$$
$$\left.\left.-\frac{BT}{4}(x'+x''-iy'-iy'')\right)\right], \tag{C.3}$$

Table 6. **Coordinate systems on the complex plane** $E(2,\mathbb{C})$

Coordinate system	Integrals of motion	Coordinates
1. Cartesian, $(x, y \in \mathbb{R})$	$I = p_1^2$	x, y
2. Polar $(\varrho > 0, \varphi \in [0, \pi))$	$I = m^2$	$x = \varrho \cos \varphi$ $x = \varrho \sin \varphi$
3. Light cone $(x, y \in \mathbb{R})$	$I = (P_1 + iP_2)^2$	$\hat{x} = x - iy$ $\hat{y} = x + iy$
4. Elliptic $(\omega > 0, \alpha \in [0, 2\pi))$	$I = M^2 - a^2 P_2^2$ $a \neq 0$	$x = \cosh\omega \cos\alpha$ $y = \sinh\omega \sin\alpha$
5. Parabolic $(\xi, \eta > 0)$	$I = \{M, P_2\}$	$x = \frac{1}{2}(\xi^2 - \eta^2)$ $y = \xi\eta$
6. Hyperbolic $(u, v > 0)$	$I = M^2 + (P_1 + iP_2)^2$	$x = \frac{u^2 + u^2v^2 + v^2}{2uv}$ $y = i\frac{u^2 - u^2v^2 + v^2}{2uv}$
7. Semihyperbolic $(w, z \in \mathbb{R})$	$I = \{M, P_1 + iP_2\} + (P_1 - iP_2)^2$	$x = \frac{1}{2}(w-z)^2 + \frac{1}{4}(w+z)$ $y = -\frac{1}{2}(w-z)^2 - \frac{1}{4}(w+z)$

$$= \left(\frac{4m}{\hbar^2 B}\right)^{4/3} \int_{\mathbb{R}} dE\, \mathrm{e}^{-iET/\hbar} \int_{\mathbb{R}} d\lambda \times$$

$$\times \mathrm{Ai}\left[\left(x' - \frac{2E+\lambda}{k}\right)\left(\frac{mB}{\hbar^2}\right)^{1/3}\right] \mathrm{Ai}\left[\left(x'' - \frac{2E+\lambda}{k}\right)\left(\frac{mB}{\hbar^2}\right)^{1/3}\right] \times$$

$$\times \mathrm{Ai}\left[i\left(y' - \frac{2E-\lambda}{k}\right)\left(\frac{mB}{\hbar^2}\right)^{1/3}\right] \mathrm{Ai}\left[i\left(y'' - \frac{2E-\lambda}{k}\right)\left(\frac{mB}{\hbar^2}\right)^{1/3}\right], \quad \text{(C.4)}$$

with the continuous spectrum $E = \hbar^2 p^2/2m$, and λ is the second separation constant.

For V_5 in the semihyperbolic coordinates we obtain for the corresponding Lagrangian ($\dot{w} = dw/dt$)

$$\mathcal{L}_E = \frac{m}{2}(w-z)(\dot{w}^2 - \dot{z}^2) - \frac{B}{2}(w+z) + E, \quad \text{(C.5)}$$

which gives after a time transformation ($\dot{w} = dw/ds$, $\dot{z} = dz/ds$ and $dt = (w-z)ds$) a transformed Lagrangian

$$\tilde{\mathcal{L}}_E = \frac{m}{2}(\dot{w}^2 - \dot{z}^2) - \frac{B}{2}(w^2 - z^2) + E(w - z). \tag{C.6}$$

Therefore the potential v_5 has been transformed into the problem of a shifted harmonic oscillator, whose solution is well known. In order to determine the path integral solution we consider the Green function of the harmonic oscillator [22], use the convolution formula for the kernel in terms of a product of two Green functions

$$\begin{aligned}
K^{(V_5)}(w'', w', z'', z'; T) &= \int\limits_{-\infty}^{\infty} \frac{dE}{2\pi\hbar}\,\mathrm{e}^{-iET/\hbar}\times \\
&\times \int\limits_{0}^{\infty} ds'' K_w(w'', w'; s'') \cdot K_z(z'', z'; s'') = \\
&= \int\limits_{-\infty}^{\infty} \frac{dE}{2\pi\hbar}\,\mathrm{e}^{-iET/\hbar}\frac{\hbar}{2\pi i}\int d\mathcal{E} G_w(E; w'', w'; -\mathcal{E}) G_z(E; z'', z'; \mathcal{E}),
\end{aligned} \tag{C.7}$$

and obtain therefore

$$\begin{aligned}
K^{(V_5)}(w'', w', z'', z'; T) &= \int\limits_{w(t')=w'}^{w(t'')=w''} \mathcal{D}w(t)\times \\
&\times \int\limits_{z(t')=z'}^{z(t'')=z''} \mathcal{D}z(t) \exp\left\{\frac{i}{\hbar}\int\limits_{t'}^{t''}\left[\frac{m}{2}(w-z)(\dot{w}^2 - \dot{z}^2) - \frac{B}{2}(w+z)\right]dt\right\} = \\
&= \frac{1}{4\pi^2}\int\limits_{-\infty}^{\infty} dE \int d\lambda \frac{m}{\pi\hbar^3}\sqrt{\frac{m}{B}}\,\Gamma^2\left(\frac{1}{2} - \frac{E+\lambda}{\hbar\omega}\right)\times \\
&\times D_{-\frac{1}{2}+\frac{+}{\hbar\omega}}\left[\sqrt{\frac{2}{\hbar}\sqrt{mB}}\left(w_> - \frac{E}{b}\right)\right] D_{-\frac{1}{2}+\frac{+}{\hbar\omega}}\left[-\sqrt{\frac{2}{\hbar}\sqrt{mB}}\left(w_< - \frac{E}{b}\right)\right]\times \\
&\times D_{-\frac{1}{2}+\frac{+}{\hbar\omega}}\left[\sqrt{\frac{2}{\hbar}\sqrt{mB}}\left(z_> - \frac{E}{b}\right)\right] D_{-\frac{1}{2}+\frac{+}{\hbar\omega}}\left[-\sqrt{\frac{2}{\hbar}\sqrt{mB}}\left(z_< - \frac{E}{b}\right)\right],
\end{aligned} \tag{C.8}$$

with the continuous spectrum $E = \hbar^2 p^2/2m$, and λ is the second separation constant. The Green function may be evaluated in terms of even and odd parabolic cylinder functions $E_\nu^{(0)}(z)$ and $E_\nu^{(1)}(z)$, e.g., [14,17,22,41], which is omitted here.

The Potential V_6. Let us consider the two Lagrangians of the potential V_6 expressed in parabolic and semihyperbolic coordinates, respectively,

$$\mathcal{L}_E = \frac{m}{2}(\xi^2+\eta^2)(\dot\xi^2+\dot\eta^2) + \sqrt{2}\,\alpha\frac{\xi - i\eta}{\xi^2+\eta^2} + E, \tag{C.9}$$

$$= \frac{m}{2}(w-z)(\dot w^2 - \dot z^2) + i\frac{\sqrt{2}\,\alpha}{w-z} + E, \tag{C.10}$$

which gives after a time transformation ($\dot\xi = d\xi/ds$, $\dot\eta = d\eta/ds$ and $dt = (\xi^2+\eta^2)ds$ in parabolic coordinates; $\dot w = dw/ds$, $\dot z = dz/ds$ and $dt = (w-z)ds$ in semihyperbolic coordinates) the transformed Lagrangians

$$\tilde{\mathcal{L}}_E = \frac{m}{2}(\dot\xi^2+\dot\eta^2) + \sqrt{2}\,\alpha(\xi - i\eta) + (\xi^2+\eta^2), \tag{C.11}$$

$$= \frac{m}{2}(\dot w^2 - \dot z^2) + i\sqrt{2}\,\alpha + E(w-z). \tag{C.12}$$

In parabolic coordinates we have a shifted harmonic oscillator and in semihyperbolic coordinates a linear potential plus a constant. The solution is consequently almost identical to the corresponding solutions for the potential V_5 with appropriate replacement of the coupling constants. See also [14,17,22,41] for more details.

The Potential V_7. Let us consider the last potential V_7. In polar coordinates we have the effective Lagrangian (note the additional $\hbar^2$-potential [22])

$$\mathcal{L} = \frac{m}{2}(\dot\varrho^2 + \varrho^2\dot\varphi^2 - \omega^2) - \frac{\hbar^2}{2mr^2}\left(\alpha\,\mathrm{e}^{-4i\varphi} - 2\beta\,\mathrm{e}^{-2i\varphi} - \frac{1}{4}\right). \tag{C.13}$$

In the variable φ we have a complex periodic Morse potential, the same kind of potentials we have encountered on D_{III} for V_3 in polar coordinates. We identify $\alpha = 4c_1^2$ and $\beta = c_2/c_1$. Furthermore we see that the remaining path integral in the variable ϱ is just a radial harmonic oscillator path integral. Putting everything together yields

$$K^{(V_7)}(\varrho'',\varrho',\varphi'',\varphi';T) = \int\limits_{\varrho(t')=\varrho'}^{\varrho(t'')=\varrho''}\mathcal{D}\varrho(t)\int\limits_{\varphi(t')=\varphi'}^{\varphi(t'')=\varphi''}\mathcal{D}\varphi(t)\times$$

$$\times\exp\left\{\frac{i}{\hbar}\int\limits_0^T\left[\frac{m}{2}(\dot\varrho^2+\varrho^2\dot\varphi^2-\omega^2\varrho^2) - \frac{\hbar^2}{2m\varrho^2}\left(\alpha\,\mathrm{e}^{-4i\varphi} - 2\beta\,\mathrm{e}^{-2i\varphi} - \frac{1}{4}\right)\right]dt\right\} =$$

$$= \sum_{l=0}^{\infty}\Phi^{(c_1,c_2)}_{[\mathrm{cMP}],l}(\varphi'')\Phi^{(c_1,c_2)}_{[\mathrm{cMP}],l}(\varphi')\frac{m\omega}{i\hbar\sin\omega T}\times$$

$$\times\exp\left[-\frac{m\omega}{2i\hbar}({\varrho'}^2+{\varrho''}^2)\cot\omega T\right]I_{l+2\frac{2}{1}+\frac{1}{2}}\left(\frac{m\omega\varrho'\varrho''}{i\hbar\sin\omega T}\right), \tag{C.14}$$

with the well-known expansion by means of the Hille–Hardy formula in terms of Laguerre polynomials for ϱ. We leave the result as it stands.

D. SUPERINTEGRABLE POTENTIALS ON $S(2,\mathbb{C})$

Let us shortly enumerate the superintegrable potentials on the complex sphere. On the real two-dimensional sphere there are two superintegrable potentials, a feature which has been already investigated, e.g., [18]. On the complex two-dimensional sphere there are four more potentials which are listed in (D.4) [27, 30, 34]. In the underlined cases we give a path integral representation. These representations remain, however, on a formal level, because the complex sphere is an abstract space and serves just as a tool to find the relevant potentials. Going to the corresponding real spaces, i.e., the sphere and the hyperboloid, respectively,

Table 7. **Coordinate systems on the complex sphere** $S(2,\mathbb{C})$

Coordinate system	Integrals of motion	Coordinates
1. Spherical $(\vartheta\in[0,\pi),\varphi\in[0,2\pi))$	$L=J_3^2$	$s_1=\sin\vartheta\cos\varphi$ $s_2=\sin\vartheta\sin\varphi$, $s_3=\cos\vartheta$
2. Elliptic	$L=J-1^2+rJ_2^2$	$s_1^2=\dfrac{(ru-1)(rv-1)}{1-r}$ $s_2^2=\dfrac{r(u-1)(v-1)}{1-r}$, $z^2=ruv$
3. Horospherical	$L=(J_1+iJ_2)^2$	$s_1=\dfrac{i}{2}\left(v+\dfrac{y^2-1}{v}\right)$ $s_2=\dfrac{i}{2}\left(v+\dfrac{y^2-1}{v}\right)$, $s_3=iy/v$
4. Degenerate Elliptic 1 $(\tau_{1,2}\in\mathbb{R})$	$L=(J_1+ij_2)^2-c^2J_3^2$	$s_1+is_3=\dfrac{1}{\cosh\tau_1\cosh\tau_2}$ $s_2-is_3=\dfrac{\cosh\tau_2}{\cosh\tau_1}+\dfrac{\cosh\tau_1}{\cosh\tau_2}-$ $-\dfrac{1}{\cosh\tau_1\cosh\tau_2}$ $s_3=\tanh\tau_1\tanh\tau_2$
5. Degenerate Elliptic 2 $(\xi,\eta>0)$	$L=J_3(J_1-iJ_2)^2$	$s_1+is_2=\dfrac{1}{\xi\eta}$ $s_1+is_2=-\dfrac{1}{4}\dfrac{(\xi^2-\eta^2)^2}{\xi\eta}$ $s_3=\dfrac{1}{2}\dfrac{\xi^2+\eta^2}{\xi\eta}$

requires the real representation of the coordinate system in question, including the corresponding path integral representation.

In Table 7 we list the five coordinate systems on the complex sphere $S(2, \mathbb{C})$ according to [27, 30, 34]. Let us note that we can also use $v = i\,\mathrm{e}^{-ix}$ as a parameterization in the horospherical system $(x, y \in \mathbb{R})$. As usual, $J_1, J_2, J-3$ are the angular momentum operators in three dimensions.

The Potential V_3. Let us start superintegrable potential on the two-dimensional complex sphere. It has the form

$$V_3(\mathbf{s}) = \frac{\alpha}{s_3^2} + \frac{\beta}{(s_1 - is_2)^2} + \gamma\frac{(s_1 + is_2)}{(s_1 - is_2)^3}, \tag{D.1}$$

$$= \frac{\alpha}{\cos^2\vartheta^2} + \beta\frac{\mathrm{e}^{-2i\varphi}}{\sin^2\vartheta} - \gamma\frac{\mathrm{e}^{-4i\varphi}}{\sin^2\vartheta}, \tag{D.2}$$

$$= \mathrm{e}^{-2ix}\left(\gamma y^2 + \frac{\alpha}{y^2} + \beta\right) - \gamma\,\mathrm{e}^{-4ix}, \tag{D.3}$$

and we have inserted spherical and horospherical coordinates on the (complex) sphere, respectively,

$$\left.\begin{array}{ll}
V_3(\mathbf{s}) = \dfrac{\alpha}{s_3^2} + \dfrac{\beta}{(s_1 - is_2)^2} + \gamma\dfrac{s_1 + is_2}{(s_1 - is_2)^3} & \begin{array}{l}\underline{\text{Spherical}}\\ \underline{\text{Horospherical}}\\ \text{Degenerate}\\ \text{elliptic I}\end{array} \\
\hline
\begin{array}{l} V_4(\mathbf{s}) = \dfrac{\alpha}{(s_1 - is_2)^2} + \dfrac{\beta s_3}{\sqrt{s_1^2 + s_2^2}} + \\ \qquad + \dfrac{i\gamma}{\sqrt{(s_1 + is_2)(s_1 - is_2)^2}} \end{array} & \begin{array}{l}\text{Spherical}\\ \text{Degenerate}\\ \text{elliptic II}\end{array} \\
\hline
\begin{array}{l} V_5(\mathbf{s}) = \dfrac{\alpha z_+ + c^2 z_-}{\sqrt{(c^2 z_- - z_+)^2 - 4c^2 z_3}} + \\ \qquad + \dfrac{\beta(z_+ - c^2 z_-)(z_+ z_- + z_3^2)}{z_3^2\sqrt{(c^2 z_- - z_+)^2 - 4c^2 z_3}} + \gamma\dfrac{z_+ z_-}{z_3^2} \\ \left(z_\pm = s_1 \pm is_2,\ z_3 = \sqrt{1 - s_1^2 - s_2^2},\ c^2 = \dfrac{1+r}{1-r}\right) \end{array} & \begin{array}{l}\text{Elliptic}\\ \\ \text{Degenerate}\\ \text{elliptic I}\end{array} \\
\hline
V_6(\mathbf{s}) = \dfrac{\alpha}{(s_1 - is_2)^2} + \dfrac{\beta s_3}{(x - iy)^3} + \gamma\dfrac{1 - 4s_3^2}{(s_1 - is_2)^4} & \begin{array}{l}\underline{\text{Horospherical}}\\ \text{Degenerate}\\ \text{elliptic II}\end{array}
\end{array}\right\}. \tag{D.4}$$

This potential has now in spherical coordinates in the φ dependence the same structure as the potential V_7 on the complex plane, thus the solution is the same ($c_{1,2}$ in the complex Morse potential appropriately). In the ϑ dependence we obtain after the separation of φ a Pöschl–Teller potential. In comparison to V_7 with the complex plane, we must therefore replace the wave functions in ϱ in terms of Laguerre polynomials by the Pöschl–Teller wave functions $\Phi_n^{\left(\tilde\alpha, l+2\frac{2}{1}+\frac{1}{2}\right)}(\vartheta)$ $\left(\tilde\alpha^2 = 2m\alpha/\hbar^2 + \dfrac{1}{4}\right)$ and we have done. Summarizing we obtain

$$K^{(V_3)}(\vartheta'',\vartheta',\varphi'',\varphi';T) = \int\limits_{\vartheta(t')=\vartheta'}^{\vartheta(t'')=\vartheta''} \mathcal{D}\vartheta(t) \int\limits_{\varphi(t')=\varphi'}^{\varphi(t'')=\varphi''} \mathcal{D}\varphi(t)\sin\vartheta\times$$

$$\times\exp\left\{\frac{i}{\hbar}\int\limits_{t'}^{t''}\left[\frac{m}{2}(\dot\vartheta^2+\sin^2\vartheta\dot\varphi^2)-\frac{\alpha}{\cos^2\vartheta}-\frac{1}{\sin^2\vartheta}\left(\beta\mathrm{e}^{-2i\varphi}-\gamma\mathrm{e}^{-4i\varphi}-\frac{1}{4}\right)\right]dt\right\} =$$

$$= (\sin\vartheta'\sin\vartheta'')^{-1/2}\sum_{n=0}^{\infty}\sum_{l=0}^{\infty}\Phi^{(c_1,c_2)}_{[\mathrm{cMP}],l}(\varphi'')\Phi^{(c_1,c_2)}_{[\mathrm{cMP}],l}(\varphi')\Phi_n^{\left(l+2\frac{2}{1}+\frac{1}{2},\tilde\alpha\right)}(\vartheta'')\times$$

$$\times\,\Phi_n^{\left(l+2\frac{2}{1}+\frac{1}{2},\tilde\alpha\right)}(\vartheta')\,\exp\left[-\frac{i}{\hbar}\frac{\hbar^2}{2m}\left(2n+l+2\frac{c_2}{c_1}+\frac{3}{2}\right)^2 T\right]. \quad \text{(D.5)}$$

In horospherical coordinates we have in the variable y a radial harmonic oscillator $\left(\text{set } \gamma = m\omega^2/2,\ \tilde\alpha^2 = 2m\alpha/\hbar^2 + \dfrac{1}{4}\right)$ and in the same way ($c_{1,2}$ in the complex Morse potential appropriately)

$$K^{(V_3)}(x'',x',y'',y';T) = \int\limits_{x(t')=x'}^{x(t'')=x''}\mathcal{D}x(t)\int\limits_{y(t')=y'}^{y(t'')=y''}\mathcal{D}y(t)\,\mathrm{e}^{2ix}\times$$

$$\times\exp\left\{\frac{i}{\hbar}\int\limits_{t'}^{t''}\left[\frac{m}{2}(\dot x+\mathrm{e}^{2ix}\dot y^2)-\mathrm{e}^{-2ix}\left(\gamma y^2+\frac{\alpha}{y^2}+\beta\right)-\gamma\mathrm{e}^{-4ix}\right]dt\right\} =$$

$$= \mathrm{e}^{-i(x'+x'')}\sum_{n=0}^{\infty}\sum_{l=0}^{\infty}\Psi_l^{(\mathrm{RHO},\tilde\alpha)}(y'')\Psi_l^{(\mathrm{RHO},\tilde\alpha)}(y')\Phi^{(c_1,c_2)}_{[\mathrm{cMP}],n}(\varphi'')\Phi^{(c_1,c_2)}_{[\mathrm{cMP}],n}(\varphi')\times$$

$$\times\exp\left[-\frac{i}{\hbar}\frac{\hbar^2}{2m}\left(n+2\frac{c_2}{c_1}+1\right)^2 T\right], \quad \text{(D.6)}$$

and the $\Psi_l^{(\mathrm{RHO},\tilde\alpha)}(y)$ are the wave functions of the radial harmonic oscillator [22].

The Potential V_6. As the last potential we consider V_6. We have (set $\gamma = -m\omega^2/8$)

$$V_6(\mathbf{s}) = \frac{\alpha}{(s_1 - is_2)^2} + \frac{\beta s_3}{(x-iy)^3} + \gamma\frac{1-4s_3^2}{(s_1-is_2)^4} \tag{D.7}$$

$$= \mathrm{e}^{-2ix}\frac{m}{2}\omega^2\left(y + \frac{i\beta}{m\omega^2}\right)^2 - \mathrm{e}^{-2ix}\left(\alpha + \frac{\beta^2}{2m\omega^2}\right) - \gamma\,\mathrm{e}^{-4ix}, \tag{D.8}$$

and we have inserted horospherical coordinates. This potential is, in the variable y, a shifted harmonic oscillator, however, the shift is a complex one. In the variable x we have the complex periodic Morse potential. Again, we encounter a complex potential, this time a $\mathcal{PT}$-symmetric harmonic oscillator with spectrum $E_l = \hbar\omega(l+1/2)$, e.g., [49]. Consequently, we have in a similar way as before ($c_{1,2}$ in the complex Morse potential appropriately, set $\kappa = i\beta/m\omega^2$):

$$K^{(V_6)}(x'',x',y'',y';T) = \int\limits_{x(t')=x'}^{x(t'')=x''}\mathcal{D}x(t)\int\limits_{y(t')=y'}^{y(t'')=y''}\mathcal{D}y(t)\,\mathrm{e}^{2ix}\times$$

$$\times\exp\left\{\frac{i}{\hbar}\int\limits_{t'}^{t''}\left[\frac{m}{2}(\dot{x}^2 + \mathrm{e}^{2ix}\dot{y}^2) - \left(\frac{m}{2}\omega^2\left(y+\frac{i\beta}{m\omega^2}\right)^2 + \right.\right.\right.$$

$$\left.\left.\left. + \left(\alpha + \frac{\beta^2}{2m\omega^2}\right)\right)\mathrm{e}^{-2ix} - \gamma\,\mathrm{e}^{-4ix}\right]dt\right\} =$$

$$= \mathrm{e}^{-i(x'+x'')}\sum_{l=0}^{\infty}\Psi_l^{(\mathrm{HO})}(y'')\Psi_l^{(\mathrm{HO})}(y')\sum_{n=0}^{\infty}\Phi^{(c_1,c_2)}_{[\mathrm{cMP}],n}(\varphi'')\Phi^{(c_1,c_2)}_{[\mathrm{cMP}],n}(\varphi')\times$$

$$\times\exp\left[-\frac{i}{\hbar}\frac{\hbar^2}{2m}\left(n + 2\frac{c_2}{c_1} + 1\right)^2 T\right], \tag{D.9}$$

and the $\Psi_l^{(\mathrm{HO},\kappa)}(y)$ are the wave functions of the shifted harmonic oscillator [22]. The representations of the potentials V_4 and V_5 in the separating coordinate systems lead to intractable powers in the various coordinates, respectively, powers of $\cosh\tau_{1,2}$, i.e., highly anharmonic terms which cannot be treated. The same holds for V_3 and V_6 in the remaining separating coordinate systems. This concludes the discussion.

REFERENCES

1. *Bender C. M. et al.* The $\mathcal{C}$ Operator in $\mathcal{PT}$-Symmetric Quantum Theories // J. Phys. A: Math. Gen. 2004. V. 37. P. 10139–10165.

2. *Böhm M., Junker G.* Path Integration over Compact and Noncompact Rotation Groups // J. Math. Phys. 1987. V. 28. P. 1978–1994.

3. *Bagchi B., Quesne C., Znojil M.* Generalized Continuity Equation and Modified Normalization in PT-Symmetric Quantum Mechanics // Mod. Phys. Lett. A. 2001. V. 16. P. 2047–2057.

4. *Cannata F., Junker J., Trost J.* Schrödinger Operators with Complex Potential but Real Spectrum // Phys. Lett. A. 1998. V. 246. P. 219–226.

5. *Daskaloyannis C., Ypsilantis K.* Unified Treatment and Classification of Superintegrable Systems with Integrals Quadratic in Momenta on a Two-Dimensional Manifold // J. Math. Phys. 2006. V. 45. P. 042904.

6. *Del Olmo M. A., Rodríguez M. A., Winternitz P.* The Conformal Group $SU(2, 2)$ and Integrable Systems on a Lorentzian Hyperboloid // Fortschr. Phys. 1996. Bd. 44. S. 199–233; Integrable Systems Based on $SU(p, q)$ Homogeneous Manifolds // J. Math. Phys. 1993. V. 34. P. 5118–5139.

7. *Evans N. W.* Superintegrability in Classical Mechanics // Phys. Rev. A. 1990. V. 41. P. 5666–5676; Group Theory of the Smorodinsky–Winternitz System // J. Math. Phys. 1991. V. 32. P. 3369–3375; Superintegrability of the Winternitz System // Phys. Lett. A. 1990. V. 147. P. 483–486.

8. *Feynman R. P., Hibbs A.* Quantum Mechanics and Path Integrals. N. Y.: McGraw Hill, 1965.

9. *Friš J. et al.* On Higher Symmetries in Quantum Mechanics // Phys. Lett. 1965. V. 16. P. 354–356;
Friš J. et al. Symmetry Groups in Classical and Quantum Mechanics // Sov. J. Nucl. Phys. 1967. V. 4. P. 444–450.

10. *Gradshteyn I. S., Ryzhik I. M.* Table of Integrals, Series, and Products. N. Y.: Acad. Press, 1980.

11. *Grosche C.* The Path Integral on the Poincaré Upper Half-Plane with a Magnetic Field and for the Morse Potential // Ann. Phys. (N. Y.) 1988. V. 187. P. 110–134.

12. *Grosche C.* The Path Integral on the Poincaré Disc, the Poincaré Upper Half-Plane and on the Hyperbolic Strip // Fortschr. Phys. 1990. Bd. 38. S. 531–569.

13. *Grosche C.* Path Integrals, Hyperbolic Spaces, and Selberg Trace Formulae. Singapore: World Sci., 1996.

14. *Grosche C.* Path Integration on Darboux Spaces // Part. Nucl. 2006. V. 37. P. 368–389.

15. *Grosche C.* Path Integral Approach for Spaces of Nonconstant Curvature in Three Dimensions. DESY Report, DESY 05-221. 2005; quant-ph/0511135; Proc. of the II Intern. Workshop on Superintegrable Systems in Classical and Quantum Mechanics, Dubna, June 27 – July 1, 2005 // Phys. At. Nucl. 2006. V. 69.

16. *Grosche C. et al.* Quantum Motion on the Three-Dimensional Sphere: The Ellipso-Cylindrical Bases // J. Phys. A: Math. Gen. 1997. V. 30. P. 1629–1657.

17. *Grosche C., Pogosyan G. S., Sissakian A. N.* Path Integral Discussion for Smorodinsky–Winternitz Potentials: I. Two- and Three-Dimensional Euclidean Space // Fortschr. Phys. 1995. Bd. 43. S. 453–521.

18. *Grosche C., Pogosyan G. S., Sissakian A. N.* Path Integral Discussion for Smorodinsky–Winternitz Potentials: II. The Two- and Three-Dimensional Sphere // Ibid. S. 523–563.

19. *Grosche C., Pogosyan G. S., Sissakian A. N.* Path Integral Approach to Superintegrable Potentials on the Two-Dimensional Hyperboloid // Part. Nucl. 1996. V. 27. P. 244–278.

20. *Grosche C., Pogosyan G. S., Sissakian A. N.* Path Integral Approach for Superintegrable Potentials on the Three-Dimensional Hyperboloid // Part. Nucl. 1997. V. 28. P. 486–519.

21. *Grosche C., Pogosyan G. S., Sissakian A. N.* Path Integral Approach for Superintegrable Potentials on Spaces of Nonconstant Curvature: I. Darboux Spaces D_{I} and D_{II}. DESY preprint, DESY 06-113. 2006; Part. Nucl. 2007. V. 38, No. 3. P. 587.

22. *Grosche C., Steiner F.* Handbook of Feynman Path Integrals. Springer Tracts in Modern Physics. Berlin; Heidelberg, 1998. V. 145.

23. *Kalnins E. G.* On the Separation of Variables for the Laplace Equation $\Delta\psi + K^2\psi = 0$ in Two- and Three-Dimensional Minkowski Space // SIAM J. Math. Anal. 1975. V. 6. P. 340–374.

24. *Kalnins E. G.* Separation of Variables for Riemannian Spaces of Constant Curvature. Essex: Longman Sci. & Technical, 1986.

25. *Kalnins E. G., Kress J. M., Miller W. (Jr.).* Jacobi, Ellipsoidal Coordinates, and Superintegrable Systems // J. Nonlin. Math. Phys. 2005. V. 12. P. 209–229.

26. *Kalnins E. G. et al.* Superintegrable Systems in Darboux Spaces // J. Math. Phys. 2003. V. 44. P. 5811–5848.

27. *Kalnins E. G. et al.* Completeness of Superintegrability in Two-Dimensional Constant-Curvature Spaces // J. Phys. A: Math. Gen. 2001. V. 34. P. 4705–4720.

28. *Kalnins E. G., Kress J. M., Winternitz P.* Superintegrability in a Two-Dimensional Space of Nonconstant Curvature // J. Math. Phys. 2002. V. 43. P. 970–983.

29. *Kalnins E. G., Miller W. (Jr.).* Lie Theory and Separation of Variables. 4. The Groups $SO(2,1)$ and $SO(3)$ // J. Math. Phys. 1974. V. 15. P. 1263–1274.

30. *Kalnins E. G., Miller W. (Jr.).* The Wave Equation, $\underset{\sim}{O}(2,2)$, and Separation of Variables on Hyperboloids // Proc. Roy. Soc. A (Edinburgh). 1977. V. 79. P. 227–256;
Kalnins E. G., Miller W. (Jr.). Lie Theory and the Wave Equation in Space-Time. 2. The Group $SO(4,\mathbb{C})$ // SIAM J. Math. Anal. 1978. V. 9. P. 12–33.

31. *Kalnins E. G. et al.* Superintegrability on the Two-Dimensional Hyperboloid II // J. Math. Phys. 1999. V. 40. P. 2291–2306.

32. *Kalnins E. G., Miller W. (Jr.), Pogosyan G. S.* Superintegrability and Associated Polynomial Solutions. Euclidean Space and the Sphere in Two Dimensions // J. Math. Phys. 1996. V. 37. P. 6439–6467.

33. *Kalnins E. G., Miller W. (Jr.), Pogosyan G. S.* Superintegrability on the Two-Dimensional Hyperboloid // J. Math. Phys. 1997. V. 38. P. 5416–5433.

34. *Kalnins E. G., Miller W. (Jr.), Pogosyan G. S.* Completeness of Multiseparable Superintegrability on the Complex 2-Sphere // J. Phys. A: Math. Gen. 2000. V. 33. P. 6791–6806.

35. *Kalnins E. G., Miller W. (Jr.), Pogosyan G. S.* Completeness of Multiseparable Superintegrability in $E_{2,C}$ // Ibid. P. 4105–4120.

36. *Kalnins E. G., Miller W. (Jr.), Pogosyan G. S.* Superintegrability on Two-Dimensional Complex Euclidean Space. Algebraic Methods in Physics: A Symp. for the 60th Birthdays of Jiří Patera and Pavel Winternitz // CRM Series in Math. Phys. Berlin; Heidelberg, 2001. P. 95–103.

37. *Kalnins E. G., Miller W. (Jr.), Pogosyan G. S.* Completeness of Multiseparable Superintegrability in Two Dimensions // Phys. At. Nucl. 2002. V. 65. P. 1033–1035.

38. *Kalnins E. G., Miller W. (Jr.), Pogosyan G. S.* Exact and Quasi-Exact Solvability of Two-Dimensional Superintegrable Quantum Systems. I. Euclidean Space. math-ph/0412035.

39. *Kleinert H.* Path Integrals in Quantum Mechanics, Statistics and Polymer Physics. Singapore: World Sci., 1990.

40. *Koenigs G.* Sur les géodésiques a intégrales quadratiques // *Darboux G.* Lecons sur la théorie générale des surface. Chelsea Publ., 1972. V. 4. P. 368–404.

41. *Lutsenko I. V. et al.* Hydrogen Atom as Indicator of Hidden Symmetry of a Ring-Shaped Potential // Theor. Math. Phys. 1990. V. 83. P. 633–639.

42. *Mostafazadeh A., Batal A.* Physical Aspects of Pseudo-Hermitian and PT-Symmetric Quantum Mechanics // J. Phys. A: Math. Gen. A. 2004. V. 37. P. 11645–11680.

43. *Olevskiĭ M. N.* Triorthogonal Systems in Spaces of Constant Curvature in which the Equation $\Delta_2 u + \lambda u = 0$ Allows the Complete Separation of Variables // Mat. Sb. 1950. V. 27. P. 379–426 (in Russian).

44. *Peak D., Inomata A.* Summation over Feynman Histories in Polar Coordinates // J. Math. Phys. 1969. V. 10. P. 1422–1428.

45. *Schulman L. S.* Techniques and Applications of Path Integration. N. Y.: John Wiley & Sons, 1981.

46. *Ushveridze A.* Quasi-Exactly Solvable Models in Quantum Mechanics. Bristol: Inst. of Phys. Publ., 1994.

47. *Winternitz P. et al.* Symmetry Groups in Classical and Quantum Mechanics // Sov. J. Nucl. Phys. 1967. V. 4. P. 444–450.

48. *Wojciechowski S.* Superintegrability of the Calogero–Moser System // Phys. Lett. A. 1983. V. 95. P. 279–281.

49. *Znojil M.* $\mathcal{PT}$-Symmetric Harmonic Oscillators // Phys. Lett. A. 1999. V. 259. P. 220–223.

50. *Znojil M.* Exact Solution for Morse Oscillator in PT-Symmetric Quantum Mechanics // Phys. Lett. A. 1999. V. 264. P. 108–111.

51. *Znojil M.* Workshop on Pseudo-Hermitian Hamiltonians in Quantum Physics. I–III // Czech. J. Phys. 2004. V. 54. P. 1–156; 1005–1148; V. 55. P. 1045–1192; http://gemma.ujf.cas.cz/%7Eznojil/conf/index.html

ФИЗИКА ЭЛЕМЕНТАРНЫХ ЧАСТИЦ И АТОМНОГО ЯДРА
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PATH-INTEGRAL APPROACH FOR SUPERINTEGRABLE POTENTIALS ON SPACES OF NONCONSTANT CURVATURE: II. DARBOUX SPACES D_{III} AND D_{IV}

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This is the second paper on the path-integral approach of superintegrable systems on Darboux spaces, spaces of nonconstant curvature. We analyze in the spaces D_{III} and D_{IV} five and, respectively, four superintegrable potentials, which were first given by Kalnins et al. We are able to evaluate the path integral in most of the separating coordinate systems, leading to expressions for the Green functions, the discrete and continuous wave functions, and the discrete energy-spectra. In some cases, however, the discrete spectrum cannot be stated explicitly, because it is determined by a higher order polynomial equation. We also show that the free motion in Darboux space of type III can also contain bound states, provided the boundary conditions are appropriate. We can state the energy spectrum and the wave functions, respectively.

Это вторая статья, посвященная приближению интегралов по путям для суперинтегрируемых систем на пространствах Дарбу, пространствах переменной кривизны. На пространствах Дарбу D_{III} и D_{IV} проводится анализ пяти и, соответственно, четырех суперинтегрируемых потенциалов, которые впервые были представлены Калнинсом и др. Нам удалось вычислить интеграл по путям в наиболее разделяющихся системах координат, что приводит к выражениям для функций Грина, волновым функциям дискретного и непрерывного спектров и дискретному спектру энергий. Однако в некоторых случаях дискретный спектр установить не удается, так как он определяется полиномиальным уравнением более высокого порядка. Показано, что свободное движение в пространстве Дарбу III типа также может содержать связанные состояния при определенных граничных условиях. Соответственно, для них можно установить спектр энергий и волновые функции.

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1. INTRODUCTION

In the previous publication [21] we have started to study superintegrable systems on spaces of nonconstant curvature, i.e., Darboux spaces. These spaces were introduced by Kalnins et al. [26, 28]. In the first paper we have studied

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the Darboux spaces $D_{\rm I}$ and $D_{\rm II}$, and we continue our study by considering the two other Darboux spaces $D_{\rm III}$ and $D_{\rm IV}$ with five and, respectively, four superintegrable potentials as determined in [26].

We find a rich structure of the spectrum of these potentials yielding bound and continuous states. As it turns out, already the free motion on $D_{\rm III}$ can give a positive continuous and an infinite negative discrete spectrum. This situation is similar to that for the quantum motion on the $SU(1,1)$ manifold [2], respectively, on the $SU(2,2)$ [6] and $SO(2,2)$ manifold [30].

The notion of superintegrable systems was introduced by Winternitz and co-workers in [9, 47], Wojciechowski [48], and was developed further later on also by Evans [7]. Superintegrable potentials have the property of finding additional constants of motion. In two dimensions one has in total three functional independent constants of motion and in three dimensions one has four (minimal superintegrable) and five (maximal superintegrable) functional independent constants of motion. Well-known examples are the Coulomb potential with its Lenz–Runge vector and the harmonic oscillator with its quadrupole moment. Another property of superintegrable potentials is that usually the corresponding equations in classical and quantum mechanics separate in more than one coordinate system.

Similar studies of the quantum motion on spaces with and without curvature have been investigated in [17] for two- and three-dimensional flat space, in [18] for the two- and three-dimensional sphere, and in [19] and [20] for the two- and three-dimensional hyperboloid. In all these cases the path integral method [8, 22, 39, 45] was applied to find the bound and continuous states, i.e., wave functions and the explicit form of the spectrum. We have not considered complexified spaces as in [37] for the two-dimensional complex sphere or in [34–36] for the two-dimensional complex Euclidean space. In particular, in [34] coordinate systems on the two-dimensional complex sphere and corresponding superintegrable potentials, and in [36] coordinate systems on the two-dimensional complex plane and corresponding superintegrable potentials were discussed. The goal of [34, 36] was to extend the notion of superintegrable potentials of real spaces to the corresponding complexified spaces. The findings were that there are, in addition to the four coordinate systems on the real two-dimensional Euclidean plane, three more coordinate systems and also three more superintegrable potentials. Similarly, in addition to the two coordinate systems on the real two-dimensional sphere there are three more coordinate systems on the complex sphere and four more superintegrable potentials. This is not surprising because the complex plane contains not only the Euclidean plane but also the pseudo-Euclidean plane (10 coordinate systems [13, 23, 24]), and the complex sphere contains not only the real sphere but also the two-dimensional hyperboloid (9 coordinate systems [13, 24, 29, 43]).

However, a complexified space is an abstract object. In order to obtain the actual spectrum of a given potential formulated in a coordinate system one has to consider a real version of the complexified space, e.g., the complex sphere: One has to determine whether one considers the potential on the real sphere or on the real hyperboloid. The complexification serves only as a tool for a unified investigation.

Further studies on superintegrability in spaces with constant curvature are due to [31, 33] (hyperboloid with new potentials), [32] (sphere and Euclidean space), [37] and [38] with a general theory about the connection of separation in nonsubgroup coordinate systems of superintegrable systems and quasi-exactly-solvable problems [46].

An extension of the study of path integration on spaces of constant curvature is the investigation of path integral formulations in spaces of nonconstant curvature. Kalnins et al. [26, 28] denoted four types of two-dimensional spaces of nonconstant curvature, labeled by $D_{\rm I}$–$D_{\rm IV}$, which are called Darboux spaces [40]. In terms of the infinitesimal distance they are described by (the coordinates (u, v) will be called the (u, v) system; the (x, y) system in turn can be called light-cone coordinates):

$$\begin{aligned} \text{(I)}\quad ds^2 &= (x+y)dxdy = \\ &= 2u(du^2+dv^2) \quad (x=u+iv, y=u-iv), \end{aligned} \tag{1.1}$$

$$\begin{aligned} \text{(II)}\quad ds^2 &= \left(\frac{a}{(x-y)^2}+b\right)dxdy = \\ &= \frac{bu^2-a}{u^2}(du^2+dv^2) \quad \left(x=\frac{1}{2}(v+iu), y=\frac{1}{2}(v-iu)\right), \end{aligned} \tag{1.2}$$

$$\begin{aligned} \text{(III)}\quad ds^2 &= \left(a\,\mathrm{e}^{-(x+y)/2}+b\,\mathrm{e}^{-x-y}\right)dxdy = \\ &= \mathrm{e}^{-2u}(b+a\,\mathrm{e}^{u})(du^2+dv^2) \quad (x=u-iv, y=u+iv), \end{aligned} \tag{1.3}$$

$$\begin{aligned} \text{(IV)}\quad ds^2 &= -\frac{a\left(\mathrm{e}^{(x-y)/2}+\mathrm{e}^{(y-x)/2}\right)+b}{\left(\mathrm{e}^{(x-y)/2}-\mathrm{e}^{(y-x)/2}\right)^2}dxdy = \\ &= \left(\frac{a_+}{\sin^2 u}+\frac{a_-}{\cos^2 u}\right)(du^2+dv^2) \quad (x=u+iv, y=u-iv), \end{aligned} \tag{1.4}$$

where a and b are additional (real) parameters ($a_\pm = (a \pm 2b)/4$). These surfaces are also called surfaces of revolution [5, 25, 26]. Kalnins et al. [26, 28] studied not only the solution of the free motion, but also placed emphasis on the superintegrable systems in these spaces.

The Gaussian curvature in a space with metric $ds^2 = g(u,v)(du^2 + dv^2)$ is given by ($g = \det\, g(u,v)$)

$$G = -\frac{1}{2g}\left(\frac{\partial^2}{\partial u^2} + \frac{\partial^2}{\partial v^2}\right)\ln\, g. \tag{1.5}$$

Equation (1.5) will be used to discuss shortly the curvature properties of the Darboux spaces, including their limiting cases of constant curvature.

In the following sections we discuss superintegrable potentials in each of the two Darboux spaces $D_{\rm III}$ and $D_{\rm IV}$, respectively. We set up the classical Lagrangian and Hamiltonian, the quantum operator, and formulate and solve (if possible) the corresponding path integral. We also discuss some of the limiting cases of the Darboux spaces, i.e., where we obtain a space of constant (zero or negative) curvature. For the Darboux space $D_{\rm III}$ the zero-curvature case $\mathbb{R}^2$ emerges. In $D_{\rm IV}$ we find a hyperboloid.

In the last section we summarize our results, where we also include the findings of our previous paper which dealt with superintegrable potentials on $D_{\rm I}$ and $D_{\rm II}$.

In the first two appendices we add some additional material about the path integral evaluation of the free motion in $D_{\rm IV}$ in degenerate elliptic coordinates. In the third appendix we summarize briefly the path integral investigation of some remaining superintegrable potentials on the two-dimensional Euclidean plane. Finally, in the fourth appendix an example of a potential on the two-dimensional complex sphere will be given.

2. SUPERINTEGRABLE POTENTIALS ON DARBOUX SPACE $D_{\rm III}$

The coordinate systems to be considered in the Darboux space $D_{\rm III}$ are as follows:

$$((u,v)\ \text{system})\quad x = v + iu, \quad y = v - iu, \tag{2.1}$$

$$(\text{Polar:})\quad \xi = \varrho\cos\varphi, \quad \eta = \varrho\sin\varphi \quad (\varrho > 0, \varphi \in [0, 2\pi]), \tag{2.2}$$

$$(\text{Parabolic:})\quad \xi = 2\,\mathrm{e}^{-u/2}\cos\frac{v}{2}, \quad \eta = 2\,\mathrm{e}^{-u/2}\sin\frac{v}{2},$$

$$u = \ln\frac{4}{\xi^2+\eta^2},\ v = \arcsin\frac{2\xi\eta}{\xi^2+\eta^2}\ (\xi\in\mathbb{R}, \eta > 0), \tag{2.3}$$

$$(\text{Elliptic:})\quad \xi = d\cosh\,\omega\cos\,\varphi,\ \eta = d\sinh\,\omega\sin\,\varphi\ (\omega > 0, \varphi\in[-\pi,\pi]), \tag{2.4}$$

$$(\text{Hyperbolic:})\quad \xi = \frac{\mu-\nu}{2\sqrt{\mu\nu}} + \sqrt{\mu\nu},\ \eta = i\left(\frac{\mu-\nu}{2\sqrt{\mu\nu}} - \sqrt{\mu\nu}\right)\ (\mu,\nu > 0). \tag{2.5}$$

For the line element we get (we also display where the metric is rescaled in such a way that we set $a = b = 1$ [26]):

$$ds^2 = \mathrm{e}^{-2u}(b + a\,\mathrm{e}^u)(du^2 + dv^2) = (\mathrm{e}^{-u} + \mathrm{e}^{-2u})(du^2 + dv^2), \quad (2.6)$$

$$\text{(Polar:)} = \left(a + \frac{b}{4}\varrho^2\right)(d\varrho^2 + \varrho^2 d\varphi^2) = \left(1 + \frac{1}{4}\varrho^2\right)(d\varrho^2 + \varrho^2 d\varphi^2), \quad (2.7)$$

$$\text{(Parabolic:)} = \left(a + \frac{b}{4}(\xi^2 + \eta^2)\right)(d\xi^2 + d\eta^2) = \\ = \left(1 + \frac{1}{4}(\xi^2 + \eta^2)\right)(d\xi^2 + d\eta^2), \quad (2.8)$$

$$\text{(Elliptic:)} = \left(a + \frac{b}{4}d^2(\sinh^2\omega + \cos^2\varphi)\right) d^2 \times \\ \times (\sinh^2\omega + \sin^2\varphi)(d\omega^2 + d\varphi^2), \quad (2.9)$$

$$\text{(Hyperbolic:)} = \left(a + \frac{b}{2}(\mu - \nu)\right)(\mu + \nu)\left(\frac{d\mu^2}{\mu^2} - \frac{d\nu^2}{\nu^2}\right). \quad (2.10)$$

For the Gaussian curvature we find

$$G = -\frac{ab\,\mathrm{e}^{-3u}}{(b\,\mathrm{e}^{-2u} + a\,\mathrm{e}^{-u})^4}. \quad (2.11)$$

For, e.g., $a = 1, b = 0$ we recover the two-dimensional flat space with the corresponding coordinate systems. To assure the positive definiteness of the metric (1.3), we can require $a, b > 0$. We introduce the following constants of motion on D_{III}:

$$X_1 = \frac{1}{4}\frac{\mathrm{e}^{2u}}{a + b\,\mathrm{e}^u}\cos v \cdot p_u^2 - \frac{1}{4}\frac{\mathrm{e}^u(e^u + 2)}{a + b\,\mathrm{e}^u}\cos v \cdot p_v^2 + \frac{1}{2}\mathrm{e}^u \sin v \cdot p_u p_v, \quad (2.12)$$

$$X_2 = \frac{1}{4}\frac{\mathrm{e}^{2u}}{a + b\,\mathrm{e}^u}\sin v \cdot p_u^2 - \frac{1}{4}\frac{\mathrm{e}^u(\mathrm{e}^u + 2)}{a + b\,\mathrm{e}^u}\sin v \cdot p_v^2 + \frac{1}{2}\mathrm{e}^u \cos v \cdot p_u p_v, \quad (2.13)$$

$$K = p_v. \quad (2.14)$$

These operators satisfy the Poisson relations

$$\{K, X_1\} = -X_2, \qquad \{K, X_2\} = X_1, \qquad \{X_1, X_2\} = K\tilde{\mathcal{H}}_0, \quad (2.15)$$

and the functional relation

$$X_1^2 + X_2^2 - \tilde{\mathcal{H}}_0^2 - \tilde{\mathcal{H}}_0 K^2 = 0. \quad (2.16)$$

Table 1. **Constants of motion and limiting cases of coordinate systems on $D_{\rm III}$**

Metric	Constant of motion	$D_{\rm III}$	E_2 $(a=1, b=0)$
$e^{-2u}(b + a\,e^u)(du^2 + dv^2)$	K^2	(u, v) system	Cartesian
$\left(a + \frac{b}{4}\varrho^2\right)(d\varrho^2 + \varrho^2 d\varphi^2$	X_2	Polar	Polar
$\left(a + \frac{b}{4}(\xi^2 + \eta^2)\right)(d\xi^2 + d\eta^2)$	X_1	Parabolic	Parabolic
$\left(a + \frac{b}{4}d^2(\sinh^2\omega + \cos^2\varphi)\right) d^2\times$ $\times(\sinh^2\omega + \sin^2\varphi)(d\omega^2 + d\varphi^2)$	$d^2X_1 + 2K^2$	Elliptic	Elliptic

The operators K, X_1, X_2 can be used to characterize the separating coordinate systems on $D_{\rm III}$, as indicated in Table 1. The corresponding quantum operators are given by

$$X_1 = \frac{1}{4}\,e^u\left[\frac{e^u\cos v}{a + b\,e^u}\cdot\partial_u^2 - \frac{e^u + 2}{a + b\,e^u}\cos v\cdot\partial_v^2 + \right.$$
$$\left. + \left(2\sin v\cdot\partial_u\partial_v + \cos v\cdot\partial_u + \sin v\cdot\partial_v\right)\right], \qquad (2.17)$$

$$X_2 = \frac{1}{4}\,e^u\left[\frac{e^u\sin v}{a + b\,e^u}\cdot\partial_u^2 - \frac{e^u + 2}{a + b\,e^u}\sin v\cdot\partial_v^2 - \right.$$
$$\left. - \left(2\cos v\cdot\partial_u\partial_v - \sin v\cdot\partial_u + \cos v\cdot\partial_v\right)\right], \qquad (2.18)$$

$$K = \partial_v. \qquad (2.19)$$

These operators satisfy the commutation relations

$$[\widehat{K}, \widehat{X}_1] = -\widehat{X}_2, \quad [\widehat{K}, \widehat{X}_2] = \widehat{X}_1, \quad [\widehat{X}_1, \widehat{X}_2] = \widehat{K}\widehat{H}_0, \qquad (2.20)$$

and the relation

$$\widehat{X}_1^2 + \widehat{X}_2^2 - \widehat{H}_0^2 - \widehat{H}_0\widehat{K}^2 + \frac{1}{4}\widehat{H}_0 = 0. \qquad (2.21)$$

(Let us note that by $\tilde{\mathcal{H}}_0$ the classical Hamiltonian without the $1/2m$ factor is meant. Keeping this factor is no problem, however, in the present form the algebra is simpler.)

We now state the superintegrable potentials on $D_{\rm III}$:

$$V_1(u,v) = \frac{2k_1\,\mathrm{e}^{-u}\cos\,v/2 + 2k_2\,\mathrm{e}^{-u}\sin v/2 + k_3}{a + \dfrac{b}{4}\,\mathrm{e}^{-u}}, \tag{2.22}$$

$$V_2(u,v) = \frac{1}{a + b\,\mathrm{e}^{-u}}\left[-\alpha + \mathrm{e}^u\frac{\hbar^2}{8m}\left(\frac{k_1^2 - 1/4}{\cos^2 v/2} + \frac{k_1^2 - 1/4}{\cos^2 v/2}\right)\right], \tag{2.23}$$

$$V_3(u,v) = \frac{1}{a + b\,\mathrm{e}^{-u}}\left[-\alpha + \frac{\hbar^2}{2m}4\,\mathrm{e}^u\Big(c_1^2\,\mathrm{e}^{-iv} - 2c_2\,\mathrm{e}^{-2iv}\Big)\right], \tag{2.24}$$

Table 2. **Separation of variables for the superintegrable potentials on $D_{\rm III}$**

Potential	Constants of motion	Separating coordinate system
V_1	$R_1 = X_1 + \dfrac{2k_1\xi(2+\eta^2) - 2k_2\eta(2+\xi^2) + k_3(\eta^2-\xi^2)}{4a + b(\xi^2+\eta^2)}$ $R_2 = X_2 + \dfrac{k_1\eta(\eta^2-\xi^2+4) + k_2\xi(\xi^2-\eta^2+4) - 2xk_3\xi\eta}{4a+b(\xi^2+\eta^2)}$	Parabolic Translated Parabolic $(\xi,\eta\to\xi\eta\pm c)$
V_2	$R_1 = X_1 +$ $+\dfrac{\hbar^2/m((k_1^2-1/4)\eta^2(\eta^2+2) - (k_2^2-1/4)\xi^2(\xi^2+2)) - \alpha(\eta^2-\xi^2)}{4a+b(\xi^2+\eta^2)}$ $R_2 = K^2 + \dfrac{\hbar^2}{8m}((k_1^2-1/4)\dfrac{\eta^2}{\xi^2} + (k_2^2-1/4)\dfrac{\xi^2}{\eta^2})$	(u,v) system Polar Parabolic
V_3	$R_1 = X_1 + iX_2 - \dfrac{-\alpha\mu^2\nu^2 + c_1^2\mu\nu - 2c_2(1+\mu-\nu)}{(a+b/2(\mu-\nu))(\mu+\nu)}$ $R_2 = K^2 - c_1^2\dfrac{\mu-\nu}{\mu\nu} + c_2\dfrac{(\mu-\nu)^2}{\mu^2\nu^2}$	Polar Hyperbolic
V_4	$R_1 = X_1 + iX_2 - K^2 -$ $-\dfrac{\mu\nu(d_1(\nu-2) + d_2(\mu+2) + m\omega^2(\nu-\mu+\mu\nu))}{(a+b/2(\mu-\nu))(\mu+\nu)}$ $R_1 = X_1 - iX_2 -$ $-\dfrac{(\mu-\nu)((\mu-\nu)(d_1\mu+d_2\nu) - m\omega^2(\mu^2+\nu^2+\mu\nu(2+\mu-\nu)))}{4(a+b/2(\mu-\nu))(\mu+\nu)}$	Hyperbolic Elliptic
V_5	$R_1 = X_1 + \dfrac{\hbar^2 v_0^2}{8m}\dfrac{\eta^2-\xi^2}{a+b/4(\xi^2+\eta^2)}$ $R_2 = X_1 - \dfrac{n\hbar^2 v_0^2}{4m}\dfrac{\xi\eta}{a+b/4(\xi^2+\eta^2)}$ $R_3 = K = p$	(u,v) system Polar Parabolic Elliptic Hyperbolic

$$V_4(\mu,\nu) = \frac{1}{\left(a+\frac{b}{2}(\mu-\nu)\right)(\mu+\nu)}\left[d_1\mu + d_2\nu + \frac{m}{2}\omega^2(\mu^2-\nu^2)\right], \quad (2.25)$$

$$V_5(u,v) = \frac{1}{a+b\,\mathrm{e}^{-u}}\frac{\hbar^2 v_0^2}{2m}. \quad (2.26)$$

In Table 2 we list the properties of these potentials on D_{III}, where the coordinate systems, where an explicit path integral solution is possible, are underlined. We see that V_5 is a special case, and it has three integrals of motion. We will threat this case in some more detail as in the other spaces, because on D_{III} the free quantum motion can give bound state solutions (provided the constants are chosen properly). This feature has not been discussed in [14].

2.1. The Superintegrable Potential V_1 on D_{III}. We state the potential V_1 in the respective coordinate systems

$$V_1(u,v) = \frac{2k_1\mathrm{e}^{-u}\cos v/2 + 2k_2\,\mathrm{e}^{-u}\,\sin v/2 + k_3}{a+\frac{b}{4}\,\mathrm{e}^{-u}}, \quad (2.27)$$

$$= \frac{k_1\xi + k_2\eta + k_3}{a+\frac{b}{4}(\xi^2+\eta^2)}, \quad (2.28)$$

$$= \frac{k_1\xi + k_2\eta + (k_1c - k_2c + k_3)}{a+\frac{b}{4}((\xi+c)^2+(\eta-c)^2)}, \quad (2.29)$$

and V_1 is also separable in translated parabolic coordinates $\xi \to \xi + c, \eta \to \eta - c$. The translated parabolic coordinates just modify the solution of a shifted harmonic oscillator, and this case we do not discuss separately.

2.1.1. Separation of V_1 in Parabolic Coordinates. The classical Lagrangian and Hamiltonian in parabolic coordinates on D_{III} are given by

$$\mathcal{L}(\xi,\dot\xi,\eta,\dot\eta) = \frac{m}{2}\left(a+\frac{b}{4}\right)(\xi^2+\eta^2))(\dot\xi^2+\dot\eta^2) - V(\xi,\eta), \quad (2.30)$$

$$\mathcal{H}(\xi,p_\xi,\eta,p_\xi) = \frac{1}{2m}\frac{1}{a+\frac{b}{4}(\xi^2+\eta^2)}(p_\xi^2+p_\eta^2) + V(\xi,\eta). \quad (2.31)$$

The canonical momenta are given by

$$p_\xi = \frac{\hbar}{i}\left(\frac{\partial}{\partial\xi} + \frac{b\xi}{a+\frac{b}{4}(\xi^2+\eta^2)}\right), \; p_\eta = \frac{\hbar}{i}\left(\frac{\partial}{\partial\eta} + \frac{b\eta}{a+\frac{b}{4}(\xi^2+\eta^2)}\right), \quad (2.32)$$

and for the quantum Hamiltonian (product ordering) we find

$$H = -\frac{\hbar^2}{2m}\frac{1}{a+\frac{b}{4}(\xi^2+\eta^2)}\left(\frac{\partial^2}{\partial\xi}+\frac{\partial^2}{\partial\eta^2}\right)+V(\xi,\eta), \tag{2.33}$$

$$= \frac{1}{2m}\sqrt{\frac{1}{a+\frac{b}{4}(\xi^2+\eta^2)}}\,(p_\xi^2+p_\eta^2)\sqrt{\frac{1}{a+\frac{b}{4}(\xi^2+\eta^2)}}+V(\xi,\eta). \tag{2.34}$$

Therefore we obtain for the path integral formulation for V_1

$$K^{(V_1)}(\xi'',\xi',\eta'',\eta';T) = \int\limits_{\xi(t')=\xi'}^{\xi(t'')=\xi''}\mathcal{D}\xi(t)\int\limits_{\eta(t')=\eta'}^{\eta(t'')=\eta''}\mathcal{D}\eta(t)\left(a+\frac{b}{4}(\xi^2+\eta^2)\right)\times$$

$$\times\exp\left\{\frac{i}{\hbar}\int\limits_0^T\left[\left(a+\frac{b}{4}(\xi^2+\eta^2)\right)(\dot\xi^2+\dot\eta^2)-\frac{k_1\xi+k_2\eta+k_3}{(a+\frac{b}{4}(\xi^2+\eta^2))}\right]dt\right\}=$$

$$=\int\limits_{-\infty}^{\infty}\frac{dE}{2\pi\hbar}\,\mathrm{e}^{-iET/\hbar}\int\limits_0^{\infty}ds''\exp\left[\frac{i}{\hbar}\left(aE-k_3-\frac{k_1^2+k_2^2}{2m\omega^2}\right)s''\right]\times$$

$$\times K^{(V_1)}(\xi'',\xi',\eta'',\eta';s''), \tag{2.35}$$

with the time-transformed path integral $K(s'')$ given by

$$K^{(V_1)}(\xi'',\xi',\eta'',\eta';s'') = \int\limits_{\xi(0)=\xi'}^{\xi(s'')=\xi''}\mathcal{D}\xi(s)\int\limits_{\eta(0)=\eta'}^{\eta(s'')=\eta''}\mathcal{D}\eta(s)\times$$

$$\times\exp\left\{\frac{i}{\hbar}\int\limits_0^{s''}\left[\frac{m}{2}\left((\dot\xi^2+\dot\eta^2)-\frac{m}{2}\omega^2(\tilde\xi^2+\tilde\eta^2)\right)\right]ds\right\}. \tag{2.36}$$

The transformed variables $\tilde\xi,\tilde\eta$ are given by $\tilde\xi=\xi+k_1/m\omega^2, \tilde\eta=\eta+k_2/m\omega^2$, and $\omega^2=-bE/2m$. Similarly as in [14] we can determine the Green function to

have the form

$$G^{(V_1)}(\xi'',\xi',\eta'',\eta';E) =$$
$$= \int d\mathcal{E}\,\frac{m}{\pi\hbar^2 b}\sqrt{-\frac{m}{2E}}\,\Gamma\biggl(\frac{1}{2}+\frac{\tilde{\mathcal{E}}}{b\hbar}\sqrt{-\frac{m}{2E}}\biggr)\Gamma\biggl(\frac{1}{2}+\frac{\mathcal{E}}{b\hbar}\sqrt{-\frac{m}{2E}}\biggr)\times$$
$$\times D_{-\frac{1}{2}+\frac{\tilde{\mathcal{E}}}{\hbar}\sqrt{-\ \ \overline{2}}}\biggl(\sqrt[4]{-\frac{8mEb^2}{\hbar^2}}\,\tilde{\xi}_>\biggr) D_{-\frac{1}{2}+\frac{\tilde{\mathcal{E}}}{\hbar}\sqrt{-\ \ \overline{2}}}\biggl(-\sqrt[4]{-\frac{8mEb^2}{\hbar^2}}\,\tilde{\xi}_<\biggr)\times$$
$$\times D_{-\frac{1}{2}+\frac{\mathcal{E}}{\ \hbar}\sqrt{-\ \ \overline{2}}}\biggl(\sqrt[4]{-\frac{8mEb^2}{\hbar^2}}\,\tilde{\eta}_>\biggr) D_{-\frac{1}{2}+\frac{\mathcal{E}}{\ \hbar}\sqrt{-\ \ \overline{2}}}\biggl(-\sqrt[4]{-\frac{8mEb^2}{\hbar^2}}\,\tilde{\eta}_<\biggr). \quad (2.37)$$

The $D_\nu(z)$ are parabolic cylinder-functions [10, p. 1064], and the $\tilde{\mathcal{E}}$ is defined by $\tilde{\mathcal{E}} = aE - k_3 - (k_1^2+k_2^2)/bE - \mathcal{E}$. On the other hand, we can insert for the discrete part of the Green function the harmonic oscillator wave functions and obtain

$$G^{(V_1)}_{\text{disc}}(\xi'',\xi',\eta'',\eta';E) = \sum_{n\ \ =0}^{\infty}\sum_{n\ \ =0}^{\infty}\frac{N^2_{n\ n}}{E_{n\ n}\ -E}\times$$
$$\times\,\Psi^{(\text{HO})}_n(\xi'')\Psi^{(\text{HO})}_n(\xi')\Psi^{(\text{HO})}_n(\eta'')\Psi^{(\text{HO})}_n(\eta'). \quad (2.38)$$

The wave functions for the harmonic oscillator are given by the well-known form in terms of Hermite-polynomials [10]

$$\Psi^{(\text{HO})}_n(x) = \biggl(\frac{m\omega}{\pi\hbar}\biggr)^{1/4}\biggl(\frac{1}{2^n n!}\biggr)^{1/2} H_n\biggl(\sqrt{\frac{m\omega}{\hbar}}\,x\biggr)\exp\biggl(-\frac{m\omega}{2\hbar}x^2\biggr). \quad (2.39)$$

$E_{n\ n}$ is determined by the equation

$$aE - k_3 - \frac{k_1^2+k_2^2}{2m\omega^2} - \hbar(n_\xi+n_\eta+1)\sqrt{-\frac{bE}{2m}} = 0, \quad (2.40)$$

which is actually an equation of the fourth order in E

$$E^4_{n\ n} + \biggl(\frac{b\hbar^2}{2ma^2}(n_\xi+n_\eta+1)^2 - \frac{2k_3}{a}\biggr)E^3_{n\ n} -$$
$$-\biggl(2\frac{k_1^2+k_2^2}{ab} - \frac{k_3^2}{a^2}\biggr)E^2_{n\ n} + 2k_3\frac{k_1^2+k_2^2}{a^2 b}E_{n\ n} - \frac{(k_1^2+k_2^2)^2}{a^2b^2} = 0. \quad (2.41)$$

We dot not solve this equation. Note that for $k_1 = k_2 = k_3 = 0$ a discrete spectrum emerges for the free motion on D_{III}, a feature which will be discussed

in more detail in the subsection for V_5. For the special case $k_1 = k_2 = 0$ we obtain the solution ($N = n_\xi + n_\eta + 1$)

$$E_{n\ n\ \pm} = -\frac{b\hbar^2 N^2}{4ma^2} + \frac{k_3}{a} \pm \frac{1}{a}\sqrt{\left(\frac{b\hbar^2 N^2}{4am}\right)^2 - \frac{bk_3\hbar^2 N^2}{2am} - k_3^2}. \tag{2.42}$$

Note that $\omega_{n\ n}$ must be taken on $\omega_{n\ n} = \sqrt{-bE_{n\ n}/2m}$. The normalization $N_{n\ n}$ is determined by the residuum in $G^{(V_1)}(E)$. If one fixes the parameters a and b and the specific surface of revolution, a more detailed investigation can be performed (special cases, limiting cases, which sign of the square-root gives a positive definite Hilbert space, etc.). Because we do not fix these parameters, we keep both signs of the square-root expression (recall that the free motion on $D_{\rm III}$ allows already a discrete spectrum reaching to $-\infty$).

Note that for the translated parabolic coordinates, the variables $\tilde{\xi}, \tilde{\eta}$ are translated by $\pm c$, respectively; and the quantity $\mathcal{E}$, by an additional $Ebc^2/2$.

2.2. The Superintegrable Potential V_2 on $D_{\rm III}$. We state the potential V_2 in the respective coordinate systems

$$V_2(u,v) = \frac{1}{a + b\,\mathrm{e}^{-u}}\left[-\alpha + \mathrm{e}^u \frac{\hbar^2}{8m}\left(\frac{k_1^2 - 1/4}{\cos^2 v/2} + \frac{k_1^2 - 1/4}{\cos^2 v/2}\right)\right], \tag{2.43}$$

$$= \frac{1}{a + \dfrac{b}{4}\varrho^2}\left[-\alpha + \frac{\hbar^2}{2m\varrho^2}\left(\frac{k_1^2 - 1/4}{\cos^2\varphi} + \frac{k_2^2 - 1/4}{\sin^2\varphi}\right)\right], \tag{2.44}$$

$$= \frac{1}{a + \dfrac{b}{4}(\xi^2 + \eta^2)}\left[-\alpha + \frac{\hbar^2}{2m}\left(\frac{k_1^2 - 1/4}{\xi^2} + \frac{k_2^2 - 1/4}{\eta^2}\right)\right], \tag{2.45}$$

$$= \frac{1}{a + b\,\mathrm{e}^{-u}}\left[-\alpha + \frac{\hbar^2}{2md^2}\left(\frac{k_1^2 - 1/4}{\cosh^2\omega\cos^2\varphi} + \frac{k_2^2 - 1/4}{\sinh^2\omega\sin^2\varphi}\right)\right]. \tag{2.46}$$

V_2 is obviously separable in elliptic coordinates, but the corresponding path integral is not solvable, so this case will be omitted.

2.2.1. Separation of V_2 in the (u,v) System. The classical Lagrangian and Hamiltonian are given by

$$\mathcal{L}(u,\dot{u},v,\dot{v}) = \frac{m}{2}\frac{b + a\,\mathrm{e}^u}{\mathrm{e}^{2u}}(\dot{u}^2 + \dot{v}^2) - V(u,v), \tag{2.47}$$

$$\mathcal{H}(u,p_u,v,p_v) = \frac{1}{2m}\frac{\mathrm{e}^{2u}}{b + a\,\mathrm{e}^u}(p_u^2 + p_v^2) + V(u,v). \tag{2.48}$$

The canonical momenta are given by

$$p_u = \frac{\hbar}{i}\left(\frac{\partial}{\partial u} - \frac{1}{2}\frac{a\,\mathrm{e}^{-u} + 2b\,\mathrm{e}^{-2u}}{a\,\mathrm{e}^{-u} + b\,\mathrm{e}^{-2u}}\right), \quad p_v = \frac{\hbar}{i}\frac{\partial}{\partial v}, \tag{2.49}$$

and for the quantum Hamiltonian we find

$$H = -\frac{\hbar^2}{2m}\frac{1}{a\,\mathrm{e}^{-u} + b\,\mathrm{e}^{-2u}}\left(\frac{\partial^2}{\partial u^2} + \frac{\partial^2}{\partial v^2}\right) + V(u,v), \tag{2.50}$$

$$= \frac{1}{2m}\sqrt{\frac{1}{a\,\mathrm{e}^{-u} + b\,\mathrm{e}^{-2u}}}\left(p_u^2 + p_v^2\right)\sqrt{\frac{1}{a\,\mathrm{e}^{-u} + b\,\mathrm{e}^{-2u}}} + V(u,v). \tag{2.51}$$

Therefore we obtain for the path integral ($f(u) = (a\,\mathrm{e}^{-u} + b\,\mathrm{e}^{-2u})$)

$$\begin{aligned}
K^{(V_2)}(u'',u',v'',v';T) &= \int\limits_{u(t')=u'}^{u(t'')=u''} \mathcal{D}u(t) \int\limits_{v(t')=v'}^{v(t'')=v''} \mathcal{D}v(t)(a\,\mathrm{e}^{-u} + b\,\mathrm{e}^{-2u})\times \\
&\times \exp\left(\frac{i}{\hbar}\int\limits_0^T \left\{(a\,\mathrm{e}^{-u} + b\,\mathrm{e}^{-2u})(\dot{u}^2 + \dot{v}^2) - \right.\right. \\
&\left.\left. - \frac{1}{a + b\,\mathrm{e}^{-u}}\left[-\alpha + \mathrm{e}^u\frac{\hbar^2}{8m}\left(\frac{k_1^2 - 1/4}{\cos^2 v/2} + \frac{k_1^2 - 1/4}{\cos^2 v/2}\right)\right]\right\}dt\right) = \\
&= \frac{1}{[f(u')f(u'')]^{1/4}}\sum_{l=0}^{\infty}\Phi_l^{(k_2,k_1)}\left(\frac{v''}{2}\right)\Phi_l^{(k_2,k_1)}\left(\frac{v'}{2}\right)\times \\
&\times \int\limits_{u(t')=u'}^{u(t'')=u''} \mathcal{D}u(t)(a\,\mathrm{e}^{-u} + b\,\mathrm{e}^{-2u})^{1/2}\exp\left(\frac{i}{\hbar}\int\limits_0^T\left\{(a\,\mathrm{e}^{-u} + b\,\mathrm{e}^{-2u})\dot{u}^2 - \right.\right. \\
&\left.\left. - \frac{1}{a + b\,\mathrm{e}^{-u}}\left[-\alpha + \mathrm{e}^u\frac{\hbar^2}{8m}\big(2l + 1 + |k_1| + |k_2|\big)\right]\right\}dt\right) = \int\limits_{-\infty}^{\infty}\frac{dE}{2\pi\hbar}\,\mathrm{e}^{-iET/\hbar}\times \\
&\times \int\limits_0^{\infty} ds''\exp\left[-\frac{i}{\hbar}\frac{\hbar^2}{8m}\big(2l + 1 + |k_1| + |k_2|\big)^2 s''\right]K_l^{(V_2)}(u'',u';s''),
\end{aligned} \tag{2.52}$$

with the time-transformed path integral $K_l(s'')$ given by

$$K_l^{(V_2)}(u'',u';s'') =$$

$$= \int\limits_{u(0)=u'}^{u(s'')=u''} \mathcal{D}u(s) \exp\left[\frac{i}{\hbar}\int\limits_0^{s''}\left(\frac{m}{2}\dot{u}^2 + Eb\,\mathrm{e}^{-2u} + (aE-\alpha)\,\mathrm{e}^{-u}\right)ds\right]. \quad (2.53)$$

The $\Phi_n^{(k_1,k_2)}(\beta)$ are the wave functions of the Pöschl–Teller potential, which are given by

$$V(x) = \frac{\hbar^2}{2m}\left(\frac{\alpha^2-1/4}{\sin^2 x} + \frac{\beta^2-1/4}{\cos^2 x}\right), \quad (2.54)$$

$$\Phi_n^{(\alpha,\beta)}(x) = \left[2(\alpha+\beta+2l+1)\frac{l!\Gamma(\alpha+\beta+l+1)}{\Gamma(\alpha+l+1)\Gamma(\beta+l+1)}\right]^{1/2} \times$$
$$\times (\sin x)^{\alpha+1/2}(\cos x)^{\beta+1/2} P_n^{(\alpha,\beta)}(\cos 2x). \quad (2.55)$$

Equation (2.53) is a path integral for the Morse potential. Inserting the corresponding solution [22] we obtain

$$G^{(V_2)}(u'',u',v'',v';E) = \sum_{l=0}^{\infty} \Phi_l^{(k_2,k_1)}\left(\frac{v''}{2}\right)\Phi_l^{(k_2,k_1)}\left(\frac{v'}{2}\right) \times$$
$$\times \sqrt{-\frac{m}{2bE}}\,\frac{m\Gamma\left(\frac{1}{2}+\lambda+\frac{aE-\alpha}{\hbar}\sqrt{-\frac{m}{2bE}}\right)}{\hbar\Gamma(1+2\lambda)\,\mathrm{e}^{(u'+u'')/2}} \times$$
$$\times W_{\frac{-}{\hbar}\sqrt{-\frac{}{2}},\lambda}\left(\frac{\sqrt{-8mbE}}{\hbar}\,\mathrm{e}^{-u}\right) M_{\frac{-}{\hbar}\sqrt{-\frac{}{2}},\lambda}\left(\frac{\sqrt{-8mbE}}{\hbar}\,\mathrm{e}^{-u}\right). \quad (2.56)$$

Inserting the bound state wave functions for the Morse potential gives the bound state contribution of $G^{(V_2)}(E)$

$$G_{\text{disc}}^{(V_2)}(u'',u',v'',v';E) = \sum_{l=0}^{\infty} \Phi_l^{(k_2,k_1)}\left(\frac{v''}{2}\right)\Phi_l^{(k_2,k_1)}\left(\frac{v'}{2}\right) \times$$
$$\times \sum_{l=0}^{\infty} \frac{N_{nl}^2}{E_{nl}-E}\Psi_n^{(\mathrm{MP})}(u'')\Psi_n^{(\mathrm{MP})}(u'), \quad (2.57)$$

$$\Psi_n^{(\mathrm{MP})}(u) = N_{nl} \left[\left(-\frac{2mbE_{nl}}{\hbar^2} \right)^{\frac{\;-\;}{\hbar}\sqrt{-\frac{\;}{2}}-n-1/2} \times \right.$$

$$\left. \times \frac{\left(\frac{aE_{nl}-\alpha}{\hbar}\sqrt{-\frac{m}{2bE_{nl}}} - 2n - 1 \right)}{\Gamma\left(\frac{aE_{nl}-\alpha}{\hbar}\sqrt{-\frac{2m}{bE_{nl}}} - n \right)} \right]^{1/2} \times$$

$$\times \exp\left[(u'+u'') \left(\frac{aE_{nl}-\alpha}{\hbar}\sqrt{-\frac{m}{2bE_{nl}}} - n - \frac{1}{2} \right) - \frac{\sqrt{-2mbE_{nl}}}{\hbar}\,\mathrm{e}^u \right] \times$$

$$\times L_n^{\left(\frac{\;-\;}{\hbar}\sqrt{-\frac{2}{\;}}-2n-1\right)} \left(\frac{-8mbE_{nl}}{\hbar}\,\mathrm{e}^u \right). \qquad (2.58)$$

The $L_n^{(\alpha)}(z)$ are Laguerre polynomials [10]. Here, the spectrum E_{nl} is determined by

$$aE_{nl} - \alpha - \hbar\sqrt{-\frac{bE_{nl}}{2m}}\,(2n + 2l + |k_1| + |k_2| + 2), \qquad (2.59)$$

which is a quadratic equation in E_{nl} with solution ($N = 2n + 2l + |k_1| + |k_2| + 2$)

$$E_{nl\pm} = \frac{1}{2a^2}\left[-\left(\frac{b\hbar^2}{2m}N^2 - 2a\alpha \right) \pm \frac{b\hbar^2}{2m}N^2\sqrt{1 - \frac{8a\alpha m}{b\hbar^2 N^2}} \right], \qquad (2.60)$$

and the normalization constants N_{nl} are determined by the residuum of (2.56). For large n, l we have

$$E_{nl-} \simeq -\frac{b\hbar^2}{m}(2n + 2l + |k_1| + |k_2| + 2)^2, \qquad (2.61)$$

$$E_{nl+} \simeq -\frac{m\alpha^2}{2b\hbar^2(2n + 2l + |k_1| + |k_2| + 2)^2}, \qquad (2.62)$$

with E_{nl+} showing a Coulomb-like behavior.

2.2.2. Separation of V_2 in Polar Coordinates. In the coordinates (ϱ, φ) the classical Lagrangian and Hamiltonian take on the form

$$\mathcal{L}(\varrho, \dot\varrho, \varphi, \dot\varphi) = \frac{m}{2}\left(a + \frac{b}{4}\varrho^2 \right)(\dot\varrho^2 + \varrho^2\dot\varphi^2) - V(\varrho, \varphi), \qquad (2.63)$$

$$\mathcal{H}(\varrho, p_\varrho, \varphi, p_\varphi) = \frac{1}{2m}\frac{1}{a + \frac{b}{4}\varrho^2}\left(p_\varrho^2 + \frac{1}{\varrho^2}p_\varphi^2 \right) + V(\varrho, \varphi). \qquad (2.64)$$

The canonical momenta are given by

$$p_\varrho = \frac{\hbar}{i}\left(\frac{\partial}{\partial\varrho} + \frac{b\varrho}{4a+b\varrho^2} + \frac{1}{2\varrho}\right), \quad p_\varphi = \frac{\hbar}{i}\frac{\partial}{\partial\varphi}. \tag{2.65}$$

Therefore the quantum Hamiltonian is given by

$$H = -\frac{\hbar^2}{2m}\frac{1}{a+\frac{b}{4}\varrho^2}\left(\frac{\partial^2}{\partial\varrho^2} + \frac{1}{\varrho}\frac{\partial}{\partial\varrho} + \frac{1}{\varrho^2}\frac{\partial^2}{\partial\varphi^2}\right) + V(\varrho,\varphi) = \tag{2.66}$$

$$= \frac{1}{2m}\sqrt{\frac{1}{a+\frac{b}{4}\varrho^2}}\left(p_\varrho^2 + \frac{1}{\varrho^2}p_\varphi^2\right)\sqrt{\frac{1}{a+\frac{b}{4}\varrho^2}} +$$

$$+ V(\varrho,\varphi) - \left(a+\frac{b}{4}\varrho^2\right)^{-1}\frac{\hbar^2}{8m\varrho^2}, \tag{2.67}$$

and in this case we have an additional quantum potential $\propto \hbar^2$. This gives for the path integral $\left(f(\varrho) = a + \frac{b}{4}\varrho^2 = \sqrt{g}\right)$

$$K^{(V_2)}(\varrho'',\varrho',\varphi'',\varphi';T) = \int\limits_{\varrho(t')=\varrho'}^{\varrho(t'')=\varrho''}\mathcal{D}\varrho(t)\int\limits_{\varphi(t')=\varphi'}^{\varphi(t'')=\varphi''}\mathcal{D}\varphi(t) f(\varrho)\varrho\times$$

$$\times\exp\left(\frac{i}{\hbar}\int_0^T\left\{\frac{m}{2}f(\varrho)(\dot\varrho^2+\varrho^2\dot\varphi^2) - \frac{1}{f(\varrho)}\times\right.\right.$$

$$\left.\left.\times\left[-\alpha + \frac{\hbar^2}{2m\varrho^2}\left(\frac{k_1^2-1/4}{\cos^2\varphi} + \frac{k_2^2-1/4}{\sin^2\varphi} - \frac{1}{4}\right)\right]\right\}dt\right) =$$

$$= \sum_{l=0}^\infty \Phi_l^{(k_2,k_1)}(\varphi'')\Phi_l^{(k_2,k_1)}(\varphi')\frac{1}{[(\varrho'\varrho'')^2 f(\varrho')f(\varrho'')]^{1/4}}\int\limits_{\varrho(t')=\varrho'}^{\varrho(t'')=\varrho''}\mathcal{D}\varrho(t) f^{1/2}(\varrho)\times$$

$$\times\exp\left\{\frac{i}{\hbar}\int_0^T\left[\frac{m}{2}f(\varrho)\dot\varrho^2 - \frac{1}{f(\varrho)}\left(-\alpha + \frac{\hbar^2}{2m}\frac{\lambda^2-1/4}{\varrho^2}\right)\right]dt\right\} =$$

$$= \frac{1}{\sqrt{\varrho'\varrho''}}\sum_{l=0}^\infty \Phi_l^{(k_2,k_1)}(\varphi'')\Phi_l^{(k_2,k_1)}(\varphi')\times$$

$$\times\int_{-\infty}^\infty \frac{dE}{2\pi\hbar}\,\mathrm{e}^{-iET/\hbar}\int_0^\infty ds''\exp\left[\frac{i}{\hbar}(aE-\alpha)s''\right]K_l^{(V_2)}(\varrho'',\varrho';s''), \tag{2.68}$$

with the time-transformed path integral $K_l(s'')$ given by ($\lambda = 2l + |k_1| + |k_2| + 1$))

$$K_l^{(V_2)}(\varrho'', \varrho'; s'') = \int\limits_{\varrho(0)=\varrho'}^{\varrho(s'')=\varrho''} \mathcal{D}\varrho(s) \times$$

$$\times \exp\left[\frac{i}{\hbar}\int\limits_0^{s''}\left(\frac{m}{2}\dot{\varrho}^2 + \frac{Eb}{4}\varrho^2 - \frac{\hbar^2}{2m}\frac{\lambda^2 - 1/4}{\varrho^2}\right)ds\right] =$$

$$= \frac{m\omega\sqrt{\varrho'\varrho''}}{i\hbar\sin\omega s''}\exp\left[-\frac{m\omega}{2i\hbar}({\varrho'}^2 + {\varrho''}^2)\cot\omega s''\right] I_\lambda\left(\frac{m\omega\varrho'\varrho''}{i\hbar\sin\omega s''}\right). \quad (2.69)$$

Performing the s'' integration yields the Green function

$$G^{(V_2)}(\varrho'', \varrho', \varphi'', \varphi'; E) = \sum_{l=0}^{\infty} \Phi_l^{(k_2,k_1)}(\varphi'')\Phi_l^{(k_2,k_1)}(\varphi') \times$$

$$\times \sqrt{-\frac{2m}{Eb}}\,\frac{\Gamma\left[\frac{1}{2}\left(1 + \lambda - \frac{1}{\hbar}(aE - \alpha)\sqrt{-2m/bE}\right)\right]}{\Gamma(1+\lambda)\sqrt{\varrho'\varrho''}} M_{\frac{-}{2\hbar}\sqrt{-\frac{2}{\ }},\frac{}{2}} \times$$

$$\times \left(\frac{m}{\hbar}\sqrt{-\frac{bE}{2m}}\,\varrho_<^2\right) M_{\frac{-}{2\hbar}\sqrt{-\frac{2}{\ }},\frac{}{2}} \left(\frac{m}{\hbar}\sqrt{-\frac{bE}{2m}}\,\varrho_>^2\right). \quad (2.70)$$

Inserting the expansion into Laguerre polynomial yields the discrete contribution of the Green function

$$G_{\text{disc}}^{(V_2)}(\varrho'', \varrho', \varphi'', \varphi'; E) = \frac{1}{\sqrt{\varrho'\varrho''}}\sum_{l=0}^{\infty} \Phi_l^{(k_2,k_1)}(\varphi'')\Phi_l^{(k_2,k_1)}(\varphi') \times$$

$$\times \sum_{n=0}^{\infty} \frac{N_{nl}^2}{E_{nl} - E}\Psi_n^{(\text{RHO},\lambda)}(\varrho'')\Psi_n^{(\text{RHO},\lambda)}(\varrho'). \quad (2.71)$$

The wave functions for the radial harmonic oscillator $V(r) = \frac{m}{2}\omega^2 - \frac{\hbar^2}{2m}\frac{\lambda^2 - 1/4}{r^2}$ have the form [22, 44]

$$\Psi_n^{(\text{RHO},\lambda)}(r) =$$

$$= \sqrt{\frac{2m}{\hbar}\frac{n!}{\Gamma(n+\lambda+1)}}\,r\left(\frac{m\omega}{\hbar}r\right)^{\lambda/2}\exp\left(-\frac{m\omega}{2\hbar}r^2\right)L_n^{(\lambda)}\left(\frac{m\omega}{\hbar}r^2\right). \quad (2.72)$$

The spectrum E_{nl} is determined by

$$aE_{nl} - \alpha - \hbar\sqrt{-\frac{bE_{nl}}{2m}}\,(2n + 2l + |k_1| + |k_2| + 2), \tag{2.73}$$

which is the same as in (2.60). In the wave functions $\Psi_n^{(\mathrm{RHO},\lambda)}(\varrho)$ the quantity ω has to be taken on $\omega = \sqrt{-bE_{nl}/2m}$, and the normalization constants N_{nl} are determined by the residuum of (2.69).

2.2.3. Separation of V_2 in Parabolic Coordinates. We insert the potential V_2 into the path integral and obtain $\big(f = a + \frac{b}{4}(\xi^2 + \eta^2)\big)$

$$\begin{aligned}
K^{(V_2)}(\xi'', \xi', \eta'', \eta'; T) = &\int\limits_{\xi(t')=\xi'}^{\xi(t'')=\xi''} \mathcal{D}\xi(t) \int\limits_{\eta(t')=\eta'}^{\eta(t'')=\eta''} \mathcal{D}\eta(t) f(\xi,\eta) \times \\
&\times \exp\left(\frac{i}{\hbar}\int\limits_0^T \left\{ f(\xi,\eta)(\dot\xi^2 + \dot\eta^2) - \frac{1}{f(\xi,\eta)} \times \right.\right. \\
&\times \left.\left. \left[-\alpha + \frac{\hbar^2}{2m}\left(\frac{k_1^2 - 1/4}{\xi^2} + \frac{k_2^2 - 1/4}{\eta^2}\right)\right]\right\} dt \right) = \\
= \int\limits_{-\infty}^{\infty} \frac{dE}{2\pi\hbar}\,\mathrm{e}^{-iET/\hbar} &\int\limits_0^{\infty} ds'' \exp\left[\frac{i}{\hbar}\big(aE - \alpha\big)s''\right] K^{(V_2)}(\xi'', \xi', \eta'', \eta'; s''),
\end{aligned} \tag{2.74}$$

with the time-transformed path integral $K^{(V_2)}(s'')$ given by $(\omega^2 = -bE/2m)$

$$\begin{aligned}
K^{(V_2)}(\xi'', \xi', \eta'', \eta'; s'') = &\int\limits_{\xi(0)=\xi'}^{\xi(s'')=\xi''} \mathcal{D}\xi(s) \int\limits_{\eta(0)=\eta'}^{\eta(s'')=\eta''} \mathcal{D}\eta(s) \times \\
&\times \exp\left\{\frac{i}{\hbar}\int\limits_0^{s''} \left[\frac{m}{2}\left((\dot\xi^2 + \dot\eta^2) - \frac{m}{2}\omega^2(\xi^2 + \eta^2)\right) - \right.\right. \\
&\left.\left. - \frac{\hbar^2}{2m}\left(\frac{k_1^2 - 1/4}{\xi^2} + \frac{k_2^2 - 1/4}{\eta^2}\right)\right] ds \right\} = \frac{m\omega\sqrt{\xi'\xi''}}{i\hbar \sin\omega s''} \times \\
&\times \exp\left[-\frac{m\omega}{i\hbar\sin\omega s''}({\xi'}^2 + {\xi''}^2 \cot\omega s'')\right] I_{k_2}\left(\frac{m\omega\xi'\xi''}{i\hbar\sin\omega s''}\right) \frac{m\omega\sqrt{\eta'\eta''}}{i\hbar\sin\omega s''} \times \\
&\times \exp\left[-\frac{m\omega}{i\hbar\sin\omega s''}({\eta'}^2 + {\eta''}^2 \cot\omega s'')\right] I_{k_1}\left(\frac{m\omega\eta'\eta''}{i\hbar\sin\omega s''}\right).
\end{aligned} \tag{2.75}$$

Performing the s'' integration yields the Green function ($\tilde{\mathcal{E}} = aE - \alpha - \mathcal{E}$)

$$G^{(V_2)}(\xi'', \xi', \eta'', \eta'; E) = \int d\mathcal{E} \sqrt{-\frac{2m}{bE}}\, \frac{\Gamma\left[\frac{1}{2}(1 + |k_1| - \mathcal{E}\sqrt{-2m/bE}/\hbar)\right]}{\hbar\Gamma(1 + |k_1|)\sqrt{\xi'\xi''}} \times$$

$$\times\, W_{\mathcal{E}\sqrt{-2m/bE}/2\hbar, |k_1|/2}\left(\frac{m}{\hbar}\sqrt{-\frac{bE}{2m}}\,\xi_>^2\right) \times$$

$$\times\, M_{\mathcal{E}\sqrt{-2m/bE}/2\hbar, |k_1|/2}\left(\frac{m}{\hbar}\sqrt{-\frac{bE}{2m}}\,\xi_<^2\right) \times$$

$$\times \sqrt{-\frac{2m}{bE}}\, \frac{\Gamma\left[\frac{1}{2}(1 + |k_2| - \tilde{\mathcal{E}}\sqrt{-2m/bE}/\hbar)\right]}{\hbar\Gamma(1 + |k_2|)\sqrt{\eta'\eta''}} \times$$

$$\times\, W_{\tilde{\mathcal{E}}\sqrt{-2m/bE}/2\hbar, |k_2|/2}\left(\frac{m}{\hbar}\sqrt{-\frac{bE}{2m}}\,\eta_>^2\right) \times$$

$$\times\, M_{\tilde{\mathcal{E}}\sqrt{-2m/bE}/2\hbar, |k_2|/2}\left(\frac{m}{\hbar}\sqrt{-\frac{bE}{2m}}\,\eta_<^2\right). \quad (2.76)$$

On the other hand, we insert the expansion of the bound states of the radial harmonic oscillator and obtain for the discrete spectrum contribution of the Green function:

$$G^{(V_2)}(\xi'', \xi', \eta'', \eta'; E) = \sum_{n\ =0}^{\infty} \sum_{n\ =0}^{\infty} \frac{N_{n\ ,n}^2}{E_{n\ ,n\ } - E} \times$$

$$\times\, \Psi_n^{(\mathrm{RHO},|k_1|)}(\xi'')\Psi_n^{(\mathrm{RHO},|k_2|)}(\xi')\Psi_n^{(\mathrm{RHO},|k_2|)}(\eta'')\Psi_n^{(\mathrm{RHO},|k_1|)}(\eta'), \quad (2.77)$$

where the energy $E_{n\ ,n\ }$ is determined by the equation

$$2n_\xi + 2n_\eta + |k_1| + |k_2| + 2 = \frac{aE_{n\ ,n\ } - \alpha}{\hbar}\sqrt{-\frac{2m}{bE_{n\ ,n}}}, \quad (2.78)$$

which is equivalent with (2.60). The normalization constants $N_{n\ n\ }$ are determined by the residuum of (2.56), and ω in the $\Psi_n^{(\mathrm{RHO},|k_2|)}\Psi_n^{(\mathrm{RHO},|k_1|)}$ has to be taken on $\omega_{n\ ,n\ } = \sqrt{-bE_{n\ ,n\ }/2m}$.

2.3. The Superintegrable Potential V_3 on $D_{\rm III}$. First we state the potential V_3 in the respective coordinate systems

$$V_3(u,v) = \frac{1}{a+b\,e^{-u}}\left[-\alpha + \frac{\hbar^2}{2m}4\,e^{u}\Big(c_1^2\,e^{-iv} - 2c_2\,e^{-2iv}\Big)\right], \tag{2.79}$$

$$= \frac{1}{a+\dfrac{b}{4}\varrho^2}\left[-\alpha + \frac{\hbar^2}{2m\varrho^2}4\Big(c_1^2\,e^{-2i\varphi} - 2c_2\,e^{-4i\varphi}\Big)\right], \tag{2.80}$$

$$= \frac{-\alpha(\mu+\nu) + c_1^2\dfrac{\mu+\nu}{\mu\nu} - c_2\dfrac{\mu^2-\nu^2}{\mu^2\nu^2}}{\left(a+\dfrac{b}{2}(\mu-\nu)\right)(\mu+\nu)}. \tag{2.81}$$

In hyperbolic coordinates no closed solution can be obtained due to the involved mixture of linear, quadratic, inverse-linear and inverse-quadratic terms. In polar coordinates the path integral in ϱ turns out to be a path integral for the radial harmonic oscillator. Note that the (u,v) system is equivalent to polar coordinates.

2.3.1. Separation of V_3 in Polar Coordinates. We insert the potential V_3 into the path integral and get $\big(f(\varrho) = a + \frac{b}{4}\varrho^2 = \sqrt{g}\big)$

$$K^{(V_3)}(\varrho'',\varrho',\varphi'',\varphi';T) = \int\limits_{\varrho(t')=\varrho'}^{\varrho(t'')=\varrho''}\mathcal{D}\varrho(t)\int\limits_{\varphi(t')=\varphi'}^{\varphi(t'')=\varphi''}\mathcal{D}\varphi(t)f(\varrho)\varrho\times$$

$$\times\exp\left(\frac{i}{\hbar}\int_0^T\left\{\frac{m}{2}f(\varrho)(\dot\varrho^2+\varrho^2\dot\varphi^2)-\right.\right.$$

$$\left.\left.-\frac{1}{f(\varrho)}\left[-\alpha+\frac{\hbar^2}{2m\varrho^2}4c_1^2\left(e^{-4i\varphi}-2\frac{c_2}{c_1^2}\,e^{-2i\varphi}-\frac{1}{4}\right)\right]\right\}dt\right) =$$

$$= \sum_{l=0}^{\infty}\Phi^{(c_1,c_2)}_{[\rm cMP],l}(\varphi'')\Phi^{(c_1,c_2)}_{[\rm cMP],l}(\varphi')\frac{1}{[(\varrho'\varrho'')^2 f(\varrho')f(\varrho'')]^{1/4}}\times$$

$$\times \int\limits_{\varrho(t')=\varrho'}^{\varrho(t'')=\varrho''} \mathcal{D}\varrho(t) f^{1/2}(\varrho) \exp\left\{ \frac{i}{\hbar} \int\limits_0^T \left[\frac{m}{2} f(\varrho)\dot{\varrho}^2 - \right.\right.$$

$$\left.\left. - \frac{1}{f(\varrho)} \left(-\alpha + \frac{\hbar^2}{2m} \frac{\left(l + \frac{2c_2}{c_1} + \frac{1}{2} \right)^2 - \frac{1}{4}}{\varrho^2} \right) \right] dt \right\} =$$

$$= \frac{1}{\sqrt{\varrho'\varrho''}} \sum_{l=0}^{\infty} \Phi_{[\mathrm{cMP}],l}^{(c_1,c_2)}(\varphi'') \Phi_{[\mathrm{cMP}],l}^{(c_1,c_2)}(\varphi') \times$$

$$\times \int\limits_{-\infty}^{\infty} \frac{dE}{2\pi\hbar} \mathrm{e}^{-iET/\hbar} \int\limits_0^{\infty} ds'' \exp\left[\frac{i}{\hbar} (aE - \alpha) s'' \right] K_l^{(V_3)}(\varrho'', \varrho'; s''), \quad (2.82)$$

with the time-transformed path integral $K_l(s'')$ given by

$$K_l^{(V_3)}(\varrho'', \varrho'; s'') =$$

$$= \int\limits_{\varrho(0)=\varrho'}^{\varrho(s'')=\varrho''} \mathcal{D}\varrho(s) \exp\left[\frac{i}{\hbar} \int\limits_0^{s''} \left(\frac{m}{2}\dot{\varrho}^2 + \frac{Eb}{4}\varrho^2 - \frac{\hbar^2}{2m} \frac{\left(l + \frac{2c_2}{c_1} + \frac{1}{2} \right)^2 - \frac{1}{4}}{\varrho^2} \right) ds \right] =$$

$$= \frac{m\omega\sqrt{\varrho'\varrho''}}{i\hbar \sin \omega s''} \exp\left[-\frac{m\omega}{2i\hbar} ({\varrho'}^2 + {\varrho''}^2) \cot \omega s'' \right] I_{l + \frac{2\ 2}{1} + \frac{1}{2}} \left(\frac{m\omega\varrho'\varrho''}{i\hbar \sin \omega s''} \right). \quad (2.83)$$

By $\Phi_{[\mathrm{cMP}],l}^{(c_1,c_2)}(\varphi)$ we denote the wave functions of the complex periodic Morse potential in the variable φ with spectrum $E_l = \hbar^2 \left(l + 2\frac{c_2}{c_1} + \frac{1}{2} \right)^2 / 2m$ [1,3,36, 42,50,51], c.f. Appendix C:

$$\Phi_{[\mathrm{cMP}],l}^{(c_1,c_2)}(\varphi) = \frac{\left(4\frac{c_2}{c_1} - 2n - 1 \right) n!}{\Gamma\left(4\frac{c_2}{c_1} - 2n \right)} \left(4\frac{c_2}{c_1} \right)^{4\frac{2}{1} - 2n - 1} \times$$

$$\times \exp\left[-2i\left(2\frac{c_2}{c_1} - n - \frac{1}{2} \right)\varphi - 2c_1 \mathrm{e}^{-2i\varphi} \right] L_n^{(4\frac{2}{1} - 2n - 1)}\left(4c_1\mathrm{e}^{-2i\varphi} \right). \quad (2.84)$$

Performing the s'' integration gives the Green function

$$
\begin{aligned}
G^{(V_3)}(\varrho'',\varrho',\varphi'',\varphi';E) &= \sum_{l=0}^{\infty}\Phi^{(c_1,c_2)}_{[\mathrm{cMP}],l}(\varphi'')\Phi^{(c_1,c_2)}_{[\mathrm{cMP}],l}(\varphi')\times\\
&\times\sqrt{-\frac{2m}{Eb}}\,\frac{\Gamma\left[\frac{1}{2}\left(l+2\frac{c_2}{c_1}+\frac{3}{2}-\frac{1}{\hbar}(aE-\alpha)\sqrt{-2m/bE}\right)\right]}{\Gamma\left(l+2\frac{c_2}{c_1}+\frac{3}{2}\right)\sqrt{\varrho'\varrho''}}\times\\
&\times M_{\frac{-}{2\hbar}\sqrt{-\frac{2}{}},\frac{1}{2}\left(l+2\frac{2}{1}+\frac{1}{2}\right)}\left(\frac{m}{\hbar}\sqrt{-\frac{bE}{2m}}\,\varrho_<^2\right)\times\\
&\times M_{\frac{-}{2\hbar}\sqrt{-\frac{2}{}},\frac{1}{2}\left(l+2\frac{2}{1}+\frac{1}{2}\right)}\left(\frac{m}{\hbar}\sqrt{-\frac{bE}{2m}}\,\varrho_>^2\right).
\end{aligned}\tag{2.85}
$$

Inserting the expansion into Laguerre polynomials yields the discrete contribution of the Green function $\left(\lambda = l+\frac{2c_2}{c_1}+\frac{1}{2}\right)$

$$
\begin{aligned}
G^{(V_3)}_{\mathrm{disc}}(\varrho'',\varrho',\varphi'',\varphi';E) &= \frac{1}{\sqrt{\varrho'\varrho''}}\sum_{l=0}^{\infty}\Phi^{(c_1,c_2)}_{[\mathrm{cMP}],l}(\varphi'')\Phi^{(c_1,c_2)}_{[\mathrm{cMP}],l}(\varphi')\times\\
&\times\sum_{n=0}^{\infty}\frac{N_{nl}^2}{E_{nl}-E}\Psi_n^{(\mathrm{RHO},\lambda)}(\varrho'')\Psi_n^{(\mathrm{RHO},\lambda)}(\varrho'),
\end{aligned}\tag{2.86}
$$

and the normalization constants N_{nl} are determined by the residuum of (2.85). Here, the spectrum E_{nl} is determined by

$$
aE_{nl}-\alpha-\hbar\sqrt{-\frac{bE_{nl}}{2m}}\left(2n+2l+\frac{c_2}{c_1}+1\right),\tag{2.87}
$$

which is quadratic equation in E_{nl} with solution $\left(N = 2n+2l+\frac{c_2}{c_1}+1\right)$

$$
E_{nl\pm} = \frac{1}{2a^2}\left[-\left(\frac{b\hbar^2}{2m}N^2-2a\alpha\right)\pm\frac{b\hbar^2}{2m}N^2\sqrt{1-\frac{8a\alpha m}{b\hbar^2N^2}}\right].\tag{2.88}
$$

In the wave functions $\Psi_n^{(\mathrm{RHO},\lambda)}(\varrho)$ the quantity ω has to be taken on $\omega = \sqrt{-bE_{nl}/2m}$. For large n, l we have

$$E_{nl-} \simeq -\frac{b\hbar^2}{m}(2n+2l+1)^2, \tag{2.89}$$

$$E_{nl+} \simeq -\frac{m\alpha^2}{2b\hbar^2(2n+2l+1)^2}, \tag{2.90}$$

with E_{nl+} showing a Coulomb-like behavior.

2.4. The Superintegrable Potential V_4 on D_{III}.

$$V_4(\mu,\nu) = \frac{1}{\left(a+\frac{b}{2}(\mu-\nu)\right)(\mu+\nu)}\left[d_1\mu + d_2\nu + \frac{m}{2}\omega^2(\mu^2-\nu^2)\right], \tag{2.91}$$

$$= \frac{1}{a+b\,\mathrm{e}^{-u}}\big[2(d_1+d_2)(\cos 2\varphi - \cosh 2\omega) +$$

$$+ 2(d_1-d_2)(2i\sin 2\varphi + \sinh 2\omega) + 2d_3(2i\sin 2\varphi + \sinh 4\omega)\big]. \tag{2.92}$$

We can evaluate the path integral in hyperbolic coordinates (application of the Morse potential); in elliptic coordinates no closed solution can be found.

2.4.1. Separation of V_4 in Hyperbolic Coordinates. The classical Lagrangian and Hamiltonian have the form

$$\mathcal{L}(\mu,\dot\mu,\nu,\dot\nu) = \frac{m}{2}\left(a+\frac{b}{2}(\mu-\nu)\right)(\mu+\nu)\left(\frac{\dot\mu^2}{\mu^2}-\frac{\dot\nu^2}{\nu^2}\right) - V(\mu,\nu), \tag{2.93}$$

$$\mathcal{H}(\mu,p_\mu,\nu,p_\nu = \frac{1}{2m}\frac{\mu^2p_\mu^2-\nu^2p_\nu^2}{\left(a+\frac{b}{2}(\mu-\nu)\right)(\mu+\nu)} + V(\mu,\nu). \tag{2.94}$$

The canonical momentum operators are given by

$$p_\mu = \frac{\hbar}{i}\left[\frac{\partial}{\partial\mu} + \frac{1}{2}\left(+\frac{1}{\mu+\nu} + \frac{b}{a+\frac{b}{2}(\mu-\nu)} - \frac{1}{\mu}\right)\right], \tag{2.95}$$

$$p_\nu = \frac{\hbar}{i}\left[\frac{\partial}{\partial\mu} + \frac{1}{2}\left(+\frac{1}{\mu+\nu} - \frac{b}{a+\frac{b}{2}(\mu-\nu)} - \frac{1}{\nu}\right)\right], \tag{2.96}$$

and the quantum Hamiltonian has the form

$$H = -\frac{\hbar^2}{2m}\frac{1}{\left(a+\frac{b}{2}(\mu-\nu)\right)(\mu+\nu)}\times$$

$$\times\left[\mu^2\left(\frac{\partial^2}{\partial\mu^2}-\frac{1}{\mu}\frac{\partial}{\partial\mu}\right)-\nu^2\left(\frac{\partial^2}{\partial\nu^2}-\frac{1}{\nu}\frac{\partial}{\partial\nu}\right)\right]+V(\mu,\nu), \tag{2.97}$$

$$=\frac{1}{2m}\left[\frac{\mu}{\sqrt{\left(a+\frac{b}{2}(\mu-\nu)\right)(\mu+\nu)}}p_\mu^2\frac{\mu}{\sqrt{\left(a+\frac{b}{2}(\mu-\nu)\right)(\mu+\nu)}}-\right.$$

$$\left.-\frac{\nu}{\sqrt{\left(a+\frac{b}{2}(\mu-\nu)\right)(\mu+\nu)}}p_\nu^2\frac{\nu}{\sqrt{\left(a+\frac{b}{2}(\mu-\nu)\right)(\mu+\nu)}}\right]+V(\mu,\nu). \tag{2.98}$$

Note that from each coordinate there comes a quantum potential $\Delta V=\hbar^2/8m$, however they are canceling each other due to the minus-sign in the metric in ν.

We insert the potential V_4 into the path integral which has the form $\Big(f(\mu,\nu)=\left(a+\frac{b}{2}(\mu-\nu)\right)(\mu+\nu)\Big)$

$$K^{(V_4)}(\mu'',\mu',\nu'',\nu';T)=\int\limits_{\mu(t')=\mu'}^{\mu(t'')=\mu''}\mathcal{D}\mu(t)\int\limits_{\nu(t')=\nu'}^{\nu(t'')=\nu''}\mathcal{D}\nu(t)\frac{f(\mu,\nu)}{\mu\nu}\times$$

$$\times\exp\left\{\frac{i}{\hbar}\int_0^T\left[\frac{m}{2}f(\mu,\nu)\left(\frac{\dot\mu^2}{\mu^2}-\frac{\dot\nu^2}{\nu^2}\right)-\right.\right.$$

$$\left.\left.-\frac{1}{f(\mu,\nu)}\left(d_1\mu+d_2\nu+\frac{m}{2}\omega^2(\mu^2-\nu^2)\right)\right]dt\right\}=$$

$$=\int_{-\infty}^{\infty}\frac{dE}{2\pi\hbar}\,\mathrm{e}^{-iET/\hbar}\int_0^\infty ds''K^{(V_4)}(\mu'',\mu',\nu'',\nu';s''), \tag{2.99}$$

and the path integral $K^{(V_4)}(s'')$ is given by

$$K^{(V_4)}(\mu'',\mu',\nu'',\nu';s'') = \int\limits_{\mu(0)=\mu'}^{\mu(s'')=\mu''} \mathcal{D}\mu(s) \int\limits_{\nu(0)=\nu'}^{\nu(s'')=\nu''} \mathcal{D}\nu(s)\frac{1}{\mu\nu}\times$$
$$\times \exp\left\{\frac{i}{\hbar}\int\limits_0^{s''}\left[\frac{m}{2}\left(\frac{\dot{\mu}^2}{\mu^2}-\frac{\dot{\nu}^2}{\nu^2}\right)+aE(\mu+\nu)+\frac{1}{2}bE(\mu^2-\nu^2)-\right.\right.$$
$$\left.\left.-\left(d_1\mu+d_2\nu+\frac{m}{2}\omega^2(\mu^2-\nu^2)\right)\right]ds\right\}. \quad (2.100)$$

Each of the last path integrals has a similar form as the one discussed in [14]. One can perform the transformation $\mu = \mathrm{e}^x$, $\nu = \mathrm{e}^y$. Then the path integration in (μ,ν) gives a path integration in (x,y) of the following form:

$$K^{(V_4)}(x'',x',y'',y';s'') = \int\limits_{x(0)=x'}^{x(s'')=x''} \mathcal{D}x(s)\times$$
$$\times \exp\left\{\frac{i}{\hbar}\int\limits_0^{s''}\left[\frac{m}{2}\dot{x}^2-\frac{1}{2}(m\omega^2-bE)\,\mathrm{e}^{2x}-(d_1-aE)\,\mathrm{e}^x\right]ds\right\}\int\limits_{y(0)=y'}^{y(s'')=y''}\mathcal{D}y(s)\times$$
$$\times \exp\left\{-\frac{i}{\hbar}\int\limits_0^{s''}\left[\frac{m}{2}\dot{y}^2-\frac{1}{2}(m\omega^2-bE)\,\mathrm{e}^{2y}-(d_2+aE)\,\mathrm{e}^y\right]ds\right\}, \quad (2.101)$$

and we find the product of two path integrals for the Morse potential. This can be evaluated now as follows. We introduce the abbreviations

$$V_0^2 = \frac{m}{\hbar^2}(m\omega^2-bE), \quad \alpha_{x,y} = -\frac{d_{1,2}\mp aE}{m\omega^2-bE}. \quad (2.102)$$

We expand each path integral first into the discrete spectrum contribution by means of the known solution of the Morse potential in terms of Laguerre polynomials with the quantum numbers n and l, respectively, and the corresponding energy spectra. The s'' integration gives the energy spectrum

$$E_{n,l} = \frac{m\omega^2}{b} - \frac{m}{4b\hbar^2}\frac{(d_1+d_2)^2}{(n+l+1)^2}, \quad (2.103)$$

together with the wave functions ($N_{n,l}$ is determined by the corresponding residuum)

$$\Psi_{n,l}(x,y) = N_{n,l}\Psi_n^{(\mathrm{MP})}(x)\cdot\Psi_n^{(\mathrm{MP})}(y), \tag{2.104}$$

$$\Psi_k^{(\mathrm{MP})}(z) = \left(\frac{2\alpha_z V_0 - 2k - 1}{k!\Gamma(2\alpha_z V_0 - k)}\right)^{1/2}\times$$
$$\times\,(2V_0)^{\alpha\ \ V_0-k-1/2}\,\mathrm{e}^{(\alpha\ \ V_0-k-1/2)z-V_0\,\mathrm{e}}\ \ L_k^{(2\alpha\ \ V_0-2k-1)}(2V_0\,\mathrm{e}^z), \tag{2.105}$$

for $z = x, y$ with $k = n, l$. The continuous spectrum is examined in an analogous way yielding

$$E = \frac{\hbar^2 p^2}{2m}, \tag{2.106}$$

with the wave functions

$$\Psi_{p,\lambda}(x,y) = \Psi_{p,\lambda}^{(\mathrm{MP})}(x)\cdot\Psi_{p,\lambda}^{(\mathrm{MP})}(y), \tag{2.107}$$

$$\Psi_{p,\lambda}^{(\mathrm{MP})}(z) = \left(\frac{p_\pm \sinh 2\pi p_\pm}{2\pi^2 V_0}\right)^{1/2}\times$$
$$\times\left|\Gamma\left(ip_\pm - \alpha_z + \frac{1}{2}\right)\right|\,\mathrm{e}^{-z}W_{\alpha\ \ V_0, ip_\pm}(2V_0\,\mathrm{e}^x), \tag{2.108}$$

with $p_\pm = p \pm \lambda$ for $z = x, y$. The entire Green function has the form

$$G(\mu'',\mu',\nu'',\nu';E) = \sum_{n,l}\frac{\Psi_{n,l}(\mu'',\nu'')\Psi_{n,l}(\mu',\nu')}{E_{n,l}-E} +$$
$$+\int dp\int d\lambda\frac{\Psi_{p,\lambda}(\mu'',\nu'')\Psi^*_{p,\lambda}(\mu',\nu')}{\dfrac{\hbar^2p^2}{2m}-E}, \tag{2.109}$$

together with the replacement $\mu = \mathrm{e}^x$, $\nu = \mathrm{e}^y$. This concludes the discussion.

2.5. The Superintegrable Potential V_5 on $D_{\rm III}$. We display the potential V_5 in the respective coordinate systems

$$V_5(u,v) = \frac{1}{a+b\,\mathrm{e}^{-u}}\frac{\hbar^2 v_0^2}{2m}, \tag{2.110}$$

$$= \frac{1}{a+\dfrac{b}{4}\varrho^2}\frac{\hbar^2 v_0^2}{2m}, \tag{2.111}$$

$$= \frac{1}{a+\dfrac{b}{4}(\xi^2+\eta^2)}\frac{\hbar^2 v_0^2}{2m}, \tag{2.112}$$

$$= \frac{1}{a+\dfrac{b}{4}d^2(\sinh^2\omega+\cos^2\varphi)}\frac{\hbar^2 v_0^2}{2m}, \tag{2.113}$$

$$= \frac{1}{\Big(a+\dfrac{b}{2}(\mu-\nu)\Big)(\mu+\nu)}\frac{\hbar^2 v_0^2}{2m}. \tag{2.114}$$

We discuss the path integral solution of V_5 in some extend, where the case of elliptic coordinates is omitted due to intractability of this system in the path integral. Provided that $b>0$, there is in the case of the free motion a discrete spectrum

$$E_N = -\frac{\hbar^2}{2m}\frac{b}{a^2}(2N+1)^2, \tag{2.115}$$

with the principal quantum number $N\in\mathbb{N}$.

2.5.1. Separation of V_5 in the (u,v) System. We insert the potential V_5 into the path integral for the (u,v) system and obtain

$$K^{(V_5)}(u'',u',v'',v';T) = \int\limits_{u(t')=u'}^{u(t'')=u''}\mathcal{D}u(t)\int\limits_{v(t')=v'}^{v(t'')=v''}\mathcal{D}v(t)(a\,\mathrm{e}^{-u}+b\,\mathrm{e}^{-2u})\times$$

$$\times\exp\left\{\frac{i}{\hbar}\int\limits_0^T\left[\frac{m}{2}(a\,\mathrm{e}^{-u}+b\,\mathrm{e}^{-2u})(\dot u^2+\dot v^2)-\frac{1}{a+b\,\mathrm{e}^{-u}}\frac{\hbar^2v_0^2}{2m}\right]dt\right\} =$$

$$= \int\limits_{-\infty}^{\infty}\frac{dE}{2\pi\hbar}\,\mathrm{e}^{-iET/\hbar}\int\limits_0^\infty ds''\mathrm{e}^{-i\hbar v_0^2 s''/2m}K^{(V_5)}(u'',u',v'',v';s''), \tag{2.116}$$

with the time-transformed path integral $K^{(V_5)}(s'')$ given by

$$K^{(V_5)}(u'',u',v'',v';s'') = \int\limits_{u(0)=u'}^{u(s'')=u''} \mathcal{D}u(s) \int\limits_{v(0)=v'}^{v(s'')=v''} \mathcal{D}v(s)\times$$

$$\times \exp\left(\frac{i}{\hbar}\int\limits_0^{s''}\left\{\frac{m}{2}(\dot{u}^2+\dot{v}^2)+Eb\left[\mathrm{e}^{-2u}+\left(\frac{aE-\hbar^2v_0^2/2m}{Eb}\right)\mathrm{e}^{-u}\right]\right\}ds\right) =$$

$$= \sum_{l=0}^{\infty}\frac{\mathrm{e}^{il(v''-v')}}{2\pi}\,\mathrm{e}^{-i\hbar l^2 s''/2m}\int\limits_{u(0)=u'}^{u(s'')=u''}\mathcal{D}u(s)\times$$

$$\times \exp\left(\frac{i}{\hbar}\int\limits_0^{s''}\left\{\frac{m}{2}\dot{u}^2+Eb\left[\mathrm{e}^{-2u}+\left(\frac{aE-\hbar^2v_0^2/2m}{Eb}\right)\mathrm{e}^{-u}\right]\right\}ds\right). \quad (2.117)$$

The path integral in u is a path integral for the Morse potential. Performing the s'' integration gives, c.f. [14], the Green function as follows ($\mathcal{E} = [Ea - (\hbar^2 v_0^2/2m)]\sqrt{-2m/bE}/2\hbar$):

$$G^{(V_5)}(u'',u',v'',v';E) = \sum_{l=-\infty}^{\infty}\frac{\mathrm{e}^{il(v''-v')}}{2\pi}\frac{m\Gamma\left(\frac{1}{2}+l-\mathcal{E}\right)}{\hbar\sqrt{-2mbE}\,\Gamma(1+2l)}\,\mathrm{e}^{(u'+u'')/2}\times$$

$$\times W_{\mathcal{E},l}\left(\frac{\sqrt{-8mbE}}{\hbar}\,\mathrm{e}^{-u}\right)M_{\mathcal{E},l}\left(\frac{\sqrt{-8mbE}}{\hbar}\,\mathrm{e}^{-u}\right). \quad (2.118)$$

The corresponding continuous part of the Green function is evaluated as [14]

$$G^{(V_5)}_{\mathrm{cont}}(u'',u',v'',v';E) = \sum_{l=-\infty}^{\infty}\frac{\mathrm{e}^{il(v''-v')}}{2\pi}\mathrm{e}^{(u'+u'')/2}\times$$

$$\times\int\limits_0^{\infty}\frac{\mathrm{e}^{\pi p/2}dp}{\frac{\hbar^2p^2}{2m}-E}\frac{\left|\Gamma\left(\frac{1}{2}+l+ip\right)\right|^2}{2\pi\Gamma^2(1+2l)}M_{ip/2,l}\Big(-2ip\,\mathrm{e}^{-u'}\Big)M_{-ip/2,l}\Big(2ip\,\mathrm{e}^{-u''}\Big). \quad (2.119)$$

In addition, we have a discrete spectrum. This is found by analyzing the poles of the Green function (2.118):

$$\frac{1}{2}+l-\frac{aE_{nl}-\frac{\hbar^2v_0^2}{2m}}{2\hbar}\sqrt{-\frac{2m}{bE_{nl}}} = -n. \quad (2.120)$$

In the case of $v_0 = 0$ this simplifies to

$$n + l + \frac{1}{2} - \frac{a}{2\hbar}\sqrt{-\frac{2m}{bE_{nl}}} = 0, \qquad (2.121)$$

with the solution

$$E_{nl} = -\frac{\hbar^2}{2m}\frac{b}{a^2}(2n + 2l + 1)^2 \qquad (2.122)$$

yielding for $b > 0$ an infinite number of bound states. For $v_0 \neq 0$, the equation for E_{nl} is a quadratic equation in E with solution

$$E_{nl\pm} = -\frac{\hbar^2}{2m}\frac{1}{2a^2}\times$$
$$\times\left[b(2n{+}2l{+}1)^2 - 2av_0^2 \pm b(2n{+}2l{+}1)^2\sqrt{1 - \frac{4av_0^2}{b(2n{+}2l{+}1)^2}}\right], \qquad (2.123)$$

$$E_{nl+} \overset{(n,l)\to\infty}{=} -\frac{\hbar^2}{2m}\frac{b}{a^2}\left[(2n + 2l + 1)^2 - 2\frac{a}{b}v_0^2\right], \qquad (2.124)$$

$$E_{nl-} \overset{(n,l)\to\infty}{=} -\frac{\hbar^2}{2bm}\frac{v_0^4}{(2n + 2l + 1)^2}. \qquad (2.125)$$

For $v_0 = 0$, there is only E_{nl+}. For $(2n{+}2l{+}1)^2 < 4av_0^2/b$, there are semibound states located approximately around $E_0 = -\hbar^2 v_0^2/2ma$.

Therefore we have for the discrete spectrum contribution

$$G^{(V_5)}_{\text{disc}}(u'', u', v'', v'; E) =$$
$$= \sum_{l=-\infty}^{\infty}\frac{e^{il(v''-v')}}{2\pi}\sum_{n=0}^{\infty}\frac{1}{E_{nl} - E}\Psi^{(V_5)}_{nl}(u'')\Psi^{(V_5)}_{nl}(u'), \qquad (2.126)$$

with the functions $\Psi^{(V_5)}_{nl}(u)$ given by ($\mathcal{E}$ as in (2.118))

$$\Psi^{(V_5)}_{nl}(u) = N_{nl}\frac{(2\mathcal{E} - 2n - 1)n!}{\Gamma(2\mathcal{E} - n)}\left(\frac{\sqrt{-8mbE_{nl}}}{\hbar}\right)^{\mathcal{E}-n-1/2}\times$$
$$\times\exp\left[\left(\mathcal{E} - n - \frac{1}{2}\right)u - \sqrt{-\frac{8mbE_{nl}}{\hbar}}\,e^u\right]\times$$
$$\times L_n^{(2\mathcal{E}-2n-1)}\left(\frac{\sqrt{-8mbE_{nl}}}{\hbar}\,e^u\right). \qquad (2.127)$$

The constant N_{nl} is determined by taking the Green function at the residuum E_{nl}. The wave functions vanish for $u \to \infty$ due to $\mathrm{e}^{-\sqrt{-8mbE}\ \mathrm{e}\ /\hbar} = \mathrm{e}^{-2b\hbar(2n+2l+1)\mathrm{e}\ /a} \to 0$, provided $b/a > 0$ for all $n \in \mathbb{N}$, which shows that the discrete spectrum is indeed infinite. The feature that an homogeneous space with curvature has at the same time a discrete and a continuous spectrum is already known from the path integration on the $SU(1,1)$ group manifold [22]. Actually, this property allows the analysis of the modified Pöschl–Teller potential with its continuous and (finite) discrete spectrum.

2.5.2. Separation of V_5 in Polar Coordinates. We insert the potential V_5 into the path integral in polar coordinates and obtain

$$K^{(V_5)}(\varrho'',\varrho',\varphi'',\varphi';T) = \int\limits_{\varrho(t')=\varrho'}^{\varrho(t'')=\varrho''} \mathcal{D}\varrho(t) \int\limits_{\varphi(t')=\varphi'}^{\varphi(t'')=\varphi''} \mathcal{D}\varphi(t) \left(a + \frac{b}{4}\varrho^2\right)\varrho \times$$
$$\times \exp\left\{\frac{i}{\hbar}\int\limits_0^T \left[\frac{m}{2}\left(a+\frac{b}{4}\varrho^2\right)(\dot\varrho^2+\varrho^2\dot\varphi^2)+\left(a+\frac{b}{4}\varrho^2\right)^{-1}\frac{\hbar^2}{2m}\left(v_0^2+\frac{1}{4\varrho^2}\right)\right]dt\right\} =$$
$$= \int\limits_{-\infty}^{\infty}\frac{dE}{2\pi\hbar}\,\mathrm{e}^{-iET/\hbar}G^{(V_5)}(\varrho'',\varrho',\varphi'',\varphi';E), \quad (2.128)$$

and the Green function is evaluated to have the form [14] $\left(\mathcal{E}=\dfrac{(aE-\hbar^2v_0^2/2m)}{\hbar\omega},\right.$ $\left.\omega^2 = -bE/2m\right)$

$$G^{(V_5)}(\varrho'',\varrho',\varphi'',\varphi';E)\sum_{l=-\infty}^{\infty}\frac{\mathrm{e}^{il(\varphi''-\varphi')}}{2\pi}\frac{1}{\varrho'\varrho''}\sqrt{-\frac{2m}{2E}}\,\frac{\Gamma\left[\frac{1}{2}(1+l-\mathcal{E})\right]}{\Gamma(1+l)}\times$$
$$\times W_{\mathcal{E}/2,\frac{}{2}}\left(\sqrt{-\frac{2mbE}{\hbar^2}}\,\varrho_>\right)M_{\mathcal{E}/2,\frac{}{2}}\left(\sqrt{-\frac{2mbE}{\hbar^2}}\,\varrho_<\right). \quad (2.129)$$

The Green function has poles which are determined by

$$2n+l+1-\frac{1}{\hbar}\left(aE_{nl}-\frac{v_0^2\hbar^2}{2m}\right)\sqrt{-\frac{2m}{Eb_{nl}}} = 0. \quad (2.130)$$

In the case of $v_0 = 0$ this simplifies to

$$(2n+l+1)-\frac{a}{\hbar}\sqrt{-\frac{2m}{E_{nl}b}} = 0, \quad (2.131)$$

with the solution

$$E_{nl} = -\frac{\hbar^2}{2m}\frac{b}{a^2}(2n+l+1)^2 \tag{2.132}$$

yielding for $b > 0$ an infinite number of bound states. For $v_0 \neq 0$ the equation for E_{nl} is a quadratic equation in E with solution

$$E_{nl\pm} = -\frac{\hbar^2}{2m}\frac{1}{2a^2}\times$$
$$\times\left[b(2n+l+1)^2 - 2av_0^2 \pm b(2n+l+1)^2\sqrt{1-\frac{4av_0^2}{b(2n+l+1)^2}}\right]. \tag{2.133}$$

The limit of $N, l \to \infty$ yields

$$E_{nl+} \simeq -\frac{\hbar^2}{2m}\left[\frac{b}{a^2}(2n+l+1)^2 + \frac{v_0^2}{a}\right], \tag{2.134}$$

$$E_{nl-} \simeq -\frac{\hbar^2}{2m}\frac{v_0^2}{4b(2n+l+1)^2}, \tag{2.135}$$

and E_{nl+} corresponds in this limit to the spectrum of the free motion.

2.5.3. Separation of V_5 in Parabolic Coordinates. We insert the potential V_5 into the path integral in parabolic coordinates and obtain

$$K^{(V_5)}(\xi'',\xi',\eta'',\eta';T) = \int\limits_{\xi(t')=\xi'}^{\xi(t'')=\xi''}\mathcal{D}\xi(t)\int\limits_{\eta(t')=\eta'}^{\eta(t'')=\eta''}\mathcal{D}\eta(t)\left(a+\frac{b}{4}(\xi^2+\eta^2)\right)\times$$
$$\times\exp\left\{\frac{i}{\hbar}\int\limits_0^T\left[\frac{m}{2}\left(a+\frac{b}{4}(\xi^2+\eta^2)\right)(\dot\xi^2+\dot\eta^2) - \frac{1}{a+\frac{b}{4}(\xi^2+\eta^2)}\frac{\hbar^2 v_0^2}{2m}\right]dt\right\} =$$
$$= \int\limits_{-\infty}^{\infty}\frac{dE}{2\pi\hbar}\,\mathrm{e}^{-iET/\hbar}G^{(V_5)}(\xi'',\xi',\eta'',\eta';E), \tag{2.136}$$

with the time-transformed path integral $K(s'')$ given by

$$K^{(V_5)}(\xi'',\xi',\eta'',\eta';s'') = \int\limits_{\xi(0)=\xi'}^{\xi(s'')=\xi''}\mathcal{D}\xi(s)\int\limits_{\eta(0)=\eta'}^{\eta(s'')=\eta''}\mathcal{D}\eta(s)\times$$
$$\times\exp\left\{\frac{i}{\hbar}\int\limits_0^{s''}\left[\frac{m}{2}(\dot\xi^2+\dot\eta^2)+E\frac{b}{4}(\xi^2+\eta^2)\right]ds+\frac{i}{\hbar}\left(aE-\frac{\hbar^2 v_0^2}{2m}\right)ds\right\}. \tag{2.137}$$

The only difference in comparison with the result in [14] is the the additional $\dfrac{\hbar^2 v_0^2}{2m}$ term in the s'' integration. In order to find the discrete spectrum we insert the solution for the harmonic oscillator and get

$$G^{(V_5)}_{\rm disc}(\xi'',\xi',\eta'',\eta';E) =$$
$$= \sum_{n\ =0}^{\infty}\sum_{n\ =0}^{\infty} \frac{N^2_{n\ n}}{E_{n\ n}-E}\Psi_n^{(\rm HO)}(\xi'')\Psi_n^{(\rm HO)}(\xi'')\Psi_n^{(\rm HO)}(\eta'')\Psi_n^{(\rm HO)}(\eta''), \quad (2.138)$$

where $E_{n\ n}$ is determined by the equation

$$(n_\xi+n_\eta+1)-\frac{1}{\hbar}\left(aE-\frac{\hbar^2 v_0^2}{2m}\right)\sqrt{-\frac{bE}{2m}}=0, \quad (2.139)$$

which is (up to a different counting in the quantum numbers) identical with (2.131). The normalization $N_{n\ n}$ is determined by the residuum in $G^{(V_5)}(E)$. We do not state the continuous spectrum part, it can be derived from [14] by the replacement $aE \to aE-\hbar^2 v_0^2/2m$.

2.5.4. Separation of V_5 in Hyperbolic Coordinates. We insert the potential V_5 into the path integral in hyperbolic coordinates and obtain: The path integral has the form

$$K^{(V_5)}(\mu'',\mu',\nu'',\nu';T) =$$
$$= \int\limits_{\mu(t')=\mu'}^{\mu(t'')=\mu''}\mathcal{D}\mu(t)\int\limits_{\nu(t')=\nu'}^{\nu(t'')=\nu''}\mathcal{D}\nu(t)\frac{\left(a+\frac{b}{2}(\mu-\nu)\right)(\mu+\nu)}{\mu\nu}\times$$
$$\times\exp\left\{\frac{i}{\hbar}\int\limits_0^T\left[\left(a+\frac{b}{2}(\mu-\nu)\right)(\mu+\nu)\left(\frac{\dot\mu^2}{\mu^2}-\frac{\dot\nu^2}{\nu^2}\right)-\right.\right.$$
$$\left.\left.-\frac{1}{\left(a+\frac{b}{2}(\mu-\nu)\right)(\mu+\nu)}\frac{\hbar^2 v_0^2}{2m}\right]dt\right\} =$$
$$= \int\limits_{-\infty}^{\infty}\frac{dE}{2\pi\hbar}\,{\rm e}^{-iET/\hbar}\int\limits_0^\infty ds''K^{(V_5)}(\mu'',\mu',\nu'',\nu';s''), \quad (2.140)$$

and the path integral $K^{(V_5)}(s'')$ is given by

$$K^{(V_5)}(\mu'',\mu',\nu'',\nu';s'') = \int\limits_{\mu(0)=\mu'}^{\mu(s'')=\mu''} \mathcal{D}\mu(s) \int\limits_{\nu(0)=\nu'}^{\nu(s'')=\nu''} \mathcal{D}\nu(s) \frac{1}{\mu\nu} \times$$
$$\times \exp\left\{ \frac{i}{\hbar} \int\limits_0^{s''} \left[\frac{m}{2}\left(\frac{\dot{\mu}^2}{\mu^2} - \frac{\dot{\nu}^2}{\nu^2}\right) + (\mu+\nu)\left(aE - \frac{\hbar^2 v_0^2}{2m}\right) + \right.\right.$$
$$\left.\left. + \frac{1}{2} bE(\mu^2-\nu^2) \right] ds \right\}. \quad (2.141)$$

Each of the last path integrals has a similar form as the one discussed in [11]. One can perform the transformation $\mu = \mathrm{e}^x$, $\nu = \mathrm{e}^y$ yielding

$$K^{(V_5)}(x'',x',y'',y';s'') =$$
$$= \int\limits_{x(0)=x'}^{x(s'')=x''} \mathcal{D}x(s) \exp\left\{ \frac{i}{\hbar} \int\limits_0^{s''} \left[\frac{m}{2}\dot{x}^2 + \left(E\frac{b}{2}\,\mathrm{e}^{2x} + \left(aE - \frac{\hbar^2 v_0^2}{2m}\right) \mathrm{e}^x \right) \right] ds \right\} \times$$
$$\times \int\limits_{y(0)=y'}^{y(s'')=y''} \mathcal{D}y(s) \exp\left\{ -\frac{i}{\hbar} \int\limits_0^{s''} \left[\frac{m}{2}\dot{y}^2 + \left(E\frac{b}{2}\,\mathrm{e}^{2y} - \left(aE - \frac{\hbar^2 v_0^2}{2m}\right) \mathrm{e}^y \right) \right] ds \right\}$$
$$(2.142)$$

and we find the product of two path integrals for the Morse potential, however more complicated as in [14]. The continuous part of the spectrum can be analyzed similarly as in [14] yielding products of M-Whittaker functions. Analyzing the discrete spectrum contribution from the Morse potential we find the quantization condition

$$(n_\xi + n_\eta + 1) - \frac{1}{\hbar}\left(aE - \frac{\hbar^2 v_0^2}{2m}\right)\sqrt{-\frac{4m}{E_{nl} b}} = 0, \quad (2.143)$$

which is up to a different counting in the quantum numbers equivalent with (2.131). This concludes the discussion.

3. SUPERINTEGRABLE POTENTIALS ON DARBOUX SPACE $D_{\rm IV}$

Finally, we consider the Darboux space $D_{\rm IV}$. We have the coordinate systems:

$$((u,v)\text{ system:})\quad x = v + iu, \quad y = v - iu \quad (u \in (0,\pi/2), v \in \mathbb{R}), \tag{3.1}$$

$$(\text{Equidistant:})\quad u = \arctan(\mathrm{e}^{\alpha}), \quad v = \frac{\beta}{2} \quad (\alpha \in \mathbb{R}, \beta \in \mathbb{R}), \tag{3.2}$$

$$(\text{Horospherical:})\quad x = \log\frac{\mu - i\nu}{2}, \quad y = \log\frac{\mu + i\nu}{2} \quad (\mu,\nu > 0), \tag{3.3}$$

$$\mu = 2\mathrm{e}^{v}\cos u, \quad \nu = -2\mathrm{e}^{v}\sin u, \tag{3.4}$$

$$(\text{Elliptic:})\quad \mu = d\cosh\omega\cos\varphi, \quad \nu = d\sinh\omega\sin\varphi \quad (\omega > 0, \varphi \in (0,\pi/2)). \tag{3.5}$$

We obtain the following forms of the line-element ($a > 2b$, $a_{\pm} = (a \pm 2b)/4$):

$$\begin{aligned} ds^2 &= \frac{2b\cos u + a}{4\sin^2 u}(du^2 + dv^2) = \\ &= \left(\frac{a_+}{\sin^2 u} + \frac{a_-}{\cos^2 u}\right)(du^2 + dv^2) \\ &\quad (\text{rescaling } u/2 \to u:), \end{aligned} \tag{3.6}$$

$$(\text{Equidistant:}) = \frac{a - 2b\tanh\alpha}{4}(d\alpha^2 + \cosh^2\alpha\, d\beta^2), \tag{3.7}$$

$$(\text{Horospherical:}) = \left(\frac{a_+}{\nu^2} + \frac{a_-}{\mu^2}\right)(d\mu^2 + d\nu^2), \tag{3.8}$$

$$\begin{aligned} (\text{Elliptic:}) &= \left(\frac{a_-}{\cosh^2\omega\cos^2\varphi} + \frac{a_+}{\sinh^2\omega\sin^2\varphi}\right) \times \\ &\times (\cosh^2\omega - \cos^2\varphi)(d\omega^2 + d\varphi^2), \\ &= \left(\frac{a_+}{\sin^2\varphi} + \frac{a_-}{\cos^2\varphi} + \frac{a_+}{\sinh^2\omega} - \frac{a_-}{\cosh^2\omega}\right) \times \\ &\times (d\omega^2 + d\varphi^2), \end{aligned} \tag{3.9}$$

$$\begin{aligned} (\text{Degenerate elliptic I:}) &= \left[a_-\left(\frac{1}{\sinh^2\hat\omega} + \frac{1}{\sin^2\hat\varphi}\right) - a_+\left(\frac{1}{\cosh^2\hat\omega} - \frac{1}{\cos^2\hat\varphi}\right)\right] \times \\ &\times (d\hat\omega^2 + d\hat\varphi^2) \quad (\gamma = 1), \end{aligned} \tag{3.10}$$

$$(\text{Degenerate elliptic II:}) = \frac{1}{4}\left(\frac{a_-}{\sinh^2\tilde\omega} + \frac{a_+}{\sin^2\tilde\varphi}\right)(d\tilde\omega^2 + d\tilde\varphi^2)\ (\gamma = 2). \tag{3.11}$$

We observe that the diagonal term in the metric corresponds in most cases to a combination of a Pöschl–Teller potential and a modified Pöschl–Teller, respectively. In particular, the (u,v) and the equidistant systems are the same, they

just differ in the parameterization. The limiting cases $a = 2b$ and $b = 0$ give particular cases for the metric on the two-dimensional hyperboloid. We have also displayed two versions of degenerate elliptic coordinates. They come from the observation that for the representatives

$$K^2, \quad X_2, \quad \gamma X_2 + K2, \quad X_1 + X_2 + \gamma K^2 \tag{3.12}$$

one can distinguish the cases $\gamma = 0$, $\gamma = 2$, and $\gamma \neq 0, 2$. For $\gamma \neq 0, 2$, one has coordinate systems which can be explicitly formulated in terms of the elliptic functions $\mathrm{sn}(\alpha, k), \mathrm{cn}(\beta, k)$, and only for a special choice of the parameter k they can be simplified in trigonometric and hyperbolic functions. Then the line element has the form

$$ds^2 = \frac{1}{4}[a_+ k^4 \mathrm{sn}^2(\alpha, k) - \mathrm{sn}^2(\beta, k) + k^2 a_-](d\alpha^2 + d\beta^2), \tag{3.13}$$

and separated equations are versions of Lame's equation, if we assume an Ansatz of the form $\Psi = A(\alpha)B(\beta)$ [28]:

$$\frac{\partial^2 A(\alpha)}{\partial \alpha^2} + \left(-\frac{1}{4}k^4 E a_+ \mathrm{sn}^2(\alpha, k) - \lambda_1\right) A(\alpha) = 0, \tag{3.14}$$

$$\frac{\partial^2 B(\beta)}{\partial \beta^2} + \left(-\frac{1}{4}k^4 E a_+ \mathrm{sn}^2(\beta, k) - \lambda_2\right) B(\beta) = 0, \tag{3.15}$$

where $\lambda_1 - \lambda_2 = -Ea_- k^2/4$. k denotes the modulus of the elliptic functions.

In particular, for the potential V_2 one has the possibilities of taking $\gamma = 0$, and $\gamma = 2$. For $\gamma = 0$, the modulus k of the elliptic functions equals $k = -i$. We do not treat V_2 in these elliptic coordinates, but only the degenerate case of $\gamma = 2$.

For the potential V_3, however, the elliptic systems with $\gamma = 1$ can be explicitly worked out. We have stated the respective line elements for these two cases. Note that for $\gamma = 2$ the coordinate transformation can be put into

$$x = \ln\Big[\tan(\tilde{\varphi} - i\tilde{\omega})\Big], \quad y = \ln\Big[\tan(\tilde{\varphi} + i\tilde{\omega})\Big] \quad (\tilde{\omega} > 0, \tilde{\varphi} \in (0, \pi/4)). \tag{3.16}$$

We do not dwell into a discussion of elliptic systems any further, for details we refer to [26]. Let us finally note that the notion *elliptic* is also used for the (ω, φ) system, and they must not be confused with the general elliptic coordinates just discussed.

Because we have not worked out the path integral for the free motion in these two further coordinate systems, this will be done in an appendix. For the

Gaussian curvature we obtain, e.g., in the (u, v) system

$$G = -\frac{\dfrac{a_+^2}{\sin^6 u} + \dfrac{a_-^2}{\cos^6 u} + \dfrac{a_- a_+}{\sin^4 u \cos^4 u}}{\left(\dfrac{a_+}{\sin^2 u} + \dfrac{a_-}{\cos^2 u}\right)^3}. \tag{3.17}$$

The case $a = 2b$ yields $a_- = 0$, and

$$G = -\frac{1}{b}, \tag{3.18}$$

and therefore again a space of constant curvature, the hyperboloid $\Lambda^{(2)}$ is given for $b > 0$. We have set the sign in the metric (1.4) in such a way that from $a = 2b > 0$ the hyperboloid $\Lambda^{(2)}$ emerges. We could also choose the metric (1.4) with the opposite sign, then $a = 2b < 0$ would give the same result. In the following it is understood that we make this restriction of positive definiteness of the metric and we do not dwell into the problem of continuation into nonpositive definiteness. Because the (u, v) coordinates and the equidistant system are the same, we do not evaluate the path integral in the equidistant system. In the following we assume $a_+ > 0$ and $a_+ > a_-$.

We introduce the following three constants of motion on D_{IV}:

$$X_1 = \mathrm{e}^{2v}(-\tilde{\mathcal{H}}_0 + \cos 2u \cdot p_u^2 + \sin 2u \cdot p_u p_v), \tag{3.19}$$

$$X_2 = \mathrm{e}^{2v}(-\tilde{\mathcal{H}}_0 + \cos 2u \cdot p_u^2 - \sin 2u \cdot p_u p_v), \tag{3.20}$$

$$K = p_v. \tag{3.21}$$

These integrals of motion satisfy the Poisson relations

$$\{K, X_1\} = 2X_1, \quad \{K, X_2\} = -2X_2, \quad \{X_1, X_2\} = -K^3 - 4aKH_0, \tag{3.22}$$

and satisfy the relation

$$X_1 X_2 - K^4 - aK^2 H_0 - H_0^2 = 0. \tag{3.23}$$

The corresponding quantum operators have the form

$$\widehat{H}_0 = \frac{\sin^2 2u}{2\cos 2u + a}(\partial_u^2 + \partial_v^2), \tag{3.24}$$

$$\widehat{X}_1 = \mathrm{e}^{2v}(-\widehat{H}_0 + \cos 2u \cdot (\partial_u^2 + \partial_v) + \sin 2u \cdot (\partial_u \partial_v + \partial_u), \tag{3.25}$$

$$\widehat{X}_2 = \mathrm{e}^{2v}(-\widehat{H}_0 + \cos 2u \cdot (\partial_u^2 - \partial_v) - \sin 2u \cdot (\partial_u \partial_v - \partial_u), \tag{3.26}$$

and the commutation relations read

$$[\widehat{K}, \widehat{X}_1] = 2\widehat{X}_1,\ [\widehat{K}, \widehat{X}_2] = -2\widehat{X}_2,\ [\widehat{X}_1, \widehat{X}_2] = -8\widehat{K}^3 - 4a\widehat{K}\widehat{H}_0 - 4\widehat{K} \tag{3.27}$$

and satisfy the operator relation

$$\frac{1}{2}\{\widehat{X}_1, \widehat{X}_2\} - \widehat{K}^4 - a\widehat{H}_0\widehat{K}^2 - 5\widehat{K}^2 - \widehat{H}_0^2 - a\widehat{H}_0 = 0. \tag{3.28}$$

In Table 3 we list the connection with these operators and the corresponding coordinate systems on $D_{\rm IV}$.

Table 3. **Constants of motion and limiting cases of coordinate systems on $D_{\rm IV}$**

Metric	Constants of motion	$D_{\rm IV}$	$\Lambda^{(2)}$ $(a = 2b)$	$\Lambda^{(2)}$ $(b = 0)$
$\dfrac{2b\cos\, u + a}{4\sin^2 u}(du^2 + dv^2)$	K^2	(u, v) system	Equidistant	Equidistant
$\left(\dfrac{a_+}{\nu^2} + \dfrac{a_-}{\mu^2}\right)(d\mu^2 + d\nu^2)$	X_2	Horospherical	Horicyclic	Semicircular parabolic
$\left(\dfrac{a_-}{\cosh^2\omega\cos^2\varphi} + \dfrac{a_+}{\sinh^2\omega\sin^2\varphi}\right)\times$ $\times(\cosh^2\omega - \cos^2\varphi)(d\omega^2 + d\varphi^2)$	$K^2 + d^2X_2$	Elliptic	Elliptic-parabolic	Hyperbolic-parabolic
$\Big[a_+k^2\Big(\mathrm{sn}^2(\alpha, k) - \mathrm{sn}^2(\beta, k)\Big) + a_-\Big]\times$ $\times\dfrac{k^2}{4}(d^2\alpha + d^2\beta)$	$X_1 + X_2 + \gamma K^2$	Elliptic	Elliptic	Elliptic

We state the superintegrable potentials on $D_{\rm IV}$:

$$V_1(u, v) = \left(\frac{a_+}{\sin^2 u} + \frac{a_-}{\cos^2 u}\right)^{-1}\times \\ \times\left[\frac{\hbar^2}{2m}\left(\frac{k^2 - 1/4}{\cos^2 u} + \frac{k^2 - 1/4}{\sin^2 u}\right) - 4\alpha e^{2v} + 8m\omega^2 e^{4v}\right], \tag{3.29}$$

$$V_2(u, v) = \left(\frac{a_+}{\sin^2 u} + \frac{a_-}{\cos^2 u}\right)^{-1}\times \\ \times\left[\frac{\hbar^2}{2m}\left(\frac{k_1^2 - 1/4}{\sinh^2 v} - \frac{k_2^2 - 1/4}{\cosh^2 v}\right) - \frac{\alpha}{4}\left(\frac{1}{\sin^2 u} + \frac{1}{\cos^2 u}\right)\right], \tag{3.30}$$

$$V_3(\tilde\omega, \tilde\varphi) = \frac{\hbar^2}{2m}\left(\frac{a_+}{\sinh^2\tilde\omega} - \frac{a_+}{\cosh^2\tilde\omega} + \frac{a_-}{\sin^2\tilde\varphi} + \frac{a_-}{\cos^2\tilde\varphi}\right)^{-1}\times \\ \times\left[\frac{c_3}{\sin^2\tilde\varphi} + \frac{c_2}{\cos^2\tilde\varphi} - \frac{c_3}{\sinh^2\tilde\omega} + \frac{c_2}{\cosh^2\tilde\omega}\right], \tag{3.31}$$

$$V_4(\mu, \nu) = \left(\frac{a_+}{\nu^2} + \frac{a_-}{\mu^2}\right)^{-1}\frac{\hbar^2}{2m}\left(k_0^2 - \frac{1}{4}\right)\left(\frac{1}{\mu^2} + \frac{1}{\nu^2}\right). \tag{3.32}$$

Table 4. **Separation of variables for the superintegrable potentials on D_{IV}**

Potential	Constants of motion	Separating coordinate system
V_1	$R_1 = K^2 - \alpha(\mu^2+\nu^2) + \frac{m}{2}\omega^2(\mu^2+\nu^2)$ $R_2 = X_2 + \frac{-2\alpha(a_+\mu^2 - a_-\nu^2) + 8(k^2-1/4)\frac{\hbar^2}{m} + 2m\omega^2(a_+\mu^4 - a_-\nu^4)}{a_+\mu^2 + a_-\nu^2}$	(u,v) system Horospherical Elliptic
V_2	$R_1 = X_1 + X_2 + (2\cos u + a)^{-1}\frac{\hbar^2}{2m}\Big[\Big(k_1^2 + k_2^2 - \frac{1}{2}\Big) -$ $-2\left(k_3^2 - \frac{1}{2}\right)\cosh 2v + (\cos 4u + 2a\cos 2u + 3)\Big(\frac{k_1^2 - 1/4}{\sinh^2 v} - \frac{k_2^2 - 1/4}{\cosh^2 v}\Big)\Big]$ $R_2 = K^2 + \frac{\hbar^2}{2m}\left(\frac{k_1^2 - 1/4}{\sinh^2 v} + \frac{k_2^2 - 1/4}{\cosh^2 v}\right)$	(u,v) system Degenerate elliptic I
V_3	$R_1 = X_1 + X_2 + 2K^2 + aH + \frac{\hbar^2}{2m}\left(\frac{a_+}{\sinh^2 2\tilde\omega} + \frac{a_-}{\sinh^2\tilde\omega}\right)^{-1}\times$ $\times\Big[\frac{a_+}{\sinh^2 2\tilde\omega}\Big(\frac{c_3}{\sin^2\tilde\varphi} + \frac{c_1}{\sin^2\tilde\varphi}\Big) + \frac{a_-}{\sinh^2 2\tilde\varphi}\Big(\frac{c_3}{\sinh^2\tilde\omega} - \frac{c_2}{\cos^2\tilde\omega}\Big)\Big]$ $R_2 = X_1 - X_2 + \frac{\hbar^2}{2m}\left(\frac{a_+}{\sinh^2 2\tilde\omega} + \frac{a_-}{\sinh^2\tilde\omega}\right)^{-1}\times$ $\times\Big[\frac{a_+}{\sinh^2 2\tilde\omega}\Big(c_1\cosh 2\tilde\omega\tan^2\tilde\varphi - c_2\cos 2\tilde\varphi -$ $-\frac{c_3(2\cos^2\tilde\varphi(\sinh^2\tilde\omega - \sin^2\tilde\varphi) + 1)}{\sin^2\tilde\varphi}\Big) +$ $+\frac{a_-}{\sin^2 2\tilde\varphi}\Big(c_2\cos 2\tilde\varphi\tanh^2\tilde\omega + c_1\cosh 2\tilde\omega -$ $-\frac{c_3(2\cosh^2\tilde\omega(\sinh^2\tilde\omega - \sin^2\tilde\varphi) + 1}{\sinh^2\tilde\omega}\Big)\Big]$	Degenerate elliptic I & II
V_4	$R_1 = X_1 + \frac{2\frac{\hbar^2}{m}(k_0^2 - 1/4)(\mu^2+\nu^2)}{a_+\mu^2 + a_-\nu^2}$ $R_2 = X_2 + \frac{32\frac{\hbar^2}{m}(k_0^2 - 1/4)}{a_+\mu^2 + a_-\nu^2}$ $R_3 = \mu p \quad + \nu p$	(u,v) system Horospherical Elliptic

In Table 4 we list the properties of these potentials on D_{IV}. We see that V_4 is a special case, and it has three integrals of motion. The variables $\tilde\omega, \tilde\varphi$ are defined by

$$x = \log\left[\tan\left(\tilde\varphi - i\tilde\omega\right)\right], \quad y = \log\left[\tan\left(\tilde\varphi + i\tilde\omega\right)\right]. \tag{3.33}$$

In terms of these coordinates the line element is given by

$$ds^2 = \frac{a+2b}{\sinh^2 2\tilde\omega} + \frac{a+2b}{\sin^2 2\tilde\varphi} = \frac{a_+}{\sinh^2\tilde\omega} - \frac{a_+}{\cosh^2\tilde\omega} - \frac{a_-}{\sin^2\tilde\varphi} + \frac{a_-}{\cos^2\tilde\varphi}. \tag{3.34}$$

3.1. The Superintegrable Potential V_1 on $D_{\rm IV}$. We start by stating the potential V_1 in the respective coordinate systems

$$V_1(u,v) = \left(\frac{a_+}{\sin^2 u} + \frac{a_-}{\cos^2 u}\right)^{-1} \times \\ \times \left[\frac{\hbar^2}{2m}\left(\frac{k^2-1/4}{\cos^2 u} + \frac{k^2-1/4}{\sin^2 u}\right) - 4\alpha e^{2v} + 8m\omega^2 e^{4v}\right], \qquad (3.35)$$

$$= \left(\frac{a_+}{\nu^2} + \frac{a_-}{\mu^2}\right)^{-1} \times \\ \times \left[-\alpha + \frac{\hbar^2}{2m}\left(\frac{k^2-1/4}{\mu^2} + \frac{k^2-1/4}{\nu^2}\right) + \frac{m}{2}\omega^2(\mu^2+\nu^2)\right], \qquad (3.36)$$

$$= d^2\left(\frac{a_+}{\sinh^2\omega\sin^2\varphi} + \frac{a_-}{\cosh^2\omega\cos^2\varphi}\right)^{-1} \times \\ \times \left[-\alpha + \frac{\hbar^2}{2m}\left(\frac{k^2-1/4}{\sinh^2\omega\sin^2\varphi} + \frac{k^2-1/4}{\cosh^2\omega\cos^2\varphi}\right) + \\ + \frac{m}{2}\omega^2 d^2(\cosh^2\omega - \sin^2\varphi)\right]. \qquad (3.37)$$

The path integral for the potential V_1 can be solved in the (u,v) system and in horospherical coordinates. We also keep the parameters k_1 and k_2 different in comparison with Kalnins et al.

3.1.1. Separation of V_1 in the (u,v) System. The classical Lagrangian and Hamiltonian are given by

$$\mathcal{L}(u,\dot u,v,\dot v) = \frac{m}{2}\frac{2b\cos 2u + a}{\sin^2 2u}(\dot u^2 + \dot v^2) + V(u,v), \qquad (3.38)$$

$$\mathcal{H}(u,p_u,v,p_v) = \frac{1}{2m}\frac{\sin^2 2u}{2b\cos 2u + a}(p_u^2 + p_v^2) + V(u,v). \qquad (3.39)$$

The canonical momentum operators are given by

$$p_u = \frac{\hbar}{i}\left(\frac{\partial}{\partial u} + 2\cot\, 2u - \frac{2b\sin 2u}{2b\cos 2u + a}\right), \quad p_v = \frac{\hbar}{i}\frac{\partial}{\partial v}, \qquad (3.40)$$

and the Hamiltonian operator has the form

$$H = -\frac{\hbar^2}{2m}\frac{\sin^2 2u}{2b\cos 2u + a}\left(\frac{\partial^2}{\partial u^2} + \frac{\partial^2}{\partial v^2}\right) + V(u,v) \qquad (3.41)$$

$$= \frac{1}{2m}\frac{\sin 2u}{\sqrt{2b\cos 2u + a}}(p_u^2 + p_v^2)\frac{\sin 2u}{\sqrt{2b\cos 2u + a}} + V(u,v). \qquad (3.42)$$

We insert V_1 into the path integral and obtain ($f = a_+/\sin^2 u + a_-/\cos^2 u$)

$$K^{(V_1)}(u'', u', v'', v'; T) = \int\limits_{u(t')=u'}^{u(t'')=u''} \mathcal{D}u(t) \int\limits_{v(t')=v'}^{v(t'')=v''} \mathcal{D}v(t) f(u) \times$$

$$\times \exp\left(\frac{i}{\hbar}\int\limits_0^T \left\{\frac{m}{2} f(\dot{u}^2+\dot{v}^2) - \frac{1}{f}\left[\frac{\hbar^2}{2m}\left(\frac{k_1^2-1/4}{\cos^2 u} - \frac{k_2^2-1/4}{\sin^2 u}\right)+\right.\right.\right.$$

$$\left.\left.\left. + 8m\omega^2\left(\mathrm{e}^{4v} - \frac{\alpha}{2m\omega^2}\mathrm{e}^{2v}\right)\right]\right\} dt\right). \quad (3.43)$$

We see that the v dependence has the form of a Morse potential:

$$V^{(\mathrm{MP})}(x) = \frac{\hbar^2 V_0^2}{2M}\left(\mathrm{e}^{2x} - 2\tilde{\alpha}\,\mathrm{e}^{x}\right), \quad (3.44)$$

where the (finite) discrete energy spectrum is given by

$$E_l = -\frac{\hbar^2}{2M}\left(\tilde{\alpha} - l - \frac{1}{2}\right)^2. \quad (3.45)$$

Proceeding in the usual way we obtain for the time-transformed path integral

$$K^{(V_1)}(u'', u', v'', v'; s'') = \int\limits_{u(0)=u'}^{u(s'')=u''} \mathcal{D}u(s) \int\limits_{v(0)=v'}^{v(s'')=v''} \mathcal{D}v(s) \times$$

$$\times \exp\left\{\frac{i}{\hbar}\int\limits_0^T \left[\frac{m}{2}(\dot{u}^2+\dot{v}^2) - \frac{\hbar^2}{2m}\left(\frac{\lambda_1^2-1/4}{\cos^2 u} - \frac{\lambda_2^2-1/4}{\sin^2 u}\right)-\right.\right.$$

$$\left.\left. - 8m\omega^2\left(\mathrm{e}^{4v} - \frac{\alpha}{2m\omega^2}\mathrm{e}^{2v}\right)\right] ds\right\} =$$

$$= \sum_n \Phi_n^{(\lambda_2,\lambda_1)}(u'')\Phi_n^{(\lambda_2,\lambda_1)}(u') \exp\left[-\frac{i}{\hbar}\frac{\hbar^2}{2m}(\lambda_1+\lambda_2+2n+1)2s''\right]\times$$

$$\times\left\{\int d\kappa \Psi_\kappa^{(\mathrm{MP})}(v'')\Phi_\kappa^{(MP)\,*}(u')\,\mathrm{e}^{-i\hbar\kappa^2 s''/2m} + \right.$$

$$\left. + \sum_l \Psi_l^{(\mathrm{MP})}(v'')\Phi_l^{(MP\,*)}(u')]\,\exp\left[\frac{i}{\hbar}\frac{\hbar^2}{2m}\left(\tilde{\alpha} - l - \frac{1}{2}\right)^2\right]\right\}. \quad (3.46)$$

Here, $\lambda_{1,2}^2 = k_{1,2}^2 - 2ma_{-,+}E/\hbar^2$, and in the variable v we have used the solution of the Morse potential and in the variable u the solution of the Pöschl–Teller potential, respectively. This form of the solution is convenient to obtain

the bound state solutions. The bound state energy levels are determined by

$$2(n+l+1)+\lambda_1+\lambda_2-\frac{\alpha}{\hbar\omega}=0. \qquad (3.47)$$

By denoting

$$N_{n,l}=\left(2(n+l+1)-\frac{\alpha}{\hbar\omega}\right)^2-(k_1^2+k_2^2) \qquad (3.48)$$

the quadratic equation in E can be solved to give (with the further abbreviation $K_a=4(a_+k_1^2+a_-k_2^2)$)

$$E_{n,l}= \\ =\frac{\hbar^2}{4mb^2}\left\{\pm\sqrt{(aN_{n,l}+K_a)^2-4b^2(N_{n,l}^2-4k_1^2k_2^2)}-(aN_{n,l}+K_a)\right\}. \qquad (3.49)$$

We keep the $\pm$-sign to allow for different boundary conditions which may depend on the parameters a and b. For instance, for $a=2b$ we get the limiting case:

$$E_{n,l}=-\frac{\hbar^2}{2ma}\left[\left((2(n+l+1)+k_1^2-\frac{\alpha}{\hbar\omega}\right)^2-k_2^2\right]. \qquad (3.50)$$

For $k_2=\pm 1/2$ it has the form of the usual zero-energy on the two-dimensional hyperboloid.

In order to obtain the continuous spectrum, the formulation in (u,v) coordinates is inconvenient. Following [12] we perform the coordinate transformation $\cos u=\tanh\tau$, and additionally we make a time-transformation with the time-transformation function $f=a_+/\sin^2 u+a_-/\cos^2 u$. Due to the coordinate transformation $\cos u=\tanh\tau$ additional quantum terms appear according to

$$\exp\left(\frac{im}{2\epsilon\hbar}\frac{\left(\Delta u^{(j)}\right)^2}{\cos u^{(j-1)}\cos u^{(j)}}\right)\doteq \\ \doteq\exp\left[\frac{im}{2\epsilon\hbar}\left(\Delta\tau^{(j)}\right)^2-i\frac{\hbar}{8m}\left(1+\frac{1}{\cosh^2\tau^{(j)}}\right)\right]. \qquad (3.51)$$

We get for the path integral (3.43)

$$K^{(V_1)}(u'',u',v'',v';T)=\int_{-\infty}^{\infty}\frac{dE}{2\pi\hbar}\,\mathrm{e}^{-iET/\hbar}\times \\ \times\int_0^\infty ds''\exp\left[\frac{i}{\hbar}\left(a_+E-\frac{\hbar^2k_2^2}{2m}\right)\right]K^{(V_1)}(\tau'',\tau',v'',v';s''), \qquad (3.52)$$

and the time-transformed path integral $K^{(V_1)}(s'')$ is given by

$$K^{(V_1)}(\tau'',\tau',v'',v';s'') = (\cosh\tau'\cosh\tau'')^{-1/2}\times \\ \times\left[\sum_l \Psi_l^{(\mathrm{MP})}(v')\Psi_l^{(\mathrm{MP})}(v'')K_l(\tau'',\tau';s'')+ \right. \\ \left. +\int d\kappa \Psi_\kappa^{(\mathrm{MP})\,*}(v')\Psi_\kappa^{(\mathrm{MP})}(v'')K_\kappa(\tau'',\tau';s'')\right], \quad (3.53)$$

$$K_{l,\kappa}^{(V_1)}(\tau'',\tau';s'') = \int\limits_{\tau(0)=\tau'}^{\tau(s'')=\tau''}\mathcal{D}\tau(s)\times \\ \times\exp\left\{\frac{i}{\hbar}\int\limits_0^{s''}\left[\frac{m}{2}\dot{\tau}^2-\frac{\hbar^2}{2m}\left(\frac{\lambda_1^2-1/4}{\sinh^2\tau}-\frac{\nu_{l,\kappa}^2-1/4}{\cosh^2\tau}\right)\right]ds\right\}. \quad (3.54)$$

The parameters $\lambda_{1,2}$ are the same as in the previous paragraph and ν is given by

$$\nu_l = \left|2l+1-\frac{\alpha}{\hbar\omega}\right| \quad \text{(discrete)}, \quad \nu_\kappa = i\kappa \quad \text{(continuous)}, \quad (3.55)$$

where discrete and continuous means the discrete and continuous contribution of the Morse potential. Of course, the analysis of the discrete spectrum gives the same result as before. The kernel $K_{l,\kappa}^{(V_1)}(s'')$ now allows us to write down the entire kernel $K^{(V_1)}(T)$ in terms of Morse wave functions and modified Pöschl–Teller wave functions in the following form:

$$K^{(V_1)}(u'',u',v'',v';T) = (\cosh\tau'\cosh\tau'')^{-1/2}\times \\ \times\left\{\sum_{ln} N_{ln}^2\Psi_l^{(\mathrm{MP})\,*}(v')\Psi_l^{(\mathrm{MP})}(v'')\Psi_n^{(\lambda_1,\nu\ )\,*}(\tau')\Psi_n^{(\lambda_1,\nu\ )}(\tau'')\,\mathrm{e}^{-iE\ \ T/\hbar}+ \right. \\ +\int dp\sum_l N_{lp}^2\Psi_l^{(\mathrm{MP})\,*}(v')\Psi_l^{(\mathrm{MP})}(v'')\Psi_p^{(\lambda_1,\nu\ )\,*}(\tau')\Psi_p^{(\lambda_1,\nu\ )}(\tau'')\,\mathrm{e}^{-iE\ \ T/\hbar}+ \\ \left. +\int dp\int d\kappa N_{\kappa p}^2\Psi_\kappa^{(\mathrm{MP})\,*}(v')\Psi_\kappa^{(\mathrm{MP})}(v'')\Psi_p^{(\lambda_1,i\kappa)\,*}(\tau')\Psi_p^{(\lambda_1,i\kappa)}(\tau'')\mathrm{e}^{-iE\ \ T/\hbar}\right\}, \quad (3.56)$$

with the proper normalization constants $N_{ln}, N_{lp}, N_{\kappa p}$, where, e.g., N_{ln} is determined by the residuum corresponding to E_{ln} in the Green function, and with the continuous spectrum

$$E_p = \frac{\hbar^2}{2ma_+}(p^2+k_2^2). \quad (3.57)$$

Note that for $k_2 = 1/2$ we obtain the well-known zero energy on the two-dimensional hyperboloid, which appears here in a natural way after performing the coordinate transformation $\cos u = \tanh \tau$.

The $\Psi_p^{(\mu,\nu)}(\omega)$ are the modified Pöschl–Teller functions, which are given by

$$\Psi_n^{(\eta,\nu)}(r) = N_n^{(\eta,\nu)}(\sinh r)^{2k_2-\frac{1}{2}}(\cosh r)^{-2k_1+\frac{3}{2}}\times \\ \times {}_2F_1(-k_1+k_2+\kappa, -k_1+k_2-\kappa+1; 2k_2; -\sinh^2 r), \tag{3.58}$$

$$N_n^{(\eta,\nu)} = \frac{1}{\Gamma(2k_2)}\left[\frac{2(2\kappa-1)\Gamma(k_1+k_2-\kappa)\Gamma(k_1+k_2+\kappa-1)}{\Gamma(k_1-k_2+\kappa)\Gamma(k_1-k_2-\kappa+1)}\right]^{1/2}. \tag{3.59}$$

The scattering states are given by

$$V(r) = \frac{\hbar^2}{2m}\left(\frac{\eta^2-1/4}{\sinh^2 r} - \frac{\nu^2-1/4}{\cosh^2 r}\right),$$

$$\Psi_p^{(\eta,\nu)}(r) = N_p^{(\eta,\nu)}(\cosh r)^{2k_1-\frac{1}{2}}(\sinh r)^{2k_2-\frac{1}{2}}\times \\ \times {}_2F_1(k_1+k_2-\kappa, k_1+k_2+\kappa-1; 2k_2; -\sinh^2 r), \tag{3.60}$$

$$N_p^{(\eta,\nu)} = \frac{1}{\Gamma(2k_2)}\sqrt{\frac{p\sinh\pi p}{2\pi^2}}\Big[\Gamma(k_1+k_2-\kappa)\Gamma(-k_1+k_2+\kappa)\times \\ \times \Gamma(k_1+k_2+\kappa-1)\Gamma(-k_1+k_2-\kappa+1)\Big]^{1/2}, \tag{3.61}$$

k_1, k_2 defined by: $k_1 = \frac{1}{2}(1\pm\nu)$, $k_2 = \frac{1}{2}(1\pm\eta)$, where the correct sign depends on the boundary conditions for $r \to 0$ and $r \to \infty$, respectively. The number N_M denotes the maximal number of states with $0, 1, \ldots, N_M < k_1 - k_2 - \frac{1}{2}$, $\kappa = k_1 - k_2 - n$ for the bound states and $\kappa = \frac{1}{2}(1+ip)$ for the scattering states; ${}_2F_1(a, b; c; z)$ is the hypergeometric function [10, p. 1057].

3.1.2. Separation of V_1 in Horospherical Coordinates. We evaluate the path integral for V_1 in horospherical coordinates. The classical Lagrangian and Hamiltonian are given by

$$\mathcal{L}(\mu,\dot\mu,\nu,\dot\nu) = \frac{m}{2}\left(\frac{a_+}{\nu^2}+\frac{a_-}{\mu^2}\right)(\dot\mu^2+\dot\nu^2) - V(\mu,\nu), \tag{3.62}$$

$$\mathcal{H}(\mu,p_\mu,\nu,p_\nu) = \frac{1}{2m}\frac{\mu^2\nu^2(p_\mu^2+p_\nu^2)}{a_+\mu^2+a_-\nu^2} + V(\mu,\nu). \tag{3.63}$$

For the canonical momentum operators we have

$$p_\mu = \frac{\hbar}{i}\left(\frac{\partial}{\partial\mu} - \frac{\nu^2 a_-/\mu}{a_+\mu^2+a_-\nu^2}\right), \tag{3.64}$$

$$p_\nu = \frac{\hbar}{i}\left(\frac{\partial}{\partial\nu} - \frac{\mu^2 a_+/\nu}{a_+\mu^2+a_-\nu^2}\right), \tag{3.65}$$

and for the quantum Hamiltonian we get

$$H = -\frac{\hbar^2}{2m}\frac{\mu^2\nu^2}{a_+\mu^2 + a_-\nu^2}\left(\frac{\partial^2}{\partial\mu^2} + \frac{\partial^2}{\partial\nu^2}\right) + V(\mu,\nu), \tag{3.66}$$

$$= \frac{1}{2m}\sqrt{\frac{\mu^2\nu^2}{a_+\mu^2 + a_-\nu^2}}\,(p_\mu^2 + p_\nu^2)\,\sqrt{\frac{\mu^2\nu^2}{a_+\mu^2 + a_-\nu^2}} + V(\mu,\nu). \tag{3.67}$$

We insert V_1 into the path integral and obtain ($f = a_+/\nu^2 + a_-/\mu^2$ and keeping to constants $k_{1,2}$)

$$K^{(V_1)}(\mu'',\mu',\nu'',\nu';T) = \int\limits_{\mu(t')=\mu'}^{\mu(t'')=\mu''}\mathcal{D}\mu(t)\int\limits_{\nu(t')=\nu'}^{\nu(t'')=\nu''}\mathcal{D}\nu(t) f(\mu,\nu)\times$$
$$\times\exp\left\{\frac{i}{\hbar}\int\limits_0^T\left[\frac{m}{2}f(\mu,\nu)(\dot\mu^2+\dot\nu^2)-\right.\right.$$
$$\left.\left.-\frac{1}{f(\mu,\nu)}\left(\frac{m}{2}\omega^2(\mu^2+\nu^2)-\alpha+\frac{\hbar^2}{2m}\left(\frac{k_1^2-1/4}{\mu^2}+\frac{k_2^2-1/4}{\nu^2}\right)\right)\right]dt\right\} =$$
$$= \int\limits_{-\infty}^{\infty}\frac{dE}{2\pi\hbar}\,\mathrm{e}^{-iET/\hbar}\int\limits_0^\infty ds''\mathrm{e}^{i\alpha s''/\hbar}K^{(V_1)}(\mu'',\mu',\nu'',\nu';s''), \tag{3.68}$$

and the time-transformed path integral $K^{(V_1)}(s'')$ is given by

$$K^{(V_1)}(\mu'',\mu',\nu'',\nu';s'') = \int\limits_{\mu(0)=\mu'}^{\mu(s'')=\mu''}\mathcal{D}\mu(s)\times$$
$$\times\exp\left\{\frac{i}{\hbar}\int\limits_0^{s''}\left[\frac{m}{2}(\dot\mu^2-\omega^2\mu^2)-\frac{\hbar^2}{2m}\frac{k_1^2-2ma_-E/\hbar^2-1/4}{\mu^2}\right]ds\right\}\times$$
$$\times\int\limits_{\nu(0)=\nu'}^{\nu(s'')=\nu''}\mathcal{D}\nu(s)\,\exp\left\{\frac{i}{\hbar}\int\limits_0^{s''}\left[\frac{m}{2}(\dot\nu^2-\omega^2\nu^2)-\frac{\hbar^2}{2m}\frac{k_2^2-2ma_+E/\hbar^2-1/4}{\nu^2}\right]ds\right\} =$$
$$= \frac{m^2\omega^2\sqrt{\mu'\mu''\nu'\nu''}}{i^2\hbar^2\sin^2\omega s''}\exp\left[-\frac{m\omega}{2i\hbar}({\mu'}^2+{\mu''}^2+{\nu'}^2+{\nu''}^2)\cot\omega s''\right]\times$$
$$\times I_{\lambda_1}\left(\frac{m\omega\mu'\mu''}{i\hbar\sin\omega s''}\right)I_{\lambda_2}\left(\frac{m\omega\nu'\nu''}{i\hbar\sin\omega s''}\right), \tag{3.69}$$

where $\lambda_{1,2} = k_{1,2}^2 - 2ma_{\mp}E/\hbar^2$. We can extract the bound state wave functions for the bound state contribution of the Green function according to:

$$G^{(V_1)}(\mu'', \mu', \nu'', \nu'; E) = \sum_{n\ \ =0}^{\infty} \sum_{n\ \ =0}^{\infty} \frac{N_{n\ \ n}^2}{E_{n\ \ n} - E} \times \\ \times \Psi_n^{(\mathrm{RHO},\lambda_1)}(\mu')\Psi_n^{(\mathrm{RHO},\lambda_1)}(\mu'')\Psi_n^{(\mathrm{RHO},\lambda_2)}(\nu')\Psi_n^{(\mathrm{RHO},\lambda_2)}(\nu''). \quad (3.70)$$

The bound states are determined by the equation

$$\frac{\alpha}{\hbar\omega} - 2(n_\mu + n_\nu + 1) = \sqrt{k_1^2 - \frac{2ma_-E}{\hbar^2}} + \sqrt{k_2^2 - \frac{2ma_+E}{\hbar^2}}. \quad (3.71)$$

This quadratic equation in E is identical with (3.47).

3.2. The Superintegrable Potential V_2 on D_{IV}. We state the potential in the respective coordinate systems

$$V_2(u,v) = \left(\frac{a_+}{\sin^2 u} + \frac{a_-}{\cos^2 u}\right)^{-1} \frac{\hbar^2}{2m}\left[\frac{k_1^2 - 1/4}{\sinh^2 v} - \frac{k_2^2 - 1/4}{\cosh^2 v} + \right. \\ \left. + \left(k_3^2 - \frac{1}{4}\right)\left(\frac{1}{\sin^2 u} + \frac{1}{\cos^2 u}\right)\right], \quad (3.72)$$

$$= 4\left(\frac{a_+}{\sinh^2 2\tilde\omega} + \frac{a_-}{\sin^2 2\tilde\varphi}\right)^{-1} \frac{\hbar^2}{2m}\left[\left(k_3^2 - \frac{1}{4}\right)\left(\frac{1}{\sinh^2 2\tilde\omega} + \frac{1}{\sin^2 2\tilde\varphi}\right) + \right. \\ \left. + \left(\frac{k_2^2 - 1/4}{\cos^2 2\tilde\varphi} - \frac{k_1^2 - 1/4}{\cosh^2 2\tilde\omega}\right)\right]. \quad (3.73)$$

It is possible to evaluate the path integral for V_2 in the (u, v) and the degenerate elliptic system with $\gamma = 2$. The elliptic system with $\gamma = 0$ is not treated.

3.2.1. Separation of V_2 in the (u, v) System. We insert V_2 into the path integral and obtain ($f = a_+/\sin^2 u + a_-/\cos^2 u$)

$$K^{(V_2)}(u'', u', v'', v'; T) = \int\limits_{u(t')=u'}^{u(t'')=u''} \mathcal{D}u(t) \int\limits_{v(t')=v'}^{v(t'')=v''} \mathcal{D}v(t) f(u) \times \\ \times \exp\left\{\frac{i}{\hbar}\int_0^T \left[\frac{m}{2} f(\dot u^2 + \dot v^2) - \frac{\hbar^2}{2mf}\left(\frac{k_1^2 - 1/4}{\sinh^2 v} - \frac{k_2^2 - 1/4}{\cosh^2 v} + \right.\right.\right. \\ \left.\left.\left. + \left(k_3^2 - \frac{1}{4}\right)\left(\frac{1}{\sin^2 u} + \frac{1}{\cos^2 u}\right)\right)\right] dt\right\}. \quad (3.74)$$

This formulation in (u, v) coordinates is inconvenient. Following the procedure as for V_1 in the (u, v) system we perform the coordinate transformation $\cos u = \tanh\tau$, and get for the path integral (3.74)

$$K^{(V_2)}(u'', u', v'', v'; T) = \int_{-\infty}^{\infty} \frac{dE}{2\pi\hbar}\,\mathrm{e}^{-iET/\hbar}\times$$
$$\times \int_0^\infty ds''\,\exp\left[\frac{i}{\hbar}\left(a_+E - \frac{\hbar^2 k_3^2}{2m}\right)\right] K(\tau'', \tau', v'', v'; s''), \quad (3.75)$$

and the time-transformed path integral $K^{(V_2)}(s'')$ is given by

$$K^{(V_2)}(\tau'', \tau', v'', v'; s'') =$$
$$= (\cosh\,\tau'\,\cosh\,\tau'')^{-1/2} \sum_{n\ =0}^{N_{\max}} \Psi_n^{(k_1,k_2)}(v')\Psi_n^{(k_1,k_2)}(v'')\times$$
$$\times \int_{\tau(0)=\tau'}^{\tau(s'')=\tau''} \mathcal{D}\tau(s)\exp\left\{\frac{i}{\hbar}\int_0^{s''}\left[\frac{m}{2}\dot\tau^2 - \frac{\hbar^2}{2m}\left(\frac{\lambda_2^2 - 1/4}{\sinh^2\tau} - \frac{\lambda_1^2 - 1/4}{\cosh^2\tau}\right)\right] ds\right\} +$$
$$+ (\cosh\tau'\cosh\tau'')^{-1/2}\int dk_v\,\Psi_k^{(k_1,k_2)}(v')\Psi_k^{(k_1,k_2)}(v'')\times$$
$$\times \int_{\tau(0)=\tau'}^{\tau(s'')=\tau''} \mathcal{D}\tau(s)\exp\left\{\frac{i}{\hbar}\int_0^{s''}\left[\frac{m}{2}\dot\tau^2 - \frac{\hbar^2}{2m}\left(\frac{\lambda_2^2 - 1/4}{\sinh^2\tau} - \frac{-k_v^2 - 1/4}{\cosh^2\tau}\right)\right] ds\right\}. \quad (3.76)$$

($\lambda_1^2 = (2n_v + |k_1| - |k_2| + 1)^2$, $\lambda_2^2 = k_3^2 - 2ma_-E/\hbar^2$).

The v-path integration gives a discrete and continuous spectrum, thus two different parts for the τ-path integration. We therefore find for the Green function

$$G^{(V_2)}(\tau'', \tau', v'', v'; E) = (\cosh\tau'\cosh\tau'')^{-1/2}\times$$
$$\times \sum_{n\ =0}^{N_{\max}} \Psi_n^{(k_1,k_2)}(v')\Psi_n^{(k_1,k_2)}(v'')\frac{m}{\hbar^2}\frac{\Gamma(m_1 - L_{\lambda_1})\Gamma(L_{\lambda_1} + m_1 + 1)}{\Gamma(m_1 + m_2 + 1)\Gamma(m_1 - m_2 + 1)}\times$$
$$\times (\cosh\tau'\cosh\tau'')^{-(k_1 - k_2)}(\tanh\tau'\tanh\tau'')^{m_1 + m_2 + 1/2}\times$$
$$\times\,{}_2F_1\left(-L_{\lambda_1} + m_1, L_{\lambda_1} + m_1 + 1; m_1 - m_2 + 1; \frac{1}{\cosh^2\tau_<}\right)\times$$

$$\times\, {}_2F_1\bigg(-L_{\lambda_1}+m_1, L_{\lambda_1}+m_1+1; m_1+m_2+1; \tanh^2\tau_>\bigg)+$$
$$+\,(\cosh\tau'\cosh\tau'')^{-1/2}\int dk_v\,\Psi_k^{(k_1,k_2)}(v')\Psi_k^{(k_1,k_2)}(v'')\times$$
$$\times\,\frac{m}{\hbar^2}\frac{\Gamma(m_1-L_k\ )\Gamma(L_k\ +m_1+1)}{\Gamma(m_1+m_2+1)\Gamma(m_1-m_2+1)}\times$$
$$\times\,(\cosh\tau'\cosh\tau'')^{-(k_1-k_2)}(\tanh\tau'\tanh\tau'')^{m_1+m_2+1/2}\times$$
$$\times\, {}_2F_1\bigg(-L_k\ +m_1, L_k\ +m_1+1; m_1-m_2+1; \frac{1}{\cosh^2\tau_<}\bigg)\times$$
$$\times\, {}_2F_1\bigg(-L_k\ +m_1, L_k\ +m_1+1; m_1+m_2+1; \tanh^2\tau_>\bigg) \qquad (3.77)$$

$\big(m_{1,2}=\frac{1}{2}(\lambda_2\pm\sqrt{2m\mathcal{E}}/\hbar)$, $L_{\lambda_1}=\frac{1}{2}(\lambda_1-1)$, $L_k\ =\frac{1}{2}(ik_v-1)$, $\mathcal{E}=a_+E-\hbar^2k_3^2/2m\big)$.

A discrete spectrum is only possible for the first summand in (3.76). First, we can analyze the discrete spectrum by looking at the poles in (3.77) which gives the equation

$$2(n_\tau+n_v)+\lambda_++\lambda_-+|k_2|-|k_1|=0 \qquad (3.78)$$

$(\lambda_\pm^2=k_3^2-2ma_\pm E/\hbar^2)$. This gives a quadratic equation in E with solution $(N_k=2n_\tau-2n_v-|k_1|+|k_2|)$

$$E_{n\ n}\ =-\frac{a\hbar^2N_k^2}{4b^2}\left(1\mp\sqrt{1+\frac{4b^2}{a^2}\left(\frac{k_3^2}{N_k^2}-1\right)}\right). \qquad (3.79)$$

The entire Green function in terms of the wave functions is given by

$$G^{(V_2)}(\tau'',\tau',v'',v';E)=(\cosh\tau'\cosh\tau'')^{-1/2}\int dp\frac{N_{pk}^2}{E_p-E}\int dk_v\times$$
$$\times\,\Psi_k^{(k_1,k_2)}(v')\Psi_k^{(k_1,k_2)}(v'')\Psi_p^{(\lambda_2,ik\ )}(\tau')\Psi_p^{(\lambda_2,ik\ )\,*}(\tau'')+$$
$$+\,(\cosh\tau'\cosh\tau'')^{-1/2}\sum_{n\ =0}^{\infty}\sum_{n\ =0}^{\infty}\Psi_n^{(k_1,k_2)}(v')\Psi_n^{(k_1,k_2)}(v'')\times$$
$$\times\left\{\sum_{n\ =0}^{N_{\max}}\frac{N_{n\ n}^2}{E_{n\ n}\ -E}\Psi_n^{(\lambda_2,\lambda_1)}(\tau')\Psi_n^{(\lambda_2,\lambda_1)}(\tau'')+\right.$$
$$\left.+\int dp\frac{N_{pn}^2}{E_p-E}\Psi_p^{(\lambda_2,\lambda_1)}(\tau')\Psi_p^{(\lambda_2,\lambda_1)\,*}(\tau'')\right\}, \qquad (3.80)$$

where $N_{n\ n}$, $N_{k\ n}$ is determined by the residuum in (3.77). The continuous spectrum has the form

$$E_p = \frac{\hbar^2}{2ma_+}(p^2 + k_3^2). \tag{3.81}$$

For $k_3 = \pm 1/2$ we obtain the usual zero-point energy on the two-dimensional hyperboloid. Reinserting $\cos u = \tanh v$ gives the Green function in the (u, v) system.

3.2.2. Separation of V_2 in Degenerate Elliptic Coordinates. We insert the potential V_2 in degenerate elliptic coordinates into the path integral and obtain $(f(\tilde{\omega}, \tilde{\varphi}) = 4(a_+/\sinh^2 2\tilde{\omega} + a_-/\sin^2 2\tilde{\varphi}))$

$$\begin{aligned} K^{(V_2)}(\tilde{\omega}'', \tilde{\omega}', \tilde{\varphi}'', \tilde{\varphi}'; T) &= \int\limits_{\tilde{\omega}(t')=\tilde{\omega}'}^{\tilde{\omega}(t'')=\tilde{\omega}''} \mathcal{D}\tilde{\omega}(t) \int\limits_{\tilde{\varphi}(t')=\tilde{\varphi}'}^{\tilde{\varphi}(t'')=\tilde{\varphi}''} \mathcal{D}\tilde{\varphi}(t) f(\tilde{\omega}, \tilde{\varphi}) \times \\ &\times \exp\left\{ \frac{i}{\hbar} \int\limits_0^T \left[\frac{m}{2} f(\tilde{\omega}, \tilde{\varphi})(\dot{\tilde{\omega}}^2 + \dot{\tilde{\varphi}}^2) 4 \frac{\hbar^2}{2mf(\tilde{\omega}, \tilde{\varphi})} \times \right.\right. \\ &\left.\left. \times \left(\frac{k_1^2 - 1/4}{\sinh^2 2\tilde{\omega}} - \frac{k_2^2 - 1/4}{\cosh^2 2\tilde{\omega}} + \frac{k_3^2 - 1/4}{\sin^2 2\tilde{\varphi}} + \frac{k_2^2 - 1/4}{\cos^2 2\tilde{\varphi}} \right) \right] dt \right\}. \end{aligned} \tag{3.82}$$

The calculation is similar as in the case of the (u, v) system: First, we rescale $2\tilde{\omega} \to \tilde{\omega}, 2\tilde{\varphi} \to \tilde{\varphi}$, then we perform the transformation $\cos\tilde{\varphi} = \tanh\tilde{\tau}$. Finally, we perform a time transformation in the path integral with the time transformation $f(\tilde{\omega}, \tilde{\varphi}) \to f(\tilde{\omega}, \tilde{\tau})$ yielding

$$\begin{aligned} &G^{(V_2)}(\tilde{\tau}'', \tilde{\tau}', \tilde{\omega}'', \tilde{\omega}'; E) = \\ &\quad = \int\limits_0^\infty ds'' \exp\left[\frac{i}{\hbar} s'' \left(Ea_- - \frac{\hbar^2 k_3^2}{2m} \right) \right] K^{(V_2)}(\tilde{\tau}'', \tilde{\tau}', \tilde{\omega}'', \tilde{\omega}'; s'') \end{aligned} \tag{3.83}$$

with the transformed path integral $K^{(V_2)}(s'')$ given by

$$\begin{aligned} &K^{(V_2)}(\tilde{\tau}'', \tilde{\tau}', \tilde{\omega}'', \tilde{\omega}'; s'') = \\ &\quad = \int\limits_{\tilde{\tau}(0)=\tilde{\tau}'}^{\tilde{\tau}(s'')=\tilde{\tau}''} \mathcal{D}\tilde{\tau}(s) \int\limits_{\tilde{\omega}(0)=\tilde{\omega}'}^{\tilde{\omega}(s'')=\tilde{\omega}''} \mathcal{D}\tilde{\omega}(s) \exp\left\{ \frac{i}{\hbar} \int\limits_0^T \left[\frac{m}{2}(\dot{\tilde{\tau}}^2 + \cosh^2\tilde{\tau}\dot{\tilde{\omega}}^2) - \right.\right. \\ &\quad \left.\left. - \frac{\hbar^2}{2m} \left(\frac{k_1^2 - 1/4}{\sinh^2\tilde{\tau}} + \frac{1}{\cosh^2\tilde{\tau}} \left(\frac{\lambda_+^2 - 1/4}{\sinh^2\tilde{\omega}} - \frac{k_2^2 - 1/4}{\cosh^2\tilde{\omega}} + \frac{1}{4} \right) \right) \right] ds \right\}. \end{aligned} \tag{3.84}$$

Again we evaluate this path integral by a successive $\tilde{\omega}$- and $\tilde{\tau}$-path integration. Performing finally the s'' integration we obtain

$$G^{(V_2)}(\tilde{\tau}'',\tilde{\tau}',\tilde{\omega}'',\tilde{\omega}';E)=(\cosh\tilde{\tau}'\cosh\tilde{\tau}'')^{-1/2}\times$$
$$\times\Bigg\{\int dp\frac{N^2_{k_{\tilde{\omega}}p}}{E_p-E}\int dk_{\tilde{\omega}}\,\Psi_p^{(k_1,ik_{\tilde{\omega}})}(\tilde{\tau}')\Psi_p^{(k_1,ik_{\tilde{\omega}})\,*}(\tilde{\tau}'')\Psi_{k_{\tilde{\omega}}}^{(\lambda_1,k_2)}(\tilde{\omega}')\Psi_{k_{\tilde{\omega}}}^{(\lambda_1,k_2)\,*}(\tilde{\omega}'')+$$
$$+\int dp\sum_{n_{\tilde{\omega}}=0}^{N_{\max}}\frac{N^2_{n_{\tilde{\omega}}p}}{E_p-E}\Psi_p^{(k_1,\epsilon\ \ {}_{\tilde{\omega}})}(\tilde{\tau}')\Psi_p^{(k_1,\epsilon\ \ {}_{\tilde{\omega}})\,*}(\tilde{\tau}'')\Psi_{n_{\tilde{\omega}}}^{(\lambda_1,k_2)}(\tilde{\omega}')\Psi_{n_{\tilde{\omega}}}^{(\lambda_1,k_2)}(\tilde{\omega}'')+$$
$$+\sum_{n_{\tilde{}}=0}^{N_{\max}}\sum_{n_{\tilde{\omega}}=0}^{N_{\max}}\frac{N^2_{n_{\tilde{}}n_{\tilde{\omega}}}}{E_{n_{\tilde{}}n_{\tilde{\omega}}}-E}\Psi_{n_{\tilde{}}}^{(k_1,\epsilon\ \ {}_{\tilde{\omega}})}(\tilde{\tau}')\Psi_{n_{\tilde{}}}^{(k_1,\epsilon\ \ {}_{\tilde{\omega}})}(\tilde{\tau}'')\Psi_{n_{\tilde{\omega}}}^{(\lambda_1,k_2)}(\tilde{\omega}')\Psi_{n_{\tilde{\omega}}}^{(\lambda_1,k_2)}(\tilde{\omega}'')\Bigg\}. \tag{3.85}$$

The normalization constants $N_{k_{\tilde{\omega}}p}, N_{k_{\tilde{\omega}}p}, N_{n_{\tilde{}}n_{\tilde{\omega}}}$ are determined by the respective residuum in $G^{(V_2)}(E)$ and the discrete spectrum is determined by the quadratic equation (3.78). The continuous spectrum has the form

$$E_p=\frac{\hbar^2}{2ma_-}(p^2+k_3^2). \tag{3.86}$$

The difference of E_p in comparison to the (u,v) system can be resolved by making in the (u,v) system the transformation $\sin u=\tanh\tau$ which changes the sign in the energy term. This concludes the discussion of V_2 on $D_{\rm IV}$.

3.3. The Superintegrable Potential V_3 on $D_{\rm IV}$. We state the potential in the respective coordinate systems

$$V_3(\tilde{\omega},\tilde{\varphi})=\frac{\hbar^2}{2m}\left(\frac{4a_+}{\sinh^2 2\tilde{\omega}}+\frac{4a_-}{\sinh^2\tilde{\varphi}}\right)^{-1}\times$$
$$\times\left[\frac{c_1}{\cos^2\tilde{\varphi}}+\frac{c_2}{\cosh^2\tilde{\omega}}+c_3\left(\frac{1}{\sin^2\tilde{\varphi}}-\frac{1}{\sinh^2\tilde{\omega}}\right)\right], \tag{3.87}$$
$$=\frac{\hbar^2}{2m}\left[a_+\left(\frac{1}{\cosh^2\hat{\omega}}-\frac{1}{\cos^2\hat{\varphi}}\right)-a_-\left(\frac{1}{\sinh^2\hat{\omega}}+\frac{1}{\sin^2\hat{\varphi}}\right)\right]^{-1}\times$$
$$\times\left[\frac{c_3}{\sinh^2\hat{\omega}}+\frac{c_2}{\cosh^2\hat{\omega}}+c_3\left(\frac{1}{\sin^2\hat{\varphi}}-\frac{1}{\cos^2\hat{\varphi}}\right)\right]. \tag{3.88}$$

It is possible to evaluate the path integral for V_3 in both separating coordinate systems. However, due to the similarity in the evaluations, only the degenerate elliptic II case will be presented.

3.3.1. Separation of V_3 in Degenerate Elliptic Coordinates II. We insert the potential V_3 in the path integral formulation for degenerate elliptic coordinates on $D_{\rm IV}$ and obtain $f(\tilde\omega,\tilde\varphi) = 4(a_+/\sinh^2 2\tilde\omega + a_-/\sin^2 2\tilde\varphi))$

$$K^{(V_3)}(\tilde\omega'',\tilde\omega',\varphi'',\varphi';T) = \int\limits_{\tilde\omega(t')=\tilde\omega'}^{\tilde\omega(t'')=\tilde\omega''} \mathcal{D}\tilde\omega(t) \int\limits_{\tilde\varphi(t')=\tilde\varphi'}^{\tilde\varphi(t'')=\tilde\varphi''} \mathcal{D}\tilde\varphi(t) f(\tilde\omega,\tilde\varphi)\times$$
$$\times \exp\left\{\frac{i}{\hbar}\int\limits_0^T \left[\frac{m}{2} f(\tilde\omega,\tilde\varphi)(\dot{\tilde\omega}^2 + \dot{\tilde\varphi}^2) - \right.\right.$$
$$\left.\left. - \frac{\hbar^2}{2mf(\tilde\omega,\tilde\varphi)}\left(\frac{c_1}{\cos^2\tilde\varphi} + \frac{c_2}{\cosh^2\tilde\omega} + c_3\left(\frac{1}{\sin^2\tilde\varphi} - \frac{1}{\sinh^2\tilde\omega}\right)\right)\right] dt\right\}. \quad (3.89)$$

In order to obtain a convenient form to evaluate (3.89) we perform the coordinate transformation $\cos\tilde\varphi = \tanh\tilde\tau$ in the same way as for V_2. Performing also the corresponding time transformation gives

$$K^{(V_3)}(\tilde\omega'',\tilde\omega',\tilde\tau'',\tilde\tau';T) = \int\limits_{-\infty}^{\infty} \frac{dE}{2\pi\hbar}\, \mathrm{e}^{-iET/\hbar}\times$$
$$\times \int\limits_0^\infty ds'' \exp\left[\frac{i}{\hbar}\left(\frac{\hbar^2}{2m}\lambda^2_{3^+_-}\right)\right] K^{(V_3)}(\tilde\omega'',\tilde\omega',\tilde\tau'',\tilde\tau';s''), \quad (3.90)$$

and the time-transformed path integral $K^{(V_3)}(s'')$ is given by

$$K^{(V_3)}(\tilde\omega'',\tilde\omega',\tilde\tau'',\tilde\tau';s'') = \int\limits_{\tilde\omega(0)=\tilde\omega'}^{\tilde\omega(s'')=\tilde\omega''} \mathcal{D}\tilde\omega(s) \int\limits_{\tilde\tau(0)=\tilde\tau'}^{\tilde\tau(s'')=\tilde\tau''} \mathcal{D}\tilde\tau(s)\cosh\tilde\tau\times$$
$$\times \exp\left\{\frac{i}{\hbar}\int\limits_0^{s''}\left[\frac{m}{2}(\dot{\tilde\tau}^2 + \cosh^2\tilde\tau\,\dot{\tilde\omega}^2) - \frac{\hbar^2}{2m}\frac{\lambda^2_{1^+_-} - 1/4}{\sinh^2\tilde\tau} - \right.\right.$$
$$\left.\left. - \frac{\hbar^2}{2m\cosh^2\tilde\tau}\left(\frac{\lambda^2_{3^-_+} - 1/4}{\sinh^2\tilde\omega} - \frac{\lambda^2_{2^-_+} - 1/4}{\cosh^2\tilde\omega} + \frac{1}{4}\right)\right] ds\right\} \quad (3.91)$$

$\big(\lambda^2_{i^\pm_\pm} = \dfrac{1}{4} \mp c_i - 2ma_\pm E/\hbar^2,\ i = 1,2,3\big)$. The latter path integral has the form of two successive modified Pöschl–Teller path integrations in $\tilde\omega$ and $\tilde\tau$. In the ω-path integration we get a contribution from the continuous and discrete spectrum. The continuous contribution gives in the $\tilde{\tau}$-path integration only a continuous part, whereas the other gives a discrete and continuous contribution in $\tilde{\tau}$. We denote the continuous parameter in $\tilde{\omega}$ by $p_{\tilde{\omega}}$, the discrete parameter in $\tilde{\omega}$ by $\epsilon_{n_{\tilde{\omega}}} = 2n_{\tilde{\omega}} + \lambda_{3^-_+} - \lambda_{2^-_+} - 1$, the continuous parameter in $\tilde{\tau}$ by p, the discrete parameter in $\tilde{\tau}$ by $\epsilon_{n_{\tilde{\tau}}} = 2n_{\tilde{\tau}} + \lambda_{1^+_-} - \epsilon_{n_{\tilde{\omega}}} - 1$, therefore:

$$\begin{aligned}
&K^{(V_3)}(\tilde{\omega}'',\tilde{\omega}',\tilde{\tau}'',\tilde{\tau}';s'') = \\
&= (\cosh\tilde{\tau}'\cosh\tilde{\tau}'')^{-1/2}\int_0^\infty dp_\omega \Psi_{p_{\tilde{\omega}}}^{(\lambda_{3^-_+},\lambda_{2^-_+})}(\tilde{\omega}')\Psi_{p_{\tilde{\omega}}}^{(\lambda_{3^-_+},\lambda_{2^-_+})*}(\tilde{\omega}'')\times \\
&\times \int_{\tilde{\tau}(0)=\tilde{\tau}'}^{\tilde{\tau}(s'')=\tilde{\tau}''} \mathcal{D}\tilde{\tau}(s)\exp\left\{\frac{i}{\hbar}\int_0^{s''}\left[\frac{m}{2}\dot{\tilde{\tau}}^2 - \frac{\hbar^2}{2m}\left(\frac{\lambda_{1^+_-}^2-1/4}{\sinh^2\tilde{\tau}} + \frac{p_{\tilde{\omega}}^2+1/4}{\cosh^2\tilde{\tau}}\right)\right]ds\right\} + \\
&+ (\cosh\tilde{\tau}'\cosh\tilde{\tau}'')^{-1/2}\sum_{n_{\tilde{\omega}}=0}^{N_{\max}} \Psi_{n_{\tilde{\omega}}}^{(\lambda_{3^-_+},\lambda_{2^-_+})}(\tilde{\omega}')\Psi_{n_{\tilde{\omega}}}^{(\lambda_{3^-_+},\lambda_{2^-_+})}(\tilde{\omega}'')\times \\
&\times \int_{\tilde{\tau}(0)=\tilde{\tau}'}^{\tilde{\tau}(s'')=\tilde{\tau}''} \mathcal{D}\tilde{\tau}(s)\exp\left\{\frac{i}{\hbar}\int_0^{s''}\left[\frac{m}{2}\dot{\tilde{\tau}}^2 - \frac{\hbar^2}{2m}\left(\frac{\lambda_{1^+_-}^2-1/4}{\sinh^2\tilde{\tau}} - \frac{\epsilon_{n_{\tilde{\omega}}}^2-1/4}{\cosh^2\tilde{\tau}}\right)\right]ds\right\} = \\
&= (\cosh\tilde{\tau}'\cosh\tilde{\tau}'')^{-1/2}\int_0^\infty dp_t\omega \Psi_{p_{\tilde{\omega}}}^{(\lambda_{3^-_+},\lambda_{2^-_+})}(\tilde{\omega}')\Psi_{p_{\tilde{\omega}}}^{(\lambda_{3^-_+},\lambda_{2^-_+})*}(\tilde{\omega}'')\times \\
&\times \int_0^\infty dp \Psi_p^{(\lambda_{1^+_-},ip_{\tilde{\omega}})}(\tilde{\omega}')\Psi_p^{(\lambda_{1^+_-},ip_{\tilde{\omega}})*}(\tilde{\omega}'')\,\mathrm{e}^{-is''\hbar p^2/2m} + \\
&+ (\cosh\tilde{\tau}'\cosh\tilde{\tau}'')^{-1/2}\sum_{n_{\tilde{\omega}}=0}^{N_{\max}} \Psi_{n_{\tilde{\omega}}}^{(\lambda_{3^-_+},\lambda_{2^-_+})}(\tilde{\omega}')\Psi_{n_{\tilde{\omega}}}^{(\lambda_{3^-_+},\lambda_{2^-_+})}(\tilde{\omega}'')\times \\
&\times \left\{\int_0^\infty dp \Psi_p^{(\lambda_{1^+_-},\epsilon_{\tilde{\omega}})}(\tilde{\omega}')\Psi_p^{(\lambda_{1^+_-},\epsilon_{\tilde{\omega}})*}(\tilde{\omega}'')\,\mathrm{e}^{-is''\hbar p^2/2m} + \right. \\
&\left. + \sum_{n_\omega=0}^{N_{\max}} \Psi_n^{(\lambda_{1^+_-},\epsilon_{\tilde{\omega}})}(\tilde{\tau}')\Psi_n^{(\lambda_{1^+_-},\epsilon_{\tilde{\omega}})}(\tilde{\tau}'')\,\mathrm{e}^{-i\hbar s''\epsilon^2_{-}/2m}\right\}. \qquad (3.92)
\end{aligned}$$

Performing the s'' integration gives the spectrum. For the continuous spectrum we obtain

$$E_p = \frac{\hbar^2}{2ma_-}\left(p^2 + \frac{1}{4} - c_3\right). \qquad (3.93)$$

The discrete spectrum is determined by

$$2(n_{\tilde\omega}+n_{\tilde\tau})+\lambda_{1^+_-}+\lambda_{3^-_+}-\lambda_{2^-_-}-2=\lambda_{3^+_-}\,. \tag{3.94}$$

This is an equation in E in the eighth order which we will not solve.

3.4. The Superintegrable Potential V_4 on $D_{\rm IV}$. We state the potential in the respective coordinate systems

$$V_4(\mu,\nu)=\left(\frac{a_+}{\sin^2 u}+\frac{a_-}{\cos^2 u}\right)^{-1}\frac{\hbar^2}{2m}\left(k_0^2-\frac{1}{4}\right)\left(\frac{1}{\sin^2 u}+\frac{1}{\cos^2 u}\right), \tag{3.95}$$

$$=\left(\frac{a_+}{\nu^2}+\frac{a_-}{\mu^2}\right)^{-1}\frac{\hbar^2}{2m}\left(k_0^2-\frac{1}{4}\right)\left(\frac{1}{\nu^2}+\frac{1}{\mu^2}\right), \tag{3.96}$$

$$=\frac{\hbar^2}{2md^2}\left(\frac{a+2b}{\sinh^2 2\omega'}+\frac{a-2b}{\sin^2 2\varphi'}\right)^{-1}\left(k_0^2-\frac{1}{4}\right)\times$$
$$\times\left(\frac{1}{\cosh^2\omega\cos^2\varphi}+\frac{1}{\sinh^2\omega\sin^2\varphi}\right). \tag{3.97}$$

It is possible to evaluate the path integral for V_4 in all the separating coordinate systems. However, we evaluate the path integral for V_4 only in the (u,v) system because V_4 is trivial.

3.4.1. Separation of V_4 in the (u,v) System. We insert V_4 into the path integral and obtain ($f=a_+/\sin^2 u+a_-/\cos^2 u$)

$$K(u'',u',v'',v';T)=\int\limits_{u(t')=u'}^{u(t'')=u''}\mathcal{D}u(t)\int\limits_{v(t')=v'}^{v(t'')=v''}\mathcal{D}v(t)f(u)\times$$
$$\times\exp\left\{\frac{i}{\hbar}\int\limits_0^T\left[\frac{m}{2}f(u)(\dot u^2+\dot v^2)-\frac{\hbar^2}{2m}\frac{k_0^2-1/4}{f(u)}\left(\frac{1}{\sin^2 u}+\frac{1}{\cos^2 u}\right)\right]dt\right\}. \tag{3.98}$$

We proceed similarly as in [14]. Because the formulation in (u,v) coordinates is inconvenient, we perform following [12] the coordinate transformation $\cos u=\tanh\tau$. Further, we separate off the v-path integration, and additionally we make a time transformation with the time-transformation function $f=a_+/\sin^2 u+a_-/\cos^2 u$. Due to the coordinate transformation $\cos u=\tanh\tau$

additional quantum terms appear according to

$$\exp\left(\frac{im}{2\epsilon\hbar}\frac{\left(\Delta u^{(j)}\right)^2}{\cos u^{(j-1)}\cos u^{(j)}}\right) \doteq$$

$$\doteq \exp\left[\frac{im}{2\epsilon\hbar}\left(\Delta\tau^{(j)}\right)^2 - i\frac{\hbar}{8m}\left(1+\frac{1}{\cosh^2\tau^{(j)}}\right)\right]. \quad (3.99)$$

We get for the path integral (3.98)

$$K(u'',u',v'',v';T) = \int\limits_{-\infty}^{\infty}\frac{dE}{2\pi\hbar}\,\mathrm{e}^{-iET/\hbar}\times$$

$$\times\int\limits_0^\infty ds''\,\exp\left[\frac{i}{\hbar}\left(a_+E-\frac{\hbar^2k_0^2}{2m}\right)\right]K(\tau'',\tau',v'',v';s''), \quad (3.100)$$

and the time-transformed path integral $K(s'')$ is given by

$$K(\tau'',\tau',v'',v';s'') =$$

$$= \int\limits_{-\infty}^{\infty} dk_v\frac{\mathrm{e}^{ik\ (v''-v')}}{2\pi}(\cosh\tau'\cosh\tau'')^{-1/2}\int\limits_{\tau(0)=\tau'}^{\tau(s'')=\tau''}\mathcal{D}\tau(s)\times$$

$$\times\exp\left\{\frac{i}{\hbar}\int\limits_0^{s''}\left[\frac{m}{2}\dot{\tau}^2-\frac{\hbar^2}{2m}\left(\frac{\lambda_0^2-1/4}{\sinh^2\tau}-\frac{-k_v^2-1/4}{\cosh^2\tau}\right)\right]ds\right\}. \quad (3.101)$$

Inserting the solution for the modified Pöschl–Teller potential and evaluating the Green function on the cut yields for the path integral solution on D_{IV} as follows ($K(u'',u',v'',v';T) = K(\tau'',\tau',v'',v';T)$):

$$K(u'',u',v'',v';T) =$$

$$= \int\limits_{-\infty}^{\infty} dk_v\int\limits_0^\infty dp\,\mathrm{e}^{-iTE\ /\hbar}\Psi_{p,k}\ (\tau'',v'')\Psi^*_{p,k}\ (\tau',v'), \quad (3.102)$$

$$\Psi_{p,k}\ (\tau,v) = \frac{\mathrm{e}^{ik\ v}}{\sqrt{2\pi a_+\cosh\tau}}\,\Psi_p^{(\lambda_0,ik)}(\tau), \quad (3.103)$$

$$E_p = \frac{\hbar^2}{2ma_+}\left(p^2+k_0^2\right), \quad (3.104)$$

where $\lambda_0^2 = k_0^2 - 2ma_eE/\hbar^2$ and the wave functions for the modified Pöschl–Teller functions. Reinserting $\cos u = \tanh \tau$ gives the solution in terms of the variable u.

We also see from this example that the introduction of a third variable w, say, to a three-dimensional version of Darboux space D_{IV} allows separation of variables, where the additional quantum number k_0 corresponds to the motion in w.

4. SUMMARY AND DISCUSSION

In this paper we have finished the discussion of superintegrable potentials on spaces of nonconstant curvature. The results are very satisfactory. There are two potentials on D_{I}, four potentials on D_{II}, five potentials on D_{III}, and four potentials on D_{IV}, respectively. We could solve many of the emerging quantum mechanical problems. To give an overview, we summarize our results in Table 5. We list for each space the corresponding potentials including the general form of the solution (if explicitly possible). We omit the trivial potentials here, because they are separable in all corresponding coordinate systems.

In the first Darboux space D_{I} the superintegrable potentials were related to the Holt potential and a shifted isotropic harmonic oscillator in two-dimensional Euclidean space. Whereas the solution in the coordinate v can be expressed in terms of the wave functions for the radial harmonic oscillator (Laguerre polynomials) and the shifted harmonic oscillator (Hermite polynomials), the solution in the coordinate u was determined by a boundary condition for u. This gave wave functions in terms of parabolic cylinder functions and a transcendental equation for the bound state energy levels. The corresponding solution in the rotated (r, q) system was similar. An explicit solution in parabolic coordinates could not be found.

In the second Darboux space there were three nontrivial superintegrable potentials. The potentials were related to the Hold potential, the isotropic singular oscillator, and the Coulomb potential in two-dimensional Euclidean space. We found combinations of polynomial wave functions for the discrete states and combinations of polynomials and Whittaker functions for the scattering states. The discrete energy spectrum for the oscillator-related potentials was usually given by a quadratic equation in the energy. For the Coulomb-related potential we found an equation in eight order in the energy, which could be studied in a special case. Also, in the semiclassical limit, we found that the energy spectra indeed had the behavior of a harmonic oscillator and a Coulomb potential, respectively.

On D_{III} we had potentials related to a linear potential, a Coulomb potential, and a shifted oscillator in two-dimensional flat space. We found for the first po-

Table 5. **Solutions of the path integration for superintegrable potentials in Darboux spaces**

Space and potential	Solution in terms of the wave functions
$D_{\rm I}$	
V_1: (u,v) Parabolic	Hermite polynomials × Parabolic cylinder functions No explicit solution
V_2: (u,v) (r,q)	Hermite polynomials × Parabolic cylinder functions Hermite polynomials × Parabolic cylinder functions
$D_{\rm II}$	
V_1: (u,v) Parabolic	Hermite polynomial × Whittaker functions* No explicit solution
V_2: (u,v) Polar Elliptic	Laguerre polynomial × Whittaker functions* Gegenbauer polynomial × Whittaker functions* No explicit solution
V_3: Polar Parabolic Elliptic	Gegenbauer polynomials × Bessel functions Product of Whittaker functions* No explicit solution
$D_{\rm III}$	
V_1: Parabolic Translated parabolic	Product of Hermite polynomials/Parabolic cylinder functions Product of Hermite polynomials/Parabolic cylinder functions
V_2: (u,v) Polar Parabolic	Gegenbauer polynomials × Whittaker functions* Gegenbauer polynomials × Whittaker functions* Product of Whittaker functions*
V_3: Polar Hyperbolic	Gegenbauer polynomials × Whittaker functions* No explicit solution
V_4: Hyperbolic Elliptic	Product of Whittaker functions* No explicit solution
$D_{\rm IV}$	
V_1: (u,v) system Horospherical Elliptic	Product of hypergeometric functions Product of Whittaker functions* No explicit solution
V_2: (u,v) Degenerate elliptic	Hypergeometric functions Hypergeometric functions
V_3: Elliptic Degenerate elliptic	Hypergeometric functions Hypergeometric functions

*The notion Whittaker functions means for a disrete spectrum Laguerre polynomials and for a continuous spectrum Whittaker functions $W_{\mu,\nu}(z)$, respectively.

tential an equation in the fourth order in the energy E, and quadratic equations in the energy E for the second and third potentials. The Coulomb-related potential showed again in the semiclassical limit the behavior of a Coulomb potential. Of some special interest was the feature of the complex periodic Morse potential for the separation of V_3 in polar coordinates. Such complex potentials have attracted in the recent years some attention, because the involved $\mathcal{PT}$ symmetry in these potentials has the consequence that they, nevertheless, have a real spectrum, e.g., [3, 4, 42, 49–51]. Such kind of potentials also appear as subsystems in the list of superintegrable potentials on the complex Euclidean plane [36].

A special feature in $D_{\rm III}$ was that for the free motion there are already positive continuous and negative infinite discrete spectra. A similar feature also exists for the free quantum motion on the $SU(1,1)$ and $SO(2,2)$ hyperboloid.

In the fourth Darboux space we found potentials which were related to the Morse and Pöschl–Teller potential, and combined modified Pöschl–Teller potentials. The modified Pöschl–Teller potentials had, of course, solutions in terms of hypergeometric functions, respectively: Jacobi polynomials (discrete spectrum) and Jacobi functions (scattering states).

We were able to solve the various path integral representations, because we have now to our disposal not only the basic path integrals for the harmonic oscillator, the linear oscillator, the radial harmonic oscillator, and the (modified) Pöschl–Teller potential, but also path-integral identities derived from path integration on harmonic spaces like the elliptic and spheroidal path-integral representations with their more complicated special functions. This includes also numerous transformation techniques to find a particular solution based on one of the basic solutions. Various Green-function analysis techniques can be applied to find an expression not only for the Green function but also for the wave functions and the energy spectrum. Usually, we stated in all cases the solution for the discrete spectrum contribution, i.e., the energy spectrum and the bound-states wave functions. However, not in all cases we stated explictly the scattering states. In the cases where we omitted the explicit representation, this can be done in a straightforward way by inserting the corresponding solution by the potential problem in question and inserting the various coupling constants and scattering quantum numbers.

Let us also note that our solutions are often on a more or less formal level. Neither have we specified an embedding space, nor have we specified boundary conditions on our spaces. For instance, in $D_{\rm I}$ boundary conditions the signature of the ambient space is very important, because choosing a positive or negative signature of the ambient space changes the boundary conditions, and hence the quantization conditions [21]. The same line of reasoning is, of course, valid in the other three Darboux spaces. We have not discussed in detail special cases of the parameters (say a and b), including the limiting cases to flat spaces or spaces

with constant (negative) curvature. Such a discussion would go far beyond the scope of this paper.

Let us finally mention an important observation due to [26]. At the end of their paper Kalnins et al. gave a list of superintegrable potentials on the two-dimensional complex plane and complex sphere. As it turns out, all of the potentials on Darboux spaces can be generated by taking a two-dimensional line element and dividing this line element by a superintegrable potential belonging to a specific class [27]. Not every class generates a new potential on a Darboux space, some are simply related by a coordinate transformation, and some potentials can be generated from the Euclidean plane as well as the complex sphere. The appearance of the complex sphere is especially obvious in the general elliptic coordinate system on $D_{\rm IV}$. Some of the various different potentials coming from the complex plane and sphere are also related by the so-called «coupling constant metamorphosis». Coupling constant metamorphosis always comes into play if the energy E of the quantum system appears in the form of $E \cdot$ metric terms. This observation leads to the notion that every nondegenerate superintegrable system in two dimensions is «Stäckel equivalent» to a superintegrable system in a two-dimensional space of constant curvature [27].

In the language of path integrals coupling constant metamorphosis comes from «time-» or «space-time» transformations (also called Duru–Kleinert transformations [39]). Here the most important example is the Coulomb problem, where by means of a space-time transformation the Coulomb coupling α just becomes a constant and the emerging harmonic oscillator problem has the frequency $\omega^2 = -2E/m$, i.e., the negative energy of the Coulomb problem appears as a harmonic oscillator frequency. As we have seen, this kind of coupling constant metamorphosis or space-time transformation, respectively, had been indispensable tools in the path integral evaluations of the free motion and for the superintegrable potentials, and we can use both notions as synonymously.

We did not go into details of three-dimensional generalization of the Darboux spaces [15]. Of course, it is possible to extend the notion of superintegrability to three-dimensional Darboux spaces. In particular, in three dimensions there are more of such potentials. In total, there are five maximally superintegrable potentials [17], the first four of them are also superintegrable, including the singular harmonic oscillator, the Holt potential and the Coulomb potential. New features will arise due to the fact that on three-dimensional generalization of the more complicated Darboux spaces $D_{\rm III}$ and $D_{\rm IV}$, coordinate systems from the three-dimensional complex sphere come into play [30]. Studies along such lines will be performed in future investigations.

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Appendices

A. PATH INTEGRAL FOR THE FREE MOTION ON $D_{\rm IV}$ IN DEGENERATE ELLIPTIC COORDINATES ($\gamma = 1$)

We start by considering the metric in elliptic coordinates ($\gamma = 1$):

$$ds^2 = \left[a_-\left(\frac{1}{\sinh^2\hat\omega} + \frac{1}{\sin^2\hat\varphi}\right) - a_+\left(\frac{1}{\cosh^2\hat\omega} - \frac{1}{\cos^2\hat\varphi}\right)\right](d\hat\omega^2 + d\hat\varphi^2). \quad \text{(A.1)}$$

We formulate the path integral in the usual way. We perform the space-time transformation with the coordinate transformation $\cos\hat\varphi = \tanh\hat\tau$ yielding

$$\begin{aligned}
K(\hat\omega'', \hat\omega', \hat\varphi'', \hat\varphi'; T) &= \int\limits_{\hat\omega(t')=\hat\omega'}^{\hat\omega(t'')=\hat\omega''} \mathcal{D}\hat\omega(t)\times \\
&\times \int\limits_{\hat\varphi(t')=\hat\varphi'}^{\hat\varphi(t'')=\hat\varphi''} \mathcal{D}\hat\varphi(t)\left[a_-\left(\frac{1}{\sinh^2\hat\omega} + \frac{1}{\sin^2\hat\varphi}\right) - a_+\left(\frac{1}{\cosh^2\hat\omega} - \frac{1}{\cos^2\hat\varphi}\right)\right]\times \\
&\times \exp\left[\frac{im}{2\hbar}\int\limits_0^T \left(\frac{a_-}{\sinh^2\hat\omega} - \frac{a_+}{\cosh^2\hat\omega} + \frac{a_-}{\sin^2\hat\varphi} - \frac{a_+}{\cos^2\hat\varphi}\right)(\dot{\hat\omega}^2 + \dot{\hat\varphi}^2)dt\right] = \\
&= \int\limits_{-\infty}^{\infty} \frac{dE}{2\pi\hbar}\,\mathrm{e}^{-iET/\hbar} \int\limits_0^{\infty} ds''\,\exp\left[\frac{i}{\hbar}\left(a_-E - \frac{\hbar^2}{8m}\right)s''\right]\times \\
&\times K(\hat\omega'', \hat\omega', \hat\tau'', \hat\tau'; s'') \quad \text{(A.2)}
\end{aligned}$$

with the transformed path integral given by

$$K(\hat\omega'',\hat\omega',\hat\tau'',\hat\tau';s'')=\int\limits_{\hat\tau(0)=\hat\tau'}^{\hat\tau(s'')=\hat\tau''}\mathcal{D}\hat\tau(s)\int\limits_{\hat\omega(0)=\hat\omega'}^{\hat\omega(s'')=\hat\omega''}\mathcal{D}\hat\omega(s)\,\cosh\hat\tau\times$$

$$\times\exp\left(\frac{i}{\hbar}\int\limits_0^{s''}\left\{\frac{m}{2}(\dot{\hat\tau}^2+\cosh^2\hat\tau\dot{\hat\omega}^2)-\frac{\hbar^2}{2m}\left[\frac{1}{\cosh^2\hat\tau}\left(\frac{\lambda_-^2+1/4}{\sinh^2\hat\omega}-\frac{\lambda_+^2+1/4}{\cosh^2\hat\omega}+\frac{1}{4}\right)-\right.\right.\right.$$

$$\left.\left.\left.-\frac{\lambda_+^2+1/4}{\sinh^2\hat\tau}\right]\right\}ds\right),\quad\text{(A.3)}$$

where $\lambda_\pm^2=\dfrac{1}{4}-2ma_\pm E/\hbar^2$. The successive path integrations are of the modified Pöschl–Teller type. Therefore the solution can be written as follows:

$$K(\hat\omega'',\hat\omega',\hat\varphi'',\hat\varphi';T)=\int dk\int p\Psi_k^{(\lambda_-,\lambda_+)}(\hat\omega'')\Psi_k^{(\lambda_-,\lambda_+)\,*}(\hat\omega')\times$$

$$\times\,\Psi_p^{(\lambda_+,ik)}(\hat\tau'')\Psi_p^{(\lambda_+,ik)\,*}(\tau')\,\mathrm{e}^{-i\hbar Tp^2/2m}\quad\text{(A.4)}$$

with the energy spectrum

$$E_p=\frac{\hbar^2}{2ma_-}\left(p^2+\frac{1}{4}\right),\quad\text{(A.5)}$$

and we can reinsert $\tanh\hat\tau\to\cos\hat\varphi$. The difference of the energy spectra in degenerate elliptic and elliptic coordinates (interchanging of a_+ and a_-) can be removed by a shift of the coordinates $\tilde\varphi$ and $\hat\varphi$ by $\pi/2$, respectively.

B. PATH INTEGRAL FOR THE FREE MOTION ON D_{IV} IN DEGENERATE ELLIPTIC COORDINATES ($\gamma=2$)

We start by considering the metric in degenerate elliptic coordinates ($\gamma=2$):

$$ds^2=\frac{1}{4}\left(\frac{a_+}{\sinh^2 2\tilde\omega}+\frac{a_-}{\sin^2 2\tilde\varphi}\right)(d\tilde\omega^2+d\tilde\varphi^2).\quad\text{(B.1)}$$

We formulate the path integral in the usual way. We scale both variables by the factor 2 and perform the space-time transformation with the coordinate transfor-

mation $\cos\tilde\varphi = \tanh\tilde\tau$ yielding $\big(\lambda^2 = \frac{1}{4} - 2ma_+E/\hbar^2\big)$:

$$K(\tilde\omega'',\tilde\omega',\tilde\varphi'',\tilde\varphi';T) = \frac{1}{2}\int\limits_{\tilde\omega(t')=\tilde\omega'}^{\tilde\omega(t'')=\tilde\omega''}\mathcal{D}\tilde\omega(t)\int\limits_{\tilde\varphi(t')=\tilde\varphi'}^{\tilde\varphi(t'')=\tilde\varphi''}\mathcal{D}\tilde\varphi(t)\bigg(\frac{a_+}{\sinh^2\tilde\omega}+\frac{a_-}{\sin^2\tilde\varphi}\bigg)\times$$
$$\times\exp\left[\frac{im}{2\hbar}\int\limits_0^T\bigg(\frac{a_+}{\sinh^2\tilde\omega}+\frac{a_-}{\sin^2\tilde\varphi}\bigg)(\dot{\tilde\omega}^2+\dot{\tilde\varphi}^2)dt\right] = \int\limits_{-\infty}^{\infty}\frac{dE}{2\pi\hbar}\,\mathrm{e}^{-iET/\hbar}\times$$
$$\times\int\limits_0^\infty ds''\,\exp\left[\frac{i}{\hbar}\bigg(a_-E-\frac{\hbar^2}{8m}\bigg)s''\right]K(\tilde\omega'',\tilde\omega',\tilde\tau'',\tilde\tau';s'') \quad \text{(B.2)}$$

with the transformed path integral given by

$$K(\tilde\omega'',\tilde\omega',\tilde\tau'',\tilde\tau';s'') = \int\limits_{\tilde\tau(0)=\tilde\tau'}^{\tilde\tau(s'')=\tilde\tau''}\mathcal{D}\tilde\tau(s)\int\limits_{\tilde\omega(0)=\tilde\omega'}^{\tilde\omega(s'')=\tilde\omega''}\mathcal{D}\tilde\omega(s)\cosh\tilde\tau\times$$
$$\times\exp\left\{\frac{i}{\hbar}\int\limits_0^{s''}\left[\frac{m}{2}(\dot{\tilde\tau}^2+\cosh^2\tilde\tau\,\dot{\tilde\omega}^2)-\frac{\hbar^2}{2m\,\cosh^2\tilde\tau}\bigg(\frac{\lambda^2+1/4}{\cosh^2\tilde\omega}+\frac{1}{4}\bigg)\right]ds\right\} =$$
$$= (\cosh\tilde\tau'\cosh\tilde\tau'')^{-1/2}\sum_{\pm}\int\limits_{\mathbb{R}}\frac{dk\,k\,\sinh\pi k}{\cosh^2\pi\lambda+\sinh^2\pi k}\times$$
$$\times P^{ik}_{i\lambda-1/2}(\pm\tanh\omega'')P^{-ik}_{i\lambda-1/2}(\pm\tanh\tilde\omega')\times$$
$$\times\sum_{\pm}\int\limits_{\mathbb{R}}\frac{dp\,p\,\sinh\pi p}{\cosh^2\pi k+\sinh^2\pi p}\times$$
$$\times P^{ip}_{ik-1/2}(\pm\tanh\tilde\tau'')P^{-ip}_{ik-1/2}(\pm\tanh\tilde\tau')\,\mathrm{e}^{-i\hbar Tp^2/2m}. \quad \text{(B.3)}$$

Therefore we obtain the wave functions and the energy spectrum, respectively,

$$\Psi_{k,p}(\tilde\tau,\tilde\omega) = \frac{1}{\sqrt{2\cosh\tilde\tau}}\bigg(\frac{k\,\sinh\pi k}{\cosh^2\pi\lambda+\sinh^2\pi k}\,\frac{p\,\sinh\pi p}{\cosh^2\pi k+\sinh^2\pi p}\bigg)^{1/2}\times$$
$$\times P^{ik}_{i\lambda-1/2}(\pm\tanh\omega)P^{ip}_{ik-1/2}(\pm\tanh\tilde\tau) \quad \text{(B.4)}$$

and $E_p = \dfrac{\hbar^2}{2ma_-}\bigg(p^2+\dfrac{1}{4}\bigg)$, and we can reinsert $\tanh\tilde\tau\to\cos\tilde\varphi$.

C. SUPERINTEGRABLE POTENTIALS ON $E(2,\mathbb{C})$

In this appendix we shortly discuss the path integral representation of superintegrable potentials on the two-dimensional complex Euclidean plane. A thorough path integral discussion on the real two-dimensional complex Euclidean plane has been done in [17], and therefore these solutions will not be repeated here, only some new due to the appearance of three more potentials V_5–V_7. In Table 6 we list the seven coordinate systems on the complex plane $E(2,\mathbb{C})$. As usual $P_1 = -i\hbar\partial_x$ and $P_2 = -i\hbar\partial_y$ denote the momentum operators, and $M = yP_1 - xP_2$ is the angular momentum. The potentials now read as follows [27, 34–36]:

$$\left.\begin{array}{ll}
V_5 = \dfrac{B}{2}(x-iy) & \underline{\text{Cartesian}} \\
 & \underline{\text{Semihyperbolic}} \\
 & \text{Light Cone} \\ \hline
V_6 = \dfrac{\alpha}{2\sqrt{x-iy}} & \underline{\text{Parabolic}} \\
 & \underline{\text{Semihyperbolic}} \\
 & \text{Light Cone} \\ \hline
V_7 = \dfrac{1}{2}\left[\alpha\dfrac{x^2+y^2}{(x+iy)^4} + \dfrac{\beta}{(x+iy)^2} + \gamma(x^2+y^2)\right] & \underline{\text{Polar}} \\
 & \text{Hyperbolic}
\end{array}\right\} . \tag{C.1}$$

In the underlined cases we give a (formal) path integral representation.

The Potential V_5. For the potential V_5 the corresponding Lagrangian has the form

$$\mathcal{L} = \frac{m}{2}(\dot{x}^2 + \dot{y}^2) - \frac{B}{2}(x - iy). \tag{C.2}$$

Thus, we identify two linear potentials [13, 45]

$$\begin{aligned}
&K^{(V_5)}(x'', x', y'', y'; T) = \\
&= \int\limits_{x(t')=x'}^{x(t'')=x''} \mathcal{D}x(t) \int\limits_{y(t')=y'}^{y(t'')=y''} \mathcal{D}y(t) \exp\left\{\frac{i}{\hbar}\int\limits_{t'}^{t''}\left[\frac{m}{2}(\dot{x}^2+\dot{y}^2) - \frac{B}{2}(x-iy)\right]dt\right\} = \\
&= \left(\frac{m}{2\pi i\hbar T}\right) \exp\left[\frac{i}{\hbar}\left(\frac{m}{2}\frac{(x''-x')^2 + (x''-x')^2}{T} - \right.\right. \\
&\qquad\qquad \left.\left. - \frac{BT}{4}(x' + x'' - iy' - iy'')\right)\right],
\end{aligned} \tag{C.3}$$

Table 6. **Coordinate systems on the complex plane** $E(2,\mathbb{C})$

Coordinate system	Integrals of motion	Coordinates
1. Cartesian, $(x, y \in \mathbb{R})$	$I = p_1^2$	x, y
2. Polar $(\varrho > 0, \varphi \in [0,\pi))$	$I = m^2$	$x = \varrho \cos\varphi$ $x = \varrho \sin\varphi$
3. Light cone $(x, y \in \mathbb{R})$	$I = (P_1 + iP_2)^2$	$\hat{x} = x - iy$ $\hat{y} = x + iy$
4. Elliptic $(\omega > 0, \alpha \in [0, 2\pi))$	$I = M^2 - a^2 P_2^2$ $a \neq 0$	$x = \cosh\omega\cos\alpha$ $y = \sinh\omega\sin\alpha$
5. Parabolic $(\xi, \eta > 0)$	$I = \{M, P_2\}$	$x = \frac{1}{2}(\xi^2 - \eta^2)$ $y = \xi\eta$
6. Hyperbolic $(u, v > 0)$	$I = M^2 + (P_1 + iP_2)^2$	$x = \frac{u^2 + u^2v^2 + v^2}{2uv}$ $y = i\frac{u^2 - u^2v^2 + v^2}{2uv}$
7. Semihyperbolic $(w, z \in \mathbb{R})$	$I = \{M, P_1 + iP_2\} + (P_1 - iP_2)^2$	$x = \frac{1}{2}(w-z)^2 + \frac{1}{4}(w+z)$ $y = -\frac{1}{2}(w-z)^2 - \frac{1}{4}(w+z)$

$$= \left(\frac{4m}{\hbar^2 B}\right)^{4/3} \int_{\mathbb{R}} dE\, \mathrm{e}^{-iET/\hbar} \int_{\mathbb{R}} d\lambda \times$$

$$\times \operatorname{Ai}\left[\left(x' - \frac{2E+\lambda}{k}\right)\left(\frac{mB}{\hbar^2}\right)^{1/3}\right] \operatorname{Ai}\left[\left(x'' - \frac{2E+\lambda}{k}\right)\left(\frac{mB}{\hbar^2}\right)^{1/3}\right] \times$$

$$\times \operatorname{Ai}\left[i\left(y' - \frac{2E-\lambda}{k}\right)\left(\frac{mB}{\hbar^2}\right)^{1/3}\right] \operatorname{Ai}\left[i\left(y'' - \frac{2E-\lambda}{k}\right)\left(\frac{mB}{\hbar^2}\right)^{1/3}\right], \quad \text{(C.4)}$$

with the continuous spectrum $E = \hbar^2 p^2/2m$, and λ is the second separation constant.

For V_5 in the semihyperbolic coordinates we obtain for the corresponding Lagrangian ($\dot{w} = dw/dt$)

$$\mathcal{L}_E = \frac{m}{2}(w - z)(\dot{w}^2 - \dot{z}^2) - \frac{B}{2}(w + z) + E, \quad \text{(C.5)}$$

which gives after a time transformation ($\dot{w} = dw/ds$, $\dot{z} = dz/ds$ and $dt = (w - z)ds$) a transformed Lagrangian

$$\tilde{\mathcal{L}}_E = \frac{m}{2}(\dot{w}^2 - \dot{z}^2) - \frac{B}{2}(w^2 - z^2) + E(w - z). \tag{C.6}$$

Therefore the potential v_5 has been transformed into the problem of a shifted harmonic oscillator, whose solution is well known. In order to determine the path integral solution we consider the Green function of the harmonic oscillator [22], use the convolution formula for the kernel in terms of a product of two Green functions

$$K^{(V_5)}(w'', w', z'', z'; T) = \int_{-\infty}^{\infty} \frac{dE}{2\pi\hbar}\, \mathrm{e}^{-iET/\hbar} \times$$
$$\times \int_0^{\infty} ds'' K_w(w'', w'; s'') \cdot K_z(z'', z'; s'') =$$
$$= \int_{-\infty}^{\infty} \frac{dE}{2\pi\hbar}\, \mathrm{e}^{-iET/\hbar} \frac{\hbar}{2\pi i} \int d\mathcal{E} G_w(E; w'', w'; -\mathcal{E}) G_z(E; z'', z'; \mathcal{E}), \tag{C.7}$$

and obtain therefore

$$K^{(V_5)}(w'', w', z'', z'; T) = \int\limits_{w(t')=w'}^{w(t'')=w''} \mathcal{D}w(t) \times$$
$$\times \int\limits_{z(t')=z'}^{z(t'')=z''} \mathcal{D}z(t) \exp\left\{ \frac{i}{\hbar} \int_{t'}^{t''} \left[\frac{m}{2}(w - z)(\dot{w}^2 - \dot{z}^2) - \frac{B}{2}(w + z) \right] dt \right\} =$$
$$= \frac{1}{4\pi^2} \int_{-\infty}^{\infty} dE \int d\lambda \frac{m}{\pi\hbar^3} \sqrt{\frac{m}{B}}\, \Gamma^2\left(\frac{1}{2} - \frac{E + \lambda}{\hbar\omega} \right) \times$$
$$\times D_{-\frac{1}{2} + \frac{+}{\hbar\omega}} \left[\sqrt{\frac{2}{\hbar}\sqrt{mB}} \left(w_> - \frac{E}{b} \right) \right] D_{-\frac{1}{2} + \frac{+}{\hbar\omega}} \left[-\sqrt{\frac{2}{\hbar}\sqrt{mB}} \left(w_< - \frac{E}{b} \right) \right] \times$$
$$\times D_{-\frac{1}{2} + \frac{+}{\hbar\omega}} \left[\sqrt{\frac{2}{\hbar}\sqrt{mB}} \left(z_> - \frac{E}{b} \right) \right] D_{-\frac{1}{2} + \frac{+}{\hbar\omega}} \left[-\sqrt{\frac{2}{\hbar}\sqrt{mB}} \left(z_< - \frac{E}{b} \right) \right], \tag{C.8}$$

with the continuous spectrum $E = \hbar^2 p^2/2m$, and λ is the second separation constant. The Green function may be evaluated in terms of even and odd parabolic cylinder functions $E_\nu^{(0)}(z)$ and $E_\nu^{(1)}(z)$, e.g., [14,17,22,41], which is omitted here.

The Potential V_6. Let us consider the two Lagrangians of the potential V_6 expressed in parabolic and semihyperbolic coordinates, respectively,

$$\mathcal{L}_E = \frac{m}{2}(\xi^2+\eta^2)(\dot{\xi}^2+\dot{\eta}^2) + \sqrt{2}\,\alpha\frac{\xi - i\eta}{\xi^2+\eta^2} + E, \tag{C.9}$$

$$= \frac{m}{2}(w-z)(\dot{w}^2-\dot{z}^2) + i\frac{\sqrt{2}\,\alpha}{w-z} + E, \tag{C.10}$$

which gives after a time transformation ($\dot{\xi} = d\xi/ds$, $\dot{\eta} = d\eta/ds$ and $dt = (\xi^2+\eta^2)ds$ in parabolic coordinates; $\dot{w} = dw/ds$, $\dot{z} = dz/ds$ and $dt = (w-z)ds$ in semihyperbolic coordinates) the transformed Lagrangians

$$\tilde{\mathcal{L}}_E = \frac{m}{2}(\dot{\xi}^2+\dot{\eta}^2) + \sqrt{2}\,\alpha(\xi - i\eta) + (\xi^2+\eta^2), \tag{C.11}$$

$$= \frac{m}{2}(\dot{w}^2-\dot{z}^2) + i\sqrt{2}\,\alpha + E(w-z). \tag{C.12}$$

In parabolic coordinates we have a shifted harmonic oscillator and in semihyperbolic coordinates a linear potential plus a constant. The solution is consequently almost identical to the corresponding solutions for the potential V_5 with appropriate replacement of the coupling constants. See also [14,17,22,41] for more details.

The Potential V_7. Let us consider the last potential V_7. In polar coordinates we have the effective Lagrangian (note the additional $\hbar^2$-potential [22])

$$\mathcal{L} = \frac{m}{2}(\dot{\varrho}^2 + \varrho^2\dot{\varphi}^2 - \omega^2) - \frac{\hbar^2}{2mr^2}\left(\alpha\,\mathrm{e}^{-4i\varphi} - 2\beta\,\mathrm{e}^{-2i\varphi} - \frac{1}{4}\right). \tag{C.13}$$

In the variable φ we have a complex periodic Morse potential, the same kind of potentials we have encountered on D_{III} for V_3 in polar coordinates. We identify $\alpha = 4c_1^2$ and $\beta = c_2/c_1$. Furthermore we see that the remaining path integral in the variable ϱ is just a radial harmonic oscillator path integral. Putting everything together yields

$$K^{(V_7)}(\varrho'',\varrho',\varphi'',\varphi';T) = \int\limits_{\varrho(t')=\varrho'}^{\varrho(t'')=\varrho''} \mathcal{D}\varrho(t) \int\limits_{\varphi(t')=\varphi'}^{\varphi(t'')=\varphi''} \mathcal{D}\varphi(t)\times$$

$$\times\exp\left\{\frac{i}{\hbar}\int\limits_0^T\left[\frac{m}{2}(\dot{\varrho}^2+\varrho^2\dot{\varphi}^2-\omega^2\varrho^2) - \frac{\hbar^2}{2m\varrho^2}\left(\alpha\,\mathrm{e}^{-4i\varphi} - 2\beta\,\mathrm{e}^{-2i\varphi} - \frac{1}{4}\right)\right]dt\right\} =$$

$$= \sum_{l=0}^{\infty}\Phi^{(c_1,c_2)}_{[\mathrm{cMP}],l}(\varphi'')\Phi^{(c_1,c_2)}_{[\mathrm{cMP}],l}(\varphi')\frac{m\omega}{i\hbar\sin\omega T}\times$$

$$\times\exp\left[-\frac{m\omega}{2i\hbar}({\varrho'}^2+{\varrho''}^2)\cot\omega T\right] I_{l+2\frac{2}{1}+\frac{1}{2}}\left(\frac{m\omega\varrho'\varrho''}{i\hbar\sin\omega T}\right), \tag{C.14}$$

with the well-known expansion by means of the Hille–Hardy formula in terms of Laguerre polynomials for ϱ. We leave the result as it stands.

D. SUPERINTEGRABLE POTENTIALS ON $S(2,\mathbb{C})$

Let us shortly enumerate the superintegrable potentials on the complex sphere. On the real two-dimensional sphere there are two superintegrable potentials, a feature which has been already investigated, e.g., [18]. On the complex two-dimensional sphere there are four more potentials which are listed in (D.4) [27, 30, 34]. In the underlined cases we give a path integral representation. These representations remain, however, on a formal level, because the complex sphere is an abstract space and serves just as a tool to find the relevant potentials. Going to the corresponding real spaces, i.e., the sphere and the hyperboloid, respectively,

Table 7. **Coordinate systems on the complex sphere $S(2,\mathbb{C})$**

Coordinate system	Integrals of motion	Coordinates
1. Spherical $(\vartheta\in[0,\pi),\varphi\in[0,2\pi))$	$L=J_3^2$	$s_1=\sin\vartheta\cos\varphi$ $s_2=\sin\vartheta\sin\varphi$, $s_3=\cos\vartheta$
2. Elliptic	$L=J-1^2+rJ_2^2$	$s_1^2=\dfrac{(ru-1)(rv-1)}{1-r}$ $s_2^2=\dfrac{r(u-1)(v-1)}{1-r}$, $z^2=ruv$
3. Horospherical	$L=(J_1+iJ_2)^2$	$s_1=\dfrac{i}{2}\left(v+\dfrac{y^2-1}{v}\right)$ $s_2=\dfrac{i}{2}\left(v+\dfrac{y^2-1}{v}\right)$, $s_3=iy/v$
4. Degenerate Elliptic 1 $(\tau_{1,2}\in\mathbb{R})$	$L=(J_1+ij_2)^2-c^2J_3^2$	$s_1+is_3=\dfrac{1}{\cosh\tau_1\cosh\tau_2}$ $s_2-is_3=\dfrac{\cosh\tau_2}{\cosh\tau_1}+\dfrac{\cosh\tau_1}{\cosh\tau_2}-$ $-\dfrac{1}{\cosh\tau_1\cosh\tau_2}$ $s_3=\tanh\tau_1\tanh\tau_2$
5. Degenerate Elliptic 2 $(\xi,\eta>0)$	$L=J_3(J_1-iJ_2)^2$	$s_1+is_2=\dfrac{1}{\xi\eta}$ $s_1+is_2=-\dfrac{1}{4}\dfrac{(\xi^2-\eta^2)^2}{\xi\eta}$ $s_3=\dfrac{1}{2}\dfrac{\xi^2+\eta^2}{\xi\eta}$

requires the real representation of the coordinate system in question, including the corresponding path integral representation.

In Table 7 we list the five coordinate systems on the complex sphere $S(2,\mathbb{C})$ according to [27, 30, 34]. Let us note that we can also use $v = i\,\mathrm{e}^{-ix}$ as a parameterization in the horospherical system $(x, y \in \mathbb{R})$. As usual, $J_1, J_2, J-3$ are the angular momentum operators in three dimensions.

The Potential V_3. Let us start superintegrable potential on the two-dimensional complex sphere. It has the form

$$V_3(\mathbf{s}) = \frac{\alpha}{s_3^2} + \frac{\beta}{(s_1 - is_2)^2} + \gamma\frac{(s_1 + is_2)}{(s_1 - is_2)^3}, \tag{D.1}$$

$$= \frac{\alpha}{\cos^2\vartheta^2} + \beta\frac{\mathrm{e}^{-2i\varphi}}{\sin^2\vartheta} - \gamma\frac{\mathrm{e}^{-4i\varphi}}{\sin^2\vartheta}, \tag{D.2}$$

$$= \mathrm{e}^{-2ix}\left(\gamma y^2 + \frac{\alpha}{y^2} + \beta\right) - \gamma\,\mathrm{e}^{-4ix}, \tag{D.3}$$

and we have inserted spherical and horospherical coordinates on the (complex) sphere, respectively,

$$\left.\begin{array}{ll}
V_3(\mathbf{s}) = \dfrac{\alpha}{s_3^2} + \dfrac{\beta}{(s_1 - is_2)^2} + \gamma\dfrac{s_1 + is_2}{(s_1 - is_2)^3} & \underline{\text{Spherical}} \\
 & \underline{\text{Horospherical}} \\
 & \text{Degenerate elliptic I} \\
\hline
V_4(\mathbf{s}) = \dfrac{\alpha}{(s_1 - is_2)^2} + \dfrac{\beta s_3}{\sqrt{s_1^2 + s_2^2}} + & \\
\qquad + \dfrac{i\gamma}{\sqrt{(s_1 + is_2)(s_1 - is_2)^2}} & \text{Spherical} \\
 & \text{Degenerate elliptic II} \\
\hline
V_5(\mathbf{s}) = \dfrac{\alpha z_+ + c^2 z_-}{\sqrt{(c^2 z_- - z_+)^2 - 4c^2 z_3}} + & \text{Elliptic} \\
\qquad + \dfrac{\beta(z_+ - c^2 z_-)(z_+ z_- + z_3^2)}{z_3^2\sqrt{(c^2 z_- - z_+)^2 - 4c^2 z_3}} + \gamma\dfrac{z_+ z_-}{z_3^2} & \\
\left(z_\pm = s_1 \pm is_2,\ z_3 = \sqrt{1 - s_1^2 - s_2^2},\ c^2 = \dfrac{1+r}{1-r}\right) & \text{Degenerate elliptic I} \\
\hline
V_6(\mathbf{s}) = \dfrac{\alpha}{(s_1 - is_2)^2} + \dfrac{\beta s_3}{(x - iy)^3} + \gamma\dfrac{1 - 4s_3^2}{(s_1 - is_2)^4} & \underline{\text{Horospherical}} \\
 & \text{Degenerate elliptic II}
\end{array}\right\} . \tag{D.4}$$

This potential has now in spherical coordinates in the φ dependence the same structure as the potential V_7 on the complex plane, thus the solution is the same ($c_{1,2}$ in the complex Morse potential appropriately). In the ϑ dependence we obtain after the separation of φ a Pöschl–Teller potential. In comparison to V_7 with the complex plane, we must therefore replace the wave functions in ϱ in terms of Laguerre polynomials by the Pöschl–Teller wave functions $\Phi_n^{\left(\tilde\alpha,l+2\frac{2}{1}+\frac{1}{2}\right)}(\vartheta)$ $\left(\tilde\alpha^2=2m\alpha/\hbar^2+\dfrac{1}{4}\right)$ and we have done. Summarizing we obtain

$$K^{(V_3)}(\vartheta'',\vartheta',\varphi'',\varphi';T)=\int\limits_{\vartheta(t')=\vartheta'}^{\vartheta(t'')=\vartheta''}\mathcal{D}\vartheta(t)\int\limits_{\varphi(t')=\varphi'}^{\varphi(t'')=\varphi''}\mathcal{D}\varphi(t)\sin\vartheta\times$$

$$\times\exp\left\{\frac{i}{\hbar}\int\limits_{t'}^{t''}\left[\frac{m}{2}(\dot\vartheta^2+\sin^2\vartheta\dot\varphi^2)-\frac{\alpha}{\cos^2\vartheta}-\frac{1}{\sin^2\vartheta}\left(\beta\mathrm{e}^{-2i\varphi}-\gamma\mathrm{e}^{-4i\varphi}-\frac{1}{4}\right)\right]dt\right\}=$$

$$=(\sin\vartheta'\sin\vartheta'')^{-1/2}\sum_{n=0}^{\infty}\sum_{l=0}^{\infty}\Phi^{(c_1,c_2)}_{[\mathrm{cMP}],l}(\varphi'')\Phi^{(c_1,c_2)}_{[\mathrm{cMP}],l}(\varphi')\Phi_n^{\left(l+2\frac{2}{1}+\frac{1}{2},\tilde\alpha\right)}(\vartheta'')\times$$

$$\times\Phi_n^{\left(l+2\frac{2}{1}+\frac{1}{2},\tilde\alpha\right)}(\vartheta')\exp\left[-\frac{i}{\hbar}\frac{\hbar^2}{2m}\left(2n+l+2\frac{c_2}{c_1}+\frac{3}{2}\right)^2T\right].\quad\text{(D.5)}$$

In horospherical coordinates we have in the variable y a radial harmonic oscillator $\left(\text{set }\gamma=m\omega^2/2,\ \tilde\alpha^2=2m\alpha/\hbar^2+\dfrac{1}{4}\right)$ and in the same way ($c_{1,2}$ in the complex Morse potential appropriately)

$$K^{(V_3)}(x'',x',y'',y';T)=\int\limits_{x(t')=x'}^{x(t'')=x''}\mathcal{D}x(t)\int\limits_{y(t')=y'}^{y(t'')=y''}\mathcal{D}y(t)\,\mathrm{e}^{2ix}\times$$

$$\times\exp\left\{\frac{i}{\hbar}\int\limits_{t'}^{t''}\left[\frac{m}{2}(\dot x+\mathrm{e}^{2ix}\dot y^2)-\mathrm{e}^{-2ix}\left(\gamma y^2+\frac{\alpha}{y^2}+\beta\right)-\gamma\mathrm{e}^{-4ix}\right]dt\right\}=$$

$$=\mathrm{e}^{-i(x'+x'')}\sum_{n=0}^{\infty}\sum_{l=0}^{\infty}\Psi_l^{(\mathrm{RHO},\tilde\alpha)}(y'')\Psi_l^{(\mathrm{RHO},\tilde\alpha)}(y')\Phi^{(c_1,c_2)}_{[\mathrm{cMP}],n}(\varphi'')\Phi^{(c_1,c_2)}_{[\mathrm{cMP}],n}(\varphi')\times$$

$$\times\exp\left[-\frac{i}{\hbar}\frac{\hbar^2}{2m}\left(n+2\frac{c_2}{c_1}+1\right)^2T\right],\quad\text{(D.6)}$$

and the $\Psi_l^{(\mathrm{RHO},\tilde\alpha)}(y)$ are the wave functions of the radial harmonic oscillator [22].

The Potential V_6. As the last potential we consider V_6. We have (set $\gamma = -m\omega^2/8$)

$$V_6(\mathbf{s}) = \frac{\alpha}{(s_1 - is_2)^2} + \frac{\beta s_3}{(x - iy)^3} + \gamma \frac{1 - 4s_3^2}{(s_1 - is_2)^4} \tag{D.7}$$

$$= \mathrm{e}^{-2ix} \frac{m}{2} \omega^2 \left(y + \frac{i\beta}{m\omega^2} \right)^2 - \mathrm{e}^{-2ix} \left(\alpha + \frac{\beta^2}{2m\omega^2} \right) - \gamma\, \mathrm{e}^{-4ix}, \tag{D.8}$$

and we have inserted horospherical coordinates. This potential is, in the variable y, a shifted harmonic oscillator, however, the shift is a complex one. In the variable x we have the complex periodic Morse potential. Again, we encounter a complex potential, this time a $\mathcal{PT}$-symmetric harmonic oscillator with spectrum $E_l = \hbar\omega(l + 1/2)$, e.g., [49]. Consequently, we have in a similar way as before ($c_{1,2}$ in the complex Morse potential appropriately, set $\kappa = i\beta/m\omega^2$):

$$K^{(V_6)}(x'', x', y'', y'; T) = \int\limits_{x(t')=x'}^{x(t'')=x''} \mathcal{D}x(t) \int\limits_{y(t')=y'}^{y(t'')=y''} \mathcal{D}y(t)\, \mathrm{e}^{2ix} \times$$

$$\times \exp\left\{ \frac{i}{\hbar} \int\limits_{t'}^{t''} \left[\frac{m}{2}(\dot{x}^2 + \mathrm{e}^{2ix}\dot{y}^2) - \left(\frac{m}{2}\omega^2 \left(y + \frac{i\beta}{m\omega^2} \right)^2 + \right.\right.\right.$$

$$\left.\left.\left. + \left(\alpha + \frac{\beta^2}{2m\omega^2} \right) \right) \mathrm{e}^{-2ix} - \gamma\, \mathrm{e}^{-4ix} \right] dt \right\} =$$

$$= \mathrm{e}^{-i(x'+x'')} \sum_{l=0}^{\infty} \Psi_l^{(\mathrm{HO})}(y'') \Psi_l^{(\mathrm{HO})}(y') \sum_{n=0}^{\infty} \Phi_{[\mathrm{cMP}],n}^{(c_1,c_2)}(\varphi'') \Phi_{[\mathrm{cMP}],n}^{(c_1,c_2)}(\varphi') \times$$

$$\times \exp\left[-\frac{i}{\hbar} \frac{\hbar^2}{2m} \left(n + 2\frac{c_2}{c_1} + 1 \right)^2 T \right], \tag{D.9}$$

and the $\Psi_l^{(\mathrm{HO},\kappa)}(y)$ are the wave functions of the shifted harmonic oscillator [22]. The representations of the potentials V_4 and V_5 in the separating coordinate systems lead to intractable powers in the various coordinates, respectively, powers of $\cosh \tau_{1,2}$, i.e., highly anharmonic terms which cannot be treated. The same holds for V_3 and V_6 in the remaining separating coordinate systems. This concludes the discussion.

REFERENCES

1. *Bender C. M. et al.* The $\mathcal{C}$ Operator in $\mathcal{PT}$-Symmetric Quantum Theories // J. Phys. A: Math. Gen. 2004. V. 37. P. 10139–10165.
2. *Böhm M., Junker G.* Path Integration over Compact and Noncompact Rotation Groups // J. Math. Phys. 1987. V. 28. P. 1978–1994.

3. *Bagchi B., Quesne C., Znojil M.* Generalized Continuity Equation and Modified Normalization in PT-Symmetric Quantum Mechanics // Mod. Phys. Lett. A. 2001. V. 16. P. 2047–2057.

4. *Cannata F., Junker J., Trost J.* Schrödinger Operators with Complex Potential but Real Spectrum // Phys. Lett. A. 1998. V. 246. P. 219–226.

5. *Daskaloyannis C., Ypsilantis K.* Unified Treatment and Classification of Superintegrable Systems with Integrals Quadratic in Momenta on a Two-Dimensional Manifold // J. Math. Phys. 2006. V. 45. P. 042904.

6. *Del Olmo M. A., Rodríguez M. A., Winternitz P.* The Conformal Group $SU(2,2)$ and Integrable Systems on a Lorentzian Hyperboloid // Fortschr. Phys. 1996. Bd. 44. S. 199–233; Integrable Systems Based on $SU(p,q)$ Homogeneous Manifolds // J. Math. Phys. 1993. V. 34. P. 5118–5139.

7. *Evans N. W.* Superintegrability in Classical Mechanics // Phys. Rev. A. 1990. V. 41. P. 5666–5676; Group Theory of the Smorodinsky–Winternitz System // J. Math. Phys. 1991. V. 32. P. 3369–3375; Superintegrability of the Winternitz System // Phys. Lett. A. 1990. V. 147. P. 483–486.

8. *Feynman R. P., Hibbs A.* Quantum Mechanics and Path Integrals. N. Y.: McGraw Hill, 1965.

9. *Friš J. et al.* On Higher Symmetries in Quantum Mechanics // Phys. Lett. 1965. V. 16. P. 354–356;
Friš J. et al. Symmetry Groups in Classical and Quantum Mechanics // Sov. J. Nucl. Phys. 1967. V. 4. P. 444–450.

10. *Gradshteyn I. S., Ryzhik I. M.* Table of Integrals, Series, and Products. N. Y.: Acad. Press, 1980.

11. *Grosche C.* The Path Integral on the Poincaré Upper Half-Plane with a Magnetic Field and for the Morse Potential // Ann. Phys. (N. Y.) 1988. V. 187. P. 110–134.

12. *Grosche C.* The Path Integral on the Poincaré Disc, the Poincaré Upper Half-Plane and on the Hyperbolic Strip // Fortschr. Phys. 1990. Bd. 38. S. 531–569.

13. *Grosche C.* Path Integrals, Hyperbolic Spaces, and Selberg Trace Formulae. Singapore: World Sci., 1996.

14. *Grosche C.* Path Integration on Darboux Spaces // Part. Nucl. 2006. V. 37. P. 368–389.

15. *Grosche C.* Path Integral Approach for Spaces of Nonconstant Curvature in Three Dimensions. DESY Report, DESY 05-221. 2005; quant-ph/0511135; Proc. of the II Intern. Workshop on Superintegrable Systems in Classical and Quantum Mechanics, Dubna, June 27 – July 1, 2005 // Phys. At. Nucl. 2006. V. 69.

16. *Grosche C. et al.* Quantum Motion on the Three-Dimensional Sphere: The Ellipso-Cylindrical Bases // J. Phys. A: Math. Gen. 1997. V. 30. P. 1629–1657.

17. *Grosche C., Pogosyan G. S., Sissakian A. N.* Path Integral Discussion for Smorodinsky–Winternitz Potentials: I. Two- and Three-Dimensional Euclidean Space // Fortschr. Phys. 1995. Bd. 43. S. 453–521.

18. *Grosche C., Pogosyan G. S., Sissakian A. N.* Path Integral Discussion for Smorodinsky–Winternitz Potentials: II. The Two- and Three-Dimensional Sphere // Ibid. S. 523–563.

19. *Grosche C., Pogosyan G. S., Sissakian A. N.* Path Integral Approach to Superintegrable Potentials on the Two-Dimensional Hyperboloid // Part. Nucl. 1996. V. 27. P. 244–278.

20. *Grosche C., Pogosyan G. S., Sissakian A. N.* Path Integral Approach for Superintegrable Potentials on the Three-Dimensional Hyperboloid // Part. Nucl. 1997. V. 28. P. 486–519.

21. *Grosche C., Pogosyan G. S., Sissakian A. N.* Path Integral Approach for Superintegrable Potentials on Spaces of Nonconstant Curvature: I. Darboux Spaces D_{I} and D_{II}. DESY preprint, DESY 06-113. 2006; Part. Nucl. 2007. V. 38, No. 3. P. 587.

22. *Grosche C., Steiner F.* Handbook of Feynman Path Integrals. Springer Tracts in Modern Physics. Berlin; Heidelberg, 1998. V. 145.

23. *Kalnins E. G.* On the Separation of Variables for the Laplace Equation $\Delta\psi + K^2\psi = 0$ in Two- and Three-Dimensional Minkowski Space // SIAM J. Math. Anal. 1975. V. 6. P. 340–374.

24. *Kalnins E. G.* Separation of Variables for Riemannian Spaces of Constant Curvature. Essex: Longman Sci. & Technical, 1986.

25. *Kalnins E. G., Kress J. M., Miller W. (Jr.).* Jacobi, Ellipsoidal Coordinates, and Superintegrable Systems // J. Nonlin. Math. Phys. 2005. V. 12. P. 209–229.

26. *Kalnins E. G. et al.* Superintegrable Systems in Darboux Spaces // J. Math. Phys. 2003. V. 44. P. 5811–5848.

27. *Kalnins E. G. et al.* Completeness of Superintegrability in Two-Dimensional Constant-Curvature Spaces // J. Phys. A: Math. Gen. 2001. V. 34. P. 4705–4720.

28. *Kalnins E. G., Kress J. M., Winternitz P.* Superintegrability in a Two-Dimensional Space of Nonconstant Curvature // J. Math. Phys. 2002. V. 43. P. 970–983.

29. *Kalnins E. G., Miller W. (Jr.).* Lie Theory and Separation of Variables. 4. The Groups $SO(2,1)$ and $SO(3)$ // J. Math. Phys. 1974. V. 15. P. 1263–1274.

30. *Kalnins E. G., Miller W. (Jr.).* The Wave Equation, $\underset{\sim}{O}(2,2)$, and Separation of Variables on Hyperboloids // Proc. Roy. Soc. A (Edinburgh). 1977. V. 79. P. 227–256;
Kalnins E. G., Miller W. (Jr.). Lie Theory and the Wave Equation in Space-Time. 2. The Group $SO(4,\mathbb{C})$ // SIAM J. Math. Anal. 1978. V. 9. P. 12–33.

31. *Kalnins E. G. et al.* Superintegrability on the Two-Dimensional Hyperboloid II // J. Math. Phys. 1999. V. 40. P. 2291–2306.

32. *Kalnins E. G., Miller W. (Jr.), Pogosyan G. S.* Superintegrability and Associated Polynomial Solutions. Euclidean Space and the Sphere in Two Dimensions // J. Math. Phys. 1996. V. 37. P. 6439–6467.

33. *Kalnins E. G., Miller W. (Jr.), Pogosyan G. S.* Superintegrability on the Two-Dimensional Hyperboloid // J. Math. Phys. 1997. V. 38. P. 5416–5433.

34. *Kalnins E. G., Miller W. (Jr.), Pogosyan G. S.* Completeness of Multiseparable Superintegrability on the Complex 2-Sphere // J. Phys. A: Math. Gen. 2000. V. 33. P. 6791–6806.

35. *Kalnins E. G., Miller W. (Jr.), Pogosyan G. S.* Completeness of Multiseparable Superintegrability in $E_{2,C}$ // Ibid. P. 4105–4120.

36. *Kalnins E. G., Miller W. (Jr.), Pogosyan G. S.* Superintegrability on Two-Dimensional Complex Euclidean Space. Algebraic Methods in Physics: A Symp. for the 60th Birthdays of Jiří Patera and Pavel Winternitz // CRM Series in Math. Phys. Berlin; Heidelberg, 2001. P. 95–103.

37. *Kalnins E. G., Miller W. (Jr.), Pogosyan G. S.* Completeness of Multiseparable Superintegrability in Two Dimensions // Phys. At. Nucl. 2002. V. 65. P. 1033–1035.

38. *Kalnins E. G., Miller W. (Jr.), Pogosyan G. S.* Exact and Quasi-Exact Solvability of Two-Dimensional Superintegrable Quantum Systems. I. Euclidean Space. math-ph/0412035.

39. *Kleinert H.* Path Integrals in Quantum Mechanics, Statistics and Polymer Physics. Singapore: World Sci., 1990.

40. *Koenigs G.* Sur les géodésiques a intégrales quadratiques // *Darboux G.* Lecons sur la théorie générale des surface. Chelsea Publ., 1972. V. 4. P. 368–404.

41. *Lutsenko I. V. et al.* Hydrogen Atom as Indicator of Hidden Symmetry of a Ring-Shaped Potential // Theor. Math. Phys. 1990. V. 83. P. 633–639.

42. *Mostafazadeh A., Batal A.* Physical Aspects of Pseudo-Hermitian and PT-Symmetric Quantum Mechanics // J. Phys. A: Math. Gen. A. 2004. V. 37. P. 11645–11680.

43. *Olevskiĭ M. N.* Triorthogonal Systems in Spaces of Constant Curvature in which the Equation $\Delta_2 u + \lambda u = 0$ Allows the Complete Separation of Variables // Mat. Sb. 1950. V. 27. P. 379–426 (in Russian).

44. *Peak D., Inomata A.* Summation over Feynman Histories in Polar Coordinates // J. Math. Phys. 1969. V. 10. P. 1422–1428.

45. *Schulman L. S.* Techniques and Applications of Path Integration. N. Y.: John Wiley & Sons, 1981.

46. *Ushveridze A.* Quasi-Exactly Solvable Models in Quantum Mechanics. Bristol: Inst. of Phys. Publ., 1994.

47. *Winternitz P. et al.* Symmetry Groups in Classical and Quantum Mechanics // Sov. J. Nucl. Phys. 1967. V. 4. P. 444–450.

48. *Wojciechowski S.* Superintegrability of the Calogero–Moser System // Phys. Lett. A. 1983. V. 95. P. 279–281.

49. *Znojil M.* $\mathcal{PT}$-Symmetric Harmonic Oscillators // Phys. Lett. A. 1999. V. 259. P. 220–223.

50. *Znojil M.* Exact Solution for Morse Oscillator in PT-Symmetric Quantum Mechanics // Phys. Lett. A. 1999. V. 264. P. 108–111.

51. *Znojil M.* Workshop on Pseudo-Hermitian Hamiltonians in Quantum Physics. I–III // Czech. J. Phys. 2004. V. 54. P. 1–156; 1005–1148; V. 55. P. 1045–1192;
http://gemma.ujf.cas.cz/%7Eznojil/conf/index.html