ФИЗИКА ЭЛЕМЕНТАРНЫХ ЧАСТИЦ И АТОМНОГО ЯДРА
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CHARACTERS OF THE POSITIVE-ENERGY UIRs OF $\mathrm{D} = 4$ CONFORMAL SUPERSYMMETRY

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Character formulae for the positive-energy unitary irreducible representations of the N-extended $\mathrm{D} = 4$ conformal superalgebras $su(2{,}2/N)$ are given. Using them we also derive decompositions of long superfields as they descend to the unitarity threshold. These results are also applicable to irreps of the complex Lie superalgebras $sl(4/N)$. Our derivations use the results from the representation theory of $su(2{,}2/N)$ developed already in the 1980s.

Представлены характеристические формулы для унитарных неприводимых представлений с положительной энергией N-расширенных ($\mathrm{D} = 4$) конформных суперапгебр $su(2{,}2/N)$. С их помощью получены разложения длинных суперполей по мере приближения их к порогу унитарности. Эти результаты также применимы к неприводимым представлениям комплексных суперапгебр Ли $sl(4/N)$. Приведенные результаты получены на основе теории представлений для суперапгебры $su(2{,}2/N)$, разработанной ранее, в 1980-х гг.

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INTRODUCTION

Recently, superconformal field theories in various dimensions are attracting more interest, cf. [1–96] and references therein. Particularly important are those for $\mathrm{D} \leqslant 6$ since in these cases the relevant superconformal algebras satisfy [97] the Haag–Lopuszanski–Sohnius theorem [98]. This makes the classification of the UIRs of these superalgebras very important. Until recently such classification was known only for the $\mathrm{D} = 4$ superconformal algebras $su(2,2/1)$ [99] and $su(2,2/N)$ [100–103] (for arbitrary N). Recently, the classification for $\mathrm{D} = 3$ (for even N), $\mathrm{D} = 5$, and $\mathrm{D} = 6$ (for $N = 1, 2$) was given in [104] (some results being conjectural), and then the $\mathrm{D} = 6$ case (for arbitrary N) was finalized in [105].

Once we know the UIRs of a (super-)algebra, the next question is to find their characters, since these give the spectrum which is important for the applications. Some results on the spectrum were given in the early papers [106–108, 102], but it is necessary to have systematic results for which the character formulae are needed. This is the question we address in this paper for the UIRs of $\mathrm{D} = 4$ conformal superalgebras $su(2,2/N)$. From the mathematical point of view this question is clear only for representations with conformal dimension above the unitarity threshold viewed as irreps of the corresponding complex superalgebra $sl(4/N)$. But for $su(2,2/N)$ even the UIRs above the unitarity threshold are truncated for small values of spin and isospin. And what is more, in the applications the most important role is played by the representations with «quantized» conformal dimensions at the unitarity threshold and at discrete points below. In the quantum field or string theory framework some of these correspond to operators with «protected» scaling dimension and therefore imply «nonrenormalization theorems» at the quantum level, cf., e.g., [22, 23].

Thus, we need detailed knowledge about the structure of the UIRs from the representation-theoretical point of view. Fortunately, such information is contained in [100–103]. Following these papers in Sec. 1 we recall the basic ingredients of the representation theory of the D $= 4$ superconformal algebras. In particular, we recall the structure of Verma modules and UIRs. Using this information we are able to derive character formulae, some of which are very explicit, cf. Sec. 2. We also pin-point the difference between character formulae for $sl(4/N)$ and $su(2,2/N)$ since for the latter we need to introduce and use the notion of counter-terms in the character formulae. The general formulae are valid for arbitrary N. For illustration we give more explicit formulae for $N = 1, 2$, but we leave the example $N = 4$ for a follow-up paper, since that would take too many pages, and the present paper is long enough. In Sec. 3 we summarize our results on the decompositions of long superfields as they descend to the unitarity threshold. These results may be applied to the problem of operators with protected dimensions.

1. REPRESENTATIONS OF D $= 4$ CONFORMAL SUPERSYMMETRY

1.1. The Setting. The superconformal algebras in D $= 4$ are $\mathcal{G} = su(2,2/N)$. The even subalgebra of $\mathcal{G}$ is the algebra $\mathcal{G}_0 = su(2,2) \oplus u(1) \oplus su(N)$. We label their physically relevant representations of $\mathcal{G}$ by the signature:

$$\chi = [\, d\, ;\, j_1\, ,\, j_2\, ;\, z\, ;\, r_1\, , \ldots , r_{N-1}\,], \tag{1.1}$$

where d is the conformal weight; j_1, j_2 are non-negative (half-)integers which are Dynkin labels of the finite-dimensional irreps of the D $= 4$ Lorentz subalgebra $so(3,1)$ of dimension $(2j_1+1)(2j_2+1)$; z represents the $u(1)$ subalgebra which is central for $\mathcal{G}_0$ (and for $N = 4$ is central for $\mathcal{G}$ itself), and $r_1, \ldots, r_{N-1}$ are non-negative integers which are Dynkin labels of the finite-dimensional irreps of the internal (or R) symmetry algebra $su(N)$.

We recall that the algebraic approach to D $= 4$ conformal supersymmetry developed in [100–103] involves two related constructions — on function spaces and as Verma modules. The first realization employs the explicit construction of induced representations of $\mathcal{G}$ (and of the corresponding supergroup $G = SU(2,2/N)$) in spaces of functions (superfields) over superspace which are called elementary representations (ER). The UIRs of $\mathcal{G}$ are realized as irreducible components of ERs, and then they coincide with the usually used superfields in indexless notation. The Verma module realization is also very useful as it provides simpler and more intuitive picture for the relation between reducible ERs, for the construction of the irreps, in particular, of the UIRs. For the latter the main tool is an adaptation of the Shapovalov form [109] to the Verma modules [102, 103]. Here we shall need only the second — Verma module — construction.

1.2. Verma Modules. To introduce Verma modules one needs the standard triangular decomposition:

$$\mathcal{G}^{\mathbb{C}} = \mathcal{G}^{+} \oplus \mathcal{H} \oplus \mathcal{G}^{-}, \tag{1.2}$$

where $\mathcal{G}^{\mathbb{C}} = sl(4/N)$ is the complexification of $\mathcal{G}$; $\mathcal{G}^{+}$, $\mathcal{G}^{-}$ are the subalgebras corresponding to the positive, negative roots of $\mathcal{G}^{\mathbb{C}}$, resp.; and $\mathcal{H}$ denotes the Cartan subalgebra of $\mathcal{G}^{\mathbb{C}}$.

We consider the lowest weight Verma modules, so that $V^{\Lambda} \cong U(\mathcal{G}^{+}) \otimes v_0$, where $U(\mathcal{G}^{+})$ is the universal enveloping algebra of $\mathcal{G}^{+}$, $\Lambda \in \mathcal{H}^*$ is the lowest weight, and v_0 is the lowest weight vector v_0 such that:

$$\begin{aligned} &X v_0 = 0, \quad X \in \mathcal{G}^{-}, \\ &H v_0 = \Lambda(H) v_0, \quad H \in \mathcal{H}. \end{aligned} \tag{1.3}$$

Further, for simplicity we omit the sign $\otimes$, i.e., we write $P v_0 \in V^{\Lambda}$ with $P \in U(\mathcal{G}^{+})$.

The lowest weight Λ is characterized by its values on the Cartan subalgebra $\mathcal{H}$, or, equivalently, by its products with the simple roots (given explicitly below). In general, these would be $3 + N$ complex numbers, however, in order to be useful for the representations of the real form $\mathcal{G}$ these values would be restricted to be real and furthermore to correspond to the signatures χ, and we shall write $\Lambda = \Lambda(\chi)$ or $\chi = \chi(\Lambda)$. Note, however, that there are Verma modules to which correspond no ERs, cf. [101] and below.

If a Verma module V^{Λ} is irreducible, then it gives the lowest weight irrep L_{Λ} with the same weight. If a Verma module V^{Λ} is reducible, then it contains a maximal invariant submodule I^{Λ}, and the lowest weight irrep L_{Λ} with the same weight is given by factorization: $L_{\Lambda} = V^{\Lambda} / I^{\Lambda}$ [110–112].

Thus, we need first to know which Verma modules are reducible. The reducibility conditions for the highest weight Verma modules over basic classical Lie superalgebra were given by Kac [112]. Translating his conditions to the lowest weight Verma modules we have [101]: A lowest weight Verma module

V^Λ is reducible only if at least one of the following conditions is true*:

$$(\rho - \Lambda, \beta) = m(\beta, \beta)/2, \quad \beta \in \Delta^+, \quad (\beta, \beta) \neq 0, \quad m \in I\!N, \tag{1.4a}$$
$$(\rho - \Lambda, \beta) = 0, \quad \beta \in \Delta^+, \quad (\beta, \beta) = 0, \tag{1.4b}$$

where Δ^+ is the positive root system of $\mathcal{G}^{\mathbb{C}}$; $\rho \in \mathcal{H}^*$ is the very important in representation theory element given by $\rho = \rho_{\bar{0}} - \rho_{\bar{1}}$, where $\rho_{\bar{0}}, \rho_{\bar{1}}$ are the half-sums of the even, odd, resp., positive roots; $(\cdot, \cdot)$ is the standard bilinear product in $\mathcal{H}^*$.

If a condition from (1.4a) is fulfilled then V^Λ contains a submodule which is a Verma module $V^{\Lambda'}$ with shifted weight given by the pair m, β: $\Lambda' = \Lambda + m\beta$. The embedding of $V^{\Lambda'}$ in V^Λ is provided by mapping the lowest weight vector v_0' of $V^{\Lambda'}$ to the singular vector $v_s^{m,\beta}$ in V^Λ which is completely determined by the conditions

$$\begin{aligned} X v_s^{m,\beta} &= 0, \quad X \in \mathcal{G}^-, \\ H v_s^{m,\beta} &= \Lambda'(H)\; v_0, \quad H \in \mathcal{H}, \quad \Lambda' = \Lambda + m\beta. \end{aligned} \tag{1.5}$$

Explicitly, $v_s^{m,\beta}$ is given by an even polynomial in the positive root generators

$$v_s^{m,\beta} = P^{m,\beta}\, v_0, \quad P^{m,\beta} \in U(\mathcal{G}^+). \tag{1.6}$$

Thus, the submodule of V^Λ which is isomorphic to $V^{\Lambda'}$ is given by $U(\mathcal{G}^+) P^{m,\beta} v_0$. (More information on the even case, following the same approach, may be seen in, e.g., [113, 114].)

If a condition from (1.4b) is fulfilled, then V^Λ contains a submodule I^β obtained from the Verma module $V^{\Lambda'}$ with shifted weight $\Lambda' = \Lambda + \beta$ as follows. In this situation V^Λ contains a singular vector

$$\begin{aligned} X v_s^\beta &= 0, \quad X \in \mathcal{G}^-, \\ H v_s^\beta &= \Lambda'(H)\; v_0, \quad H \in \mathcal{H}, \quad \Lambda' = \Lambda + \beta. \end{aligned} \tag{1.7}$$

Explicitly, v_s^β is given by an odd polynomial in the positive root generators

$$v_s^\beta = P^\beta\, v_0, \quad P^\beta \in U(\mathcal{G}^+). \tag{1.8}$$

Then we have

$$I^\beta = U(\mathcal{G}^+)\, P^\beta\, v_0 \tag{1.9}$$

*Many statements below are true for any basic classical Lie superalgebra and would require changes only for the superalgebras $osp(1/2N)$.

which is smaller than $V^{\Lambda'} = U(\mathcal{G}^+)\, v'_0$ since this polynomial is Grassmannian:

$$(P^\beta)^2 = 0. \tag{1.10}$$

To describe this situation we say that $V^{\Lambda'}$ is *oddly embedded* in V^Λ.

Note, however, that the above formulae describe also more general situations when the difference $\Lambda' - \Lambda = \beta$ is not a root, as used in [101] and below.

The weight shifts $\Lambda' = \Lambda + \beta$, when β is an odd root, are called *odd reflections* in [101] (see also [115]) and for future reference will be denoted as

$$\hat{s}_\beta \cdot \Lambda \equiv \Lambda + \beta, \quad (\beta, \beta) = 0, \quad (\Lambda, \beta) \neq 0. \tag{1.11}$$

Each such odd reflection generates an infinite discrete Abelian group

$$\tilde{W}_\beta \equiv \{(\hat{s}_\beta)^n | n \in \mathbb{Z}\}, \quad \ell((\hat{s}_\beta)^n) = n, \tag{1.12}$$

where the unit element is obviously obtained for $n = 0$, and $(\hat{s}_\beta)^{-n}$ is the inverse of $(\hat{s}_\beta)^n$, and for future use we have also defined the length function $\ell(\cdot)$ on the elements of $\tilde{W}_\beta$. This group acts on the weights Λ extending (1.11):

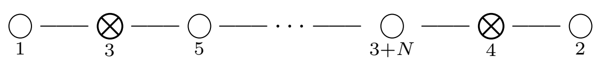
$$(\hat{s}_\beta)^n \cdot \Lambda = \Lambda + n\beta, \quad n \in \mathbb{Z}, \quad (\beta, \beta) = 0, \quad (\Lambda, \beta) \neq 0. \tag{1.13}$$

This is related to the fact that there is a doubly-infinite chain of oddly embedded Verma modules whenever a Verma module is reducible w.r.t. an odd root. This is explained in detail and used for the classification of the Verma modules in [100] and shall be used below.

Further, to be more explicit we need to recall the root system of $\mathcal{G}^{\mathbb{C}}$ — for definiteness — as used in [101]. The positive root system Δ^+ is comprised from α_{ij}, $1 \leqslant i < j \leqslant 4 + N$. The even positive root system $\Delta_{\bar{0}}^+$ is comprised from α_{ij}, with $i, j \leqslant 4$ and $i, j \geqslant 5$; the odd positive root system $\Delta_{\bar{1}}^+$ is comprised from α_{ij}, with $i \leqslant 4, j \geqslant 5$. The simple roots are chosen as in (1.4) of [101]:

$$\gamma_1 = \alpha_{12}, \gamma_2 = \alpha_{34}, \gamma_3 = \alpha_{25}, \gamma_4 = \alpha_{4,4+N}, \gamma_k = \alpha_{k,k+1}, 5 \leqslant k \leqslant 3 + N. \tag{1.14}$$

Thus, the Dynkin diagram is



(1.15)

This is a nondistinguished simple root system with two odd simple roots (for the various root systems of the basic classical superalgebras we refer to [116]).

Let $\Lambda = \Lambda(\chi)$. The products of Λ with the simple roots are [101]:

$$(\Lambda, \gamma_a) = -\,2j_a, \qquad a = 1, 2, \tag{1.16a}$$

$$(\Lambda, \gamma_3) = \frac{1}{2}(d + z') + j_1 - \frac{m}{N} + 1, \tag{1.16b}$$

$$(\Lambda, \gamma_4) = \frac{1}{2}(d - z') + j_2 - m_1 + \frac{m}{N} + 1, \tag{1.16c}$$

$$z' \equiv z(1 - \delta_{N4}),$$

$$(\Lambda, \gamma_j) = r_{N+4-j}, \quad 5 \leqslant j \leqslant 3 + N. \tag{1.16d}$$

These formulae give the correspondence between signatures χ and the lowest weights $\Lambda(\chi)^*$.

In the case of even roots $\beta \in \Delta^+_{\bar{0}}$ there are six roots α_{ij}, $j \leqslant 4$, coming from the $sl(4)$ factor (which is complexification of $su(2,2)$) and $N(N-1)/2$ roots α_{ij}, $5 \leqslant i$, coming form the $sl(N)$ factor (complexification of $su(N)$).

The reducibility conditions, w.r.t. the positive roots coming from $sl(4)(su(2,2))$, coming from (1.4) (denoting $m \to n_{ij}$ for $\beta \to \alpha_{ij}$) are

$$n_{12} = 1 + 2j_1 \equiv n_1, \tag{1.17a}$$

$$n_{23} = 1 - d - j_1 - j_2 \equiv n_2, \tag{1.17b}$$

$$n_{34} = 1 + 2j_2 \equiv n_3, \tag{1.17c}$$

$$n_{13} = 2 - d + j_1 - j_2 = n_1 + n_2, \tag{1.17d}$$

$$n_{24} = 2 - d - j_1 + j_2 = n_2 + n_3, \tag{1.17e}$$

$$n_{14} = 3 - d + j_1 + j_2 = n_1 + n_2 + n_3. \tag{1.17f}$$

Thus, reducibility conditions (1.17a), (1.17c) are fulfilled automatically for $\Lambda(\chi)$ with χ from (1.1) since we always have: $n_1, n_3 \in I\!N$. There are no such conditions for the ERs since they are induced from the finite-dimensional irreps of the Lorentz subalgebra (parameterized by j_1, j_2). However, to these two conditions there correspond differential operators of order $1 + 2j_1$ and $1 + 2j_2$ (as we mentioned above) and these annihilate all functions of the ERs with signature χ.

The reducibility conditions w.r.t. the positive roots coming from $sl(N)$ $(su(N))$ are all fulfilled for $\Lambda(\chi)$ with χ from (1.1). In particular, for the simple roots from those condition (1.4) is fulfilled with $\beta \to \gamma_j$, $m = 1 + r_{N+4-j}$,

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*For $N = 4$ the factor $u(1)$ in $\mathcal{G}_0$ becomes central in $\mathcal{G}$ and $\mathcal{G}^{\mathbb{C}}$. Consequently, the representation parameter z cannot come from the products of Λ with the simple roots, as indicated in (1.16). In that case the lowest weight is actually given by the sum $\Lambda + \tilde{\Lambda}$, where $\tilde{\Lambda}$ carries the representation parameter z. This is explained in detail in [101] and further we shall not comment more on it, but the peculiarities for $N = 4$ will be evident in the formulae.

for every $j = 5, 6, \ldots, N+3$. There are no such conditions for the ERs since they are induced from the finite-dimensional UIRs of $su(N)$. However, to these $N-1$ conditions there correspond $N-1$ differential operators of orders $1+r_k$ (as we mentioned), and the functions of our ERs are annihilated by all these operators [101]*.

For future use we note also the following decompositions:

$$\Lambda = \sum_{j=1}^{N+3} \lambda_j \alpha_{j,j+1} = \Lambda^s + \Lambda^z + \Lambda^u, \tag{1.18a}$$

$$\Lambda^s \equiv \sum_{j=1}^{3} \lambda_j \alpha_{j,j+1}, \quad \Lambda^z \equiv \lambda_4 \alpha_{45}, \quad \Lambda^u \equiv \sum_{j=5}^{N+3} \lambda_j \alpha_{j,j+1}, \tag{1.18b}$$

which actually employ the distinguished root system with one odd root α_{45}.

The reducibility conditions for the $4N$ odd positive roots of $\mathcal{G}$ are [102, 101]:

$$d = d^1_{Nk} - z\delta_{N4}, \quad d^1_{Nk} \equiv 4 - 2k + 2j_2 + z + 2m_k - 2m/N, \tag{1.19a.k}$$

$$d = d^2_{Nk} - z\delta_{N4}, \quad d^2_{Nk} \equiv 2 - 2k - 2j_2 + z + 2m_k - 2m/N, \tag{1.19b.k}$$

$$d = d^3_{Nk} + z\delta_{N4}, \quad d^3_{Nk} \equiv 2 + 2k - 2N + 2j_1 - z - 2m_k + 2m/N, \tag{1.19c.k}$$

$$d = d^4_{Nk} + z\delta_{N4}, \quad d^4_{Nk} \equiv 2k - 2N - 2j_1 - z - 2m_k + 2m/N, \tag{1.19d.k}$$

where in all four cases of (1.19) $k = 1, \ldots, N$, $m_N \equiv 0$, and

$$m_k \equiv \sum_{i=k}^{N-1} r_i, \quad m \equiv \sum_{k=1}^{N-1} m_k = \sum_{k=1}^{N-1} k r_k, \tag{1.20}$$

m_k is the number of cells of the kth row of the standard Young tableau; m is the total number of cells. Condition (1.19a.k) corresponds to the root $\alpha_{3,N+5-k}$, (1.19b.k) corresponds to the root $a_{4,N+5-k}$, (1.19c.k) corresponds to the root $a_{1,N+5-k}$, (1.19d.k) corresponds to the root $a_{2,N+5-k}$.

Note that for a fixed module and fixed $i = 1, 2, 3, 4$ only one of the odd N conditions involving d^i_{Nk} may be satisfied. Thus, no more than four (two, for $N = 1$) of the conditions (1.19) may hold for a given Verma module.

Remark. *Note that for $n_2 \in I\!N$ (cf. (1.17)) the corresponding irreps of su(2,2) are finite-dimensional (the necessary and sufficient condition for this is: $n_1, n_2, n_3 \in I\!N$). Then the irreducible LWM L_Λ of su(2,2/N) are also finite-dimensional (and nonunitary) independently of whether the corresponding Verma*

*Note that there are actually as many operators as positive roots of $sl(N)$ but all are expressed in terms of the $N-1$ above corresponding to the simple roots [101].

module V^Λ is reducible w.r.t. any odd root. If V^Λ is not reducible w.r.t. any odd root, then these finite-dimensional irreps are called «typical» [112], otherwise, the irreps are called «atypical» [112]. In our considerations $n_2 \notin \mathbb{N}$ in all cases, except the trivial 1-dimensional UIR (for which $n_2 = 1$, cf. below). $\diamondsuit$

We shall consider quotients of Verma modules factoring out the *even* submodules for which the reducibility conditions are always fulfilled. Before this we recall the root vectors following [101]. The positive (negative) root vectors corresponding to α_{ij}, $(-\alpha_{ij})$ are denoted by X^+_{ij}, (X^-_{ij}). In the $su(2,2/N)$ matrix notation the convention of [101], (1.7), is

$$X^+_{ij} = \begin{cases} e_{ji} & \text{for} \quad (i,j) = (3,4), (3,j), (4,j), \quad 5 \leqslant j \leqslant N+4, \\ e_{ij} & \text{otherwise} \end{cases} \tag{1.21}$$

$$X^-_{ij} = {}^t\,(X^+_{ij}),$$

where e_{ij} are $(N+4) \times (N+4)$ matrices with all elements zero except the element equal to 1 on the intersection of the ith row and jth column. The simple root vectors X^+_i follow the notation of the simple roots γ_i (1.14):

$$X^+_1 \equiv X^+_{12},\ \ X^+_2 \equiv X^+_{34},\ \ X^+_3 \equiv X^+_{25},\ \ X^+_4 \equiv X^+_{4,4+N},\ \ X^+_k \equiv X^+_{k,k+1}, \quad 5 \leqslant k \leqslant 3+N. \tag{1.22}$$

The mentioned submodules are generated by the singular vectors related to the even simple roots $\gamma_1, \gamma_2, \gamma_5, \ldots, \gamma_{N+3}$ [101]:

$$v^1_s = (X^+_1)^{1+2j_1}\, v_0, \tag{1.23a}$$

$$v^2_s = (X^+_2)^{1+2j_2}\, v_0, \tag{1.23b}$$

$$v^j_s = (X^+_j)^{1+r_{N+4-j}}\, v_0, \quad j = 5, \ldots, N+3 \tag{1.23c}$$

(for $N = 1$ (1.23c) being empty). The corresponding submodules are $I^\Lambda_k = U(\mathcal{G}^+)\, v^k_s$, and the invariant submodule to be factored out is

$$I^\Lambda_c = \bigcup_k I^\Lambda_k. \tag{1.24}$$

Thus, instead of V^Λ we shall consider the factor-modules

$$\tilde{V}^\Lambda = V^\Lambda \,/\, I^\Lambda_c \tag{1.25}$$

which are closer to the structure of the ERs. In the factorized modules the singular vectors (1.23) become null conditions, i.e., denoting by $\widetilde{|\Lambda\rangle}$ the lowest weight vector of $\tilde{V}^\Lambda$, we have

$$(X^+_1)^{1+2j_1}\, \widetilde{|\Lambda\rangle} = 0, \tag{1.26a}$$

$$(X^+_2)^{1+2j_2}\, \widetilde{|\Lambda\rangle} = 0, \tag{1.26b}$$

$$(X^+_j)^{1+r_{N+4-j}}\, \widetilde{|\Lambda\rangle} = 0, \quad j = 5, \ldots, N+3. \tag{1.26c}$$

1.3. Singular Vectors and Invariant Submodules at the Unitary Reduction Points. We first recall the result of [102] (cf. part (i) of the Theorem there) that the following is the complete list of the lowest weight (positive energy) UIRs of $su(2,2/N)$:

$$d \geqslant d_{\max} = \max(d^1_{N1}, d^3_{NN}), \tag{1.27a}$$

$$d = d^4_{NN} \geqslant d^1_{N1}, j_1 = 0, \tag{1.27b}$$

$$d = d^2_{N1} \geqslant d^3_{NN}, j_2 = 0, \tag{1.27c}$$

$$d = d^2_{N1} = d^4_{NN}, j_1 = j_2 = 0, \tag{1.27d}$$

where $d_{\max}$ is the threshold of the continuous unitary spectrum*. Note that in case (d) we have $d = m_1$, $z = 2m/N - m_1$, and that it is trivial for $N = 1$ since then the internal symmetry algebra $su(N)$ is trivial and by definition $m_1 = m = 0$ (the resulting irrep is 1-dimensional with $d = z = j_1 = j_2 = 0$). The UIRs for $N = 1$ were first given in [99].

Next we note that if $d > d_{\max}$, the factorized Verma modules are irreducible and coincide with the UIRs L_Λ. These UIRs are called *long* in the modern literature, cf., e.g., [8, 17, 23, 32–35]. Analogously, we shall use for the cases when $d = d_{\max}$, i.e., (1.27a), the terminology of *semishort* UIRs, introduced in [8, 23], while the cases (1.27b)–(1.27d) are also called *short* UIRs, cf., e.g., [17, 23, 32–35].

Next consider in more detail the UIRs at the four distinguished reduction points determining the list above:

$$\begin{aligned} d^1_{N1} &= 2 + 2j_2 + z + 2m_1 - 2m/N, \\ d^2_{N1} &= z + 2m_1 - 2m/N \quad (j_2 = 0), \\ d^3_{NN} &= 2 + 2j_1 - z + 2m/N, \\ d^4_{NN} &= -z + 2m/N \quad (j_1 = 0). \end{aligned} \tag{1.28}$$

*Note that from (1.27a) follows:

$$d_{\max} \geqslant 2 + j_1 + j_2 + m_1,$$

the equality being achieved only when $d^1_{N1} = d^3_{NN}$, while from (1.27b), (1.27c) follows:

$$d \geqslant 1 + j_1 + j_2 + m_1, \quad j_1 j_2 = 0,$$

the equality being achieved only when $d^4_{NN} = d^1_{N1}$, or $d^2_{N1} = d^3_{NN}$, for (1.27b), (1.27c), resp. Recalling the unitarity conditions [117] for the conformal algebra $su(2,2)$:

$$d \geqslant 2 + j_1 + j_2, \quad j_1 j_2 > 0,$$

$$d \geqslant 1 + j_1 + j_2, \quad j_1 j_2 = 0,$$

we see that the superconformal unitarity conditions are more stringent than the conformal ones.

First, we recall the singular vectors corresponding to these points. The above reducibilities occur for the following odd roots, resp.:

$$\alpha_{3,4+N}, \quad \alpha_{4,4+N}, \quad \alpha_{15}, \quad \alpha_{25}. \tag{1.29}$$

The second and the fourth are the two odd simple roots:

$$\gamma_3 = \alpha_{25}, \quad \gamma_4 = \alpha_{4,4+N}, \tag{1.30}$$

and the other two are simply related to these:

$$\alpha_{15} = \alpha_{12} + \alpha_{25} = \gamma_1 + \gamma_3, \qquad \alpha_{3,4+N} = \alpha_{34} + \alpha_{4,4+N} = \gamma_2 + \gamma_4. \tag{1.31}$$

Thus, the corresponding singular vectors are

$$\begin{aligned}
v^1_{\rm odd} =& P_{3,4+N}\, v_0 = \left(X^+_4 X^+_2 (h_2 - 1) - X^+_2 X^+_4 h_2\right)\, v_0 = & (1.32\text{a})\\
&= \left(2j_2 X^+_2 X^+_4 - (2j_2 + 1) X^+_4 X^+_2\right)\, v_0 = &\\
&= \left(2j_2 X^+_{3,4+N} - X^+_4 X^+_2\right)\, v_0, \qquad d = d^1_{N1}, & (1.32\text{a}')\\
v^2_{\rm odd} =& X^+_4\, v_0, \qquad d = d^2_{N1}, & (1.32\text{b})\\
v^3_{\rm odd} =& P_{15}\, v_0 = \left(X^+_3 X^+_1 (h_1 - 1) - X^+_1 X^+_3 h_1\right)\, v_0 = & (1.32\text{c})\\
&= \left(2j_1 X^+_1 X^+_3 - (2j_1 + 1) X^+_3 X^+_1\right)\, v_0 = &\\
&= \left(2j_1 X^+_{15} - X^+_3 X^+_1\right)\, v_0, \qquad d = d^3_{NN}, & (1.32\text{c}')\\
v^4_{\rm odd} =& X^+_3\, v_0, \qquad d = d^4_{NN}, & (1.32\text{d})
\end{aligned}$$

where $X^+_{3,4+N} = [X^+_2, X^+_4]$ is the odd-root vector corresponding to the root $\alpha_{3,4+N}$, $X^+_{15} = [X^+_1, X^+_3]$ is the odd-root vector corresponding to the root α_{15}, h_1, $h_2 \in \mathcal{H}$ are Cartan generators corresponding to the roots γ_1, γ_2, (cf. [101]), and passing from the (1.32a), (1.32c) to the next line we have used the fact that $h_2\, v_0 = -2j_2\, v_0$ ($h_1\, v_0 = -2j_1\, v_0$), consistently with (1.16b), (1.16a). These vectors are given in (8.9a), (8.7b), (8.8a), (8.7a), resp., of [101].

These singular vectors carry over for the factorized Verma modules $\tilde{V}^\Lambda$:

$$\begin{aligned}
\tilde{v}^1_{\rm odd} =& P_{3,4+N}\, |\widetilde{\Lambda}\rangle = \left(X^+_4 X^+_2 (h_2 - 1) - X^+_2 X^+_4 h_2\right)\, |\widetilde{\Lambda}\rangle = & (1.33\text{a})\\
&= \left(2j_2 X^+_{3,4+N} - X^+_4 X^+_2\right)\, |\widetilde{\Lambda}\rangle, \qquad d = d^1_{N1}, & (1.33\text{a}')\\
\tilde{v}^2_{\rm odd} =& X^+_4\, |\widetilde{\Lambda}\rangle, \qquad d = d^2_{N1}, & (1.33\text{b})\\
\tilde{v}^3_{\rm odd} =& P_{15}\, |\widetilde{\Lambda}\rangle = \left(X^+_3 X^+_1 (h_1 - 1) - X^+_1 X^+_3 h_1\right)\, |\widetilde{\Lambda}\rangle = & (1.33\text{c})\\
&= \left(2j_1 X^+_{15} - X^+_3 X^+_1\right)\, |\widetilde{\Lambda}\rangle, \qquad d = d^3_{NN}, & (1.33\text{c}')\\
\tilde{v}^4_{\rm odd} =& X^+_3\, |\widetilde{\Lambda}\rangle, \qquad d = d^4_{NN}. & (1.33\text{d})
\end{aligned}$$

For $j_1 = 0$, $j_2 = 0$, resp., the vector $v^3_{\rm odd}$, $v^1_{\rm odd}$, resp., is a descendant of the singular vector v^1_s, v^2_s, resp., cf. (1.23a), (1.23b), resp. In the same situations the tilde counterparts $\tilde v^1_s$, $\tilde v^2_s$ are just zero — cf. (1.26a), (1.26b), resp. However, then there is another independent singular vector of $\tilde V^\Lambda$ in both cases. For $j_1 = 0$, it corresponds to the sum of two roots: $\alpha_{15} + \alpha_{25}$ (which sum is not a root!) and is given by formula (D.1) of [101]:

$$\tilde v^{34} = X^+_3\, X^+_1\, X^+_3\, |\widetilde{\Lambda}\rangle = X^+_3\, X^+_{15}\, |\widetilde{\Lambda}\rangle, \quad d = d^3_{NN}, \quad j_1 = 0. \qquad (1.34)$$

Checking singularity we see at once that $X^-_k\, \tilde v^{34} = 0$ for $k \neq 3$. It remains to calculate the action of X^-_3:

$$X^-_3\, \tilde v^{34} = h_3\, X^+_1\, X^+_3\, |\widetilde{\Lambda}\rangle - X^+_3\, X^+_1\, h_3\, |\widetilde{\Lambda}\rangle = $$
$$= X^+_1\, X^+_3\, (h_3 - 1)\, |\widetilde{\Lambda}\rangle - X^+_3\, X^+_1\, h_3\, |\widetilde{\Lambda}\rangle = 0,$$

$h_3\,, h_4\, \in \mathcal{H}$ are Cartan generators corresponding to the roots $\gamma_3\,, \gamma_4$, (cf. [101]), the first term is zero since $\Lambda(h_3) - 1 = \frac{1}{2}(d - d^3_{NN}) = 0$, while the second term is zero due to (1.26a) for $j_1 = 0$.

For $j_2 = 0$, there is a singular vector corresponding to the sum of two roots: $\alpha_{3,4+N} + \alpha_{4,4+N}$ (which sum is not a root) and is given in [101] (cf. the formula before (D.4) there):

$$\tilde v^{12} = X^+_4\, X^+_2\, X^+_4\, |\widetilde{\Lambda}\rangle = X^+_4\, X^+_{3,4+N}\, |\widetilde{\Lambda}\rangle, \quad d = d^1_{N1}, \quad j_2 = 0. \qquad (1.35)$$

As above, one checks that $X^-_k\, v^{12} = 0$ for $k \neq 4$ and then calculates:

$$X^-_4\, \tilde v^{12} = h_4\, X^+_2\, X^+_4\, |\widetilde{\Lambda}\rangle - X^+_4\, X^+_2\, h_4\, |\widetilde{\Lambda}\rangle = $$
$$= X^+_2\, X^+_4\, (h_4 - 1)\, |\widetilde{\Lambda}\rangle - X^+_4\, X^+_2\, h_4\, |\widetilde{\Lambda}\rangle = 0$$

using $\Lambda(h_4) - 1 = \frac{1}{2}(d - d^1_{N1}) = 0$, and (1.26b) for $j_2 = 0$.

To the above two singular vectors in the ER picture there correspond second-order superdifferential operators given explicitly in formulae (11a), (11b) of [102] and in formulae (D3), (D5) of [105]*.

*Note that w.r.t. V^Λ the analogues of the vectors $\tilde v^{34}$ and $\tilde v^{12}$ are not singular, but subsingular vectors. Indeed, consider the vector in V^Λ given by the same $U(\mathcal{G}^+)$ monomial as $\tilde v^{34}$: $v^{34} = X^+_3\, X^+_1\, X^+_3$. Clearly, $X^-_k\, v^{34} = 0$ for $k \neq 3$. It remains to calculate the action of X^-_3:

$$X^-_3\, v^{34} = h_3\, X^+_1\, X^+_3\, v_0 - X^+_3\, X^+_1\, h_3\, v_0 = X^+_1\, X^+_3\, (h_3 - 1)\, v_0 - X^+_3\, X^+_1\, h_3\, v_0 = -X^+_3\, X^+_1\, v_0,$$

where the first term is zero as above, while the second term is a descendant of the singular vector $v^1_s = X^+_1\, v_0$ (cf. (1.23a) for $j_1 = 0$), which fulfills the definition of subsingular vector. Analogously, for the vector $v^{12} = X^+_4\, X^+_2\, X^+_4$ we have $X^-_k\, v^{12} = 0$ for $k \neq 4$, and

$$X^-_4\, v^{12} = X^-_4\, X^+_4\, X^+_2\, X^+_4 = -X^+_4\, X^+_2\, v_0,$$

(using $\Lambda(h_4) - 1$), which is a descendant of the singular vector $v^2_s = X^+_2\, v_0$, cf. (1.23b) for $j_2 = 0$.

From the expressions of the singular vectors follow, using (1.9), the explicit formulae for the corresponding invariant submodules I^β of the modules $\tilde{V}^\Lambda$ as follows:

$$I^1 = U(\mathcal{G}^+)\, P_{3,4+N}\, |\widetilde{\Lambda}\rangle = U(\mathcal{G}^+)\, \left(X_4^+ X_2^+ (h_2 - 1) - X_2^+ X_4^+ h_2\right)\, |\widetilde{\Lambda}\rangle = \tag{1.36a}$$

$$= U(\mathcal{G}^+)\, \left(2j_2 X_{3,4+N}^+ - X_4^+ X_2^+\right)\, |\widetilde{\Lambda}\rangle, \quad d = d_{N1}^1, \quad j_2 > 0, \tag{1.36a$'$}$$

$$I^2 = U(\mathcal{G}^+)\, X_4^+\, |\widetilde{\Lambda}\rangle, \qquad d = d_{N1}^2, \tag{1.36b}$$

$$I^3 = U(\mathcal{G}^+)\, P_{15}\, |\widetilde{\Lambda}\rangle = U(\mathcal{G}^+)\, \left(X_3^+ X_1^+ (h_1 - 1) - X_1^+ X_3^+ h_1\right)\, |\widetilde{\Lambda}\rangle = \tag{1.36c}$$

$$= U(\mathcal{G}^+)\, \left(2j_1 X_{15}^+ - X_3^+ X_1^+\right)\, |\widetilde{\Lambda}\rangle, \qquad d = d_{NN}^3, \quad j_1 > 0, \tag{1.36c$'$}$$

$$I^4 = U(\mathcal{G}^+)\, X_3^+\, |\widetilde{\Lambda}\rangle, \quad d = d_{NN}^4, \tag{1.36d}$$

$$I^{12} = U(\mathcal{G}^+)\, \tilde{v}^{12} = X_4^+\, X_2^+\, X_4^+\, |\widetilde{\Lambda}\rangle, \qquad d = d_{N1}^1, \quad j_2 = 0, \tag{1.36e}$$

$$I^{34} = U(\mathcal{G}^+)\, \tilde{v}^{34} = X_3^+\, X_1^+\, X_3^+\, |\widetilde{\Lambda}\rangle, \qquad d = d_{NN}^3, \quad j_1 = 0. \tag{1.36f}$$

Sometimes we shall indicate the signature $\chi(\Lambda)$, writing, e.g., $I^1(\chi)$; sometimes we shall indicate also the resulting signature, writing, e.g., $I^1(\chi, \chi')$ — this is a redundancy since it is determined by what is displayed already: $\chi' = \chi(\Lambda + \beta)$, but will be useful to see immediately in the concrete situations without calculation.

The invariant submodules were used in [102] in the construction of the UIRs, as we shall recall below.

1.4. Structure of Single-Reducibility-Condition Verma Modules and UIRs. We discuss now the reducibility of Verma modules at the four distinguished points (1.28). We note a partial ordering of these four points:

$$d_{N1}^1 > d_{N1}^2, \qquad d_{NN}^3 > d_{NN}^4, \tag{1.37}$$

or more precisely:

$$d_{N1}^1 = d_{N1}^2 + 2\ (j_2 = 0); \qquad d_{NN}^3 = d_{NN}^4 + 2\ (j_1 = 0). \tag{1.38}$$

Due to this ordering at most two of these four points may coincide. Thus, we have two possible situations: of Verma modules (or ERs) reducible at one and at two reduction points from (1.28).

In this Subsection we deal with the situations in which *no two* of the points in (1.28) coincide. According to [102] (Theorem) there are four such situations involving UIRs:

$$d = d_{\max} = d^1_{N1} > d^3_{NN}, \tag{1.39a}$$

$$d = d^2_{N1} > d^3_{NN}, \quad j_2 = 0, \tag{1.39b}$$

$$d = d_{\max} = d^3_{NN} > d^1_{N1}, \tag{1.39c}$$

$$d = d^4_{NN} > d^1_{N1}, \quad j_1 = 0. \tag{1.39d}$$

We shall call these cases *single-reducibility-condition (SRC)* Verma modules or UIRs, depending on the context. In addition, as already stated, we use for the cases when $d = d_{\max}$, i.e., (1.39a), (1.39c), the terminology of semishort UIRs [8, 23], while the cases (1.39b), (1.39d) are also called short UIRs [17, 23, 32–35].

As we see, the SRC cases have supplementary conditions as specified. And due to the inequalities there are the following additional restrictions which are correspondingly given as

$$z > j_1 - j_2 - m_1 + 2m/N, \tag{1.39a$'$}$$

$$z > j_1 + 1 - m_1 + 2m/N, \tag{1.39b$'$}$$

$$z < j_1 - j_2 - m_1 + 2m/N, \tag{1.39c$'$}$$

$$z < -1 - j_2 - m_1 + 2m/N. \tag{1.39d$'$}$$

Using these inequalities, the unitarity conditions (1.39) may be rewritten more explicitly:

$$d = d^1_{N1} = d^a \equiv 2 + 2j_2 + z + 2m_1 - 2m/N > 2 + j_1 + j_2 + m_1, \tag{1.39a$''$}$$

$$d = d^2_{N1} = z + 2m_1 - 2m/N > j_1 + 1 + m_1, \qquad j_2 = 0, \tag{1.39b$''$}$$

$$d = d^3_{NN} = d^c \equiv 2 + 2j_1 - z + 2m/N > 2 + j_1 + j_2 + m_1, \tag{1.39c$''$}$$

$$d = d^4_{NN} = -z + 2m/N > 1 + j_2 + m_1, \qquad j_2 = 0, \tag{1.39d$''$}$$

where we have introduced notation d^a, d^c to designate two of the SRC cases.

To finalize the structure we should check the even reducibility conditions (1.17b), (1.17d)–(1.17f). It is enough to note that the conditions on d in (1.39a$''$), (1.39c$''$):

$$d > 2 + j_1 + j_2 + m_1$$

and in (1.39b$''$), (1.39d$''$):

$$d > 1 + j_1 + j_2 + m_1 (j_1 j_2 = 0)$$

are incompatible with (1.17b), (1.17d)–(1.17f), except in two cases. The exceptions are in cases (1.39b$''$), (1.39d$''$) when $d = 2 + j_1 + j_2 = z$ and $j_1 j_2 = 0$. In these cases we have $n_{14} = 1$ in (1.17f) and there exists a Verma submodule $V^{\Lambda+\alpha_{14}}$. However, the $su(2,2)$ signature $\chi_0(\Lambda + \alpha_{14})$ is unphysical:

$[j_1-1/2,-1/2;3+j_1]$ for $j_2=0$, and $[-1/2,j_2-1/2;3+j_1]$ for $j_1=0$. Thus, there is no such submodule of $\tilde V^\Lambda$.

Thus, the factorized Verma modules $\tilde V^\Lambda$ with the unitary signatures from (1.39) have only one invariant (odd) submodule which has to be factorized in order to obtain the UIRs. These odd embeddings are given explicitly as

$$\tilde V^\Lambda \to \tilde V^{\Lambda+\beta}, \tag{1.40}$$

where we use the convention [100] that arrows point to the oddly embedded module, and there are the following cases for β:

$$\begin{aligned}
\beta &= \alpha_{3,4+N}, \quad \text{for} \quad (1.39\text{a}), \quad j_2>0, && (1.41\text{a})\\
&= \alpha_{4,4+N}, \quad \text{for} \quad (1.39\text{b}), && (1.41\text{b})\\
&= \alpha_{15}, \quad \text{for} \quad (1.39\text{c}), \quad j_1>0, && (1.41\text{c})\\
&= \alpha_{25}, \quad \text{for} \quad (1.39\text{d}), && (1.41\text{d})\\
&= \alpha_{3,4+N}+\alpha_{4,4+N}, \quad \text{for} \quad (1.39\text{a}), \quad j_2=0, && (1.41\text{e})\\
&= \alpha_{15}+\alpha_{25}, \quad \text{for} \quad (1.39\text{c}), \quad j_1=0. && (1.41\text{f})
\end{aligned}$$

This diagram gives the UIR L_Λ contained in $\tilde V^\Lambda$ as follows:

$$L_\Lambda = \tilde V^\Lambda / I^\beta, \tag{1.42}$$

where I^β is given by I^1, I^2, I^3, I^4, I^{12}, I^{34}, resp. (cf. (1.36), in the cases (1.41a), (1.41b), (1.41c), (1.41d), (1.41e), (1.41f), resp.

It is useful to record the signatures of the shifted lowest weights, i.e., $\chi' = \chi(\Lambda+\beta)$. In fact, for future use we give the signature changes for arbitrary roots. The explicit formulae are [100, 101]:

$$\begin{aligned}
&\beta = \alpha_{3,N+5-k} : \chi' = \\
&= \left[d+\frac{1}{2};\, j_1,j_2-\frac{1}{2};\, z+\epsilon_N;\, r_1,\ldots,r_{k-1}-1,r_k+1,\ldots,r_{N-1}\right], && (1.43\text{a})\\
&\qquad j_2>0, \quad r_{k-1}>0, && (1.43\text{a}')\\
&\beta = \alpha_{4,N+5-k} : \chi' = \\
&= \left[d+\frac{1}{2};\, j_1,j_2+\frac{1}{2};\, z+\epsilon_N;\, r_1,\ldots,r_{k-1}-1,r_k+1,\ldots,r_{N-1}\right], && (1.43\text{b})\\
&\qquad r_{k-1}>0, && (1.43\text{b}')
\end{aligned}$$

$$\beta = \alpha_{1,N+5-k} : \chi' =$$
$$= \left[d + \frac{1}{2};\, j_1 - \frac{1}{2}, j_2;\, z - \epsilon_N;\, r_1, \ldots, r_{k-1} + 1, r_k - 1, \ldots, r_{N-1} \right], \quad (1.43c)$$
$$j_1 > 0, \quad r_k > 0, \quad (1.43c')$$
$$\beta = \alpha_{2,N+5-k} : \chi' =$$
$$= \left[d + \frac{1}{2};\, j_1 + \frac{1}{2}, j_2;\, z - \epsilon_N;\, r_1, \ldots, r_{k-1} + 1, r_k - 1, \ldots, r_{N-1} \right], \quad (1.43d)$$
$$r_k > 0, \quad (1.43d')$$
$$k = 1, \ldots, N, \qquad \epsilon_N \equiv \frac{2}{N} - \frac{1}{2}. \quad (1.44)$$

For each fixed χ the lowest weight $\Lambda(\chi')$ fulfills the same odd reducibility condition as $\Lambda(\chi')$. We need also the special cases used in (1.41e), (1.41f):

$$\beta_{12} = \alpha_{3,4+N} + \alpha_{4,4+N} : \chi'_{12} = [d + 1;\, j_1, 0;\, z + 2\epsilon_N\, ;\, r_1 + 2, r_2, \ldots, r_{N-1}], \quad (1.43e)$$
$$j_2 = 0, \; d = d^1_{N1},$$
$$\beta_{34} = \alpha_{15} + \alpha_{25} : \chi'_{34} = [d + 1;\, 0, j_2;\, z - 2\epsilon_N\, ;\, r_1, \ldots, r_{N-2}, r_{N-1} + 2], \quad (1.43f)$$
$$j_1 = 0, \; d = d^3_{NN}.$$

The lowest weight $\Lambda(\chi'_{12})$ fulfils (1.39b), while the lowest weight $\Lambda(\chi'_{34})$ fulfils (1.39d).

The embedding diagram (1.40) is a piece of a much richer picture [100]. Indeed, notice that if (1.4b) is fulfilled for some odd root β, then it is fulfilled also for an infinite number of Verma modules $V_\ell = V^{\Lambda + \ell\beta}$ for all $\ell \in \mathbb{Z}$. These modules form an infinite chain complex of oddly embedded modules:

$$\cdots \longrightarrow V_{-1} \longrightarrow V_0 \longrightarrow V_1 \longrightarrow \cdots \quad (1.45)$$

Because of (1.10) this is an exact sequence with one nilpotent operator involved in the whole chain. Of course, once we restrict to the factorized modules $\tilde{V}^\Lambda$, the diagram will be shortened — this is evident from the signature changes (1.43a)–(1.43d). In fact, there are only a finite number of factorized modules for $N > 1$, while for $N = 1$ the diagram continues to be infinite to the left. Furthermore, when $\beta = \beta_{12}, \beta_{34}$, from the end of the restricted chain one transmutes — via the embeddings (1.36e) (1.36f), resp. — to the chain with $\beta = \alpha_{4,N+4}, \alpha_{25}$, resp. More explicitly, when $\beta = \beta_{12}, \beta_{34}$, then the module V_1 plays the role of V_0 with $\beta = \alpha_{4,N+4}, \alpha_{25}$. All this is explained in detail in [100]. Furthermore, when a factorized Verma module $\tilde{V}^\Lambda = \tilde{V}^\Lambda_0$ contains an UIR, then not all modules $\tilde{V}^\Lambda_\ell$

would contain an UIR [101, 102]. From all this, all which is important from the view of modern applications can be summarized as follows:

• The semishort SRC UIRs (cf. (1.39a), (1.39c)) are obtained by factorizing a Verma submodule $\tilde{V}^{\Lambda+\beta}$ containing either another semishort SRC UIR of the same type (cf. (1.41a), (1.41c)) or containing a short SRC UIR of a different type (cf. (1.41e), (1.41f)). In contrast, short SRC UIRs (cf. (1.39b), (1.39d)) are obtained by factorizing a Verma submodule $\tilde{V}^{\Lambda+\beta}$ whose irreducible factor-module is not unitary (cf. (1.41b), (1.41d)).

1.5. Structure of Double-Reducibility-Condition Verma Modules and UIRs. We consider now the situations in which *two* of the points in (1.28) coincide. According to [102] (Theorem) there are four such situations involving UIRs:

$$d = d_{\max} = d^{ac} \equiv d^1_{N1} = d^3_{NN}, \tag{1.46a}$$

$$d = d^1_{N1} = d^4_{NN}, \quad j_1 = 0, \tag{1.46b}$$

$$d = d^2_{N1} = d^3_{NN}, \quad j_2 = 0, \tag{1.46c}$$

$$d = d^2_{N1} = d^4_{NN}, \quad j_1 = j_2 = 0. \tag{1.46d}$$

We shall call these *double-reducibility-condition (DRC)* Verma modules or UIRs. As in the previous Subsection we shall use for the cases when $d = d_{\max}$, i.e., (1.46a), also the terminology of semishort UIRs [8, 23], while the cases (1.46b)–(1.46d) shall also be called short UIRs [17, 23, 32–35].

For later use we list more explicitly the values of d and z:

$$\begin{aligned} &d = d^{ac} = d^1_{N1} = d^3_{NN} = 2 + j_1 + j_2 + m_1, \\ &z = j_1 - j_2 + 2m/N - m_1; \end{aligned} \tag{1.46a$'$}$$

$$\begin{aligned} &d = d^1_{N1} = d^4_{NN} = 1 + j_2 + m_1, \quad j_1 = 0, \\ &z = -1 - j_2 + 2m/N - m_1; \end{aligned} \tag{1.46b$'$}$$

$$\begin{aligned} &d = d^2_{N1} = d^3_{NN} = 1 + j_1 + m_1, \quad j_2 = 0, \\ &z = 1 + j_1 + 2m/N - m_1; \end{aligned} \tag{1.46c$'$}$$

$$\begin{aligned} &d = d^2_{N1} = d^4_{NN} = m_1, \quad j_1 = j_2 = 0, \\ &z = 2m/N - m_1. \end{aligned} \tag{1.46d$'$}$$

We noted already that for $N = 1$ the last case (1.46d), (1.46d′), is trivial. Note also that for $N = 2$ we have: $2m/N - m_1 = m - m_1 = 0$.

To finalize the structure we should check the even reducibility conditions (1.17b), (1.17d), (1.17e), (1.17f). It is enough to note that the values of d in (1.46) are incompatible with (1.17b), (1.17d), (1.17e), (1.17f), except in a few cases. The exceptions are:

$$d = d^1_{N1} = d^3_{NN} = 2 + j_1 + j_2, \qquad m_1 = 0, \tag{1.47a}$$

$$d = d^1_{N1} = d^4_{NN} = 1 + j_2 + m_1, j_1 = 0, \qquad m_1 = 0, 1, \tag{1.47b}$$

$$d = d^2_{N1} = d^3_{NN} = 1 + j_1 + m_1, j_2 = 0, \qquad m_1 = 0, 1, \tag{1.47c}$$

$$d = d^2_{N1} = d^4_{NN} = m_1, \qquad j_1 = j_2 = 0, \quad m_1 = 0, 1, 2, \tag{1.47d}$$

• In case (1.47a) we have $n_{14} = 1$ in (1.17f) and there exists a Verma submodule $V^{\Lambda+\alpha_{14}}$ with $su(2,2)$ signature $\chi_0(\Lambda+\alpha_{14}) = [j_1-1/2, j_2-1/2; 3+j_1+j_2]$. As we can see this signature is unphysical for $j_1 j_2 = 0$. Thus, there is the even submodule $\tilde{V}^{\Lambda+\alpha_{14}}$ of $\tilde{V}^{\Lambda}$ only if $j_1 j_2 \neq 0$.

• In case (1.47b) there are three subcases:

$m_1 = 0$, $j_2 = 1/2$; then $d = 3/2$, $n_{24} = 1$, $n_{14} = 2$. The signatures of the embedded submodules of V^{Λ} are: $\chi_0(\Lambda + \alpha_{24}) = [1/2, 0; 5/2]$, $\chi_0(\Lambda + 2\alpha_{14}) = [-1, -1/2; 7/2]$. Thus, there is only the even submodule $\tilde{V}^{\Lambda+\alpha_{24}}$ of $\tilde{V}$.

$m_1 = 0$, $j_2 = 0$; then $d = 1$, $n_{13} = 1$, $n_{24} = 1$, $n_{14} = 2$. The signatures of the embedded submodules of V^{Λ} are: $\chi_0(\Lambda + \alpha_{13}) = [-1/2, 1/2; 2]$, $\chi_0(\Lambda + \alpha_{24}) = [1/2, -1/2; 2]$, $\chi_0(\Lambda + 2\alpha_{14}) = [-1, -1; 3]$, and all are unphysical. However, the Verma module V^{Λ} has a subsingular vector of weight $\alpha_{23} + \alpha_{14}$, cf. [118], and thus, the factorized Verma module $\tilde{V}^{\Lambda}$ has the submodule $\tilde{V}^{\Lambda+\alpha_{23}+\alpha_{14}}$.

$m_1 = 1$; then $n_{14} = 1$, but as above there is no nontrivial even submodule of $\tilde{V}^{\Lambda}$.

• The case (1.47c) is dual to (1.47b) so we list shortly the three subcases:

$m_1 = 0$, $j_1 = 1/2$; then $d = 3/2$, $n_{13} = 1$, $n_{14} = 2$. There is only the even submodule $\tilde{V}^{\Lambda+\alpha_{13}}$ of $\tilde{V}$.

$m_1 = 0$, $j_1 = 0$; then $d = 1$, $n_{13} = 1$, $n_{24} = 1$, $n_{14} = 2$. This subcase coincides with the second subcase of (1.47b).

$m_1 = 1$; then $n_{14} = 1$ and as above there is no nontrivial submodule of $\tilde{V}^{\Lambda}$.

• In case (1.47d) there are again three subcases:

$m_1 = 0$; then all quantum numbers in the signature are zero and the UIR is the one-dimensional trivial irrep.

$m_1 = 1$; then $d = 1$, $n_{13} = 1$, $n_{24} = 1$, $n_{14} = 2$. Though this subcase has nontrivial isospin from $su(2,2)$ point of view, it has the same structure as the second subcase of (1.47b) and the factorized Verma module $\tilde{V}^{\Lambda}$ has the submodule $\tilde{V}^{\Lambda+\alpha_{23}+\alpha_{14}}$.

$m_1 = 2$; then $d = 2$, $n_{14} = 1$ and as above there is no nontrivial even submodule of $\tilde{V}^{\Lambda}$.

The embedding diagrams for the corresponding modules $\tilde{V}^{\Lambda}$ when there are no even embeddings are:

$$\begin{array}{c} \tilde{V}^{\Lambda+\beta'} \\ \uparrow \\ \tilde{V}^{\Lambda} \rightarrow \tilde{V}^{\Lambda+\beta}, \end{array} \tag{1.48}$$

where

$$\begin{aligned}
(\beta,\beta') &= (\alpha_{15},\alpha_{3,4+N}), \quad \text{for (1.46a)}, \quad m_1 j_1 j_2 > 0, && (1.49a)\\
&= (\alpha_{15},\alpha_{3,4+N}+\alpha_{3,4+N}), \quad \text{for (1.46a)}, \quad j_1 > 0, \quad j_2 = 0, && (1.49b)\\
&= (\alpha_{15}+\alpha_{25},\alpha_{3,4+N}), \quad \text{for (1.46a)}, \quad j_1 = 0, \quad j_2 > 0, && (1.49c)\\
&= (\alpha_{15}+\alpha_{25},\alpha_{3,4+N}+\alpha_{3,4+N}), \quad \text{for (1.46a)}, \quad j_1 = j_2 = 0, && (1.49d)\\
&= (\alpha_{25},\alpha_{3,4+N}), \ \text{for (1.46b)}, \quad j_2 > 0, \quad 2j_2 + m_1 \geqslant 2, && (1.49e)\\
&= (\alpha_{25},\alpha_{3,4+N}+\alpha_{4,4+N}), \quad \text{for (1.46b)}, \quad j_2 = 0, \quad m_1 > 0, && (1.49f)\\
&= (\alpha_{15},\alpha_{4,4+N}), \quad \text{for (1.46c)}, \quad j_1 > 0, \quad 2j_1 + m_1 \geqslant 2, && (1.49g)\\
&= (\alpha_{15}+\alpha_{25},\alpha_{4,4+N}), \quad \text{for (1.46c)}, \quad j_1 = 0, \quad m_1 > 0, && (1.49h)\\
&= (\alpha_{25},\alpha_{4,4+N}), \ \text{for (1.46d)}, \quad m_1 \neq 1. && (1.49i)
\end{aligned}$$

This diagram gives the UIR L_Λ contained in $\tilde{V}^\Lambda$ as follows:

$$L_\Lambda = \tilde{V}^\Lambda / I^{\beta,\beta'}, \quad I^{\beta,\beta'} = I^\beta \cup I^{\beta'}, \tag{1.50}$$

where I^β, $I^{\beta'}$ are given in (1.36), according to the cases in (1.49).

The embedding diagrams for the corresponding modules $\tilde{V}^\Lambda$, when there are even embeddings are:

$$\begin{array}{ccccc} & & \tilde{V}^{\Lambda+\beta'} & & \\ & & \uparrow & & \\ \tilde{V}^{\Lambda+\beta_e} & \leftarrow & \tilde{V}^{\Lambda} & \rightarrow & \tilde{V}^{\Lambda+\beta}, \end{array} \tag{1.51}$$

where

$$\begin{aligned}
(\beta,\beta',\beta_e) &= (\alpha_{15},\alpha_{3,4+N},\alpha_{14}), \quad \text{for (1.46a)}, \quad j_1 j_2 > 0, \quad m_1 = 0, && (1.52a)\\
&= (\alpha_{25},\alpha_{3,4+N},\alpha_{24}), \quad \text{for (1.46b)}, \quad j_2 = 1/2, \quad m_1 = 0, && (1.52b)\\
&= (\alpha_{25},\alpha_{3,4+N}+\alpha_{4,4+N},\alpha_{23}+\alpha_{14}), \quad \text{for (1.46b)}, \quad j_2 = m_1 = 0, && (1.52c)\\
&= (\alpha_{15},\alpha_{4,4+N},\alpha_{13}), \quad \text{for (1.46c)}, \quad j_1 = 1/2, \ m_1 = 0, && (1.52d)\\
&= (\alpha_{15}+\alpha_{25},\alpha_{4,4+N},\alpha_{23}+\alpha_{14}), \quad \text{for (1.46c)}, \quad j_1 = m_1 = 0, && (1.52e)\\
&= (\alpha_{25},\alpha_{4,4+N},\alpha_{23}+\alpha_{14}), \quad \text{for (1.46d)}, \quad m_1 = 1. && (1.52f)
\end{aligned}$$

This diagram gives the UIR L_Λ contained in $\tilde{V}^\Lambda$ as follows:

$$L_\Lambda = \tilde{V}^\Lambda / I^{\beta,\beta',\beta_e}, \quad I^{\beta,\beta'} = I^\beta \cup I^{\beta'} \cup \tilde{V}^{\Lambda+\beta_e}. \tag{1.53}$$

Naturally, the two odd embeddings in (1.48) or (1.51) are the combination of the different cases of (1.40). Similarly, like (1.40) is a piece of the richer picture (1.45), here we have the following analogues of (1.45) [100]*

$$\begin{array}{ccccc} & \vdots & & & \\ & \uparrow & & & \\ & V_{01} & & & \\ & \uparrow & & & \\ \cdots \to & V_{00} \to & V_{10} \to \cdots & & \\ & \uparrow & & & \\ & \vdots & & & \end{array} \qquad N = 1, \tag{1.54}$$

where $V_{k\ell} \equiv V^{\Lambda+k\beta+\ell\beta'}$, and β, β' are the roots appearing in (1.49a), (1.49e), (1.49g), (1.49i) (or (1.52a), (1.52b), (1.52d), (1.52f))

$$\begin{array}{ccc} & \vdots & \vdots \\ & \uparrow & \uparrow \\ \cdots \to & V_{10} \to & V_{11} \to \cdots \\ & \uparrow & \uparrow \\ \cdots \to & V_{00} \to & V_{01} \to \cdots \\ & \uparrow & \uparrow \\ & \vdots & \vdots \end{array} \qquad N > 1, \tag{1.55}$$

The difference between the cases $N = 1$ and $N > 1$ is due to the fact that if (1.4b) is fulfilled for V_{00} w.r.t. two odd roots β, β', then for $N > 1$ it is fulfilled also for all Verma modules $V_{k\ell}$ again w.r.t. these odd roots β, β', while for $N = 1$ it is fulfilled only for V_{k0} w.r.t. the odd root β and only for $V_{0\ell}$ w.r.t. the odd root β'.

In the cases (1.49b), (1.49c), (1.49d), (1.49f), (1.49h) (or (1.52c), (1.52e)) we have the same diagrams though their parameterization is more involved [100] (cf. also what we said about transmutation for the single chains after (1.45)).

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*These diagrams are essential parts of much richer diagrams (which we do not need since we consider only UIRs-related modules) which are explicitly described for any N in [100], and shown there in Fig. 1 (for $N = 1$) and Fig. 2 (for $N = 2$).

However, for the modules with $0 \leqslant k, \ell \leqslant 1$ (which we use) we have simply as before $V_{k\ell} = V^{\Lambda + k\beta + \ell\beta'}$ for the appropriate β, β'.

The richer structure for $N > 1$ has practical consequences for the calculation of the character formulae, cf. the next Section.

2. CHARACTER FORMULAE OF POSITIVE-ENERGY UIRs

2.1. Character Formulae: Generalities. In the beginning of this Subsection we follow [110]. Let $\hat{\mathcal{G}}$ be a simple Lie algebra of rank ℓ with Cartan subalgebra $\hat{\mathcal{H}}$, root system $\hat{\Delta}$, simple root system $\hat{\pi}$. Let Γ (resp. Γ_+) be the set of all integral, (resp. integral dominant), elements of $\hat{\mathcal{H}}^*$, i.e., $\lambda \in \hat{\mathcal{H}}^*$ such that $(\lambda, \alpha_i^\vee) \in \mathbb{Z}$, (resp. $\mathbb{Z}_+$), for all simple roots α_i, ($\alpha_i^\vee \equiv 2\alpha_i/(\alpha_i, \alpha_i)$). Let V be the lowest weight module with the lowest weight Λ and the lowest weight vector v_0. It has the following decomposition:

$$V = \underset{\mu \in \Gamma_+}{\oplus} V_\mu, \quad V_\mu = \{u \in V | Hu = (\lambda + \mu)(H)u, \ \forall H \in \mathcal{H}\}. \tag{2.1}$$

(Note that $V_0 = \mathbb{C} v_0$.) Let $E(\mathcal{H}^*)$ be the associative Abelian algebra consisting of the series $\sum_{\mu \in \mathcal{H}^*} c_\mu e(\mu)$, where $c_\mu \in \mathbb{C}, c_\mu = 0$ for μ outside the union of a finite number of sets of the form $\mathrm{D}(\lambda) = \{\mu \in \mathcal{H}^* | \mu \geqslant \lambda\}$, using some ordering of $\mathcal{H}^*$, e.g., the lexicographic one; the formal exponents $e(\mu)$ have the properties: $e(0) = 1, e(\mu)e(\nu) = e(\mu + \nu)$.

Then the (formal) character of V is defined by

$$\mathrm{ch}_0\, V = \sum_{\mu \in \Gamma_+} (\dim V_\mu) e(\Lambda + \mu) = e(\Lambda) \sum_{\mu \in \Gamma_+} (\dim V_\mu) e(\mu) \tag{2.2}$$

(we shall use subscript «0» for the even case).

For a Verma module, i.e., $V = V^\Lambda$, one has $\dim V_\mu = P(\mu)$, where $P(\mu)$ is a generalized partition function, $P(\mu) = \#$ of ways μ can be presented as a sum of positive roots β, each root taken with its multiplicity $\dim C_\beta$ ($= 1$ here), $P(0) \equiv 1$. Thus, the character formula for Verma modules is

$$\mathrm{ch}_0\, V^\Lambda = e(\Lambda) \sum_{\mu \in \Gamma_+} P(\mu) e(\mu) = e(\Lambda) \prod_{\alpha \in \Delta^+} (1 - e(\alpha))^{-1}. \tag{2.3}$$

Further we recall the standard reflections in $\hat{\mathcal{H}}^*$:

$$s_\alpha(\lambda) = \lambda - (\lambda, \alpha^\vee)\alpha, \quad \lambda \in \hat{\mathcal{H}}^*, \quad \alpha \in \hat{\Delta}. \tag{2.4}$$

The Weyl group W is generated by the simple reflections $s_i \equiv s_{\alpha_i}$, $\alpha_i \in \hat{\pi}$. Thus every element $w \in W$ can be written as the product of simple reflections. It is said that w is written in a reduced form if it is written with the minimal possible number of simple reflections; the number of reflections of a reduced form of w is called the length of w, denoted by $\ell(w)$.

The Weyl character formula for the finite-dimensional irreducible LWM L_Λ over $\hat{\mathcal{G}}$, i.e., when $\Lambda \in -\Gamma_+$, has the form*:

$$\mathrm{ch}_0\, L_\Lambda = \sum_{w\in W} (-1)^{\ell(w)} \mathrm{ch}_0\, V^{w\cdot\Lambda}, \quad \Lambda \in -\Gamma_+, \tag{2.5}$$

where the dot $\cdot$ action is defined by $w \cdot \lambda = w(\lambda - \rho) + \rho$. For future reference we note:

$$s_\alpha \cdot \Lambda = \Lambda + n_\alpha \alpha, \tag{2.6}$$

where

$$n_\alpha = n_\alpha(\Lambda) \doteq (\rho - \Lambda, \alpha^\vee) = (\rho - \Lambda)(H_\alpha), \quad \alpha \in \Delta^+. \tag{2.7}$$

In the case of basic classical Lie superalgebras the first character formulae were given by Kac [112, 120]**. For all such superalgebras (except $osp(1/2N)$) the character formula for Verma modules is [112, 120]:

$$\mathrm{ch}\, V^\Lambda = e(\Lambda) \left(\prod_{\alpha\in\Delta^+_{\bar 0}} (1 - e(\alpha))^{-1} \right) \left(\prod_{\alpha\in\Delta^+_{\bar 1}} (1 + e(\alpha)) \right). \tag{2.8}$$

Note that the factor $\prod_{\alpha\in\Delta^+_{\bar 0}} (1 - e(\alpha))^{-1}$ represents the states of the even sector: $V_0^\Lambda \equiv U((\mathcal{G}^{\mathbb{C}}_+)_{(0)})\, v_0$ (as above in the even case), while $\prod_{\alpha\in\Delta^+_{\bar 1}} (1+e(\alpha))$ represents the states of the odd sector: $\hat{V}^\Lambda \equiv \big(U(\mathcal{G}^{\mathbb{C}}_+)/U((\mathcal{G}^{\mathbb{C}}_+)_{(0)})\big)\, v_0$. Thus, we may introduce a character for $\hat{V}^\Lambda$ as follows:

$$\mathrm{ch}\, \hat{V}^\Lambda \equiv \prod_{\alpha\in\Delta^+_{\bar 1}} (1 + e(\alpha)). \tag{2.9}$$

In our case, $\hat{V}^\Lambda$ may be viewed as a result of all possible applications of the $4N$ odd generators $X^+_{a,4+k}$ on v_0, i.e., $\hat{V}^\Lambda$ has 2^{4N} states (including the vacuum).

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*A more general character formula involves the Kazhdan–Lusztig polynomials $P_{y,w}(u)$, $y, w \in W$ [119].

** Kac considers the highest weight modules, but his results are immediately transferable to the lowest weight modules.

Explicitly, the basis of $\hat{V}^\Lambda$ may be chosen as in [103]:

$$\Psi_{\bar\varepsilon} = \left(\prod_{k=N}^{1} (X^+_{1,4+k})^{\varepsilon_{1,4+k}}\right)\left(\prod_{k=N}^{1} (X^+_{2,4+k})^{\varepsilon_{2,4+k}}\right) \times \\ \times \left(\prod_{k=1}^{N} (X^+_{3,4+k})^{\varepsilon_{3,4+k}}\right)\left(\prod_{k=1}^{N} (X^+_{4,4+k})^{\varepsilon_{4,4+k}}\right) v_0, \quad \varepsilon_{aj} = 0,1, \tag{2.10}$$

where $\bar\varepsilon$ denotes the set of all ε_{ij}*. Thus, the character of $\hat{V}^\Lambda$ may be written as

$$\mathrm{ch}\,\hat{V}^\Lambda = \sum_{\bar\varepsilon} e(\Psi_{\bar\varepsilon}) = \tag{2.11a}$$

$$= \sum_{\bar\varepsilon} \left(\prod_{k=1}^{N} e(\alpha_{1,4+k})^{\varepsilon_{1,4+k}}\right)\left(\prod_{k=1}^{N} e(\alpha_{2,4+k})^{\varepsilon_{2,4+k}}\right) \times \\ \times \left(\prod_{k=1}^{N} e(\alpha_{3,4+k})^{\varepsilon_{3,4+k}}\right)\left(\prod_{k=1}^{N} e(\alpha_{4,4+k})^{\varepsilon_{4,4+k}}\right) = \tag{2.11b}$$

$$= \sum_{\bar\varepsilon} e\left(\sum_{k=1}^{N}\sum_{a=1}^{4} \varepsilon_{a,4+k}\,\alpha_{a,4+k}\right) \tag{2.11c}$$

(note that in the above formula there is no actual dependence on Λ.)

We shall use the above to write for the character of V^Λ:

$$\mathrm{ch}\,V^\Lambda = \mathrm{ch}\,\hat{V}^\Lambda \cdot \mathrm{ch}_0\,V_0^\Lambda = \\ = \sum_{\bar\varepsilon} e\left(\sum_{k=1}^{N}\sum_{a=1}^{4} \varepsilon_{a,4+k}\,\alpha_{a,4+k}\right) e(\Lambda)\left(\prod_{\alpha\in\Delta_{\bar 0}^+} (1-e(\alpha))^{-1}\right) = \\ = \sum_{\bar\varepsilon} e\left(\Lambda + \sum_{k=1}^{N}\sum_{a=1}^{4} \varepsilon_{a,4+k}\,\alpha_{a,4+k}\right)\left(\prod_{\alpha\in\Delta_{\bar 0}^+} (1-e(\alpha))^{-1}\right) = \\ = \sum_{\bar\varepsilon} \mathrm{ch}_0\,V_0^{\Lambda + \sum_{k=1}^{N}\sum_{a=1}^{4} \varepsilon_{a,4+k}\,\alpha_{a,4+k}}, \tag{2.12}$$

*The order chosen in (2.10) was important in the proof of unitarity in [102, 103] and for that purposes one may choose also an order in which the vectors on the first row are exchanged with the vectors on the second row. For our purposes the order is important as far as to avoid impossible states — this is much of the analysis done in the next Subsections.

where $\mathrm{ch}_0\, V_0^\Lambda$ is the character obtained by restriction of V^Λ to V_0^Λ:

$$\mathrm{ch}_0\, V_0^\Lambda = e(\Lambda^z)\, \mathrm{ch}_0\, V^{\Lambda^s} \cdot \mathrm{ch}_0\, V^{\Lambda^u}, \tag{2.13}$$

where we use the decomposition $\Lambda = \Lambda^s + \Lambda^z + \Lambda^u$ from (1.18a); and V^{Λ^s}, V^{Λ^u}, resp., are Verma modules over the complexifications of $su(2,2)$, $su(N)$, resp., cf. Appendix C.

Analogously, for the factorized Verma modules $\tilde{V}^\Lambda$ the character formula is

$$\mathrm{ch}\, \tilde{V}^\Lambda = \mathrm{ch}\, \hat{V}^\Lambda \cdot \mathrm{ch}_0\, \tilde{V}_0^\Lambda = \sum_{\bar{\varepsilon}} \mathrm{ch}_0\, \tilde{V}_0^{\Lambda + \sum_{k=1}^{N} \sum_{a=1}^{4} \varepsilon_{a,4+k}\, \alpha_{a,4+k}}, \tag{2.14}$$

where $\mathrm{ch}_0\, \tilde{V}_0^\Lambda$ is the character obtained by restriction of $\tilde{V}^\Lambda$ to $\tilde{V}_0^\Lambda \equiv U((\mathcal{G}_+^{\mathbb{C}})_{(0)})$ $\widetilde{|\Lambda\rangle}$, or more explicitly:

$$\mathrm{ch}_0\, \tilde{V}_0^\Lambda = e(\Lambda^z) \mathrm{ch}_0\, L_{\Lambda^s} \cdot \mathrm{ch}_0\, L_{\Lambda^u}, \tag{2.15}$$

where we use the decomposition $\Lambda = \Lambda^s + \Lambda^z + \Lambda^u$ from (1.18a) and character formulae (C.2)–(C.4) for the irreps of the even subalgebra (from Appendix C).

Formula (2.14) represents the expansion of the corresponding superfield in components, and each component has its own even character. We see that this expansion is given exactly by the expansion of the odd character (2.11).

We have already displayed how the UIRs L_Λ are obtained as factor-modules of the (even-submodules-factorized) Verma modules $\tilde{V}^\Lambda$. Of course, this factorization means that the odd singular vectors of $\tilde{V}^\Lambda$ from (1.33) are becoming null conditions in L_Λ. However, this is not enough to determine the character formulae even when considering our UIRs as irreps of the complexification $sl(4/N)$. The latter is a well known feature even in the bosonic case. Here the situation is much more complicated and much more refined analysis is necessary.

The most important aspect of this analysis is the determination of the superfield content. (This analysis was used in [102, 103] but was not explicated enough.) This is given by the positive norm states $\hat{L}_\Lambda$ among all states in the odd sector $\hat{V}^\Lambda$. Of course, $\hat{L}_\Lambda$ may have less than 2^{4N} states.

For future use we introduce notation for the levels of the different chiralities ε_i and the overall level ε

$$\varepsilon_i = \sum_{k=1}^{N} \varepsilon_{i,4+k}, \quad i = 1,2,3,4, \qquad \varepsilon = \varepsilon_1 + \varepsilon_2 + \varepsilon_3 + \varepsilon_4. \tag{2.16}$$

The odd null conditions entwine with the even null conditions as we shall see. The even null conditions follow from the even singular vectors in (1.23) (alternatively, one may say that they carry over from the even null conditions

(1.26) of $\tilde{V}^\Lambda$). We write down the even null conditions first since they hold for any positive energy UIR:

$$(X_1^+)^{1+2j_1} \,|\Lambda\rangle = 0, \tag{2.17a}$$

$$(X_2^+)^{1+2j_2} \,|\Lambda\rangle = 0, \tag{2.17b}$$

$$(X_j^+)^{1+r_{N+4-j}} \,|\Lambda\rangle = 0, \quad j = 5, \ldots, N+3 \tag{2.17c}$$

((2.17c) being empty for $N = 1$), where by $|\Lambda\rangle$ we shall denote the lowest weight vector of the UIR L_Λ.

2.2. Character Formulae for the Long UIRs. As we mentioned if $d > d_{\max}$, there are no further reducibilities, and the UIRs $L_\Lambda = \tilde{V}^\Lambda$ are called *long* since $\hat{L}_\Lambda$ may have the maximally possible number of states 2^{4N} (including the vacuum state).

However, the actual number of states may be less than 2^{4N} states due to the fact that — depending on the values of j_a and r_k — not all actions of the odd generators on the vacuum would be allowed. The latter is obvious from formulae (1.43). Using the latter we can give the resulting signature of the state $\Psi_{\bar{\varepsilon}}$:

$$\chi\left(\Psi_{\bar{\varepsilon}}\right) = \Big[d + \frac{1}{2}\varepsilon;\; j_1 + \frac{1}{2}(\varepsilon_2 - \varepsilon_1),\; j_2 + \frac{1}{2}(\varepsilon_4 - \varepsilon_3);\; z + \\ + \epsilon_N(\varepsilon_3 + \varepsilon_4 - \varepsilon_1 - \varepsilon_2); \ldots, r_i + \varepsilon_{1,N+4-i} - \varepsilon_{1,N+5-i} + \varepsilon_{2,N+4-i} - \\ - \varepsilon_{2,N+5-i} - \varepsilon_{3,N+4-i} + \varepsilon_{3,N+5-i} - \varepsilon_{4,N+4-i} + \varepsilon_{4,N+5-i}, \ldots\Big]. \tag{2.18}$$

Thus, only if $j_1, j_2 \geqslant N/2$ and $r_i \geqslant 4$ (for all i), the number of states is 2^{4N} [102], and the character formula for the irreducible lowest weight module is (2.14):

$$\mathrm{ch}\, L_\Lambda = \mathrm{ch}\, \tilde{V}^\Lambda = \mathrm{ch}\, \hat{V}^\Lambda \cdot \mathrm{ch}_0\, \tilde{V}_0^\Lambda, \quad d > d_{\max}, \tag{2.19a}$$

$$j_1, j_2 \geqslant N/2, \quad r_i \geqslant 4, \quad i = 1, \ldots, N-1. \tag{2.20a}$$

The general formula for $\mathrm{ch}\, L_\Lambda$ shall be written in a similar fashion:

$$\mathrm{ch}\, L_\Lambda = \mathrm{ch}\, \hat{L}_\Lambda \cdot \mathrm{ch}_0\, \tilde{V}_0^\Lambda. \tag{2.21}$$

Moreover, from now on we shall write only the formulae for $\mathrm{ch}\, \hat{L}_\Lambda$. Thus, formula (2.19) shall be written equivalently as

$$\mathrm{ch}\, \hat{L}_\Lambda = \mathrm{ch}\, \hat{V}^\Lambda, \quad j_1, j_2 \geqslant N/2, \quad r_i \geqslant 4, \forall\, i. \tag{2.22}$$

As we have noted after (2.14) we do not lose information using this factorized form which has the advantage of brevity.

If the auxiliary conditions (2.19b) are not fulfilled, then a careful analysis is necessary. To simplify the exposition we classify the states by the following quantities:

$$\begin{aligned}
&\varepsilon^c_j \equiv \varepsilon_1 - \varepsilon_2, \\
&\varepsilon^a_j \equiv \varepsilon_3 - \varepsilon_4, \\
&\varepsilon^i_r \equiv \varepsilon_{1,5+i} + \varepsilon_{2,5+i} + \varepsilon_{3,4+i} + \varepsilon_{4,4+i} - \varepsilon_{1,4+i} - \varepsilon_{2,4+i} - \varepsilon_{3,5+i} - \varepsilon_{4,5+i}, \\
&\quad i = 1, \dots, N-1.
\end{aligned} \tag{2.22}$$

This gives the following necessary conditions on ε_{ij} for a state to be allowed:

$$\varepsilon^c_j \leqslant 2j_1, \tag{2.23a}$$

$$\varepsilon^a_j \leqslant 2j_2, \tag{2.23b}$$

$$\varepsilon^i_r \leqslant r_{N-i}, \quad i = 1, \dots, N-1. \tag{2.23c}$$

These conditions are also sufficient only for $N = 1$ (when (2.23c) is absent). The exact conditions are:

Criterion. The necessary and sufficient conditions for the state $\Psi_{\bar\varepsilon}$ of level ε to be allowed are that conditions (2.23) are fulfilled and that the state is a descendant of an allowed state of level $\varepsilon - 1$. $\diamondsuit$

The second part of the Criterion will take care first of all of chiral (or antichiral) states when some ε_{aj} contribute to opposing sides of the inequalities in (2.23a) and (2.23c) (or (2.23b) and (2.23c)). This phenomenon happens for $j_1 = r_i = 0$ (or $j_2 = r_i = 0$).

We shall give now the most important such occurrences. Take first chiral states, i.e., all $\varepsilon_{3,4+k} = \varepsilon_{3,4+k} = 0$. Fix $i = 1, \dots, N-1$. It is easy to see that the following states are not allowed:

$$\begin{gathered}
\psi_{ij} = \phi_{ij}\,|\Lambda\rangle = X^+_{1,i+4}\,X^+_{2,i+5}\,X^+_{a_1,i+6}\,\dots\,X^+_{a_{j-1},i+4+j}\,|\Lambda\rangle, \quad a_n = 1,2, \\
j = 1, \dots, N-i, \quad j_1 = r_{N-i} = \dots = r_{N-i-j+1} = 0, \\
\text{in addition, for } N > 2, \quad i > 1 \text{ holds } r_{N-i+1} \neq 0.
\end{gathered} \tag{2.24}$$

D e m o n s t r a t i o n. *Naturally, this statement is nontrivial only when these states are allowed by condition (2.23a) (i.e., the number of a_n being equal to 2 is not less than the number of a_n being equal to 1), thus we restrict to those. By design these states fulfil also (2.23c) ((2.23b) is not relevant) however, they are not descendants of allowed states. First, all states $\hat\psi_{ij} = X^+_{2,i+5}\,X^+_{a_1,i+6}\,\dots\,X^+_{a_{j-1},i+4+j}\,|\Lambda\rangle$ violate (2.23c) with $r_{N-i} = 0$. Next, the state ψ_{i1} is not allowed since in addition to $\hat\psi_{11}$ also the state $X^+_{1,i+4}\,|\Lambda\rangle$ is not allowed (it violates (2.23a) with $j_1 = 0$). Due to this, the state ψ_{i2} is not* *descendant of any allowed states, and so on, for all ψ_{ij}. Note that the last part of the proof trivializes unless all $a_n = 2$.* ■

Remark. *The additional condition on the last line of (2.24) is there, since if $r_{N-i+1} = 0$, the states $\psi_{ij}\,|\Lambda\rangle$ (for $i > 1$) violate (2.23c) with $r_{N-i+1} = 0$ and are excluded without use of the Criterion.* ◇

Consider now antichiral states, i.e., such that $\varepsilon_{1,4+k} = \varepsilon_{2,4+k} = 0$, for all $k = 1, \ldots, N$. Fix $i = 1, \ldots, N-1$. Then the following antichiral states are not allowed:

$$\psi'_{ij} = \phi'_{ij}\,|\Lambda\rangle = X^+_{3,i+5}\,X^+_{4,i+4}\,X^+_{b_1,i+3}\,\cdots\,X^+_{b_{j-1},i+5-j}\,|\Lambda\rangle, \quad b_n = 3,4, \tag{2.25}$$
$$j = 1, \ldots, i, \quad j_2 = r_{N-i} = \ldots = r_{N-i+j-1} = 0,$$
$$\text{in addition,} \quad \text{for} \quad N > 2,\, i > 1 \quad \text{holds} \quad r_{N-i-1} \neq 0.$$

Furthermore, any combinations of ϕ_{ij} and $\phi'_{i'j'}$ are not allowed.

Note that for $N \geqslant 4$ the states in (2.24), (2.25) do not exhaust the states forbidden by our Criterion. For example, for $N = 4$ there are the following forbidden states:

$$\psi_4 = \phi_4\,|\Lambda\rangle = X^+_{28}\,X^+_{17}\,X^+_{16}\,X^+_{25}\,|\Lambda\rangle, \quad j_1 = r_1 = r_2 = r_3 = 0, \tag{2.24$'$}$$
$$\psi'_4 = \phi'_4\,|\Lambda\rangle = X^+_{45}\,X^+_{36}\,X^+_{37}\,X^+_{48}\,|\Lambda\rangle, \quad j_2 = r_1 = r_2 = r_3 = 0. \tag{2.25$'$}$$

Summarizing the discussion so far, the general character formula may be written as follows:

$$\mathrm{ch}\,\hat{L}_\Lambda = \mathrm{ch}\,\hat{V}^\Lambda - \mathcal{R}_{\mathrm{long}}, \qquad d > d_{\max}, \tag{2.26}$$
$$\mathcal{R} = e(\hat{V}^\Lambda_{\mathrm{excl}}) = \sum_{\substack{\text{excluded}\\ \text{states}}} e(\Psi_{\bar{\varepsilon}}),$$

where the counter-terms denoted by $\mathcal{R}$ are determined by $\hat{V}^\Lambda_{\mathrm{excl}}$ which is the collection of all states (i.e., collection of ε_{jk}) which violate the conditions (2.23), or are impossible in the sense of (2.24) and/or (2.25). Of course, each excluded state is accounted for only once even if it is not allowed for several reasons*.

Finally, we consider two important conjugate special cases.

First, the chiral sector of R-symmetry scalars with $j_1 = 0$. Taking into account (2.23a), (2.23c) ((2.23b) is trivially satisfied for chiral states) and our

*We should stress that the necessity of the counter-terms above is related to the fact that our representations of $su(2,2/N)$ have physical meaning and the states of $\hat{L}_\Lambda$ represent components of a superfield. There are no counter-terms when we consider these UIRs as irreps of $sl(4/N)$. Thus, formula (2.26) and almost all character formulae derived further in this Section are character formulae of $sl(4/N)$ by just dropping the counter-term $\mathcal{R}$, cf. the next Section.

Criterion, it is easy to see that the appearance of the generators $X^+_{1,4+k}$ is restricted as follows. The generator X^+_{15} may appear only in the state

$$X^+_{15}\, X^+_{25}\, |\Lambda\rangle \tag{2.27}$$

and its descendants. The generator X^+_{16} may only appear either in states descendant to the state (2.27) or in the state

$$X^+_{16}\, X^+_{25}\, |\Lambda\rangle \tag{2.28}$$

and its descendants including only generators $X^+_{a,5+\ell}$, $a = 1, 2$, $\ell > 1$. Further, the restrictions are described recursively, namely, fix ℓ such that $1 < \ell \leqslant N - 1$. The generator $X^+_{1,5+\ell}$ may only appear either in states containing generators $X^+_{1,5+j}$, where $0 \leqslant j < \ell$, or in the state

$$X^+_{1,5+\ell}\, X^+_{2,4+\ell}\, X^+_{2,3+\ell} \cdots X^+_{2,5}\, |\Lambda\rangle \tag{2.29}$$

and its descendants including only generators $X^+_{a,5+\ell'}$, $a = 1, 2$, $\ell' > \ell$.

The chiral part of the basis is further restricted. Namely, there are only N chiral states that can be built from the generators $X^+_{2,4+k}$ alone, given as follows:

$$X^+_{2,4+k} \cdots X^+_{25}\, |\Lambda\rangle, \quad k = 1, \dots N, \quad j_1 = r_i = 0, \forall\, i. \tag{2.30}$$

This follows from (2.23c) which in this case is reduced to $\varepsilon_{1i} \leqslant \varepsilon_{1,i+1}$ for $i = 1, \dots, N - 1$.

Second, the antichiral sector of R-symmetry scalars with $j_2 = 0$. Taking into account (2.23b), (2.23c) ((2.23a) is trivially satisfied for antichiral states) and our Criterion, it is easy to see that the appearance of the generators $X^+_{3,4+k}$ is restricted as follows. The generator $X^+_{3,4+N}$ may only appear in the state

$$X^+_{3,4+N}\, X^+_{4,4+N}\, |\Lambda\rangle \tag{2.31}$$

and its descendants. The generator $X^+_{3,3+N}$ may only appear either in states descendant to the state (2.31) or in the state

$$X^+_{3,3+N}\, X^+_{4,4+N}\, |\Lambda\rangle \tag{2.32}$$

and its descendants including only generators $X^+_{a,4+N-\ell}$, $a = 3, 4$, $\ell > 1$. Further, fix ℓ such that $1 < \ell \leqslant N - 1$. The generator $X^+_{3,4+N-\ell}$ may only appear either in states containing generators $X^+_{3,4+N-j}$, where $0 \leqslant j < \ell$, or in the state

$$X^+_{3,4+N-\ell}\, X^+_{4,5+N-\ell}\, X^+_{4,6+N-\ell} \cdots X^+_{4,4+N}\, |\Lambda\rangle \tag{2.33}$$

and its descendants including only generators $X^+_{a,4+N-\ell'}$, $a = 3, 4$, $\ell' > \ell$.

The antichiral part of the basis is further restricted. Namely, there are only N antichiral states that can be built from the generators $X^+_{4,4+k}$ alone, given as follows:

$$X^+_{4,5+N-k}\, X^+_{4,6+N-k}\, \cdots\, X^+_{4,4+N}\, |\Lambda\rangle, \quad k = 1, \ldots N, \quad j_2 = r_i = 0, \forall\, i. \quad (2.34)$$

This follows from (2.23c) which for such states becomes $\varepsilon_{4,4+N-i} \leqslant \varepsilon_{4,5+N-i}$ for $i = 1, \ldots, N-1$.

2.3. Character Formulae of SRC UIRs. Here we consider the four SRC cases.

a) $d = d^1_{N1} = d^a \equiv 2 + 2j_2 + z + 2m_1 - 2m/N > d^3_{NN}$.

• Let first $j_2 > 0$. In these semishort SRC cases there holds the odd null condition (following from the singular vector (8.9a) of [101], cf. also (1.32a′), (1.33a′), (1.36a)):

$$\begin{aligned} P_{3,4+N}\, |\Lambda\rangle &= \left(X^+_4 X^+_2 (h_2 - 1) - X^+_2 X^+_4 h_2\right)\, |\Lambda\rangle = \\ &= \left(2j_2 X^+_{3,4+N} - X^+_4 X^+_2\right)\, |\Lambda\rangle = 0, \quad (2.35) \end{aligned}$$

where $X^+_{3,4+N} = [X^+_2, X^+_4]$. Clearly, condition (2.35) means that the generator $X^+_{3,4+N}$ is eliminated from the basis that is built on the lowest weight vector $|\Lambda\rangle$. Thus, for $N = 1$ and if $r_1 > 0$ for $N > 1$, the character formula is

$$\begin{gathered} \mathrm{ch}\, \hat{L}_\Lambda = \prod_{\substack{\alpha \in \Delta^+_{\bar{1}} \\ \alpha \neq \alpha_{3,4+N}}} (1 + e(\alpha)) - \mathcal{R}, \\ d = d_{\max} = d^1_{N1} > d^3_{NN}, \quad j_2 r_1 > 0. \end{gathered} \quad (2.36)$$

There are no counter-terms when $j_1 \geqslant N/2$, $j_2 \geqslant (N-1)/2$ and $r_i \geqslant 4$ (for all i), and then the number of states is 2^{4N-1}. The change of statement (as compared to the long superfields) w.r.t. j_2 comes because of the elimination of the generator $X^+_{3,4+N}$.

R e m a r k. *For the finite-dimensional irreps of sl(4/N) (in fact, of all basic classical Lie superalgebras) such situations are called «singly atypical» and the character formulae look exactly as (2.36) with* $\mathcal{R} = 0$, *cf. [121–123]**. ◇

When there are no counter-terms (also for the complex $sl(4/N)$ case) this formula follows easily from (1.42). Indeed, in the case at hand $I^\beta = I^1$ (cf. (1.36a)); then from $L_\Lambda = \tilde{V}^\Lambda / I^1$ follows:

$$\mathrm{ch}\, L_\Lambda = \mathrm{ch}\, \tilde{V}^\Lambda - \mathrm{ch}\, I^1, \quad \text{or equivalently, } \mathrm{ch}\, \hat{L}_\Lambda = \mathrm{ch}\, \hat{V}^\Lambda - \mathrm{ch}\, \hat{I}^1, \quad (2.37)$$

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*For character formulae of finite-dimensional irreps beyond the «singly atypical» case cf. [124–127] and references therein.

where $\hat{I}^1$ is the projection of I^1 to the odd sector. Naively, the character of $\hat{I}^1$ should be given by the character of $\hat{V}^{\Lambda+\alpha_{3,4+N}}$, however, as discussed in general (cf. (1.9)), I^1 is smaller than $\hat{V}^{\Lambda+\alpha_{3,4+N}}$ and its character is given with a prefactor*:

$$\mathrm{ch}\,\hat{I}^1 = \frac{1}{1+e(\alpha_{3,4+N})}\mathrm{ch}\,\hat{V}^{\Lambda+\alpha_{3,4+N}} = \frac{e(\alpha_{3,4+N})}{1+e(\alpha_{3,4+N})}\mathrm{ch}\,\hat{V}^{\Lambda}. \tag{2.38}$$

Now (2.36) (with $\mathcal{R}=0$) follows from the combination of (2.37) and (2.38).

Formula (2.36) may also be described by using the odd reflection (1.11) with $\beta = \alpha_{3,4+N}$:

$$\mathrm{ch}\,\hat{L}_\Lambda = \mathrm{ch}\,\hat{V}^{\Lambda} - \frac{1}{1+e(\alpha_{3,4+N})}\mathrm{ch}\,\hat{V}^{\hat{s}_{\alpha_{3,4+N}}\cdot\Lambda} - \mathcal{R} = \tag{2.39a}$$

$$= \mathrm{ch}\,\hat{V}^{\Lambda} - \hat{s}_{\alpha_{3,4+N}} \cdot \mathrm{ch}\,\hat{V}^{\Lambda} - \mathcal{R} = \tag{2.39b}$$

$$= \sum_{\hat{s}\in\hat{W}_{\alpha_{3,4+N}}} (-1)^{\ell(\hat{s})}\hat{s}\cdot\mathrm{ch}\,\hat{V}^{\Lambda} - \mathcal{R}, \tag{2.39c}$$

where $\hat{W}_\beta \equiv \{1, \hat{s}_\beta\}$ is a two-element semigroup restriction of $\tilde{W}_\beta$, and we have formalized further by introducing notation for the action of an odd reflection on characters:

$$\hat{s}_\beta \cdot \mathrm{ch}\,V^{\Lambda} = \frac{1}{1+e(\beta)}\mathrm{ch}\,V^{\hat{s}_\beta\cdot\Lambda} = \frac{1}{1+e(\beta)}\mathrm{ch}\,V^{\Lambda+\beta} = \frac{e(\beta)}{1+e(\beta)}\mathrm{ch}\,V^{\Lambda}. \tag{2.40}$$

It is natural to introduce the restriction $\hat{W}_\beta$ since only the identity element of $\tilde{W}_\beta$ and the generator $\hat{s}_\beta$ act nontrivially because the action $\hat{s}_\beta$ on characters is nilpotent:

$$(\hat{s}_\beta)^2 \cdot \mathrm{ch}\,V^{\Lambda} = 0. \tag{2.41}$$

(*This is because the odd embeddings are nilpotent, cf. (1.9), and the action of* $(\hat{s}_\beta)^n, n<0$ *on the characters is also trivial, since the embeddings are in the opposite direction, e.g.,* V^Λ *is oddly embedded in* $V^{\Lambda-\beta}$*, cf. [100] and (1.45).*)

In fact, we shall need more general formula for the action of odd reflections on polynomials $\mathcal{P}$ from $E(\mathcal{H}^*)$. Thus, instead of (2.40) we shall define the action of $\hat{s}_\beta$ on $\mathcal{P}$ as a homogeneity operator treating $e(\beta)$ as a variable

$$\hat{s}_\beta \cdot \mathcal{P} \equiv e(\beta)\,\frac{\partial}{\partial e(\beta)}\,\mathcal{P}, \tag{2.42}$$

*This technique was applied first when deriving the characters of the $N=2$ super-Virasoro algebras, cf. [128].

where β may be a root or the sum of roots. Obviously, if $\mathcal{P}$ is a monomial which contains a multiplicative factor $1 + e(\beta)$, this action is equivalent to the action in (2.40), but it is more general since it acts on arbitrary polynomials $\mathcal{P}$ which we need to describe our results below.

In particular, we shall show that in many cases character formulae (2.36), (2.39) may be written as follows:

$$\mathrm{ch}\,\hat{L}_\Lambda = \sum_{\hat{s}\in\hat{W}_\beta} (-1)^{\ell(\hat{s})}\hat{s}\left(\mathrm{ch}\,\hat{V}^\Lambda - \mathcal{R}_{\mathrm{long}}\right), \tag{2.43}$$

where $\mathcal{R}_{\mathrm{long}}$ represents the counter-terms for the long superfields for the same values of j_1 and r_i as Λ, while the value of j_2 is zero when j_2 from Λ is zero, otherwise it has to be the generic value $j_2 \geqslant N/2$. (As we know, restriction (2.23b) trivializes for $j_2 \geqslant N/2$ and thus the structure of the irrep is the same for any such generic value.)

Writing (2.36) as (2.39) (or (2.43)) may look as a complicated way to describe the cancellation of a factor from the character formula for $\hat{V}^\Lambda$, however, first of all it is related to the structure of $\tilde{V}^\Lambda$ given by (1.42) and furthermore may be interpreted — when there are no counter-terms — as the following decomposition:

$$\hat{V}^\Lambda = \hat{L}_\Lambda \oplus \hat{L}_{\Lambda+\beta}, \tag{2.44}$$

for $\beta = \alpha_{3,4+N}$. Indeed, for generic signatures $\hat{L}_{\Lambda+\beta}$ is isomorphic to $\hat{L}_\Lambda$ as a vector space (this is due to the fact that $V^{\Lambda+\beta}$ has the same reducibilities as V^Λ, cf. Sec. 1), they differ only by the vacuum state. Thus, when there are no counter-terms, both $\hat{L}_\Lambda$ and $\hat{L}_{\Lambda+\beta}$ have the same 2^{4N-1} states. If we describe them for shortness as

$$\Phi_i\,|\Lambda\rangle, \qquad \Phi_i\,|\Lambda+\beta\rangle, \tag{2.45}$$

where none of Φ_i contains $X^+_{3,4+N}$ and recall that the embedding of $\hat{V}^{\Lambda+\beta}$ into $\hat{V}^\Lambda$ is given essentially by the generator $X^+_{3,4+N}$ (cf. (1.36a)), then we see that after the embedding the states in (2.45) restore all 2^{4N} states in $\hat{V}^\Lambda$:

$$\Phi_i\,|\widetilde{\Lambda}\rangle, \qquad X^+_{3,4+N}\,\Phi_i\,|\widetilde{\Lambda}\rangle. \tag{2.46}$$

It is more important that there is a similar decomposition valid for many cases beyond the generic, i.e., we have

$$\left(\hat{L}_{\mathrm{long}}\right)\Big|_{d=d^a} = \hat{L}_\Lambda \oplus \hat{L}_{\Lambda+\alpha_{3,4+N}}, \qquad N=1 \text{ or } r_1>0 \text{ for } N>1, \tag{2.47}$$

where $\hat{L}_{\mathrm{long}}$ is a long superfield with the same values of j_1 and r_i as Λ, while the value of j_2 has to be specified as above for $\mathcal{R}_{\mathrm{long}}$, and equality is as vector spaces.

For $N > 1$, there are possible additional truncations of the basis. To make the exposition easier we need additional notation. Let i_0 be an integer such that $0 \leqslant i_0 \leqslant N-1$, and $r_i = 0$ for $i \leqslant i_0$, and if $i_0 < N-1$, then $r_{i_0+1} > 0$*.

Let now $N > 1$ and $i_0 > 0$; then the generators $X^+_{3,4+N-i}$, $i = 1, \ldots, i_0$, are eliminated from the basis.

D e m o n s t r a t i o n. *First, we consider the vector*

$$P_{3,3+N}\, v_0 = \left(2j_2 X^+_{3,3+N} - X^+_{4,3+N}\, X^+_2\right) v_0 = \\ = 2j_2 \left(X^+_{3,4+N} X^+_{3+N} - X^+_{3+N} X^+_{3,4+N}\right) v_0 - \\ - \left(X^+_4 X^+_{3+N} - X^+_{3+N} X^+_4\right) X^+_2\, v_0 = \left(P_{3,4+N}\, X^+_{3+N} - X^+_{3+N}\, P_{3,4+N}\right) v_0. \tag{2.48}$$

For $r_1 = 0$, it is descendant of (1.23c) and (1.32) and leads to the null condition

$$P_{3,3+N}\, |\Lambda\rangle = \left(2j_2 X^+_{3,3+N} - X^+_{4,3+N}\, X^+_2\right) |\Lambda\rangle = 0, \tag{2.49}$$

which naturally follows from (2.17c) and (2.35), and which means that the generator $X^+_{3,3+N}$ is eliminated from the basis. Analogously, we define the vectors

$$P_{3,4+N-i}\, v_0 = \left(2j_2 X^+_{3,4+N-i} - X^+_{4,4+N-i} X^+_2\right) v_0, \tag{2.50}$$

which are recursively related:

$$P_{3,4+N-i}\, v_0 = 2j_2 \left(X^+_{3,5+N-i} X^+_{4+N-i} - X^+_{4+N-i} X^+_{3,5+N-i}\right) v_0 - \\ - \left(X^+_{4,5+N-i} X^+_{4+N-i} - X^+_{4+N-i} X^+_{4,5+N-i}\right) X^+_2 v_0 = \\ = \left(P_{3,5+N-i} X^+_{4+N-i} - X^+_{4+N-i}\, P_{3,5+N-i}\right) v_0. \tag{2.51}$$

Thus, in the situation: $r_i = 0$, $i = 1, \ldots, i_0$, there are the following null conditions:

$$P_{3,4+N-i}\, |\Lambda\rangle = \left(2j_2 X^+_{3,4+N-i} - X^+_{4,4+N-i}\, X^+_2\right) |\Lambda\rangle = 0, \ \ r_j = 0, 1 \leqslant j \leqslant i \leqslant i_0. \tag{2.52}$$

These are recursively descendant null conditions, which means that a condition for fixed i is a descendant of the one for $i-1$ (since $X^+_{4+N-i}\, |\Lambda\rangle = 0$ due to (2.17c)). Conditions (2.52) mean that the generators $X^+_{3,4+N-i}$, $i = 1, \ldots, i_0$, are eliminated from the basis. ■

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*This is formally valid for $N = 1$ with $i_0 = 0$ since $r_0 \equiv 0$ by convention. This shall be used to make certain statements valid for general N.

From the above follows that for $i_0 > 0$ the decomposition (2.47) cannot hold. Indeed, the generators $X^+_{3,4+N-i}$, $i = 1, \ldots, i_0$, are eliminated from the irrep $\hat{L}_\Lambda$ due to the fact that we are at a reducibility point, but there is no reason for them to be eliminated from the long superfield. Certainly, some of these generators are present in the second term $\hat{L}_{\Lambda+\alpha_{3,4+N}}$ in (2.47), but that would be only those which in the long superfield were in states of the kind: $\Phi\, X^+_{3,4+N}\, |\Lambda\rangle$, and, certainly, such states do not exhaust the occurrence of the discussed generators in the long superfield. Symbolically, instead of the decomposition (2.47) we shall write:

$$\left(\hat{L}_{\text{long}}\right)\Big|_{d=d^a} = \hat{L}_\Lambda \;\oplus\; \hat{L}_{\Lambda+\alpha_{3,4+N}} \oplus\; \hat{L}'_\Lambda, \quad N > 1, \quad i_0 > 0, \tag{2.53}$$

where we have represented the excess states by the last term with prime. With the prime we stress that this is not a genuine irrep, but just a book-keeping device. Formulae as (2.53) in which not all terms are genuine irreps shall be called *quasi-decompositions*.

The corresponding character formula is

$$\text{ch}\,\hat{L}_\Lambda = \prod_{\substack{\alpha\in\Delta^+_{\bar{1}} \\ \alpha\neq\alpha_{3,5+N-k} \\ k=1,\ldots,1+i_0}} (1+e(\alpha)) - \mathcal{R} = \tag{2.54a}$$

$$= \sum_{\hat{s}\in\hat{W}^a_{i_0}} (-1)^{\ell(\hat{s})}\; \hat{s}\cdot \text{ch}\, V^\Lambda - \mathcal{R} = \tag{2.54b}$$

$$= \sum_{\hat{s}\in\hat{W}^a_{i_0}} (-1)^{\ell(\hat{s})}\; \hat{s}\cdot \left(\text{ch}\,\hat{V}^\Lambda - \mathcal{R}_{\text{long}}\right), \tag{2.54c}$$

$$\hat{W}^a_{i_0} \equiv \hat{W}_{\alpha_{3,N+4}} \times \hat{W}_{\alpha_{3,N+3}} \times \cdots \times \hat{W}_{\alpha_{3,N+4-i_0}}, \tag{2.54d}$$

$$d = d_{\max} = d^1_{N1} > d^3_{NN}, \quad j_2 > 0, \quad r_i = 0, \quad i \leqslant i_0.$$

The restrictions (2.23) used to determine the counter-terms are, of course, with $\varepsilon_{3,5+N-k} = 0$, $k = 1, \ldots, 1+i_0$. Formulae (2.36), (2.39), (2.43) are special cases of (2.54a)–(2.54c), resp., for $i_0 = 0$. The maximal number of states in $\hat{L}_\Lambda$ is 2^{4N-1-i_0}. This is the number of states that is obtained from the action of the Weyl group $\hat{W}^a_{i_0}$ on $\text{ch}\,\hat{V}^\Lambda$, while the actual counter-term is obtained from the action of the Weyl group on $\mathcal{R}_{\text{long}}$.

In the extreme case of R-symmetry scalars: $i_0 = N-1$, i.e., $r_i = 0$, $i = 1, \ldots, N-1$, or, equivalently, $m_1 = 0 = m$, all the N generators $X^+_{3,4+k}$ are eliminated. The character formula is again (2.54) taken with $i_0 = N-1$.

• Let now $j_2 = 0$. Then all null conditions above (valid for $j_2 > 0$) follow from (1.26b), so these conditions do not mean elimination of the mentioned vectors. As we know in this situation we have the singular vector (1.35) which leads to the following null condition:

$$X^+_{3,4+N}\, X^+_{4,4+N}\, |\Lambda\rangle = X^+_4\, X^+_2\, X^+_4\, |\Lambda\rangle = 0. \tag{2.55}$$

The state in (2.55) and all of its 2^{4N-2} descendants are zero for any N. Thus, the character formula is similar to (2.39), but with $\alpha_{3,4+N}$ is replaced by $\beta_{12} = \alpha_{3,4+N} + \alpha_{4,4+N}$, (cf. (1.43e)):

$$\mathrm{ch}\,\hat{L}_\Lambda = \sum_{\hat{s}\in\hat{W}_{\beta_{12}}} (-1)^{\ell(\hat{s})}\ \hat{s}\,\mathrm{ch}\,\hat{V}^\Lambda - \mathcal{R} = \tag{2.56a}$$

$$= \sum_{\hat{s}\in\hat{W}_{\beta_{12}}} (-1)^{\ell(\hat{s})}\ \hat{s}\left(\mathrm{ch}\,\hat{V}^\Lambda - \mathcal{R}_{\mathrm{long}}\right), \quad N = 1 \text{ or } r_1 > 0, \tag{2.56b}$$

where $\hat{W}_{\beta_{12}} \equiv \{1, \beta_{12}\}$.

Note that for $N = 1$ formula (2.56) is equivalent to (2.36) since due to (2.23b) the generator $X^+_{3,4+N}$ could appear only together with $X^+_{4,4+N}$ but the resulting state (2.55) is zero.

Here holds a decomposition similar to (2.47):

$$\left(\hat{L}_{\mathrm{long}}\right)\Big|_{d=d^a} = \hat{L}_\Lambda \oplus \hat{L}_{\Lambda+\beta_{12}}, \qquad N = 1 \text{ or } r_1 > 0 \text{ for } N > 1, \tag{2.57}$$

where $\hat{L}_{\mathrm{long}}$ is with the same values of $j_1, j_2 (= 0), r_i$ as Λ. Note, however, that the UIR $\hat{L}_{\Lambda+\beta_{12}}$ belongs to type b) below.

There are more eliminations for $N > 1$ when $i_0 > 0$. For instance, we can show that all states as in (2.33) considered for $\ell = 1, \ldots, i_0$ are not allowed.

D e m o n s t r a t i o n. *We show this by induction. Consider first the case* $\ell = 1$*:*

$$X^+_{3,3+N}\, X^+_{4,4+N}\, |\Lambda\rangle = \left(X^+_{3+N} X^+_{3,4+N} - X^+_{3,4+N} X^+_{3+N}\right)\, X^+_{4,4+N}\, |\Lambda\rangle =$$
$$= -X^+_{3,4+N}\, X^+_{3+N}\, X^+_{4,4+N}\, |\Lambda\rangle = -X^+_{3,4+N}\, X^+_{4,3+N}\, |\Lambda\rangle, \tag{2.58}$$

where the first term is zero due to (2.55), and the second term is transformed by pulling X^+_{3+N} *to the right, where it annihilates the vacuum (due to (2.17c) with* $j = N + 3$ *for* $r_1 = 0$*), and the resulting state is the forbidden* $\psi'_{N-1,1}$ *from (2.25). Thus, the above state is not allowed.*

Now fix k *such that* $1 < k \leqslant i_0$ *and suppose that we have already shown that all states in (2.33) for* $\ell < k$ *are not allowed, and we shall show this for* $\ell = k$*.*

Indeed, this state is not allowed:

$$X^+_{3,4+N-k}\,X^+_{4,5+N-k}\,X^+_{4,6+N-k}\,\cdots\,X^+_{4,4+N}\,|\Lambda\rangle = \\ = -X^+_{4,5+N-k}\,X^+_{3,4+N-k}\,X^+_{4,6+N-k}\,\cdots\,X^+_{4,4+N}\,|\Lambda\rangle = \\ = -X^+_{4,5+N-k}\left(X^+_{4+N-k}X^+_{3,5+N-k} - X^+_{3,5+N-k}X^+_{4+N-k}\right)\times \\ \times\, X^+_{4,6+N-k}\,\cdots\,X^+_{4,4+N}\,|\Lambda\rangle, \quad (2.59)$$

where the first term on the last line is a state descendant of (2.33) with $k \to k-1$, *which is not allowed by the induction hypothesis and the second term is zero due to pulling* X^+_{4+N-k} *to the right, where it annihilates the vacuum (due to (2.17c) with* $j = N+4-k$ *for* $r_k = 0$*).* ■

From the above follows that if $i_0 > 0$ the decomposition (2.57) does not hold. Instead, there is a quasi-decomposition similar to (2.53).

We can be more explicit in the case when all $r_i = 0$. In that case all the vectors $X^+_{3,5+N-k}$ are eliminated from all antichiral states.

D e m o n s t r a t i o n. *We show this by induction in* k *starting with* $k = 1, 2$. *Take first the generator* $X^+_{3,4+N}$. *As we know, when* $j_2 = r_i = 0$, $\forall\, i$, *the only antichiral state containing it in a long superfield is the state (2.31) and its descendants. However, here all these possible states are zero due to (2.55). Thus, there are no antichiral states containing* $X^+_{3,4+N}$.

Take next the vector $X^+_{3,3+N}$. *As we know, the only antichiral states containing it in a long superfield are the states (2.31), (2.32), and their descendants. The first is zero, while the second is not allowed as we showed above. Thus, the vector* $X^+_{3,3+N}$ *is eliminated from all antichiral states.*

Now fix ℓ *such that* $1 < \ell \leqslant N-1$ *and suppose that we have already shown elimination of* $X^+_{3,5+N-k}$ *for* $k = 1, \ldots, \ell$, *from all antichiral states. We want to show elimination for* $k = \ell + 1$, *i.e., of the generator* $X^+_{3,4+N-\ell}$. *As we know from the similar consideration of long superfields, all antichiral states including* $X^+_{3,4+N-\ell}$ *and which are not yet excluded may be written as the state (2.33) and its descendants including only generators* $X^+_{a,4+N-\ell'}$, $a = 3, 4$, $\ell' > \ell$. *However, above we have shown that this state is not allowed. Thus, all generators* $X^+_{3,4+k}$ *for* $k = 1, \ldots, N$ *are eliminated from the antichiral part of the basis.* ■

The antichiral part of the basis is further restricted. As we know, when $j_2 = r_i = 0$, $\forall\, i$, there are only N antichiral states that can be built from the generators $X^+_{4,4+k}$ alone, given in (2.34). Thus the corresponding character formula is

$$\operatorname{ch}\hat{L}_\Lambda = \sum_{k=1}^{N}\prod_{i=1}^{k} e(\alpha_{4,5+N-i}) + \prod_{\substack{\alpha\in\Delta_{\bar{1}}^{+}\\ \varepsilon_1+\varepsilon_2>0}} (1+e(\alpha)) - \mathcal{R}, \tag{2.60}$$
$$d = d_{\max} = d_{N1}^{1} > d_{NN}^{3}, \quad j_2 = 0, \quad r_i = 0, \forall\, i.$$

b) $d = d_{N1}^{2} = z + 2m_1 - 2m/N > d_{NN}^{3}, \quad j_2 = 0.$

In these short single-reducibility-condition cases, there holds the odd null condition (following from the singular vector (1.32b) or (1.33b))

$$X_4^+ \,|\Lambda\rangle = X_{4,4+N}^+ \,|\Lambda\rangle = 0. \tag{2.61}$$

Since $j_2 = 0$ from (1.26b) and (1.61) follows the additional null condition:

$$X_{3,4+N}^+ \,|\Lambda\rangle = [X_2^+, X_4^+]\,|\Lambda\rangle = 0. \tag{2.62}$$

For $N > 1$ and $r_1 > 2$ each of these UIRs enters as the second term in decomposition (2.57), when the first term is an UIR of type **a** with $j_2 = 0$, as explained above.

Further, for $N > 1$ there are additional null conditions if $r_i = 0$, $i \leqslant i_0$. Indeed, let $r_1 = 0$, then from (1.26c) and (2.62) follow the additional null conditions:

$$X_{4,3+N}^+ \,|\Lambda\rangle = [X_{4,4+N}^+, X_{3+N}^+]\,|\Lambda\rangle = 0, \quad r_1 = 0, \tag{2.63a}$$
$$X_{3,3+N}^+ \,|\Lambda\rangle = [X_{3,4+N}^+, X_{3+N}^+]\,|\Lambda\rangle = 0, \quad r_1 = 0. \tag{2.63b}$$

Analogously, in the situation: $r_i = 0$, $i = 1, \ldots, i_0$, there are recursive null conditions:

$$X_{3,4+N-i}^+ \,|\Lambda\rangle = [X_{3,5+N-i}^+, X_{4+N-i}^+]\,|\Lambda\rangle = 0, \quad r_j = 0, \quad 1 \leqslant j \leqslant i \leqslant i_0, \tag{2.64a}$$
$$X_{4,4+N-i}^+ \,|\Lambda\rangle = [X_{4,5+N-i}^+, X_{4+N-i}^+]\,|\Lambda\rangle = 0, \quad r_j = 0, \quad 1 \leqslant j \leqslant i \leqslant i_0, \tag{2.64b}$$

Thus, $2(1+i_0)$ generators $X_{3,5+N-k}^+$, $X_{4,5+N-k}^+$, $k = 1, \ldots, 1+i_0$ are eliminated. The maximal number of states in $\hat{L}_\Lambda$ is $2^{4N-2-2i_0}$.

The corresponding character formula is

$$\mathrm{ch}\,\hat{L}_\Lambda = \prod_{\substack{\alpha\in\Delta_{\bar 1}^+ \\ \alpha\neq\alpha_{j,5+N-k} \\ j=3,4\,,k=1,\ldots,1+i_0}} (1+e(\alpha)) - \mathcal{R} = \tag{2.65a}$$

$$= \sum_{\hat{s}\in\hat{W}^b_{i_0}} (-1)^{\ell(\hat{s})}\hat{s}\cdot\mathrm{ch}\,\hat{V}^\Lambda - \mathcal{R}, \tag{2.65b}$$

$$\begin{aligned} \hat{W}^b_{i_0} &\equiv \hat{W}_{\alpha_{3,N+4}}\times\hat{W}_{\alpha_{3,N+3}}\times\cdots\times\hat{W}_{\alpha_{3,N+4-i_0}}\times \\ &\times\hat{W}_{\alpha_{4,N+4}}\times\hat{W}_{\alpha_{4,N+3}}\times\cdots\times\hat{W}_{\alpha_{4,N+4-i_0}}, \\ d &= d^2_{N1} > d^3_{NN}, \quad j_2=0, \quad r_i=0, \quad i\leqslant i_0, \end{aligned} \tag{2.65c}$$

where determining the counter-terms we use $\varepsilon_{j,5+N-k}=0$, $j=3,4$, $k=1,\ldots,1+i_0$.

In the case of R-symmetry scalars ($i_0=N-1$) we have

$$X^+_{3,4+k}\,|\Lambda\rangle = 0, \quad X^+_{4,4+k}\,|\Lambda\rangle = 0, \qquad k=1,\ldots,N, \quad r_i=0, \forall\, i. \tag{2.66}$$

The character formula is (2.65) taken with $1+i_0=N$. These UIRs should be called chiral since all antichiral generators are eliminated.

The next two cases are conjugates of the first two and the exposition will be compact.

c) $d=d^3_{NN}=d^c\equiv 2+2j_1-z+2m/N > d^1_{N1}$.

• Let first $j_1>0$. In these semishort SRC cases, the odd null condition holds (following from the singular vector (8.8a) of [101]), here cf. (1.32c′) or (1.33c′)):

$$P_{15}\,|\Lambda\rangle = \left(2j_1X^+_{15} - X^+_3X^+_1\right)\,|\Lambda\rangle = 0, \tag{2.67}$$

where $X^+_{15}=[X^+_1,X^+_3]$. Clearly, condition (2.67) means that the generator X^+_{15} is eliminated from the basis.

Let now i'_0 be an integer such that $0\leqslant i'_0\leqslant N-1$, and $r_{N-i}=0$ for $i\leqslant i'_0$, and if $i'_0<N-1$, then $r_{N-1-i'_0}>0$*. For $N>1$ and $i'_0>0$ there are additional truncations due to the vectors (cf. (C.7) of [101]):

$$\begin{aligned} P_{1,5+i}\,v_0 &= \left(2j_1X^+_{1,5+i} - X^+_{2,5+i}\right)X^+_1\,v_0 = 2j_1\left(X^+_{1,4+i}X^+_{4+i} - X^+_{4+i}X^+_{1,4+i}\right)\times \\ &\times v_0 - \left(X^+_{2,4+i}X^+_{4+i} - X^+_{4+i}X^+_{2,4+i}\right)X^+_1\,v_0 = \left(P_{1,4+i}\,X^+_{4+i} - X^+_{4+i}\,P_{1,4+i}\right)v_0, \end{aligned} \tag{2.68}$$

*This is formally valid for $N=1$ with $i'_0=0$ since $r_N=0$ by convention.

which produced recursive null conditions:

$$P_{1,5+i}\,|\Lambda\rangle = \left(2j_1 X^+_{1,5+i} - X^+_{2,5+i} X^+_1\right)\,|\Lambda\rangle = 0, \quad r_{N-j} = 0, \quad 1 \leqslant j \leqslant i \leqslant i'_0, \tag{2.69}$$

which means that the generators $X^+_{1,5+i}$ are eliminated from the basis.

The corresponding character formula is

$$\mathrm{ch}\,\hat{L}_\Lambda = \prod_{\substack{\alpha\in\Delta_{\bar{1}}^+ \\ \alpha\neq\alpha_{1,4+k} \\ k=1,\ldots,1+i'_0}} (1+e(\alpha)) - \mathcal{R} = \tag{2.70a}$$

$$= \sum_{\hat{s}\in\hat{W}^c_{i'_0}} (-1)^{\ell(\hat{s})}\ \hat{s}\cdot \mathrm{ch}\,\hat{V}^\Lambda - \mathcal{R} = \tag{2.70b}$$

$$= \sum_{\hat{s}\in\hat{W}^c_{i'_0}} (-1)^{\ell(\hat{s})}\ \hat{s}\cdot \left(\mathrm{ch}\,\hat{V}^\Lambda - \mathcal{R}_{\mathrm{long}}\right), \tag{2.70c}$$

$$\hat{W}^c_{i'_0} \equiv \hat{W}_{\alpha_{15}} \times \hat{W}_{\alpha_{16}} \times \cdots \times \hat{W}_{\alpha_{1,5+i'_0}}, \tag{2.70d}$$

$$d = d_{\max} = d^3_{NN} > d^1_{N1}, \quad j_1 > 0, \quad r_{N-i} = 0, \quad i \leqslant i'_0 \leqslant N-1.$$

This formula is valid also for $N = 1$ or when $r_{N-1} > 0$ by setting $i'_0 = 0$. The maximal number of states in $\hat{L}_\Lambda$ is $2^{4N-1-i'_0}$. The restrictions (2.23) used for the counter-terms are with $\varepsilon_{1,N-i} = 0$, $i = 0, 1, \ldots, i'_0$.

When $i'_0 = 0$, decomposition similar to (2.47) holds:

$$\left(\hat{L}_{\mathrm{long}}\right)\Big|_{d=d^c} = \hat{L}_\Lambda \oplus \hat{L}_{\Lambda+\alpha_{15}}, \quad N = 1 \text{ or } r_{N-1} > 0 \text{ for } N > 1, \tag{2.71}$$

where $\hat{L}_{\mathrm{long}}$ is a long superfield with the same values of j_2 and r_i as Λ, while the value of j_1 is zero when j_1 from Λ is zero, otherwise it has to be the generic value $j_1 \geqslant N/2$. From the above follows also that when $i'_0 > 0$, the decomposition (2.71) does not hold.

In the case of R-symmetry scalars ($i'_0 = N-1$) all the N generators $X^+_{1,4+k}$ ($k = 1, \ldots, N$) are eliminated. The maximal number of states in $\hat{L}_\Lambda$ is 2^{3N}.

• Let now $j_1 = 0$. Then the null conditions above all follow from (1.26a) so these conditions do not mean elimination of the mentioned vectors. In this situation we have the singular vector (1.34) which leads to the following null condition:

$$X^+_{15}\, X^+_{25}\,|\Lambda\rangle = X^+_3\, X^+_1\, X^+_3\,|\Lambda\rangle = 0. \tag{2.72}$$

The state in (2.72) and all of its 2^{4N-2} descendants are zero for any N. Thus, for $N = 1$ or if $r_{N-1} > 0$, the character formula is as (2.70) for $i'_0 = 0$, but with

α_{15} replaced by $\beta_{34} = \alpha_{25} + \alpha_{25}$ (cf. (1.43f)):

$$\mathrm{ch}\,\hat{L}_\Lambda = \sum_{\hat{s}\in\hat{W}_{\beta_{34}}} (-1)^{\ell(\hat{s})}\ \hat{s}\cdot \mathrm{ch}\,\hat{V}^\Lambda - \mathcal{R} = \tag{2.73a}$$

$$= \sum_{\hat{s}\in\hat{W}_{\beta_{34}}} (-1)^{\ell(\hat{s})}\ \hat{s}\cdot \left(\mathrm{ch}\,\hat{V}^\Lambda - \mathcal{R}_{\mathrm{long}}\right), \quad N=1 \text{ or } r_{N-1}>0, \tag{2.73b}$$

where $\hat{W}_{\beta_{34}} \equiv \{1, \beta_{34}\}$.

For $N = 1$ formula (2.73) is equivalent to (2.70) for $i'_0 = 0$ since due to (2.23a) the generator X^+_{15} could appear only together with X^+_{25} but the resulting state (2.72) is zero.

For $i'_0 = 0$ there holds the decomposition

$$\left(\hat{L}_{\mathrm{long}}\right)\Big|_{d=d^c} = \hat{L}_\Lambda \oplus \hat{L}_{\Lambda+\beta_{34}}, \qquad N=1 \text{ or } r_{N-1}>0 \text{ for } N>1, \tag{2.74}$$

where $\hat{L}_{\mathrm{long}}$ is with the same values of $j_1(=0), j_2, r_i$ as Λ. Note, however, that the UIR $\hat{L}_{\Lambda+\beta_{34}}$ belongs to type d) below.

There are more eliminations for $N > 1$ when $i'_0 > 0$. For instance, we can show that all states as in (2.29) considered for $\ell = 1, \ldots, i'_0$ are not allowed.

D e m o n s t r a t i o n. *We show this by induction. Consider first the case $\ell = 1$:*

$$X^+_{16}\, X^+_{25}\, |\Lambda\rangle = \left(X^+_{15} X^+_5 - X^+_5 X^+_{15}\right)\, X^+_{25}\, |\Lambda\rangle = \\ = X^+_{15}\, X^+_5\, X^+_{25}\, |\Lambda\rangle = X^+_{15}\, X^+_{26}\, |\Lambda\rangle, \tag{2.75}$$

where the second term is zero due to (2.72) and the first term is transformed by pulling X^+_5 to the right, where it annihilates the vacuum (due to (2.17c) with $j = 1$ for $r_{N-1} = 0$), and the resulting state is the forbidden ψ_{11}. Thus, the above state is not allowed. Further, we proceed by induction similarly to the conjugate case, cf. (2.59). ■

From the above follows that when $i'_0 > 0$, the decomposition (2.74) does not hold.

We can be more explicit in the case when all $r_i = 0$. In that case all the generators $X^+_{1,4+k}$ (for $k = 1, \ldots, N$) are eliminated from all chiral states.

D e m o n s t r a t i o n. *Take first the vector X^+_{15}. As we know, when $j_1 = r_i = 0$, $\forall\, i$, the only chiral state containing it in a long superfield is the state (2.27) and its descendants. However, here all these possible states are zero due to (2.72). Thus, there are no chiral states containing X^+_{15}.*

Take next the vector X^+_{16}. As we know, the only chiral states containing it in a long superfield are the states (2.27), (2.28) and their descendants. The first is zero, while the second is not allowed as we showed above. Thus, the vector X^+_{16} is eliminated from all chiral states.

Now fix ℓ such that $1 < \ell \leqslant N-1$ and suppose that we have already shown elimination of $X^+_{1,4+k}$ for $k = 1, \dots, \ell$, from all antichiral states. We want to show elimination of $X^+_{1,4+k}$ for $k = \ell + 1$. As we know from the similar consideration of long superfields all chiral states including $X^+_{1,5+\ell}$ and which are not yet excluded may be written as the state (2.29) and its descendants including only generators $X^+_{a,5+\ell'}$, $a = 1, 2$, $\ell' > \ell$. Then it is shown (analogously to (2.33)) that this state is also not allowed. Thus, all generators $X^+_{1,4+k}$ for $k = 1, \dots, N$ are eliminated from the chiral part of the basis. ■

The chiral part of the basis is further restricted. As we know, when $j_1 = r_i = 0$, $\forall\, i$, there are only N chiral states that can be built from the generators $X^+_{2,4+k}$ alone, given in (2.30). Thus, the corresponding character formula is

$$\operatorname{ch} \hat{L}_\Lambda = \sum_{k=1}^{N} \prod_{i=1}^{k} e(\alpha_{2,4+i}) + \prod_{\substack{\alpha\in\Delta_{\bar{1}}^+ \\ \varepsilon_3+\varepsilon_4>0}} (1 + e(\alpha)) - \mathcal{R}, \tag{2.76}$$
$$d = d_{\max} = d^3_{NN} > d^1_{N1}, \quad j_1 = 0, \quad r_i = 0, \forall\, i.$$

d) $d = d^4_{NN} = -z + 2m/N > d^1_{N1}$, $j_1 = 0$.

In these short single-reducibility-condition cases, the odd null condition holds (following from the singular vector (1.32d) or (1.33d)):

$$X^+_3 \,|\Lambda\rangle = X^+_{25} \,|\Lambda\rangle = 0. \tag{2.77}$$

Since $j_1 = 0$, from (1.26a) and (2.77) the additional null condition follows:

$$X^+_{15} \,|\Lambda\rangle = [X^+_1, X^+_3] \,|\Lambda\rangle = 0. \tag{2.78}$$

For $N > 1$ and $r_{N-1} > 2$ each of these UIRs enters as the second term in decomposition (2.74), when the first term is an UIR of type c) with $j_1 = 0$, as explained above.

Further, for $N > 1$ there are additional null conditions if $r_{N-i} = 0$, $i \leqslant i'_0$. These are recursive null conditions:

$$X^+_{1,5+i} \,|\Lambda\rangle = [X^+_{1,4+i}, X^+_{4+i}] \,|\Lambda\rangle = 0, \quad r_{N-j} = 0, \quad 1 \leqslant j \leqslant i \leqslant i'_0, \tag{2.79a}$$
$$X^+_{2,5+i} \,|\Lambda\rangle = [X^+_{2,4+i}, X^+_{4+i}] \,|\Lambda\rangle = 0, \quad r_{N-j} = 0, \quad 1 \leqslant j \leqslant i \leqslant i'_0. \tag{2.79b}$$

Thus, $2(1 + i'_0)$ generators $X^+_{1,4+k}$, $X^+_{2,4+k}$, $k = 1, \dots, 1 + i'_0$, are eliminated. The maximal number of states in $\hat{L}_\Lambda$ is $2^{4N-2-2i'_0}$.

The corresponding character formula is

$$\operatorname{ch} \hat{L}_\Lambda = \prod_{\substack{\alpha\in\Delta_1^+ \\ \alpha\neq\alpha_{j,4+k} \\ j=1,2,k=1,\dots,1+i_0'}} (1+e(\alpha)) - \mathcal{R} = \tag{2.80a}$$

$$= \sum_{\hat{s}\in\hat{W}^d_{i_0'}} (-1)^{\ell(\hat{s})}\ \hat{s}\cdot\operatorname{ch}\hat{V}^\Lambda - \mathcal{R}, \tag{2.80b}$$

$$\hat{W}^d_{i_0'} \equiv \hat{W}_{\alpha_{15}} \times \hat{W}_{\alpha_{16}} \times \dots \times \hat{W}_{\alpha_{1,5+i_0}} \times \hat{W}_{\alpha_{25}} \times \hat{W}_{\alpha_{26}} \times \dots \times \hat{W}_{\alpha_{2,5+i_0}}, \tag{2.80c}$$

$$d = d^4_{NN} > d^1_{N1}, \quad j_1 = 0, \quad r_{N-i} = 0, \quad i \leqslant i_0' \leqslant N-1, \quad r_{N-1-i_0'} > 0,$$

where $\mathcal{R}$ designates the counter-terms due to our Criterion, in particular, due to (2.23) taken with $\varepsilon_{j,4+k} = 0$, $j = 1,2$, $k = 1,\dots,1+i_0'$.

In the case of R-symmetry scalars we have

$$X^+_{1,4+k}\,|\Lambda\rangle = 0, \quad X^+_{2,4+k}\,|\Lambda\rangle = 0, \qquad k = 1,\dots,N, \quad r_i = 0, \forall\, i. \tag{2.81}$$

The character formula is (2.80) taken with $1 + i_0' = N$. These are chiral UIRs conjugate to the antichiral ones in (2.66).

2.4. Character Formulae of DRC UIRs. Each of the DRC cases is the obvious combination of two SRC cases and some results follow from this. In fact, in the generic cases, we can give a general character formula which follows directly from embedding diagram (1.55).

So let first $N > 1$ and $r_1 r_N - 1 > 0$ (i.e., $i_0 = i_0' = 0$). Then the following character formula holds:

$$\operatorname{ch}\hat{L}_\Lambda = \sum_{\hat{s}\in\hat{W}_{\beta,\beta'}} (-1)^{\ell(\hat{s})}\ \hat{s}\cdot\operatorname{ch}\hat{V}^\Lambda - \mathcal{R} = \tag{2.82a}$$

$$= \operatorname{ch}\hat{V}^\Lambda - \frac{1}{1+e(\beta)}\operatorname{ch}\hat{V}^{\Lambda+\beta} - \frac{1}{1+e(\beta')}\operatorname{ch}\hat{V}^{\Lambda+\beta'} + \\ + \frac{1}{(1+e(\beta))\,(1+e(\beta'))}\operatorname{ch}\hat{V}^{\Lambda+\beta+\beta'} - \mathcal{R}, \tag{2.82b}$$

$$\hat{W}_{\beta,\beta'} \equiv \hat{W}_\beta \times \hat{W}_{\beta'}. \tag{2.82c}$$

The above formula is proved similarly to what we had in the SRC cases. It reflects the contribution of the modules on embedding diagram (1.55). In fact, the two terms with minus sign on the first line of (2.82b) take into account the factorization of the oddly embedded submodules $I^\beta, I^{\beta'}$, cf. (1.50), coming from the modules V_{10}, V_{01}, resp. There can be no contribution of the modules along the same lines of embeddings V_{k0}, $V_{0\ell}$, $k,\ell > 1$, due to the Grassmannian nature of the odd embeddings involved. Consequently, all modules $V_{k\ell}$ for $k,\ell > 1$ cannot contribute to the character formula of UIR in V_{00}. Only the module V_{11} can contribute since it is also a nonzero submodule of V_{00}. However, since it is oddly embedded in V_{00} via both submodules V_{10}, V_{01}, its contribution is taken out *two times* — once with I^{β}, and a second time with $I^{\beta'}$. Thus, we need the term with plus sign on the second line of (2.82b) to restore its contribution once*. We cannot apply the same kind of arguments for $N = 1$, nevertheless, formula (2.82) holds also then for the case (1.49a), cf. Appendix A.1.

In accord with (2.82) for $N > 1$ and $d = d^{ac}$ the following decomposition holds:

$$\left(\hat{L}_{\rm long}\right)\Big|_{d=d^{ac}} = \hat{L}_{\Lambda} \oplus \hat{L}_{\Lambda+\beta} \oplus \hat{L}_{\Lambda+\beta'} \oplus \hat{L}_{\Lambda+\beta+\beta'}, \qquad r_1 r_{N-1} > 0, \quad (2.83)$$

where $\hat{L}_{\rm long}$ is a long superfield with the same values of r_i as Λ, while the value of j_1 (resp. j_2) is zero when j_1 (resp. j_2) from Λ is zero, otherwise it has to be the generic value $j_1 \geqslant N/2$ (resp. $j_2 \geqslant N/2$).

Next we consider the four DRC cases separately.

ac) $d = d_{\max} = d^1_{N1} = d^3_{NN} = d^{ac} \equiv 2 + j_1 + j_2 + m_1$, $z = j_1 - j_2 + 2m/N - m_1$.

In these semishort DRC cases, the two null conditions (2.35) and (2.67) hold. In addition, for $N > 1$, if $r_i = 0$, $i = 1,\ldots,i_0$, there hold (2.52) and if $r_{N-i} = 0$, $i = 1,\ldots,i'_0$, there hold (2.69).

There are two basic situations. The first is when $i_0 + i'_0 \leqslant N - 2$. (This situation is not applicable for $N = 1$.) This means that not all r_i are zero and all eliminations are as described separately for cases a) and c). These semishort UIRs may be called Grassmann-analytic following [23], since odd generators from different chiralities are eliminated. The maximal number of states in $\hat{L}_{\Lambda}$ is $2^{4N-2-i_0-i'_0}$.

The second is when $i_0 + i'_0 \leqslant N-2$ does not hold which means that all r_i are zero (R-symmetry scalars, $m_1 = 0 = m$), and in fact we have $i_0 = i'_0 = N-1$, and all generators $X^+_{1,4+k}$ and $X^+_{3,4+k}$ are eliminated. The maximal number of states in $\hat{L}_{\Lambda}$ is 2^{2N}.

Note that below only one case is applicable for $N = 1$.

• For $j_1 j_2 > 0$ the corresponding character formulae are combinations of (2.54) and (2.70):

*For more complicated application of similar arguments we refer to [128].

$$\operatorname{ch}\hat{L}_\Lambda = \prod_{\substack{\alpha\in\Delta_{\bar{1}}^+ \\ \alpha\neq\alpha_{1,4+k} \\ k=1,\ldots,1+i_0' \\ \alpha\neq\alpha_{3,5+N-j} \\ j=1,\ldots,1+i_0}} (1+e(\alpha)) - \mathcal{R} = \tag{2.84a}$$

$$= \sum_{\hat{s}\in\hat{W}^{ac}_{i_0,i_0'}} (-1)^{\ell(\hat{s})}\ \hat{s}\cdot\operatorname{ch}\hat{V}^\Lambda - \mathcal{R}, \tag{2.84b}$$

$$= \sum_{\hat{s}\in\hat{W}^{ac}_{i_0,i_0'}} (-1)^{\ell(\hat{s})}\ \hat{s}\cdot\left(\operatorname{ch}\hat{V}^\Lambda - \mathcal{R}_{\rm long}\right), \tag{2.84c}$$

$$\hat{W}^{ac}_{i_0,i_0'} \equiv \hat{W}^a_{i_0}\times\hat{W}^c_{i_0'}, \tag{2.84d}$$

$$d = d_{\max} = d^1_{N1} = d^3_{NN} = 2 + j_1 + j_2 + m_1, \quad j_1 j_2 > 0,$$

either

$$i_0 + i_0' \leqslant N-2, \quad r_i = 0, \quad i = 1,2,\ldots,i_0, N-i_0', N-i_0'+1,\ldots,N-1,$$
$$r_i > 0, \quad i = i_0+1, N-i_0'-1, \text{ or } i_0 = i_0' = N-1, \quad r_i = 0, \forall\, i.$$

The last subcase is of R-symmetry scalars. It is also the only formula in the case under consideration — ac) — valid for $N=1$ (where there are no counterterms since (2.23a), (2.23b) bring no restrictions, cf. also Appendix A.1).

For $N>1$ and $i_0 = i_0' = 0$, formula (2.84) is equivalent to (2.82) with $\beta = \alpha_{15}$, $\beta' = \alpha_{3,4+N}$. Also (2.83) holds with these β, β':

$$\left(\hat{L}_{\rm long}\right)\Big|_{d=d^{ac}} = \hat{L}_\Lambda \oplus \hat{L}_{\Lambda+\alpha_{15}} \oplus \hat{L}_{\Lambda+\alpha_{3,4+N}} \oplus \hat{L}_{\Lambda+\alpha_{15}+\alpha_{3,4+N}}, \quad r_1 r_{N-1} > 0, \tag{2.85}$$

and with $\hat{L}_{\rm long}$ being a long superfield with the same values of r_i as Λ and with $j_1, j_2 \geqslant N/2$.

All formulae below to the end of case ac) are for $N>1$.

• For $j_1 > 0, j_2 = 0$ the corresponding character formulae are combinations of (2.56) and (2.70):

$$\operatorname{ch}\hat{L}_\Lambda = \sum_{\hat{s}\in\hat{W}^{a'c}_{i_0'}} (-1)^{\ell(\hat{s})}\ \hat{s}\cdot\operatorname{ch}\hat{V}^\Lambda - \mathcal{R} = \tag{2.86a}$$

$$= \sum_{\hat{s}\in\hat{W}^{a'c}_{i_0'}} (-1)^{\ell(\hat{s})}\ \hat{s}\cdot\left(\operatorname{ch}\hat{V}^\Lambda - \mathcal{R}_{\rm long}\right), \tag{2.86b}$$

$$\hat{W}^{a'c}_{i_0'} \equiv \hat{W}_{\beta_{12}}\times\hat{W}^c_{i_0'}, \qquad \beta_{12} = \alpha_{3,4+N} + \alpha_{4,4+N}, \tag{2.86c}$$

$$d = d_{\max} = d^1_{N1} = d^3_{NN} = 2 + j_1 + m_1, \quad j_1 > 0, \quad j_2 = 0, \quad r_1 > 0.$$

For $i_0 = i'_0 = 0$, decomposition (2.83) holds:

$$\left(\hat{L}_{\rm long}\right)\Big|_{d=d^{ac}} = \hat{L}_\Lambda \oplus \hat{L}_{\Lambda+\alpha_{15}} \oplus \hat{L}_{\Lambda+\beta_{12}} \oplus \hat{L}_{\Lambda+\alpha_{15}+\beta_{12}}, \qquad r_1 r_{N-1} > 0, \tag{2.87}$$

where $\hat{L}_{\rm long}$ is a long superfield with the same values of $j_2(=0), r_i$ as Λ and with $j_1 \geqslant N/2$. Note that the UIR $\hat{L}_{\Lambda+\alpha_{15}}$ is also of the type ac) under consideration, while the last two UIRs are short from type bc) considered below.

For R-symmetry scalars we combine (2.60) and (2.70a):

$$\mathrm{ch}\,\hat{L}_\Lambda = \sum_{k=1}^{N}\prod_{i=1}^{k} e(\alpha_{4,5+N-i}) + \prod_{\substack{\alpha\in\Delta_{\bar 1}^+ \\ \alpha\neq\alpha_{1,4+k}, \\ k=1,\dots,N \\ \varepsilon_2>0}} (1+e(\alpha)) - \mathcal{R}, \tag{2.88}$$

$$d = d_{\max} = d^1_{N1} = d^3_{NN} = 2 + j_1, \quad j_1 > 0, \quad j_2 = 0, \quad r_i = 0, \ \forall\, i.$$

• For $j_1 = 0, j_2 > 0$ the corresponding character formulae are combinations of (2.73) and (2.54):

$$\mathrm{ch}\,\hat{L}_\Lambda = \sum_{\hat{s}\in\hat{W}^{ac'}_{i_0}} (-1)^{\ell(\hat{s})}\ \hat{s}\cdot \mathrm{ch}\,\hat{V}^\Lambda - \mathcal{R} = \tag{2.89a}$$

$$= \sum_{\hat{s}\in\hat{W}^{ac'}_{i_0}} (-1)^{\ell(\hat{s})}\ \hat{s}\cdot \left(\mathrm{ch}\,\hat{V}^\Lambda - \mathcal{R}_{\rm long}\right), \tag{2.89b}$$

$$\hat{W}^{ac'}_{i_0} \equiv \hat{W}_{\beta_{34}} \times \hat{W}^{\alpha}_{i_0}, \quad \beta_{34} = \alpha_{15} + \alpha_{25}, \tag{2.89c}$$

$$d = d_{\max} = d^1_{N1} = d^3_{NN} = 2 + j_2 + m_1, \quad j_1 = 0, \quad j_2 > 0, \quad r_{N-1} > 0.$$

For $i_0 = i'_0 = 0$ there holds decomposition (2.83):

$$\left(\hat{L}_{\rm long}\right)\Big|_{d=d^{ac}} = \hat{L}_\Lambda \oplus \hat{L}_{\Lambda+\alpha_{3,4+N}} \oplus \hat{L}_{\Lambda+\beta_{34}} \oplus \hat{L}_{\Lambda+\alpha_{3,4+N}+\beta_{34}}, \quad r_1 r_{N-1} > 0, \tag{2.90}$$

where $\hat{L}_{\rm long}$ is a long superfield with the same values of $j_1(=0), r_i$ as Λ and with $j_2 \geqslant N/2$. Note that the UIR $\hat{L}_{\Lambda+\alpha_{3,4+N}}$ is again of the type ac) under consideration, while the last two UIRs are actually from type ad) considered below.

For R-symmetry scalars we combine (2.54a) and (2.76):

$$\mathrm{ch}\,\hat{L}_\Lambda = \sum_{k=1}^{N} \prod_{i=1}^{k} e(\alpha_{2,4+i}) + \prod_{\substack{\alpha\in\Delta_{\bar 1}^+ \\ \alpha\neq\alpha_{3,4+k} \\ k=1,\dots,N \\ \varepsilon_4>0}} (1+e(\alpha)) - \mathcal{R}, \tag{2.91}$$

$$d = d_{\max} = d^1_{N1} = d^3_{NN} = 2 + j_2,$$
$$j_1 = 0, \quad j_2 > 0, \quad r_i = 0, \forall\, i.$$

• For $j_1 = j_2 = 0$ the corresponding character formulae are combinations of (2.56) and (2.73):

$$\mathrm{ch}\,\hat{L}_\Lambda = \sum_{\hat{s}\in\hat{W}^{a'c'}_{i'_0}} (-1)^{\ell(\hat{s})}\ \hat{s}\cdot \mathrm{ch}\,\hat{V}^\Lambda - \mathcal{R} = \tag{2.92a}$$

$$= \sum_{\hat{s}\in\hat{W}^{a'c'}_{i'_0}} (-1)^{\ell(\hat{s})}\ \hat{s}\cdot \left(\mathrm{ch}\,\hat{V}^\Lambda - \mathcal{R}_{\mathrm{long}}\right), \tag{2.92b}$$

$$\hat{W}^{a'c'}_{i'_0} \equiv \hat{W}_{\beta_{12}} \times \hat{W}_{\beta_{34}}, \tag{2.92c}$$
$$d = d_{\max} = d^1_{N1} = d^3_{NN} = 2 + m_1,$$
$$j_1 = j_2 = 0, \quad r_1 r_{N-1} > 0.$$

For $i_0 = i'_0 = 0$, decomposition (2.83) holds:

$$\left(\hat{L}_{\mathrm{long}}\right)\Big|_{d=d^{ac}} = \hat{L}_\Lambda \oplus \hat{L}_{\Lambda+\beta_{12}} \oplus \hat{L}_{\Lambda+\beta_{34}} \oplus \hat{L}_{\Lambda+\beta_{12}+\beta_{34}}, \quad r_1 r_{N-1} > 0, \tag{2.93}$$

where $\hat{L}_{\mathrm{long}}$ is a long superfield with the same values of $j_1(=0), j_2(=0), r_i$ as Λ. Note that the UIR $\hat{L}_{\Lambda+\beta_{12}}$ is of the bc) type, $\hat{L}_{\Lambda+\beta_{34}}$ is of the ad) type, $\hat{L}_{\Lambda+\beta_{12}+\beta_{34}}$ is of the bd) type, these three being considered below.

For R-symmetry scalars we combine (2.60) and (2.76):

$$\mathrm{ch}\,\hat{L}_\Lambda = \sum_{k=1}^{N} \prod_{i=1}^{k} e(\alpha_{2,4+i}) + \sum_{k=1}^{N} \prod_{i=1}^{k} e(\alpha_{4,5+N-i}) + \prod_{\substack{\alpha\in\Delta_{\bar 1}^+ \\ \varepsilon_1+\varepsilon_2>0 \\ \varepsilon_3+\varepsilon_4>0}} (1+e(\alpha)) - \mathcal{R}, \tag{2.94}$$

$$d = d_{\max} = d^1_{N1} = d^3_{NN} = 2, \quad z = 0,$$
$$j_1 = j_2 = 0, \quad r_i = 0,\ \forall\, i.$$

ad) $d = d^1_{N1} = d^4_{NN} = 1 + j_2 + m_1, \quad j_1 = 0, \quad z = 2m/N - m_1 - 1 - j_2.$

In these short DRC cases, three null conditions (2.35), (2.77) and (2.78) hold. In addition, for $N > 1$, if $r_i = 0$, $i = 1, \ldots, i_0$, there hold (2.52), and if $r_{N-i} = 0$, $i = 1, \ldots, i'_0$, there hold (2.79).

If $i_0 + i'_0 \leqslant N - 2$, all eliminations are as described separately for cases a) and d). All these are Grassmann-analytic UIRs. The maximal number of states in $\hat{L}_\Lambda$ is $2^{4N-3-i_0-2i'_0}$. Interesting subcases are the so-called BPS states, cf. [16, 20, 23, 35, 41, 42, 44, 53]. They are characterized by the number κ of odd generators which annihilate them — then the corresponding case is called $\dfrac{k}{4N}$-BPS state. For example, consider $N = 4$ and $\dfrac{1}{4}$-BPS cases with $z = 0 \Rightarrow d = 2m/N$. One such case is obtained for $i_0 = 1, i'_0 = 0, j_2 > 0$, then $d = \dfrac{1}{2}(2r_2 + 3r_3)$, $r_1 = 0$, $r_2 > 0$, $r_3 = 2(1 + j_2)$.

For $j_2 m_1 > 0$ the corresponding character formula is a combination of (2.54) and (2.80):

$$\operatorname{ch} \hat{L}_\Lambda = \prod_{\substack{\alpha \in \Delta_{\bar{1}}^+ \\ \alpha \neq \alpha_{3,5+N-k}, \\ k=1,\ldots,1+i_0 \\ \alpha \neq \alpha_{a,4+j}, \\ a=1,2,\ j=1,\ldots,1+i'_0}} (1 + e(\alpha)) - \mathcal{R} = \tag{2.95a}$$

$$= \sum_{\hat{s} \in \hat{W}^{ad}_{i_0,i'_0}} (-1)^{\ell(\hat{s})} \hat{s} \cdot \operatorname{ch} \hat{V}^\Lambda - \mathcal{R}, \tag{2.95b}$$

$$\hat{W}^{ad}_{i_0,i'_0} \equiv \hat{W}^a_{i_0} \times \hat{W}^d_{i'_0}, \quad d = d^1_{N1} = d^4_{NN} = 1 + j_2 + m_1, \tag{2.95c}$$

$$j_1 = 0, j_2 > 0, i_0 + i'_0 \leqslant N - 2,$$

$$r_i = 0, \quad i = 1, 2, \ldots, i_0, \quad N - i'_0, \quad N - i'_0 + 1, \ldots, N - 1,$$

$$r_i > 0, \quad i = i_0 + 1, \quad N - i'_0 - 1.$$

For $i_0 = i'_0 = 0$ some of these UIRs appear (up to two times) in the decomposition (2.90). More precisely, those with $r_i > 2\delta_{i,N-1}$, $i = 1, N-1$, appear as the term $\hat{L}_{\Lambda+\beta_{34}}$, while those with $r_i > \delta_{i1} + 2\delta_{i,N-1}$, $i = 1, N-1$, appear also as the term $\hat{L}_{\Lambda+\alpha_{3,4+N}+\beta_{34}}$.

For $j_2 = 0, m_1 > 0$ the corresponding character formula is a combination of (2.56) and (2.80b):

$$\operatorname{ch} \hat{L}_\Lambda = \sum_{\hat{s} \in \hat{W}^{a'd}_{i'_0}} (-1)^{\ell(\hat{s})}\ \hat{s} \cdot \operatorname{ch} \hat{V}^\Lambda - \mathcal{R}, \tag{2.96a}$$

$$\hat{W}^{a'd}_{i'_0} \equiv \hat{W}_{\beta_{12}} \times \hat{W}^d_{i'_0}, \tag{2.96b}$$

where $\beta_{12} = \alpha_{3,4+N} + \alpha_{4,4+N}$. For $i_0 = i_0' = 0$ some of these UIRs appear in the decomposition (2.93) or (2.90). More precisely, those with $r_i > 2\delta_{i,N-1}$, $i = 1, N-1$, appear as the term $\hat{L}_{\Lambda+\beta_{34}}$ of (2.93), while those with $r_i > \delta_{i1} + 2\delta_{i,N-1}$, $i = 1, N-1$, appear as the term $\hat{L}_{\Lambda+\alpha_{3,4+N}+\beta_{34}}$ of (2.90) but only when $j_2 = 1/2$ in Λ there.

In the case of R-symmetry scalars we have $i_0 = i_0' = N-1$, $\kappa = 3N$ and all generators $X^+_{1,4+k}$, $X^+_{2,4+k}$, $X^+_{3,4+k}$ are eliminated. Here holds $d = -z = 1 + j_2$. These antichiral irreps form one of the three series of *massless* UIRs; they are denoted χ^+_s, $s = j_2 = 0, 1/2, 1, \ldots$, in Sec. 3 of [102]. Besides the vacuum, they contain only N states in $\hat{L}_\Lambda$ given by (2.34) for $k = 1, \ldots, N$. These should be called ultrashort UIRs. The character formula can be written in the most explicit way:

$$\begin{gathered}\operatorname{ch} \hat{L}_\Lambda = 1 + \sum_{k=1}^{N} \prod_{i=1}^{k} e(\alpha_{4,5+N-i}), \\ d = d^1_{N1} = d^4_{NN} = 1 + j_2 = -z, \quad j_1 = 0, \quad r_i = 0, \forall\, i,\end{gathered} \tag{2.97}$$

and it is valid for any j_2. In the case under consideration — ad) — only the last character formula is valid for $N = 1$ (cf. Appendix A.1).

The next case is conjugate to the previous one.

bc) $d = d^2_{N1} = d^3_{NN} = 1 + j_1 + m_1, j_2 = 0$, $z = 2m/N - m_1 + 1 + j_1$.

In these short DRC cases, three null conditions (2.61), (2.62) and (2.67) hold. In addition, for $N > 1$, if $r_i = 0$, $i = 1, \ldots, i_0$, there hold (2.64), and if $r_{N-i} = 0$, $i = 1, \ldots, i_0'$, there hold (2.69).

If $i_0 + i_0' \leqslant N - 2$, all eliminations are as described separately for cases b) and c). These are also Grassmann-analytic UIRs. The maximal number of states in $\hat{L}_\Lambda$ is $2^{4N-3-2i_0-i_0'}$. Here for $N = 4$ one $\dfrac{1}{4}$-BPS case is obtained for $i_0 = 0, i_0' = 1, j_1 > 0$, then $d = \dfrac{1}{2}(2r_2 + 3r_1)$, $r_1 = 2(1 + j_1)$, $r_2 > 0$, $r_3 = 0$.

For $j_1 m_1 > 0$ the corresponding character formula is a combination of (2.65) and (2.70):

$$\operatorname{ch} \hat{L}_\Lambda = \prod_{\substack{\alpha \in \Delta^+_{\bar{1}} \\ \alpha \neq \alpha_{1,4+k}, \\ k=1,\ldots,1+i_0' \\ \alpha \neq \alpha_{a,5+N-j}, \\ a=3,4, j=1,\ldots,1+i_0}} (1 + e(\alpha)) - \mathcal{R} = \tag{2.98a}$$

$$= \sum_{\hat{s}\in\hat{W}^{bc}_{i_0,i'_0}} (-1)^{\ell(\hat{s})}\ \hat{s}\cdot \mathrm{ch}\,\hat{V}^\Lambda - \mathcal{R}, \tag{2.98b}$$

$$\hat{W}^{bc}_{i_0,i'_0} \equiv \hat{W}^b_{i_0} \times \hat{W}^c_{i'_0}, \quad d = d^2_{N1} = d^3_{NN} = 1 + j_1 + m_1, \tag{2.98c}$$

$$j_1 > 0, \quad j_2 = 0, \quad i_0 + i'_0 \leqslant N-2,$$
$$r_i = 0,\ i = 1,2,\ldots,i_0, N-i'_0, N-i'_0+1,\ldots,N-1,$$
$$r_i > 0,\ i = i_0+1, N-i'_0-1.$$

For $i_0 = i'_0 = 0$ some of these UIRs appear in the decomposition (2.87). More precisely, those with $r_i > 2\delta_{i1}$, $i = 1, N-1$, appear as the term $\hat{L}_{\Lambda+\beta_{12}}$, while those with $r_i > 2\delta_{i1} + \delta_{i,N-1}$, $i = 1, N-1$, appear as the term $\hat{L}_{\Lambda+\alpha_{15}+\beta_{12}}$.

For $j_1 = 0, m_1 > 0$ the corresponding character formula is a combination of (2.73) and (2.65b):

$$\mathrm{ch}\,\hat{L}_\Lambda = \sum_{\hat{s}\in\hat{W}^{bc'}_{i_0}} (-1)^{\ell(\hat{s})} \hat{s}\,\mathrm{ch}\,\hat{V}^\Lambda - \mathcal{R}, \tag{2.99a}$$

$$\hat{W}^{bc'}_{i_0} \equiv \hat{W}_{\beta_{34}} \times \hat{W}^b_{i_0}, \tag{2.99b}$$

where $\beta_{34} = \alpha_{15} + \alpha_{25}$. For $i_0 = i'_0 = 0$ some of these UIRs appear in the decomposition (2.93) or (2.87). More precisely, those with $r_i > 2\delta_{i1}$, $i = 1, N-1$, appear as the term $\hat{L}_{\Lambda+\beta_{12}}$ of (2.93), while those with $r_i > 2\delta_{i1}+\delta_{i,N-1}$, $i = 1, N-1$, appear as the term $\hat{L}_{\Lambda+\alpha_{15}+\beta_{12}}$ of (2.87) but only when $j_1 = 1/2$ in Λ there.

In the case of R-symmetry scalars we have $i_0 = i'_0 = N-1$, $\kappa = 3N$ and all generators $X^+_{1,4+k}$, $X^+_{3,4+k}$, $X^+_{4,4+k}$ are eliminated. These chiral irreps form another series of *massless* UIRs, conjugate to the first above; they are denoted χ_s, $s = j_1 = 0, 1/2, 1, \ldots$, in Sec. 3 of [102]. Besides the vacuum they contain only N states in $\hat{L}_\Lambda$ given by (2.30) for $k = 1,\ldots,N$. These should also be called ultrashort UIRs. The character formula is

$$\mathrm{ch}\,\hat{L}_\Lambda = 1 + \sum_{k=1}^{N} \prod_{i=1}^{k} e(\alpha_{2,4+i}), \tag{2.100}$$

$$d = d^2_{N1} = d^3_{NN} = 1 + j_1 = z, \quad j_2 = 0, \quad r_i = 0, \forall\, i,$$

and it is valid for any j_1. In the case under consideration — bc) — only the last character formula is valid for $N = 1$ (cf. Appendix A.1).

bd) $d = d^2_{N1} = d^4_{NN} = m_1, j_1 = j_2 = 0, z = 2m/N - m_1$.

In these short DRC cases, four null conditions (2.61), (2.62), (2.77), and (2.78) hold.

For $N = 1$ this is the trivial irrep with $d = z = 0$. This follows from the fact that since $d = j_1 = j_2 = 0$, the even reducibility condition (1.17b) also holds (and consequently (1.17d)–(1.17f)). Thus, we have the null conditions: $X_k^+ |\Lambda\rangle = 0$ for all simple root generators (and consequently for all generators) and the irrep consists only of the vacuum $|\Lambda\rangle$.

For $N > 1$ the situation is nontrivial. In addition to the mentioned conditions, and if $r_i = 0$, $i = 1, \ldots, i_0$, there hold (2.64) and if $r_{N-i} = 0$, $i = 1, \ldots, i'_0$, there hold (2.79).

If $i_0 + i'_0 \leqslant N - 2$, all eliminations are as described separately for cases b) and d). These are also Grassmann-analytic UIRs. The maximal number of states in $\hat{L}_\Lambda$ is $2^{4N-4-2i_0-2i'_0}$. For $N = 4$, for the BPS cases we take $z = \dfrac{1}{2}(r_3 - r_1) = 0 \Rightarrow d = 2r_1 + r_2$. In the $\dfrac{1}{4}$-BPS case we have $i_0 = i'_0 = 0$, $r_1 = r_3 > 0$.

For $i_0 = i'_0 = 0$ some of these UIRs appear in the decomposition (2.93). More precisely, those with $r_i > 2\delta_{i1} + 2\delta_{i,N-1}$, $i = 1, N-1$ appear as the term $\hat{L}_{\Lambda+\beta_{12}+\beta_{34}}$.

Most interesting is the case $i_0 + i'_0 = N - 2$, then there is only one nonzero r_i, namely, $r_{1+i_0} = r_{N-1-i'_0} > 0$, while the rest r_i are zero. Thus, the Young tableau parameters are: $m_1 = r_{1+i_0}$, $m = (1 + i_0)r_{1+i_0}$.

An important subcase is when $d = m_1 = 1$, then $m = i_0 + 1 = N - 1 - i'_0$, $r_i = \delta_{mi}$, and these irreps form the third series of *massless* UIRs. In Sec. 3 of [102] they are parameterized by $n \in \mathbb{N}$, $\dfrac{1}{2}N \leqslant n < N$, and denoted by χ'_n, $n = m$ $(z = 2n/N - 1)$, χ'^{+}_n, $n = N - m$ $(z = 1 - 2n/N)$. Note that for even N there is the coincidence: $\chi'_n = \chi'^{+}_n$, where $n = m = N - m = N/2$. Here we shall parameterize these UIRs by the parameter $i_0 = 0, 1, \ldots, N-2$.

Another subcase here are $\dfrac{1}{2}$-BPS states for even N with $z = 0 \Rightarrow d = m_1 = 2m/N \Rightarrow i_0 = i'_0 = N/2 - 1 \Rightarrow m_1 = r_{N/2}$, $m = \dfrac{N}{2} r_{N/2}$. These are also massless only if $r_{N/2} = 1$, which is the self-conjugate case: χ'_n, $n = N/2$. For $N = 4$ we have: $i_0 = i'_0 = 1$, $r_1 = r_3 = 0$, $r_2 > 0$, which is also massless if $r_2 = 1$.

Finally, in the case of R-symmetry scalars we have $i_0 = i'_0 = N - 1$ and all $4N$ odd generators $X^+_{1,4+k}$, $X^+_{2,4+k}$, $X^+_{3,4+k}$, $X^+_{4,4+k}$ are eliminated. More than this, all quantum numbers are zero (cf. (1.46d), (1.46d′)), and this is the trivial irrep. The latter follows exactly as explained above for the case $N = 1$.

For $m_1 > 0$ the corresponding character formula is a combination of (2.65) and (2.80):

$$\operatorname{ch}\hat{L}_\Lambda = \prod_{\substack{\alpha\in\Delta_{\bar 1}^+ \\ \alpha\neq\alpha_{j,5+N-k} \\ j=3,4\ ,k=1,\ldots,1+i_0 \\ \alpha\neq\alpha_{j',4+k'} \\ j'=1,2,k'=1,\ldots,1+i'_0}} (1+e(\alpha)) - \mathcal{R} = \tag{2.101a}$$

$$= \sum_{\hat{s}\in\hat{W}^{bd}_{i_0,i'_0}} (-1)^{\ell(\hat{s})}\ \hat{s}\cdot\operatorname{ch}\hat{V}^\Lambda - \mathcal{R}, \tag{2.101b}$$

$$\hat{W}^{bd}_{i_0,i'_0} \equiv \hat{W}^{b}_{i_0}\ \times \hat{W}^{d}_{i'_0}, d = d^2_{N1} = d^4_{NN} = m_1, \tag{2.101c}$$

$$j_1 = j_2 = 0, \quad i_0 + i'_0 \leqslant N-2,$$

$$r_i = 0, \quad i = 1,2,\ldots,i_0, N-i'_0, N-i'_0+1,\ldots,N-1,$$

$$r_i > 0, \quad i = i_0+1, N-i'_0-1,$$

where $\mathcal{R}$ designates the counter-terms due to our Criterion, in particular, due to (2.23) taken with $\varepsilon_{j,5+N-k} = 0$, $j = 3,4$, $k = 1,\ldots,1+i_0$, $\varepsilon_{j',4+k'} = 0$, $j' = 1,2$, $k' = 1,\ldots,1+i'_0$.

Also for the third series of massless UIRs we can give a much more explicit character formula without counter-terms. Fix the parameter $i_0 = 0,1,\ldots,N-2$. Then there are only the following states in $\hat{L}_\Lambda$:

$$X^+_{2,N+4-j}\cdots X^+_{2,N+4-i_0}\,|\Lambda\rangle, \quad j = 0,1,\ldots,i_0, \tag{2.102a}$$

$$X^+_{4,4+k}\cdots X^+_{4,N+3-i_0}\,|\Lambda\rangle, \quad k = 1,\ldots,N-1-i_0, \tag{2.102b}$$

altogether N states besides the vacuum.

D e m o n s t r a t i o n. *Indeed, besides (2.102) no other states involving generators $X^+_{a,4+k}$ for $a = 2,4$ are possible due to the restrictions (2.23). Note that the generators of the latter kind which do not appear in (2.102) are eliminated due to (2.61), (2.64b) and (2.77), (2.79b). We have to discuss the generators $X^+_{a,4+k}$ for $a = 1,3$. Part of them are eliminated due to (2.64a) and (2.79a). The rest are: $X^+_{1,N+4-j}$, $j = 0,1,\ldots,i_0$ and $X^+_{4,4+k}$, $k = 1,\ldots,N-1-i_0$. They cannot act on the vacuum, so they can only act on some of the states in (2.102a) or (2.102b), resp. For two of these: $X^+_{1,N+4-i_0}$ and $X^+_{3,N+3-i_0}$ it is easy to see that they cannot act on any state. For the rest: $X^+_{1,N+4-j}$, $j = 0,1,\ldots,i_0-1$ and $X^+_{4,4+k}$, $k = 1,\ldots,N-2-i_0$, the only possibility for action which cannot be excluded in an obvious way, is:*

$$X^+_{1,N+4-j}\,X^+_{2,N+3-j}\cdots X^+_{2,N+4-i_0}\,|\Lambda\rangle, \quad j = 0,1,\ldots,i_0-1, \tag{2.103a}$$

$$X^+_{3,4+k}\,X^+_{4,5+k}\cdots X^+_{4,N+3-i_0}\,|\Lambda\rangle, \quad k = 1,\ldots,N-2-i_0. \tag{2.103b}$$

However, all these states are not allowed. This is shown also for the states (2.33) and (2.29). Thus, besides the vacuum, $\hat{L}_\Lambda$ contains only the N states given in (2.102). ■

The corresponding character formula for the massless UIRs of this series is therefore:

$$\mathrm{ch}\,\hat{L}_\Lambda = 1 + \sum_{j=0}^{i_0} \prod_{i=j}^{i_0} e(\alpha_{2,N+4-i}) + \sum_{k=1}^{N-1-i_0} \prod_{i=k}^{N-1-i_0} e(\alpha_{4,4+i}), \tag{2.104}$$

$$d = d^2_{N1} = d^4_{NN} = m_1 = 1, \quad i_0 = 0, 1, \ldots, N-2,$$
$$z = 2(i_0+1)/N - 1, \quad j_1 = j_2 = 0, \quad r_i = \delta_{i,i_0+1}.$$

Remark. *In this paper we use the Verma (factor-)module realization of the UIRs. We give here a short remark on what happens with the ER realization of the UIRs. As we know, cf. [101], the ERs are superfields depending on the Minkowski space-time and on the $4N$ Grassmann coordinates θ^i_a, $\bar\theta^k_b$, $a, b = 1, 2$, $i, k = 1, \ldots, N^*$. There is 1-to-1 correspondence in these dependences and the odd null conditions. Namely, if the condition $X^+_{a,4+k}\,|\Lambda\rangle = 0$, $a = 1, 2$, holds, then the superfields of the corresponding ER do not depend on the variable θ^k_a, while if the condition $X^+_{a,4+k}\,|\Lambda\rangle = 0$, $a = 3, 4$, holds, then the superfields of the corresponding ER do not depend on the variable $\bar\theta^k_{a-2}$. These statements were used in the proof of unitarity for the ERs picture, cf. [103], but were not explicated. They were analyzed in detail in the papers [16, 17, 20, 23], using the notions of «harmonic superspace analyticity» and the Grassmann analyticity.* ◇

3. DISCUSSION AND OUTLOOK

First, we summarize the results on decompositions of long irreps as they descend to the unitarity threshold.

In the SRC cases we have embedding formula (1.40), and UIRs are given by formula (1.42). Starting from this, in Subsection (2.3) we have established that for $d = d_{\max}$ there hold the following decompositions:

$$\left(\hat{L}_{\mathrm{long}}\right)\Big|_{d=d_{\max}} = \hat{L}_\Lambda \oplus \hat{L}_{\Lambda+\beta}, \tag{3.1}$$

where there are two possibilities for Λ and four possibilities for β as given in (1.41a), (1.41c), (1.41e), (1.41f), however, for $N > 1$ there are additional

*A mathematically precise formulation is given in [101], while for the even case we refer to [113, 114].

conditions on r_i. In more detail, Λ and β are specified as follows:

$$d = d_{\max} = d^a = d^1_{N1} > d^3_{NN}, \quad r_1 > 0, \tag{3.2a}$$

$$\beta = \alpha_{3,4+N}, \quad j_2 > 0, \tag{3.2a$'$}$$

$$\beta = \alpha_{3,4+N} + \alpha_{4,4+N}, \quad j_2 = 0, \tag{3.2a$''$}$$

$$d = d_{\max} = d^c = d^3_{NN} > d^1_{N1}, \quad r_{N-1} > 0, \tag{3.2b}$$

$$\beta = \alpha_{15}, \quad j_1 > 0, \tag{3.2b$'$}$$

$$\beta = \alpha_{15} + \alpha_{25}, \quad j_1 = 0. \tag{3.2b$''$}$$

The corresponding four decompositions are given in formulae (2.47), (2.57), (2.71), (2.74), resp., and in each case it is explained how $\hat{L}_{\rm long}$ is specified. It is also noted that in cases (3.2a$''$), (3.2b$''$) the UIRs $\hat{L}_{\Lambda+\beta}$ are short from types given in (1.39b), (1.39d), resp., and with $r_1 > 2$, $r_{N-1} > 2$, resp.

In the DRC cases we have embedding formulae (1.48), (1.54), (1.55), and UIRs are given by formula (1.50). Starting from this, in Subsec. 2.4 we have established that for $N > 1$ and $d = d_{\max} = d^{ac}$ there hold the following decompositions:

$$\left(\hat{L}_{\rm long}\right)\Big|_{d=d^{ac}} = \hat{L}_\Lambda \oplus \hat{L}_{\Lambda+\beta} \oplus \hat{L}_{\Lambda+\beta'} \oplus \hat{L}_{\Lambda+\beta+\beta'}, \qquad r_1 r_{N-1} > 0, \tag{3.3}$$

where Λ is the semishort DRC designated as type ac), and there are four possibilities for β, β' as given in (1.49a)–(1.49d). The corresponding four decompositions are given in formulae (2.85), (2.87), (2.90), (2.93), resp., and in each case it is explained how $\hat{L}_{\rm long}$ is specified. Note that in (2.85) all UIRs are semishort. In (2.87) the first two UIRs are semishort, the last two UIRs are short of type bc). From the latter two, the first is with $r_1 > 2, r_{N-1} > 0$ ($r_1 > 2$ if $N = 2$), the second is with $r_1 > 2, r_{N-1} > 1$ ($r_1 > 3$ if $N = 2$). In (2.90) the first two UIRs are semishort, the last two UIRs are short of type ad). From the latter two, the first is with $r_1 > 0, r_{N-1} > 2$ ($r_1 > 2$ if $N = 2$), the second is with $r_1 > 1, r_{N-1} > 2$ ($r_1 > 3$ if $N = 2$). In (2.93) the first UIR is the semishort, the other three UIRs are short of types bc), ad), bd), resp. From the latter three, the first is with $r_1 > 2, r_{N-1} > 0$, the second is with $r_1 > 0, r_{N-1} > 2$, the third is with $r_1, r_{N-1} > 2$ ($r_1 > 4$ if $N = 2$).

Summarizing the above, we note first that for $N = 1$ all SRC cases enter some decomposition (3.1), while no DRC cases enter any decomposition (3.3). For $N > 1$ the situation is more diverse and so we give the list of UIRs that do *not* enter decompositions (3.1) and (3.3):

SRC Cases:

a) $d = d_{\max} = d^a = d^1_{N1} = 2 + 2j_2 + z + 2m_1 - 2m/N > d^3_{NN}$,
j_1, j_2 arbitrary, $r_1 = 0$.

b) $d = d^2_{N1} = z + 2m_1 - 2m/N > d^3_{NN}, j_2 = 0,$
j_1 arbitrary, $r_1 \leqslant 2$.
c) $d = d_{\max} = d^c = d^3_{NN} = 2 + 2j_1 - z + 2m/N > d^1_{N1},$
j_1, j_2 arbitrary, $r_{N-1} = 0$.
d) $d = d^4_{NN} = -z + 2m/N > d^1_{N1}, j_1 = 0,$
j_1 arbitrary, $r_{N-1} \leqslant 2$.

DRC Cases:
all nontrivial cases for $N = 1$, while for $N > 1$ the list is:
ac) $d = d_{\max} = d^{ac} = d^1_{N1} = d^3_{NN} = 2 + j_1 + j_2 + m_1$, $z = j_1 - j_2 + 2m/N - m_1$,
j_1, j_2 arbitrary, $r_1 r_{N-1} = 0$.
ad) $d = d^1_{N1} = d^4_{NN} = 1 + j_2 + m_1, j_1 = 0$, $z = -1 - j_2 + 2m/N - m_1$,
j_2 arbitrary, $r_{N-1} \leqslant 2$, $r_1 = 0$ for $N > 2$.
bc) $d = d^2_{N1} = d^3_{NN} = 1 + j_1 + m_1, j_2 = 0$, $z = 1 + j_1 + 2m/N - m_1$,
j_1 arbitrary, $r_1 \leqslant 2$, $r_{N-1} = 0$ for $N > 2$.
bd) $d = d^2_{N1} = d^4_{NN} = m_1, j_1 = j_2 = 0$, $z = 2m/N - m_1$,
$r_1, r_{N-1} \leqslant 2$ for $N > 2$, $r_1 \leqslant 4$ for $N = 2$.

We would like to point out possible application of our results to current developments in the conformal field theory. Recently, there is interest in superfields with conformal dimensions which are protected from renormalization in the sense that they cannot develop anomalous dimensions [23, 31–34, 51]. Initially, the idea was that this happens because the representations, under which they transform, determine these dimensions uniquely. Later, it was argued that one can tell which operators will be protected in the quantum theory simply by looking at the representations they transform under and whether they can be written in terms of single trace 1/2 BPS operators (chiral primaries or CPOs) on analytic superspace [34]. In [51] it was shown how, at the unitarity threshold, a long multiplet can be decomposed into four semishort multiplets, and decompositions similar to (2.83), i.e., involving the modules in (1.55) (as given in [100]), were considered for $N = 2, 4$. However, the decompositions of [51] are justified on the dimensions of the finite-dimensional irreps of the Lorentz and $su(N)$ subalgebras involved in the superfields involved in the decompositions, and in particular, the latter hold also when $r_1 r_{N-1} = 0$.

Independently of the above, we would like to make a *mathematical* remark. As a by-product of our analysis we have obtained character formulae for the complex Lie superalgebras $sl(4/N)$. The point is that our character formulae have as starting point character formulae of Verma modules and factor-modules over $sl(4/N)$. Thus, almost all character formulae in Section 2, more precisely, formulae (2.26), (2.36), (2.39), (2.43), (2.54), (2.56), (2.65), (2.70), (2.73), (2.80), (2.82), (2.84), (2.86), (2.89), (2.92), (2.95), (2.96), (2.97a), (2.98), (2.99), (2.100a), (2.101), become character formulae for $sl(4/N)$ for the same values of the representation parameters by just discarding the counter-terms $\mathcal{R}$, $\mathcal{R}_{\rm long}$, resp.

Finally, let us mention that we explicate our results for $N = 1, 2$ in Appendix A. There we display explicitly all decompositions (3.1), (3.3), and when these do not hold, all quasi-decompositions (like (2.53)) that replace them. We leave similar detailed discussion for $N = 4$ for the follow-up paper.

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Appendix A

EXPLICIT CHARACTER FORMULAE FOR $N = 1, 2$

A.1. $N = 1$. For $N = 1$ the displayed results are almost explicit, so we can allow telegram style.

Long Superfields. If $j_1 j_2 > 0$, then $\hat{L}_\Lambda$ has the maximum possible number of states: sixteen. The character formula is (2.19).

If $j_1 = 0, j_2 > 0$, then the generator X^+_{15} can appear only together with the generator X^+_{25}, and $\hat{L}_\Lambda$ has 12 states = 3(chiral)×4(antichiral) states*. The character formula is (2.26) with

$$\mathcal{R} = e(\alpha_{15})(1 + e(\alpha_{35}))(1 + e(\alpha_{45})). \tag{A.1}$$

The next case is conjugate. If $j_1 > 0, j_2 = 0$, then the generator X^+_{35} can appear only together with the generator X^+_{45}, and $\hat{L}_\Lambda$ has 12 states. The character formula is (2.26) with

$$\mathcal{R} = e(\alpha_{35})(1 + e(\alpha_{15}))(1 + e(\alpha_{25})). \tag{A.2}$$

The next case combines the previous two. If $j_1 = j_2 = 0$, then the generator X^+_{15} can appear only together with the generator X^+_{25}, the generator X^+_{35} can appear only together with the generator X^+_{45}, and $\hat{L}_\Lambda$ has 9 states = 3(chiral)×3(antichiral) states. The character formula is (2.26) with

$$\mathcal{R} = e(\alpha_{15})(1 + e(\alpha_{35}))(1 + e(\alpha_{45})) + e(\alpha_{35})(1 + e(\alpha_{15}))(1 + e(\alpha_{25}))- \\ - e(\alpha_{15})e(\alpha_{35}), \tag{A.3}$$

*In statements like this, each sector includes the vacuum.

i.e., we combine the counter-terms of the previous two cases, but need to subtract a counter-term that is counted twice.

SRC Cases

a) $d = d_{\max} = d^1_{11} = 2 + 2j_2 + z > d^3_{11}$.

• $j_1 > 0$. The generator X^+_{35} is eliminated (though for different reasons for $j_2 > 0$ and $j_2 = 0$, cf. (2.35), resp., (2.55)) and there are only 8 states*. Then the character formula is (2.36) (or equivalently (2.39)) without counter-terms:

$$\operatorname{ch}\hat{L}_\Lambda = \prod_{\substack{\alpha\in\Delta^+_{\bar 1}\\ \alpha\neq\alpha_{35}}} (1 + e(\alpha)), \quad d = d_{\max} = d^1_{11} > d^3_{11}, \quad j_1 > 0. \tag{A.4}$$

For $j_2 > 0$ the decomposition (2.47) is fulfilled with $\hat{L}_{\rm long}$ having 16 states as the maximal long superfield with $j_1 j_2 > 0$, while $\hat{L}_{\Lambda+\alpha_{35}}$ has 8 states (being of the same type as $\hat{L}$).

For $j_2 = 0$ the decomposition (2.57) is fulfilled with $\hat{L}_{\rm long}$ having 12 states as the long superfield with $j_1 > 0, j_2 = 0$, $\beta = \alpha_{35} + \alpha_{45}$, and $\hat{L}_{\Lambda+\alpha_{35}+\alpha_{45}}$ having 4 states — it actually belongs to case b) below (for $j_1 > 0, j_2 = 0$).

• $j_1 = 0$. The generator X^+_{35} is eliminated, the generator X^+_{15} can appear only together with the generator X^+_{25} and there are only 6 states. Then the character formula is (2.36) (equivalently (2.39)):

$$\operatorname{ch}\hat{L}_\Lambda = \prod_{\substack{\alpha\in\Delta^+_{\bar 1}\\ \alpha\neq\alpha_{35}}} (1 + e(\alpha)) - \mathcal{R}, \qquad R = e(\alpha_{15})(1 + e(\alpha_{45})), d = d_{\max} = d^1_{11} > d^3_{11}, \quad j_1 = 0. \tag{A.5}$$

This formula is equivalent also to (2.43), noting:

$$\mathcal{R} = e(\alpha_{15})(1 + e(\alpha_{45})) = (1 - \hat{s}_{\alpha_{35}})\mathcal{R}_{\rm long}, \tag{A.6}$$

taking $\mathcal{R}_{\rm long}$ from (A.1).

For $j_2 > 0$ the decomposition (2.47) is fulfilled with $\hat{L}_{\rm long}$ having 12 states as the long superfield with $j_1 = 0, j_2 > 0$, while $\hat{L}_{\Lambda+\alpha_{35}}$ has 6 states (being of the same type as $\hat{L}$).

For $j_2 = 0$ the decomposition (2.57) is fulfilled with $\hat{L}_{\rm long}$ having 9 states as the long superfield with $j_1 = j_2 = 0$, while $\hat{L}_{\Lambda+\alpha_{35}+\alpha_{45}}$ has 3 states — it actually belongs to the next case b), cf. below (for $j_1 = j_2 = 0$).

b) $d = d^2_{11} = z > d^3_{11}, j_2 = 0$.

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*For brevity, here and often below we shall say «there are M states» meaning «there are M states in $\hat{L}_\Lambda$».

• $j_1 > 0$. The generators X^+_{35} and X^+_{45} are eliminated and there are only 4 states. Then the character formula is (2.65) (for $i_0 = 0$) without counter-terms:

$$\mathrm{ch}\,\hat{L}_\Lambda = (1 + e(\alpha_{15}))(1 + e(\alpha_{25})), \quad d = d^2_{11} > d^3_{11}, \quad j_1 > 0, \quad j_2 = 0. \tag{A.7}$$

These UIRs and the next subcase enter formula (2.47) together with UIRs of case a) as we have shown above.

• $j_1 = 0$. The generators X^+_{35} and X^+_{45} are eliminated, the generator X^+_{15} can appear only together with the generator X^+_{25}, and there are only 3 states. Then the character formula is (2.65) (for $i_0 = 0$) with counter-term $\mathcal{R} = e(\alpha_{15})$:

$$\mathrm{ch}\,\hat{L}_\Lambda = 1 + e(\alpha_{25}) + e(\alpha_{15})e(\alpha_{25}), \quad d = d^2_{11} > d^3_{11}, \quad j_1 = j_2 = 0. \tag{A.8}$$

Here holds also an analog of (2.43) with $\hat{W}_\beta$ replaced by $\hat{W}^b_0$ and $\mathcal{R}_{\mathrm{long}}$ from (A.1).

c) $d = d_{\max} = d^3_{11} = 2 + 2j_1 - z > d^1_{11}$.

• $j_2 > 0$. The generator X^+_{15} is eliminated (though for different reasons for $j_1 > 0$ and $j_1 = 0$, cf. (2.67), resp., (2.72)) and there are only 8 states. Then the character formula is (2.70) without counter-terms:

$$\mathrm{ch}\,\hat{L}_\Lambda = \prod_{\substack{\alpha\in\Delta^+_{\bar{1}}\\ \alpha\neq\alpha_{15}}} (1 + e(\alpha)), \quad d = d_{\max} = d^3_{11} > d^1_{11}, \quad j_2 > 0. \tag{A.9}$$

For $j_1 > 0$ the decomposition (2.71) is fulfilled with $\hat{L}_{\mathrm{long}}$ having 16 states as the maximal long superfield with $j_1 j_2 > 0$. For $j_1 = 0$ the decomposition (2.74) is fulfilled with $\hat{L}_{\mathrm{long}}$ having 12 states as the long superfield with $j_1 = 0, j_2 > 0$, and $L_{\Lambda+\alpha_{15}+\alpha_{25}}$ having 4 states — it actually belongs to the next case d), cf. below (for $j_1 = 0, j_2 > 0$).

• $j_2 = 0$. The generator X^+_{15} is eliminated, the generator X^+_{35} can appear only together with the generator X^+_{45} and there are only 6 states. Then the character formula is (2.70):

$$\begin{gathered}\mathrm{ch}\,\hat{L}_\Lambda = \prod_{\substack{\alpha\in\Delta^+_{\bar{1}}\\ \alpha\neq\alpha_{15}}} (1 + e(\alpha)) - \mathcal{R},\\ \mathcal{R} = e(\alpha_{35})(1 + e(\alpha_{25})), \quad d = d_{\max} = d^3_{11} > d^1_{11}, \quad j_2 = 0.\end{gathered} \tag{A.10}$$

This formula is equivalent also to (2.43) with $\mathcal{R}_{\mathrm{long}}$ from (A.2).

For $j_1 > 0$ the decomposition (2.71) is fulfilled with $\hat{L}_{\mathrm{long}}$ having 12 states as the long superfield with $j_1 > 0, j_2 = 0$. For $j_1 = 0$ the decomposition (2.74) is fulfilled with $\hat{L}_{\mathrm{long}}$ having 9 states as the long superfield with $j_1 = j_2 = 0$, and $L_{\Lambda+\alpha_{15}+\alpha_{25}}$ having 3 states — it actually belongs to the next case d), cf. below, (for $j_1 = j_2 = 0$).

d) $d = d^4_{11} = -z > d^1_{11}, j_1 = 0$.

• $j_2 > 0$. The generators X^+_{15} and X^+_{25} are eliminated and there are only 4 states. Then the character formula is (2.80) (for $i'_0 = 0$) without counter-terms:

$$\mathrm{ch}\,\hat{L}_\Lambda = (1+e(\alpha_{35}))(1+e(\alpha_{45})), \quad d = d^4_{11} > d^1_{11}, \quad j_1 = 0, \quad j_2 > 0. \tag{A.11}$$

These UIRs and the next subcase enter formula (2.71) together with UIRs of case c) as we have shown above.

• $j_2 = 0$. The generators X^+_{15} and X^+_{25} are eliminated, the generator X^+_{35} can appear only together with the generator X^+_{45}, and there are only 3 states. Then the character formula is (2.80) (for $i'_0 = 0$) with counter-term $\mathcal{R} = e(\alpha_{35})$:

$$\mathrm{ch}\,\hat{L}_\Lambda = 1 + e(\alpha_{45}) + e(\alpha_{35})e(\alpha_{45}), \quad d = d^4_{11} > d^1_{11}, \quad j_1 = j_2 = 0. \tag{A.12}$$

Here holds also an analog of (2.43) with $\hat{W}_\beta$ replaced by $\hat{W}^d_0$ and $\mathcal{R}_{\rm long}$ from (A.2).

DRC Cases

ac) $d = d_{\max} = d^1_{11} = d^3_{11} = d^{ac} = 2 + j_1 + j_2$, $z = j_1 - j_2$.

The generators X^+_{15} and X^+_{35} are eliminated (though for different reasons for $j_1 > 0$ and $j_1 = 0$, resp., for $j_2 > 0$ and $j_2 = 0$). There are only 4 states and the character formula is (2.84) (for $i_0 = i'_0 = 0$) without counter-terms:

$$\mathrm{ch}\,\hat{L}_\Lambda = (1 + e(\alpha_{25}))(1 + e(\alpha_{45})) = \mathrm{ch}\,\hat{V}^\Lambda - \frac{1}{1 + e(\alpha_{15})}\mathrm{ch}\,\hat{V}^{\Lambda+\alpha_{15}} - \\ - \frac{1}{1 + e(\alpha_{35})}\mathrm{ch}\,\hat{V}^{\Lambda+\alpha_{35}} + \frac{1}{(1 + e(\alpha_{15}))(1 + e(\alpha_{35}))}\mathrm{ch}\,\hat{V}^{\Lambda+\alpha_{15}+\alpha_{35}}, \tag{A.13}$$

where the terms with minus may be interpreted as taking out states, while the last term indicates adding back what was taken two times. This may be written also in the form of the following pseudodecomposition:

$$\left(\hat{L}_{\rm long}\right)\Big|_{d=d^{ac}} = \hat{L}_\Lambda \oplus \hat{L}_{\Lambda+\alpha_{15}} \oplus \hat{L}_{\Lambda+\alpha_{35}} \ominus \hat{L}_{\Lambda+\alpha_{15}+\alpha_{35}}, \tag{A.14}$$

where $\hat{L}_{\Lambda+\alpha_{15}}$, $\hat{L}_{\Lambda+\alpha_{35}}$ are SRC UIRs with 8 states each described above in cases c), a), resp. They are embedded in $\hat{V}^\Lambda$ via the generators X^+_{15}, X^+_{35}, resp. Together with $\hat{L}_\Lambda$ this brings in terms which have to be taken out with the last term in which the representation denoted $\hat{L}_{\Lambda+\alpha_{15}+\alpha_{35}}$ is supposed to have the same 4 states as $\hat{L}_\Lambda$ and these excessive states are de-embedded via the composition of the other two maps, i.e., via the product of generators $X^+_{15}\,X^+_{35}$.

ad) $d = d^1_{11} = d^4_{11} = 1 + j_2 = -z, j_1 = 0$.

The generators X^+_{15}, X^+_{25}, and X^+_{35} are eliminated (for the latter for different reasons for $j_2 > 0$ and $j_2 = 0$). These are the first series of massless UIRs, and everything is already explicit in the general formulae. There are only 2 states and the character formula is (2.97) for $N = 1$:

$$\mathrm{ch}\,\hat{L}_\Lambda = 1 + e(\alpha_{45}). \tag{A.15}$$

bc) $d = d^2_{11} = d^3_{11} = 1 + j_1 = z, j_2 = 0$.

The generators X^+_{15}, X^+_{35}, and X^+_{45} are eliminated (for the first for different reasons for $j_1 > 0$ and $j_1 = 0$). These are the second series of massless UIRs. There are only 2 states and the character formula is (2.100) for $N = 1$:

$$\mathrm{ch}\,\hat{L}_\Lambda = 1 + e(\alpha_{25}). \tag{A.16}$$

bd) $d = d^2_{11} = d^4_{11} = j_1 = j_2 = z = 0$.

As we explained in detail, this is the trivial one-dimensional irrep consisting of the vacuum.

A.2. $N = 2$**.**

Long Superfields. We first write down conditions (2.23) explicitly for $N = 2$:

$$\varepsilon_{15} + \varepsilon_{16} \leqslant \varepsilon_{25} + \varepsilon_{26} + 2j_1, \tag{A.17a}$$
$$\varepsilon_{35} + \varepsilon_{36} \leqslant \varepsilon_{45} + \varepsilon_{46} + 2j_2, \tag{A.17b}$$
$$\varepsilon_{16} + \varepsilon_{26} + \varepsilon_{35} + \varepsilon_{45} \leqslant \varepsilon_{15} + \varepsilon_{25} + \varepsilon_{36} + \varepsilon_{46} + r_1. \tag{A.17c}$$

To simplify the exposition we classify the generators by their contribution to (A.17). Namely, the chiral and antichiral operators

$$\begin{aligned} \Phi^c &= (X^+_{16})^{\varepsilon_{16}}\,(X^+_{15})^{\varepsilon_{15}}\,(X^+_{26})^{\varepsilon_{26}}\,(X^+_{25})^{\varepsilon_{25}}, \\ \Phi^a &= (X^+_{35})^{\varepsilon_{35}}\,(X^+_{36})^{\varepsilon_{36}}\,(X^+_{45})^{\varepsilon_{45}}\,(X^+_{46})^{\varepsilon_{46}} \end{aligned} \tag{A.18}$$

will be distinguished by the values (cf. also (2.22)):

$$\begin{aligned} \varepsilon^c_j &= \varepsilon_{25} + \varepsilon_{26} - \varepsilon_{15} - \varepsilon_{16}, \\ \varepsilon^a_j &= \varepsilon_{45} + \varepsilon_{46} - \varepsilon_{35} - \varepsilon_{36}, \\ \varepsilon^c_r &\equiv \varepsilon_{15} + \varepsilon_{25} - \varepsilon_{16} - \varepsilon_{26}, \\ \varepsilon^a_r &\equiv \varepsilon_{36} + \varepsilon_{46} - \varepsilon_{35} - \varepsilon_{45}, \\ \varepsilon_r &\equiv \varepsilon^1_r = \varepsilon^c_r + \varepsilon^a_r. \end{aligned} \tag{A.19}$$

Explicitly, the chiral operators are arranged as follows:

$$\begin{aligned}
&X^+_{15}\,X^+_{25}, \quad \varepsilon^c_r = 2, \quad \varepsilon^c_j = 0,\\
&X^+_{25}, \quad X^+_{26}\,X^+_{15}\,X^+_{25}, \quad \varepsilon^c_r = 1, \quad \varepsilon^c_j = 1,\\
&X^+_{15}, \quad X^+_{16}\,X^+_{15}\,X^+_{25}, \quad \varepsilon^c_r = 1, \quad \varepsilon^c_j = -1,\\
&X^+_{26}\,X^+_{25}, \quad \varepsilon^c_r = 0, \quad \varepsilon^c_j = 2,\\
&1, \quad X^+_{16}\,X^+_{25}, \quad X^+_{26}\,X^+_{15}, \quad X^+_{16}\,X^+_{26}\,X^+_{15}\,X^+_{25}, \quad \varepsilon^c_r = 0, \quad \varepsilon^c_j = 0,\\
&X^+_{16}\,X^+_{15}, \quad \varepsilon^c_r = 0, \quad \varepsilon^c_j = -2,\\
&X^+_{26}, \quad X^+_{26}\,X^+_{16}\,X^+_{25}, \quad \varepsilon^c_r = -1, \quad \varepsilon^c_j = 1,\\
&X^+_{16}, \quad X^+_{16}\,X^+_{15}\,X^+_{26}, \quad \varepsilon^c_r = -1, \quad \varepsilon^c_j = -1,\\
&X^+_{16}\,X^+_{26}, \quad \varepsilon^c_r = -2, \quad \varepsilon^c_j = 0,
\end{aligned} \tag{A.20}$$

while the antichiral operators are arranged as follows:

$$\begin{aligned}
&X^+_{36}\,X^+_{46}, \quad \varepsilon^a_r = 2, \quad \varepsilon^a_j = 0,\\
&X^+_{46}, \quad X^+_{45}\,X^+_{36}\,X^+_{46}, \quad \varepsilon^a_r = 1, \quad \varepsilon^a_j = 1,\\
&X^+_{36}, \quad X^+_{35}\,X^+_{36}\,X^+_{46}, \quad \varepsilon^a_r = 1, \quad \varepsilon^a_j = -1,\\
&X^+_{45}\,X^+_{46}, \quad \varepsilon^a_r = 0, \quad \varepsilon^a_j = 2,\\
&1, \quad X^+_{35}\,X^+_{46}, \quad X^+_{45}\,X^+_{36}, \quad X^+_{35}\,X^+_{45}\,X^+_{36}\,X^+_{46}, \quad \varepsilon^a_r = 0, \quad \varepsilon^a_j = 0,\\
&X^+_{35}\,X^+_{36}, \quad \varepsilon^a_r = 0, \quad \varepsilon^a_j = -2,\\
&X^+_{45}, \quad X^+_{45}\,X^+_{35}\,X^+_{46}, \quad \varepsilon^a_r = -1, \quad \varepsilon^a_j = 1,\\
&X^+_{35}, \quad X^+_{35}\,X^+_{36}\,X^+_{45}, \quad \varepsilon^a_r = -1, \quad \varepsilon^a_j = -1,\\
&X^+_{35}\,X^+_{45}, \quad \varepsilon^a_r = -2, \quad \varepsilon^a_j = 0.
\end{aligned} \tag{A.21}$$

The same arrangement applies to the states obtained by applying the operators on the vacuum (for which all these indices naturally have zero value). We have added also the identity operator 1 in order to be able to take into account the vacuum automatically.

The allowed states satisfy: $\varepsilon^c_j + 2j_1 \geqslant 0$, $\varepsilon^a_j + 2j_2 \geqslant 0$, $\varepsilon_r + r_1 \geqslant 0$, cf. (2.23). Now we are ready to classify the allowed states depending on the values of j_1, j_2, r_1. Actually what we do below amounts to giving explicitly formula (2.26).

• First, we give the possible states when $j_1, j_2 \geqslant 1$:

$$\Phi^c\,\Phi^a\,|\Lambda\rangle, \quad j_1, j_2 \geqslant 1, \quad r_1 \geqslant 4, \quad 256 \text{ states}; \tag{A.22a}$$

$$\Phi^c\,\Phi^a\,|\Lambda\rangle, \quad j_1, j_2 \geqslant 1, \quad r_1 = 3, \quad 255 \text{ states},$$

$$\text{excluding the state } X^+_{16}\, X^+_{26}\, X^+_{35}\, X^+_{45},\ \text{with } \varepsilon_r = -4; \tag{A.22b}$$

$$\Phi^c\,\Phi^a\,|\Lambda\rangle,\quad j_1, j_2 \geqslant 1,\quad r_1 = 2,\quad 247 \text{ states,}$$
$$\text{excluding the 9 states with } \varepsilon_r \leqslant -3; \tag{A.22c}$$

$$\Phi^c\,\Phi^a\,|\Lambda\rangle,\quad j_1, j_2 \geqslant 1,\quad r_1 = 1,\quad 219 \text{ states,}$$
$$\text{excluding the 37 states with } \varepsilon_r \leqslant -2; \tag{A.22d}$$

$$\Phi^c\,\Phi^a\,|\Lambda\rangle,\quad j_1, j_2 \geqslant 1,\quad r_1 = 0,\quad 163 \text{ states,}$$
$$\text{excluding the 93 states with } \varepsilon_r \leqslant -1. \tag{A.22e}$$

Further we classify the states when $j_1, j_2 \geqslant 1$ is not fulfilled using the five cases in (A.22) as a reference point.

• $j_1 \geqslant 1$, $j_2 = 1/2$. With respect to (A.22) we exclude 16 states with $\varepsilon^a_j = -2$, (so $\varepsilon^a_j + 2j_2 = -1$):

$$X^+_{35}\, X^+_{36}\, \Phi^c\,|\Lambda\rangle. \tag{A.23}$$

However, for (A.22d), (A.22e) the case when $\Phi^c = X^+_{16}\, X^+_{26}$ (with $\varepsilon^c_r = -2$) is already taken out, and for (A.22e) the four cases of Φ^c with $\varepsilon^c_r = -1$ are already taken out. Thus, altogether, in the five cases corresponding to (A.22a)–(A.22e) we take out $16, 16, 16, 15, 11$ states and so there remain now $240, 239, 231, 204, 152$ states.

• $j_1 = 1/2$, $j_2 \geqslant 1$. This is the case conjugate to the previous one. With respect to (A.22) we exclude 16 states with $\varepsilon^c_j = -2$ (so $\varepsilon^c_j + 2j_1 = -1$):

$$X^+_{16}\, X^+_{15}\, \Phi^a\,|\Lambda\rangle. \tag{A.24}$$

Noting the double-counting for the five cases Φ^a with $\varepsilon^a_r = -2, -1$, in the cases corresponding to (A.22a)–(A.22e) we have now $240, 239, 231, 204, 152$ states.

• $j_1 = j_2 = 1/2$. This is a combination of the previous two cases. With respect to (A.22) we exclude the states we have excluded in both the cases, which would double the numbers (to $32, 32, 32, 30, 22$), however, we have to take into account that the state $X^+_{16}\, X^+_{15}\, X^+_{35}\, X^+_{36}\,|\Lambda\rangle$ is counted two times. Thus, altogether, in the five cases corresponding to (A.22a)–(A.22e) we take out $31, 31, 31, 29, 21$ states and so there remain now $225, 224, 216, 190, 142$ states.

• $j_1 \geqslant 1$, $j_2 = 0$. In addition to the states excluded in the case $j_1 \geqslant 1$, $j_2 = 1/2$, we exclude 64 states with $\varepsilon^a_j = -1$ (so $\varepsilon^a_j + 2j_2 = -1$):

$$X^+_{36}\, \Phi^c\,|\Lambda\rangle,\quad X^+_{35}\, X^+_{36}\, X^+_{46}\, \Phi^c\,|\Lambda\rangle, \tag{A.25a}$$
$$X^+_{35}\, \Phi^c\,|\Lambda\rangle,\quad X^+_{35}\, X^+_{36}\, X^+_{45}\, \Phi^c\,|\Lambda\rangle. \tag{A.25b}$$

We have to take into account that certain states were already taken out, namely, the following:

— for (A.22c)–(A.22e) the two cases (A.25b) with $\Phi^c = X^+_{16}\,X^+_{26}$ (so that $\varepsilon_r = -3$);

— for (A.22d), (A.22e) the eight cases obtained by combining (A.25b) with Φ^c with $\varepsilon^c_r = -1$ (so that $\varepsilon_r = -2$);

— for (A.22e) the twelve cases obtained by combining (A.25b) with Φ^c with $\varepsilon^c_r = 0$ (so that $\varepsilon_r = -1$);

— for (A.22e) the two cases (A.25a) with $\Phi^c = X^+_{16}\,X^+_{26}$ (so that $\varepsilon_r = -1$).

Altogether, for (A.22c)–(A.22e) the overcounting is by $2, 10, 24$ states. Thus, the states we actually take out w.r.t. the case $j_1 \geqslant 1$, $j_2 = 1/2$ are $64, 64, 62, 54, 40$. Finally, for (A.22e) we have to take out the impossible state $X^+_{36}\,X^+_{45}\,|\Lambda\rangle$, cf. (2.25). Altogether the states remaining in the cases corresponding to (A.22a)–(A.22e) are $176, 175, 169, 150, 111$, resp.

• $j_1 = 0$, $j_2 \geqslant 1$. This is the case conjugate to the previous one. In addition to the states excluded in the case $j_1 = 1/2$, $j_2 \geqslant 1$, we exclude 64 states with $\varepsilon^c_j = -1$, (so $\varepsilon^c_j + 2j_1 = -1$):

$$X^+_{15}\,\Phi^a\,|\Lambda\rangle, \quad X^+_{16}\,X^+_{15}\,X^+_{25}\,\Phi^a\,|\Lambda\rangle, \tag{A.26a}$$

$$X^+_{16}\,\Phi^a\,|\Lambda\rangle, \quad X^+_{16}\,X^+_{15}\,X^+_{26}\,\Phi^a\,|\Lambda\rangle. \tag{A.26b}$$

We have to take into account that certain states were already taken out, namely, the following:

— for (A.22c)–(A.22e) the two cases (A.26b) with $\Phi^a = X^+_{35}\,X^+_{45}$ (so that $\varepsilon_r = -3$);

— for (A.22d), (A.22e) the eight cases obtained by combining (A.26b) with Φ^a with $\varepsilon^a_r = -1$ (so that $\varepsilon_r = -2$);

— for (A.22e) the twelve cases obtained by combining (A.26b) with Φ^a with $\varepsilon^a_r = 0$ (so that $\varepsilon_r = -1$);

— for (A.22e) the two cases (A.26a) with $\Phi^a = X^+_{35}\,X^+_{45}$ (so that $\varepsilon_r = -1$).

Altogether, excluding also the impossible state $X^+_{15}\,X^+_{26}\,|\Lambda\rangle$ (when $r_1 = 0$, cf. (2.24)), in the five cases corresponding to (A.22a)–(A.22e) we have now $176, 175, 169, 150, 111$ states.

• $j_1 = 1/2$, $j_2 = 0$. This is a combination of previous cases, so w.r.t. (A.22) we exclude the states in (A.23), (A.24), (A.25). Due to overlaps there are five states which are counted two times — those in (A.23), (A.25) when $\Phi^c = X^+_{16}\,X^+_{15}$. Thus, w.r.t. the case $j_1 \geqslant 1$, $j_2 = 1/2$ we would take out eleven states. However, from those cases the state (A.24) with $\Phi^a = X^+_{35}\,X^+_{45}$ was taken out in (A.22d), (A.22e) and the states (A.24) with $\Phi^a = X^+_{45}$ $\Phi^a = X^+_{45}\,X^+_{35}\,X^+_{46}$ were taken out in (A.22e). Thus, w.r.t. the case $j_1 \geqslant 1$, $j_2 = 1/2$ we take out $11, 11, 11, 10, 8$ states, and in the cases corresponding to (A.22a)–(A.22e) there are $165, 164, 158, 140$, 103 states.

• $j_1 = 0$, $j_2 = 1/2$. This case is conjugate to the previous one and so w.r.t. (A.22) we exclude the states in (A.23), (A.24), (A.26). With respect to the case $j_1 = 1/2$, $j_2 \geqslant 1$ we take out $11, 11, 11, 10, 8$ states. Thus, in the cases corresponding to (A.22a)–(A.22e) there are $165, 164, 158, 140, 103$ states.

• $j_1 = j_2 = 0$. This is a combination of previous cases so we exclude the states in (A.23), (A.24), (A.25), (A.26). Due to overlaps of (A.26) with (A.23) and (A.24) w.r.t. the case $j_1 = 1/2$, $j_2 = 0$ we would take out 44 states (instead of 64). However, from those cases the two states (A.26b) with $\Phi^a = X^+_{35} X^+_{45}$ were taken out in (A.22c)–(A.22e), the four states obtained from (A.26b) with $\Phi^a = X^+_{45}$, $\Phi^a = X^+_{45} X^+_{35} X^+_{46}$ were taken out in (A.22d), (A.22e), the two states (A.26b) with $\Phi^a = X^+_{45} X^+_{46}$ were taken out in (A.22e), the eight states obtained from (A.26b) with $\Phi^a = 1$, $\Phi^a = X^+_{35} X^+_{46}$, $\Phi^a = X^+_{45} X^+_{36}$, $\Phi^a = X^+_{35} X^+_{45} X^+_{36} X^+_{46}$ were taken out in (A.22e), the two states (A.26a) with $\Phi^a = X^+_{35} X^+_{45}$ were taken out in (A.22e). Thus, the states we actually take out w.r.t. the case $j_1 = 1/2$, $j_2 = 0$ are $44, 44, 42, 38, 26$. For (A.22e) we have also to take out two impossible states: (2.24) and its combination with (2.25):

$$X^+_{15} X^+_{26} X^+_{36} X^+_{45} |\Lambda\rangle. \tag{A.27}$$

Altogether the states remaining in the cases corresponding to (A.22a)–(A.22e) are $121, 120, 116, 102, 75$ states.

Thus, the smallest $N = 2$ long superfield has 75 states in $\hat{L}_\Lambda$. Since above the states we described by exclusion we would like to list these 75 states. First, there are 6 chiral states:

$$X^+_{25} |\Lambda\rangle, \quad X^+_{15} X^+_{25} |\Lambda\rangle, \quad X^+_{26} X^+_{15} X^+_{25} |\Lambda\rangle, \tag{A.28a}$$

$$X^+_{26} X^+_{25} |\Lambda\rangle, \quad X^+_{16} X^+_{25} |\Lambda\rangle, \quad X^+_{16} X^+_{26} X^+_{15} X^+_{25} |\Lambda\rangle \tag{A.28b}$$

and 6 antichiral states:

$$X^+_{46} |\Lambda\rangle, \quad X^+_{36} X^+_{46} |\Lambda\rangle, \quad X^+_{45} X^+_{36} X^+_{46} |\Lambda\rangle, \tag{A.29a}$$

$$X^+_{45} X^+_{46} |\Lambda\rangle, \quad X^+_{35} X^+_{46} |\Lambda\rangle, \quad X^+_{35} X^+_{45} X^+_{36} X^+_{46} |\Lambda\rangle. \tag{A.29b}$$

Now let $\Phi_c |\Lambda\rangle$, $\Phi_a |\Lambda\rangle$ denote any of the six states in (A.28), (A.29), resp., $\Phi'_c |\Lambda\rangle$, $\Phi'_a |\Lambda\rangle$ denote any of the three states in (A.28a), (A.29a), resp. Then, there are the following states:

$$|\Lambda\rangle, \quad \Phi_c \Phi_a |\Lambda\rangle, \tag{A.30a}$$

$$X^+_{15} X^+_{26} \Phi_a |\Lambda\rangle, \tag{A.30b}$$

$$X^+_{36} X^+_{45} \Phi_c |\Lambda\rangle, \tag{A.30c}$$

$$X^+_{26} \Phi'_a |\Lambda\rangle, \quad X^+_{16} X^+_{26} X^+_{25} \Phi'_a |\Lambda\rangle, \tag{A.30d}$$

$$X^+_{45} \Phi'_c |\Lambda\rangle, \quad X^+_{35} X^+_{45} X^+_{46} \Phi'_c |\Lambda\rangle, \tag{A.30e}$$

$$X^+_{35} X^+_{45} X^+_{15} X^+_{25} |\Lambda\rangle, \quad X^+_{16} X^+_{26} X^+_{36} X^+_{46} |\Lambda\rangle. \tag{A.30f}$$

Obviously, there are 63 states in (A.30) $(37+6+6+6+6+2)$ and altogether 75 states in (A.29), (A.28), and (A.30).

SRC Cases. Here we consider the SRC cases similarly to the long superfields taking again the five cases in (A.22) as reference point.

a) $d = d_{\max} = d^1_{21} = 2 + 2j_2 + z + r_1 > d^3_{22}$.

The maximal number of states is $128 = 16$(chiral)$\times 8$(antichiral), achieved for $j_1 \geqslant 1, r_1 \geqslant 4$.

• $j_2 > 0$. Here there hold the character formulae (2.36), or equivalently (2.39) or (2.43) when $r_1 > 0$, while for $r_1 = 0$ the character formula is (2.54) (for $i_0 = 1$). We give more detailed description.

The generator X^+_{36} is eliminated. The eight states in the antichiral sector are obtained by applying to the vacuum the following operators:

$$\begin{array}{lll}
X^+_{46}, & \varepsilon^a_r = 1, & \varepsilon^a_j = 1, \\
X^+_{45}\,X^+_{46}, & \varepsilon^a_r = 0, & \varepsilon^a_j = 2, \\
1, \quad X^+_{35}\,X^+_{46}, & \varepsilon^a_r = 0, & \varepsilon^a_j = 0, \\
X^+_{45}, \quad X^+_{45}\,X^+_{35}\,X^+_{46}, & \varepsilon^a_r = -1, & \varepsilon^a_j = 1, \\
X^+_{35}, & \varepsilon^a_r = -1, & \varepsilon^a_j = -1, \\
X^+_{35}\,X^+_{45}, & \varepsilon^a_r = -2, & \varepsilon^a_j = 0.
\end{array} \qquad \text{(A.31a)}$$

The above is equivalent to the antichiral part of character formula (2.36):

$$(1 + e(\alpha_{46}))(1 + e(\alpha_{35}))(1 + e(\alpha_{45})), \qquad \text{(A.31b)}$$

however, the more detailed description in (A.31a) is necessary to obtain the results on the counter-terms. In particular, for $r_1 = 1$ the last operator does not contribute to the antichiral sector, while for $r_1 = 0$ only the first three operators contribute to the antichiral sector, and the generator X^+_{35} is also eliminated from the whole basis.

In summary, the results are: When $j_1 \geqslant 1$, correspondingly to the cases in (A.22a)–(A.22e) we have now $128, 127, 120, 99, 42$ states. When $j_1 = 1/2$, correspondingly to the cases in (A.22a)–(A.22e) we have now $120, 119, 112, 92$, 39 states. When $j_1 = 0$, correspondingly to the cases in (A.22a)–(A.22e) we have now $88, 87, 82, 68, 28$ states.

When $r_1 > 0$, there holds formula (2.47) with $\beta = \alpha_{36}$, where $\hat{L}_{\rm long}$ is a long superfield with the same values of j_1 and r_i as Λ, and with $j_2 \geqslant 1$. Note that when the weight Λ corresponds to cases (A.22a)–(A.22e), then the weight $\Lambda + \alpha_{36}$ corresponds to cases (A.22a), (A.22a)–(A.22d) (since the value of r_1 is increased by 1). Thus, when $j_1 \geqslant 1$, the UIR $\hat{L}_{\Lambda+\alpha_{36}}$ has $128, 128, 127, 120$, 99 states, when $j_1 = 1/2$, it has $120, 120, 119, 112, 92$ states, when $j_1 = 0$, it has $88, 88, 87, 82, 68$ states. Summed together with the numbers for the UIR $\hat{L}_\Lambda$

from above we obtain the following contributions to $\hat{L}_{\rm long}$: when $j_1 \geqslant 1$, there are $256, 255, 247, 219, 141$ states, when $j_1 = 1/2$, there are $240, 239, 231, 204,$ 131 states, when $j_1 = 0$, there are $176, 175, 169, 150, 96$ states. Except the last cases (in which $r_1 = 0$) these cases match exactly (not only by numbers) the cases of long superfields for the corresponding values of $j_1 = 1, 1/2, 0$ and $j_2 \geqslant 1$.

When $r_1 = 0$, the long superfields have $163, 152, 111$ states, i.e., a mismatch of $22, 21, 15$ states. All these extra states contain the generator X^+_{35} and do not contain the generator X^+_{36}. Explicitly, when $j_1 \geqslant 1$ the 22 states are:

$$X^+_{35}\,\Phi^c_1\,\widetilde{|\Lambda\rangle}, \quad X^+_{35}\,X^+_{45}\,X^+_{46}\,\Phi^c_1\,\widetilde{|\Lambda\rangle}, \quad X^+_{35}\,X^+_{46}\,\Phi^c_2\,\widetilde{|\Lambda\rangle}, \quad X^+_{35}\,X^+_{45}\,X^+_{15}\,X^+_{25}\,\widetilde{|\Lambda\rangle}, \tag{A.32}$$

where Φ^c_1 denotes the 5 chiral operators of the first three rows of (A.20), Φ^c_2 denotes the 11 chiral operators of the first six rows of (A.20). When $j_1 = 1/2$, the 21 states are as in (A.32) except the state $X^+_{35}\,X^+_{46}\,X^+_{16}\,X^+_{15}\,\widetilde{|\Lambda\rangle}$ which is not in the long superfield (since $\varepsilon^c_j + 2j_1 = -1$). When $j_1 = 0$, the 16 states are as in (A.32) except the state excluded for $j_1 = 1/2$ and six states which are obtained for $\Phi^c_1 = \Phi^c_2 = X^+_{16}, X^+_{16}X^+_{15}\,X^+_{25}$ (i.e., excluding the third row of (A.20), since for them $\varepsilon^c_j + 2j_1 = \varepsilon^c_j = -1$). Altogether, instead of the decomposition (2.47) we have the quasi-decomposition (2.53):

$$\left(\hat{L}_{\rm long}\right)\Big|_{d=d^a} = \hat{L}_\Lambda \oplus \hat{L}_{\Lambda+\alpha_{36}} \oplus \hat{L}'_{\Lambda+\alpha_{35}}, \quad r_1 = 0. \tag{A.33}$$

The 28 states of the minimal case are given as follows. There are two antichiral states:

$$X^+_{46}\,|\Lambda\rangle, \quad X^+_{45}\,X^+_{46}\,|\Lambda\rangle \tag{A.34}$$

and six chiral states (just as in (A.28)):

$$X^+_{25}\,|\Lambda\rangle, \quad X^+_{15}\,X^+_{25}\,|\Lambda\rangle, \quad X^+_{26}\,X^+_{15}\,X^+_{25}\,|\Lambda\rangle, \tag{A.35a}$$

$$X^+_{26}\,X^+_{25}\,|\Lambda\rangle, \quad X^+_{16}\,X^+_{25}\,|\Lambda\rangle, \quad X^+_{16}\,X^+_{26}\,X^+_{15}\,X^+_{25}\,|\Lambda\rangle. \tag{A.35b}$$

Combining the chiral and antichiral states would give further 12 states. The rest of the states are obtained by combining these states with impossible states from the opposite chirality, yet obtaining allowed states. Explicitly, the list looks like this. Let $\Phi_a\,|\Lambda\rangle$, $\Phi_c\,|\Lambda\rangle$ denote any of the states in (A.34), (A.35), resp., $\Phi'_c\,|\Lambda\rangle$ denote any of the three states in (A.35a), resp. Thus, there are the following states:

$$|\Lambda\rangle, \quad \Phi_c\,\Phi_a\,|\Lambda\rangle, \tag{A.36a}$$

$$X^+_{15}\,X^+_{26}\,\Phi_a\,|\Lambda\rangle, \tag{A.36b}$$

$$X^+_{26}\,X^+_{46}\,|\Lambda\rangle, \quad X^+_{26}\,X^+_{16}\,X^+_{25}\,X^+_{46}\,|\Lambda\rangle, \tag{A.36c}$$

$$X^+_{45}\,\Phi'_c\,|\Lambda\rangle. \tag{A.36d}$$

Obviously, there are 20 states in (A.36) $(13+2+2+3)$. Altogether, there are 28 states in (A.34), (A.35), and (A.36). This list amounts to giving explicitly character formula (2.54) (for $N=2$, $i_0=N-1=1$) without counter-terms. This superfield and its conjugate (considered below) are the shortest semishort SRC $N=2$ superfields.

• $j_2=0$. Here there holds character formula (2.56) and for $r_1=0$ there holds also character formula (2.60). A more detailed description follows.

The state $X_{36}^+ X_{46}^+ |\Lambda\rangle$ and its descendants are eliminated (due to (2.55)). This elimination is described by the second term in character formula (2.56a). The eight states in the antichiral sector here come from:

$$\begin{aligned} &X_{46}^+, && \varepsilon_r^a=1, \quad \varepsilon_j^a=1,\\ &X_{45}^+ X_{46}^+, && \varepsilon_r^a=0, \quad \varepsilon_j^a=2,\\ &1, \quad X_{35}^+ X_{46}^+, \quad X_{45}^+ X_{36}^+, && \varepsilon_r^a=0, \quad \varepsilon_j^a=0,\\ &X_{45}^+, \quad X_{45}^+ X_{35}^+ X_{46}^+, && \varepsilon_r^a=-1, \quad \varepsilon_j^a=1,\\ &X_{35}^+ X_{45}^+, && \varepsilon_r^a=-2, \quad \varepsilon_j^a=0. \end{aligned} \tag{A.37}$$

The above eight differ from (A.31) by one operator: X_{35}^+ is replaced here by $X_{45}^+ X_{36}^+$. For $r_1=1$ the last operator does not contribute to the antichiral sector. Whenever $r_1=0$, the generators X_{35}^+ and X_{36}^+ are eliminated from the antichiral part of the basis, which is further restricted due to (2.23c) and there are only two antichiral states as given in (A.34).

In summary, when $j_1 \geqslant 1$, correspondingly to the cases in (A.22a)–(A.22e) we have now $128, 127, 121, 103, 68$ states. When $j_1=1/2$, correspondingly to the cases in (A.22a)–(A.22e) we have now $120, 119, 113, 96, 63$ states. When $j_1=0$, correspondingly to the cases in (A.22a)–(A.22e) we have now $88, 87, 83, 70,$ 45 states.

We know that when $r_1>0$, there holds formula (2.57) for $\hat{L}_{\rm long}$ with the same values of $j_1, j_2(=0), r_1$ as Λ and with $\beta=\beta_{12}=\alpha_{36}+\alpha_{46}$. In more detail, when the weight Λ corresponds to cases (A.22a)–(A.22e), then the weight $\Lambda+\beta_{12}$ corresponds to cases (A.22a), (A.22a), (A.22a), (A.22b), (A.22c) (since the value of r_1 is increased by 2) and furthermore $\hat{L}_{\Lambda+\beta_{12}}$ is actually a SRC of type b), see below from where we take the numbers: When $j_1 \geqslant 1$, the UIR $\hat{L}_{\Lambda+\beta_{12}}$ has $48, 48, 48, 47, 42$ states, when $j_1=1/2$, it has $45, 45, 45, 44, 39$ states, when $j_1=0$, it has $33, 33, 33, 32, 29$ states. Summed together with the numbers for the UIR $\hat{L}_\Lambda$ from above we obtain the following contributions to $\hat{L}_{\rm long}$: when $j_1 \geqslant 1$, there are $176, 175, 169, 150, 110$ states, when $j_1=1/2$, there are $165, 164, 158, 140, 102$ states, when $j_1=0$, there are $121, 120, 116, 102, 74$ states. Except the last cases (when $r_1=0$) these cases match exactly the cases of long superfields for the corresponding values of $j_1=1, 1/2, 0$ and $j_2=0$. For completeness one may check that the states of $\hat{L}_{\Lambda+\beta_{12}}$ appear in $\hat{L}_{\rm long}$ being multiplied by $X^+_{36}\,X^+_{46}$. In the cases when $r_1 = 0$, there is a mismatch of one state and that extra state is $X^+_{35}\,X^+_{46}\,\widetilde{|\Lambda\rangle}$ which is excluded from $\hat{L}_\Lambda$ as explained in general, cf. (2.58). (It is also excluded in case b) below.) Thus, instead of (2.57) we have the quasi-decomposition:

$$\left(\hat{L}_{\rm long}\right)\Big|_{d=d^a} = \hat{L}_\Lambda \oplus \hat{L}_{\Lambda+\beta_{12}} \oplus \hat{L}'_{\Lambda+\alpha_{35}+\alpha_{46}}, \qquad r_1 = 0, \tag{A.38}$$

where as in (A.33) we have put a prime on the last term indicating that this is not a genuine irrep.

b) $d = d^2_{21} = z + r_1 > d^3_{22}, j_2 = 0$.

The character formula is (2.65). The generators X^+_{36} and X^+_{46} are eliminated due to (2.61) and (2.62). Due to (2.23b) there are at most two antichiral states:

$$X^+_{45}\,|\Lambda\rangle, \quad X^+_{35}\,X^+_{45}\,|\Lambda\rangle. \tag{A.39}$$

Thus, the maximal number of states is $48(16\times 3)$ achieved for $r_1 \geqslant 4$, $j_1 \geqslant 1$. These states are given explicitly as

$$\Psi_{\bar\varepsilon} = (X^+_{16})^{\varepsilon_{16}}\,(X^+_{15})^{\varepsilon_{15}}\,(X^+_{26})^{\varepsilon_{26}}\,(X^+_{25})^{\varepsilon_{25}}\,(X^+_{35})^{\varepsilon_{35}}\,(X^+_{45})^{\varepsilon_{45}}\,|\Lambda\rangle, \tag{A.40}$$
$$\varepsilon_{aj} = 0{,}1; \quad \varepsilon_{35} \leqslant \varepsilon_{45}; \quad r_1 \geqslant 4, \quad j_1 \geqslant 1.$$

In summary, when $j_1 \geqslant 1$, we have correspondingly to the cases in (A.22a)–(A.22e) $48, 47, 42, 31, 10$ states. When $j_1 = 1/2$, we have correspondingly to the cases in (A.22a)–(A.22e) $45, 44, 39, 29, 9$ states. When $j_1 = 0$, we have correspondingly to the cases in (A.22a)–(A.22e) $33, 32, 29, 23, 7$ states. The cases when $r_1 > 2$ were included in decompositions (2.57) in the previous case a). (The cases when $r_1 = 2$ were included in quasi-decompositions (A.38) in the previous case a).)

The minimal number, when $r_1 > 0$, is 23 achieved for $r_1 = 1$, $j_1 = 0$. Besides the obvious states, which include $X^+_{45}\,|\Lambda\rangle$, nine chiral states, their combinations and the vacuum, there are the following states:

$$X^+_{35}\,X^+_{45}\,\Phi'\,|\Lambda\rangle, \tag{A.41}$$
$$\Phi' = X^+_{25}, X^+_{15}\,X^+_{25}, X^+_{26}\,X^+_{15}\,X^+_{25}.$$

Whenever $r_1 = 0$, the generators X^+_{35} and X^+_{45} are also eliminated from the basis due to (2.63). Thus, these UIRs are chiral. Due to (2.23c) and excluding the state (2.24) there are $10, 9, 7$ states for $j_1 \geqslant 1, 1/2, 0$, resp. (as stated above). These states explicitly are

$$|\Lambda\rangle, \quad X^+_{25}\,|\Lambda\rangle, \quad X^+_{15}\,X^+_{25}\,|\Lambda\rangle, \quad X^+_{16}\,X^+_{25}\,|\Lambda\rangle, \quad X^+_{26}\,X^+_{25}\,|\Lambda\rangle,$$
$$X^+_{26}\,X^+_{15}\,X^+_{25}\,|\Lambda\rangle, \quad X^+_{16}\,X^+_{26}\,X^+_{15}\,X^+_{25}\,|\Lambda\rangle, \quad j_1 \geqslant 0, \tag{A.42a}$$
$$X^+_{15}\,|\Lambda\rangle, X^+_{16}\,X^+_{15}\,X^+_{25}\,|\Lambda\rangle, \quad j_1 \geqslant 1/2, \tag{A.42b}$$
$$X^+_{16}\,X^+_{15}\,|\Lambda\rangle, \quad j_1 \geqslant 1. \tag{A.42c}$$

For $j_1 = 0$, the superfield in (A.42a) and its conjugate (considered below) are the shortest short SRC $N = 2$ superfields.

c) $d = d_{\max} = d^3_{22} = 2 + 2j_1 - z + r_1 > d^1_{21}$.

This case is conjugate to a) and the maximal number of states is $128 = 8$ (chiral) $\times 16$ (antichiral) achieved for $j_2 \geqslant 1, r_1 \geqslant 4$.

• $j_1 > 0$. The generator X^+_{15} is eliminated. The eight states in the chiral sector are obtained from the following operators:

$$\begin{aligned}
&X^+_{25}, && \varepsilon^c_r = 1, \quad \varepsilon^c_j = 1,\\
&X^+_{26}\,X^+_{25}, && \varepsilon^c_r = 0, \quad \varepsilon^c_j = 2,\\
&1, \quad X^+_{16}\,X^+_{25}, && \varepsilon^c_r = 0, \quad \varepsilon^c_j = 0,\\
&X^+_{26}, X^+_{26}\,X^+_{16}\,X^+_{25}, && \varepsilon^c_r = -1, \quad \varepsilon^c_j = 1,\\
&X^+_{16}, && \varepsilon^c_r = -1, \quad \varepsilon^c_j = -1,\\
&X^+_{16}\,X^+_{26}, && \varepsilon^c_r = -2, \quad \varepsilon^c_j = 0.
\end{aligned} \tag{A.43}$$

In summary, when $j_2 \geqslant 1$, correspondingly to the cases in (A.22a)–(A.22e) we have now $128, 127, 120, 99, 42$ states. When $j_2 = 1/2$, correspondingly to the cases in (A.22a)–(A.22e) we have now $120, 119, 112, 92, 39$ states. When $j_2 = 0$, correspondingly to the cases in (A.22a)–(A.22e) we have now $88, 87, 82, 68$, 28 states. Whenever $r_1 = 0$, the generator X^+_{16} is also eliminated from the basis.

When $r_1 > 0$, there holds decomposition (2.71) with $\beta = \alpha_{15}$. When $r_1 = 0$, there holds the quasi-decomposition:

$$\left(\hat{L}_{\rm long}\right)\Big|_{d=d^c} = \hat{L}_\Lambda \oplus \hat{L}_{\Lambda+\alpha_{15}} \oplus\ \hat{L}'_{\Lambda+\alpha_{16}}, \quad r_1 = 0, \tag{A.44}$$

cf. (A.33). We omit most details since all results and formulae are by conjugation from case a) (when $j_2 \neq 0$).

We still give the 28 states of the minimal case. There are two chiral states:

$$X^+_{25}\,|\Lambda\rangle, \quad X^+_{26}\,X^+_{25}\,|\Lambda\rangle \tag{A.45}$$

and six antichiral states (just as in (A.29)):

$$X^+_{46}\,|\Lambda\rangle, \quad X^+_{36}\,X^+_{46}\,|\Lambda\rangle, \quad X^+_{45}\,X^+_{36}\,X^+_{46}\,|\Lambda\rangle, \tag{A.46a}$$

$$X^+_{45}\,X^+_{46}\,|\Lambda\rangle, \quad X^+_{35}\,X^+_{46}\,|\Lambda\rangle, \quad X^+_{35}\,X^+_{45}\,X^+_{36}\,X^+_{46}\,|\Lambda\rangle. \tag{A.46b}$$

The rest of the states are obtained as follows. Let $\hat{\Phi}_a\,|\Lambda\rangle$, $\hat{\Phi}_c\,|\Lambda\rangle$ denote any of the states in (A.45), (A.46), resp., $\hat{\Phi}'_c\,|\Lambda\rangle$ denote any of the three states in

(A.46a), resp. Thus, there are the following states:

$$|\Lambda\rangle, \quad \hat{\Phi}_c\,\hat{\Phi}_a\,|\Lambda\rangle, \tag{A.47a}$$

$$X^+_{36}\,X^+_{45}\,\hat{\Phi}_c\,|\Lambda\rangle, \tag{A.47b}$$

$$X^+_{45}\,X^+_{25}\,|\Lambda\rangle, \quad X^+_{45}\,X^+_{35}\,X^+_{46}\,X^+_{25}\,|\Lambda\rangle, \tag{A.47c}$$

$$X^+_{26}\,\hat{\Phi}'_c\,|\Lambda\rangle. \tag{A.47d}$$

This superfield and its conjugate (considered in a)) are the shortest semishort SRC $N = 2$ superfields.

• $j_1 = 0$. The state $X^+_{15}\,X^+_{25}\,|\Lambda\rangle$ and its descendants are eliminated (due to (2.55)). The eight states in the chiral sector here come from:

$$\begin{array}{lll} X^+_{25}, & \varepsilon^c_r = 1, & \varepsilon^c_j = 1, \\ X^+_{26}\,X^+_{25}, & \varepsilon^c_r = 0, & \varepsilon^c_j = 2, \\ 1, \quad X^+_{16}\,X^+_{25}, \quad X^+_{26}\,X^+_{15}, & \varepsilon^c_r = 0, & \varepsilon^c_j = 0, \\ X^+_{26}, \quad X^+_{26}\,X^+_{16}\,X^+_{25}, & \varepsilon^c_r = -1, & \varepsilon^c_j = 1, \\ X^+_{16}\,X^+_{26}, & \varepsilon^c_r = -2, & \varepsilon^c_j = 0. \end{array} \tag{A.48}$$

The above eight states differ from (A.43) by one operator: X^+_{16} is replaced here by $X^+_{26}\,X^+_{15}$. In summary, when $j_2 \geqslant 1$, correspondingly to the cases in (A.22a)–(A.22e) we have now $128, 127, 121, 103, 68$ cases. When $j_2 = 1/2$, correspondingly to the cases in (A.22a)–(A.22e) we have now $120, 119, 113, 96,$ 63 cases. When $j_2 = 0$, correspondingly to the cases in (A.22a)–(A.22e) we have now $88, 87, 83, 70, 45$ cases. Whenever $r_1 = 0$, the generators X^+_{16} and X^+_{15} are eliminated from the chiral part of the basis, which is further restricted due to (2.23c) and there are only two chiral states as in (A.45).

When $r_1 > 0$, there holds formula (2.74) for $\hat{L}_{\rm long}$ with the same values of $j_1(=0), j_2, r_1$ as for Λ and with $\beta = \beta_{34} = \alpha_{15} + \alpha_{25}$. When $r_1 = 0$, this decomposition is spoiled by one state $X^+_{16}\,X^+_{25}\,\widetilde{|\Lambda\rangle}$ which is excluded from $\hat{L}_\Lambda$ as explained in general, cf. (2.75), and instead of (2.74) we have the quasi-decomposition:

$$\left(\hat{L}_{\rm long}\right)\Big|_{d=d^c} = \hat{L}_\Lambda \oplus \hat{L}_{\Lambda+\beta_{34}} \oplus \hat{L}'_{\Lambda+\alpha_{16}+\alpha_{25}}, \quad r_1 = 0. \tag{A.49}$$

d) $d = d^4_{22} = -z + r_1 > d^1_{21}, \quad j_1 = 0.$

This case is conjugate to b).

The generators X^+_{15} and X^+_{25} are eliminated due to (2.77) and (2.78). Due to (2.23b) there are at most two chiral states depending on the value of r_1:

$$\begin{array}{rl} X^+_{26}\,|\Lambda\rangle, & r_1 \geqslant 1, \\ X^+_{16}\,X^+_{26}\,|\Lambda\rangle, & r_1 \geqslant 2. \end{array} \tag{A.50}$$

The maximal number of states is $48(3 \times 16)$ achieved for $r_1 \geqslant 4$, $j_2 \geqslant 1$. These states are given explicitly as

$$\begin{gathered}\Psi_{\bar{\varepsilon}} = (X_{35}^+)^{\varepsilon_{35}}\,(X_{36}^+)^{\varepsilon_{36}}\,(X_{45}^+)^{\varepsilon_{45}}\,(X_{46}^+)^{\varepsilon_{46}}\,(X_{16}^+)^{\varepsilon_{16}}\,(X_{26}^+)^{\varepsilon_{26}}\,|\Lambda\rangle, \\ \varepsilon_{aj} = 0,1; \quad \varepsilon_{35} \leqslant \varepsilon_{45}; \quad r_1 \geqslant 2, \quad j_2 \geqslant 1. \end{gathered} \tag{A.51}$$

In summary, when $j_2 \geqslant 1$, correspondingly to the cases in (A.22a)–(A.22e) we have now $48, 47, 42, 31, 10$ states. When $j_2 = 1/2$, correspondingly to the cases in (A.22a)–(A.22e) we have now $45, 44, 39, 29, 9$ states. When $j_2 = 0$, correspondingly to the cases in (A.22a)–(A.22e) we have now $33, 32, 29, 23,$ 7 states. The cases, when $r_1 > 2$, were included in decompositions (2.74) in the previous case c). (The cases, when $r_1 = 2$, were included in decompositions (A.49) in the previous case c).)

The minimal number, when $r_1 > 0$, is 23 achieved for $r_1 = 1$, $j_2 = 0$. Besides the obvious states which include $X_{26}^+\,|\Lambda\rangle$, nine antichiral states, their combinations and the vacuum, there are the following states:

$$\begin{gathered} X_{16}^+\,X_{26}^+\,\Phi'\,|\Lambda\rangle, \\ \Phi' = X_{46}^+, X_{36}^+\,X_{46}^+, X_{45}^+\,X_{36}^+\,X_{46}^+. \end{gathered} \tag{A.52}$$

Whenever $r_1 = 0$, the generators X_{16}^+ and X_{26}^+ are also eliminated from the basis due to (2.63). Thus, these UIRs are antichiral. Due to (2.23c) and excluding the state (2.25) there are $10, 9, 7$ states for $j_2 \geqslant 1, 1/2, 0$, resp. These states explicitly are

$$\begin{gathered} |\Lambda\rangle, X_{46}^+\,|\Lambda\rangle, \quad X_{36}^+\,X_{46}^+\,|\Lambda\rangle, \quad X_{35}^+\,X_{46}^+\,|\Lambda\rangle, \quad X_{45}^+\,X_{46}^+\,|\Lambda\rangle, \\ X_{45}^+\,X_{36}^+\,X_{46}^+\,|\Lambda\rangle, \quad X_{35}^+\,X_{45}^+\,X_{36}^+\,X_{46}^+\,|\Lambda\rangle, \quad j_2 \geqslant 0, \end{gathered} \tag{A.53a}$$

$$X_{36}^+\,|\Lambda\rangle, \quad X_{35}^+\,X_{36}^+\,X_{46}^+\,|\Lambda\rangle, \quad j_2 > 0, \tag{A.53b}$$

$$X_{35}^+\,X_{36}^+\,|\Lambda\rangle, \quad j_2 \geqslant 1. \tag{A.53c}$$

For $j_2 = 0$ the superfield in (A.53a) and its conjugate (considered above) are the shortest short SRC $N = 2$ superfields.

DRC Cases. Here we consider the DRC cases taking again the five cases of long superfields in (A.22) as a reference point.

ac) $d = d_{\max} = d_{21}^1 = d_{22}^3 = 2 + j_1 + j_2 + r_1$, $z = j_1 - j_2$.

The maximal number of states is $64 = 8$(chiral)$\times 8$(antichiral), achieved for $r_1 \geqslant 4$. The 8 antichiral, chiral states are as described in a), c), resp. (differing for $j_2 > 0$ and $j_2 = 0$, $j_1 > 0$ and $j_1 = 0$, resp.).

• $j_1 j_2 > 0$. Here character formulae (2.84) hold (without counterterms for $r_1 \geqslant 4$). The states $X_{15}^+\,|\Lambda\rangle$, $X_{36}^+\,|\Lambda\rangle$ and their descendants are eliminated. Correspondingly to the cases in (A.22a)–(A.22e) we have now $64, 63, 57, 42, 11$ states. In the last case (where $r_1 = 0$), we eliminate also the generators X_{35}^+ and X_{16}^+.

For $r_1 > 0$, there holds decomposition (2.83) with $\beta = \alpha_{15}$, $\beta' = \alpha_{36}$ as stated in the general exposition. We would like to demonstrate this and also see how it breaks down for $r_1 = 0$, thus, we include for the moment the case $r_1 = 0$. Referring to (2.83) we note when the weight Λ corresponds to cases (A.22a)–(A.22e) then the weights $\Lambda + \alpha_{15}$, $\Lambda + \alpha_{36}$ correspond to cases (A.22a), (A.22a), (A.22b), (A.22c), (A.22d) (since the value of r_1 is increased by 1), i.e., the corresponding UIRs have $64, 64, 63, 57, 42$ states each. The weight $\Lambda + \alpha_{15} + \alpha_{36}$ corresponds to cases (A.22a), (A.22a), (A.22a), (A.22b), (A.22c) (since the value of r_1 is increased by 2), i.e., the corresponding UIRs have $64, 64, 64, 63, 57$ states. Summed together with the numbers for the UIR $\hat{L}_\Lambda$ from above we obtain the following contributions to $\hat{L}_{\text{long}}$: $256, 255, 247, 219, 152$. Except the last case (in which $r_1 = 0$) these cases match exactly the cases of long superfields for the case $j_1, j_2 \geqslant 1$.

When $r_1 = 0$, the long superfields for the cases $j_1, j_2 \geqslant 1$ have 163 states, i.e., a mismatch of 11 states*. These extra states contain either the generator X^+_{16} or X^+_{35} or both, and they do not contain either X^+_{15} or X^+_{36}. Explicitly, these extra states are

$$\begin{aligned}
&X^+_{16}\, X^+_{46}\, |\Lambda\rangle, \quad X^+_{16}\, X^+_{25}\, |\Lambda\rangle, \quad X^+_{16}\, X^+_{25}\, X^+_{46}\, |\Lambda\rangle, \\
&X^+_{16}\, X^+_{26}\, X^+_{25}\, X^+_{46}\, |\Lambda\rangle, \quad X^+_{16}\, X^+_{25}\, X^+_{45}\, X^+_{46}\, |\Lambda\rangle, \\
&X^+_{35}\, X^+_{46}\, |\Lambda\rangle, \quad X^+_{35}\, X^+_{25}\, |\Lambda\rangle, \quad X^+_{35}\, X^+_{25}\, X^+_{46}\, |\Lambda\rangle, \\
&X^+_{35}\, X^+_{26}\, X^+_{25}\, X^+_{46}\, |\Lambda\rangle, \quad X^+_{35}\, X^+_{25}\, X^+_{45}\, X^+_{46}\, |\Lambda\rangle, \\
&X^+_{16}\, X^+_{35}\, X^+_{25}\, X^+_{46}\, |\Lambda\rangle.
\end{aligned} \tag{A.54}$$

Altogether, instead of (2.83) we may write:

$$\begin{aligned}
\left(\hat{L}_{\text{long}}\right)\Big|_{d=d^{ac}} = \hat{L}_\Lambda \oplus \hat{L}_{\Lambda+\alpha_{15}} \oplus \hat{L}_{\Lambda+\alpha_{36}} \oplus \hat{L}_{\Lambda+\alpha_{15}+\alpha_{36}} \oplus \\
\oplus \hat{L}'_{\Lambda+\alpha_{16}} \oplus \hat{L}'_{\Lambda+\alpha_{35}} \oplus\ \hat{L}'_{\Lambda+\alpha_{16}+\alpha_{35}}, \quad r_1 = 0,
\end{aligned} \tag{A.55}$$

where we have represented the extra states by the last three terms (corresponding to the first and second line of (A.54), the third and fourth line of (A.54), the fifth line of (A.54), resp.), and we have put primes on these since they are not genuine irreps.

*The reader may wonder whether the long superfield with $j_1 = 1/2, j_2 \geqslant 1, r_1 = 0$ may not be used since it has 152 states, however, this is only a coincidence of the total number.

Finally, we give the 11 states of the UIR at $r_1 = 0$:

$$\begin{aligned} &|\Lambda\rangle, \quad X^+_{25}\, X^+_{46}\, |\Lambda\rangle, \quad X^+_{26}\, X^+_{45}\, X^+_{25}\, X^+_{46}\, |\Lambda\rangle, \\ &X^+_{46}\, |\Lambda\rangle, \quad X^+_{45}\, X^+_{46}\, |\Lambda\rangle, \\ &X^+_{25}\, |\Lambda\rangle, \quad X^+_{26}\, X^+_{25}\, |\Lambda\rangle, \\ &\Phi^0_c\, X^+_{46}\, |\Lambda\rangle, \quad \Phi^0_c = X^+_{26}, \quad X^+_{26}\, X^+_{25}, \\ &\Phi^0_a\, X^+_{25}\, |\Lambda\rangle, \quad \Phi^0_a = X^+_{45}, \quad X^+_{45}\, X^+_{46}. \end{aligned} \tag{A.56}$$

This superfield is the shortest semishort $N = 2$ superfield.

• $j_1 > 0, j_2 = 0$. Here there hold character formulae (2.86) (without counterterms for $r_1 \geqslant 4$). The states $X^+_{36}\, X^+_{46}\, |\Lambda\rangle$, $X^+_{15}\, |\Lambda\rangle$ and their descendants are eliminated. Correspondingly to the cases in (A.22a)–(A.22e) we have now $64, 63, 58, 45, 16$ states. In the last case, where $r_1 = 0$, we eliminate the generator X^+_{16} and exclude the generators $X^+_{3,4+k}$ from the antichiral sector.

For $r_1 > 0$, there holds decomposition (2.87). Note that when the weight Λ corresponds to cases (A.22a)–(A.22e), then the weight $\Lambda + \alpha_{15}$ corresponds to cases (A.22a), (A.22a), (A.22b), (A.22c), (A.22d) (since the value of r_1 is increased by 1), i.e., the corresponding UIRs have $64, 64, 63, 58, 45$ states. The weight $\Lambda + \beta_{12}$ corresponds to cases (A.22a), (A.22a), (A.22a), (A.22b), (A.22c) (since the value of r_1 is increased by 2), but from type bc) considered below, i.e., the corresponding UIRs have $24, 24, 24, 23, 19$ states. The weight $\Lambda + \alpha_{15} + \beta_{12}$ corresponds to cases (A.22a), (A.22a), (A.22a), (A.22a), (A.22b) (since the value of r_1 is increased by 3), also from type bc), i.e., the corresponding UIRs have $24, 24, 24, 24, 23$ states. Summed together with the numbers for the UIR $\hat{L}_\Lambda$ from above we obtain the following contributions to $\hat{L}_{\rm long}$: $176, 175, 169, 150, 103$. Except the last case (in which $r_1 = 0$) these cases match exactly the cases of long superfields for the cases when $j_1 \geqslant 1, j_2 = 0$.

When $r_1 = 0$, the corresponding long superfields have 111 states, i.e., there is a mismatch of 8 states*. These extra states contain either the generator X^+_{16} or X^+_{35} or both, and they do not contain X^+_{15}. Explicitly, they are

$$\begin{aligned} &X^+_{16}\, X^+_{46}\, |\Lambda\rangle, \quad X^+_{16}\, X^+_{25}\, |\Lambda\rangle, \quad X^+_{16}\, X^+_{25}\, X^+_{46}\, |\Lambda\rangle, X^+_{16}\, X^+_{26}\, X^+_{25}\, X^+_{46}\, |\Lambda\rangle, \\ &X^+_{16}\, X^+_{25}\, X^+_{45}\, X^+_{46}\, |\Lambda\rangle, \quad X^+_{16}\, X^+_{25}\, X^+_{36}\, X^+_{45}\, |\Lambda\rangle, \\ &X^+_{35}\, X^+_{46}\, |\Lambda\rangle, \\ &X^+_{16}\, X^+_{35}\, X^+_{25}\, X^+_{46}\, |\Lambda\rangle. \end{aligned} \tag{A.57}$$

*Again the long superfield with correct number of states 103 (with $j_1 = 1/2, j_2 = r_1 = 0$) does not fit.

Altogether, instead of (2.87) we may write:

$$\left(\hat{L}_{\text{long}}\right)\Big|_{d=d^{ac}} = \hat{L}_\Lambda \oplus \hat{L}_{\Lambda+\alpha_{15}} \oplus \hat{L}_{\Lambda+\beta_{12}} \oplus \hat{L}_{\Lambda+\alpha_{15}+\beta_{12}} \oplus \\ \oplus \hat{L}'_{\Lambda+\alpha_{16}} \oplus \hat{L}'_{\Lambda+\alpha_{35}} \oplus \hat{L}'_{\Lambda+\alpha_{16}+\alpha_{35}}, \quad r_1 = 0, \quad \text{(A.58)}$$

where (as in (A.55)) we have represented the extra states by the last three terms (corresponding to the first and second line of (A.57), third line of (A.57), fourth line of (A.57), resp.), and we have put primes on these since they are not genuine irreps.

Finally, we give the 16 states of the UIR at $r_1 = 0$:

$$\begin{aligned}
&|\Lambda\rangle, && X^+_{25}\, X^+_{46}\, |\Lambda\rangle, \\
&X^+_{46}\, |\Lambda\rangle, && X^+_{45}\, X^+_{46}\, |\Lambda\rangle, \\
&X^+_{25}\, |\Lambda\rangle, && X^+_{26}\, X^+_{25}\, |\Lambda\rangle, \\
&\Phi^0_a\, X^+_{25}\, |\Lambda\rangle, && \Phi^0_a = X^+_{45}\, X^+_{46}, X^+_{35}\, X^+_{46}, X^+_{36}\, X^+_{45}, \\
&\Phi^-_a\, X^+_{25}\, |\Lambda\rangle, && \Phi^-_a = X^+_{45}, X^+_{35}\, X^+_{45}\, X^+_{46}, \\
&\hat{\Phi}_c\, X^+_{46}\, |\Lambda\rangle, && \hat{\Phi}_c = X^+_{26}, X^+_{26}\, X^+_{25}, \\
&\Phi^0_a\, X^+_{45}\, X^+_{25}\, |\Lambda\rangle.
\end{aligned} \quad \text{(A.59)}$$

The states of (A.56) are a subset of (A.59).

The next case is conjugate to the preceding.

• $j_1 = 0, j_2 > 0$. Here character formulae (2.89) hold (without counterterms for $r_1 \geqslant 4$). The states $X^+_{15}\, X^+_{25}\, |\Lambda\rangle$, $X^+_{36}\, |\Lambda\rangle$ and their descendants are eliminated. Correspondingly to the cases in (A.22a)–(A.22e) we have now $64, 63, 58, 45, 16$ states. In the last case, when $r_1 = 0$, we eliminate the generator X^+_{35} and exclude the generators $X^+_{1,4+k}$ from the chiral sector.

For $r_1 > 0$, decomposition (2.90) holds. When $r_1 = 0$, the corresponding long superfields have 111 states, i.e., there is a mismatch of 8 states. These extra states contain either the generator X^+_{16} or X^+_{35} or both, and they do not contain X^+_{36}. Explicitly, these extra states are

$$\begin{aligned}
&X^+_{35}\, X^+_{25}\, |\Lambda\rangle, \quad X^+_{35}\, X^+_{46}\, |\Lambda\rangle, \quad X^+_{35}\, X^+_{46}\, X^+_{25}\, |\Lambda\rangle, \quad X^+_{35}\, X^+_{45}\, X^+_{46}\, X^+_{25}\, |\Lambda\rangle, \\
&X^+_{35}\, X^+_{46}\, X^+_{26}\, X^+_{25}\, |\Lambda\rangle, \quad X^+_{35}\, X^+_{46}\, X^+_{15}\, X^+_{26}\, |\Lambda\rangle, \\
&X^+_{16}\, X^+_{25}\, |\Lambda\rangle, \\
&X^+_{35}\, X^+_{16}\, X^+_{46}\, X^+_{25}\, |\Lambda\rangle.
\end{aligned} \quad \text{(A.60)}$$

Altogether, instead of (2.90) we may write:

$$\left(\hat{L}_{\text{long}}\right)\Big|_{d=d^{ac}} = \hat{L}_\Lambda \oplus \hat{L}_{\Lambda+\alpha_{36}} \oplus \hat{L}_{\Lambda+\beta_{34}} \oplus \hat{L}_{\Lambda+\alpha_{36}+\beta_{34}} \oplus \\ \oplus \hat{L}'_{\Lambda+\alpha_{16}} \oplus \hat{L}'_{\Lambda+\alpha_{35}} \oplus \hat{L}'_{\Lambda+\alpha_{16}+\alpha_{35}}, \quad r_1 = 0. \quad \text{(A.61)}$$

Finally, we give the 16 states of the UIR at $r_1 = 0$:

$$\begin{array}{ll} |\Lambda\rangle, & X^+_{25}\, X^+_{46}\, |\Lambda\rangle, \\ X^+_{46}\, |\Lambda\rangle, & X^+_{45}\, X^+_{46}\, |\Lambda\rangle, \\ X^+_{25}\, |\Lambda\rangle, & X^+_{26}\, X^+_{25}\, |\Lambda\rangle, \\ \Phi^0_c\, X^+_{46}\, |\Lambda\rangle, & \Phi^0_c = X^+_{26}\, X^+_{25}, X^+_{16}\, X^+_{25}, X^+_{15}\, X^+_{26}, \\ \Phi^-_c\, X^+_{46}\, |\Lambda\rangle, & \Phi^-_c = X^+_{26}, X^+_{16}\, X^+_{26}\, X^+_{25}, \\ \hat{\Phi}_a\, X^+_{25}\, |\Lambda\rangle, & \hat{\Phi}_a = X^+_{45}, X^+_{45}\, X^+_{46}, \\ \Phi^0_c\, X^+_{45}\, X^+_{46}\, |\Lambda\rangle. & \end{array} \tag{A.62}$$

The states of (A.56), are a subset of (A.62).

• $j_1 = j_2 = 0$. Here character formulae (2.92) hold (without counterterms for $r_1 \geqslant 4$). The states $X^+_{15}\, X^+_{25}\, |\Lambda\rangle$, $X^+_{36}\, X^+_{46}\, |\Lambda\rangle$ and their descendants are eliminated. Correspondingly to the cases in (A.22a)–(A.22e) we have now $64, 63, 59, 47, 24$ states. In the last case, when $r_1 = 0$, we exclude the generators $X^+_{3,4+k}$ from the antichiral sector and the generators $X^+_{1,4+k}$ from the chiral sector and also the combination of impossible states (A.27) as explained in the general exposition.

For $r_1 > 0$, decomposition (2.93) holds. Note that when the weight Λ corresponds to cases (A.22a)–(A.22e), then the weights $\Lambda + \beta_{12}$, $\Lambda + \beta_{34}$ correspond to cases (A.22a), (A.22a), (A.22a), (A.22b), (A.22c) (since the value of r_1 is increased by 2), but from types bc), ad), resp., considered below, i.e., the corresponding UIRs have $24, 24, 24, 23, 20$ states each. The weight $\Lambda + \beta_{12} + \beta_{34}$ corresponds to cases (A.22a), (A.22a), (A.22a), (A.22a), (A.22a) (since the value of r_1 is increased by 4), but from type bd), i.e., the corresponding UIRs have $9, 9, 9, 9, 9$ states. Summed together with the numbers for the UIR $\hat{L}_\Lambda$ from above we obtain the following contributions to $\hat{L}_{\rm long}$: $121, 120, 116, 102, 73$. Except the last case (in which $r_1 = 0$) these cases match exactly the cases of long superfields for the cases when $j_1 = j_2 = 0$.

When $r_1 = 0$, the corresponding long superfields have 75 states, i.e., there is a mismatch of 2 states. These extra states are

$$X^+_{16}\, X^+_{25}\, |\Lambda\rangle, \quad X^+_{35}\, X^+_{46}\, |\Lambda\rangle. \tag{A.63}$$

Altogether, instead of (2.93) we may write:

$$\left(\hat{L}_{\rm long}\right)\Big|_{d=d^{ac}} = \hat{L}_\Lambda \oplus \hat{L}_{\Lambda+\beta_{12}} \oplus \hat{L}_{\Lambda+\beta_{34}} \oplus \hat{L}_{\Lambda+\beta_{12}+\beta_{34}} \oplus \\ \oplus \hat{L}'_{\Lambda+\alpha_{16}} \oplus \hat{L}'_{\Lambda+\alpha_{35}}, \quad r_1 = 0. \tag{A.64}$$

Finally, we give the 24 states of the UIR at $r_1 = 0$:

$$\begin{aligned}
&|\Lambda\rangle, && X^+_{25}\, X^+_{46}\,|\Lambda\rangle,\\
&X^+_{46}\,|\Lambda\rangle, && X^+_{45}\, X^+_{46}\,|\Lambda\rangle,\\
&X^+_{25}\,|\Lambda\rangle, && X^+_{26}\, X^+_{25}\,|\Lambda\rangle,\\
&\Phi^0_c\, X^+_{46}\,|\Lambda\rangle, && \Phi^0_c = X^+_{26}\, X^+_{25}, X^+_{16}\, X^+_{25}, X^+_{15}\, X^+_{26},\\
&\Phi^-_c\, X^+_{46}\,|\Lambda\rangle, && \Phi^-_c = X^+_{26}, X^+_{16}\, X^+_{26}\, X^+_{25},\\
&\Phi^0_a\, X^+_{25}\,|\Lambda\rangle, && \Phi^0_a = X^+_{45}\, X^+_{46}, X^+_{35}\, X^+_{46}, X^+_{36}\, X^+_{45},\\
&\Phi^-_a\, X^+_{25}\,|\Lambda\rangle, && \Phi^-_a = X^+_{45}, X^+_{35}\, X^+_{45}\, X^+_{46},\\
&\Phi^0_c\, \Phi^0_a\,|\Lambda\rangle && \text{excluding the state: } X^+_{15}\, X^+_{26}\, X^+_{36}\, X^+_{45}\,|\Lambda\rangle.
\end{aligned} \tag{A.65}$$

The states of (A.59), (A.62) are subsets of (A.65).

ad) $d = d^1_{21} = d^4_{22} = 1 + j_2 + r_1, j_1 = 0,\ z = -1 - j_2$.

Here character formulae (2.95) hold when $j_2 r_1 > 0$, (2.96) when $j_2 = 0, r_1 > 0$ (both these cases without counterterms for $r_1 \geqslant 4$), and finally when $r_1 = 0$, (2.97) holds independently of the value of j_2 — these are the antichiral massless UIRs.

The generators X^+_{15}, X^+_{25}, and in addition X^+_{36} for $j_2 > 0$ (resp., the state $X^+_{36}\, X^+_{46}\,|\Lambda\rangle$, and its descendants for $j_2 = 0$) are eliminated. The maximal number of states is $24 = 3$(chiral)$\times 8$(antichiral), achieved for $r_1 \geqslant 4$. The chiral sector for $r_1 > 0$ consists of the two states in (A.50) and the vacuum, while the antichiral sector is given by (A.31) for $j_2 > 0$ and by (A.37) for $j_2 = 0$.

The 24 states for $j_2 > 0$ are given explicitly as

$$\begin{aligned}
&|\Lambda\rangle, X^+_{46}\,|\Lambda\rangle,\ \ X^+_{45}\, X^+_{46}\,|\Lambda\rangle,\ \ r_1 \geqslant 0,\\
&X^+_{35}\, X^+_{46}\,|\Lambda\rangle,\ \ X^+_{26}\, X^+_{46}\,|\Lambda\rangle,\ \ r_1 \geqslant 1,\\
&X^+_{45}\,|\Lambda\rangle,\ \ X^+_{45}\, X^+_{35}\, X^+_{46}\,|\Lambda\rangle,\ \ X^+_{26}\, X^+_{45}\, X^+_{46}\,|\Lambda\rangle,\ \ X^+_{35}\,|\Lambda\rangle,\ \ r_1 \geqslant 1,\\
&X^+_{26}\,|\Lambda\rangle,\ \ X^+_{26}\, X^+_{35}\, X^+_{46}\,|\Lambda\rangle,\ \ X^+_{16}\, X^+_{26}\, X^+_{46}\,|\Lambda\rangle,\ \ r_1 \geqslant 1,\\
&X^+_{26}\, X^+_{45}\,|\Lambda\rangle,\ X^+_{26}\, X^+_{45}\, X^+_{35}\, X^+_{46}\,|\Lambda\rangle,\ X^+_{35}\, X^+_{45}\,|\Lambda\rangle,\ X^+_{16}\, X^+_{26}\, X^+_{45}\, X^+_{46}\,|\Lambda\rangle,\ r_1 \geqslant 2,\\
&X^+_{16}\, X^+_{26}\,|\Lambda\rangle,\ \ X^+_{16}\, X^+_{26}\, X^+_{35}\, X^+_{46}\,|\Lambda\rangle,\ \ X^+_{26}\, X^+_{35}\,|\Lambda\rangle,\ \ r_1 \geqslant 2,\\
&X^+_{26}\, X^+_{35}\, X^+_{45}\,|\Lambda\rangle, X^+_{16}\, X^+_{26}\, X^+_{45}\,|\Lambda\rangle, X^+_{16}\, X^+_{26}\, X^+_{45}\, X^+_{35}\, X^+_{46}\,|\Lambda\rangle, X^+_{16}\, X^+_{26}\, X^+_{35}\,|\Lambda\rangle,\\
&r_1 \geqslant 3,\\
&X^+_{16}\, X^+_{26}\, X^+_{35}\, X^+_{45}\,|\Lambda\rangle,\quad r_1 \geqslant 4.
\end{aligned} \tag{A.66}$$

Thus, correspondingly to the cases in (A.22a)–(A.22e) we have now 24, 23, 19, 12, 3 states.

The irreps with $r_1 > 2$ appear (two times if $r_1 > 3$) in decomposition (2.90) as explained in detail in the main text for type ad). (The irreps with $r_1 = 2$ have appeared in quasi-decomposition (A.61).)

The 24 states for $j_2 = 0$ are given explicitly as

$$|\Lambda\rangle,\ X^+_{46}\,|\Lambda\rangle,\ X^+_{45}\,X^+_{46}\,|\Lambda\rangle, \qquad r_1 \geqslant 0,$$
$$X^+_{35}\,X^+_{46}\,|\Lambda\rangle,\ X^+_{26}\,X^+_{46}\,|\Lambda\rangle,\ X^+_{45}\,X^+_{36}\,|\Lambda\rangle, \qquad r_1 \geqslant 1,$$
$$X^+_{45}\,|\Lambda\rangle,\ X^+_{45}\,X^+_{35}\,X^+_{46}\,|\Lambda\rangle,\ X^+_{26}\,X^+_{45}\,X^+_{46}\,|\Lambda\rangle, \qquad r_1 \geqslant 1,$$
$$X^+_{26}\,|\Lambda\rangle,\ X^+_{26}\,X^+_{35}\,X^+_{46}\,|\Lambda\rangle,\ X^+_{16}\,X^+_{26}\,X^+_{46}\,|\Lambda\rangle,\ X^+_{26}\,X^+_{45}\,X^+_{36}\,|\Lambda\rangle,\ r_1 \geqslant 1, \quad \text{(A.67)}$$
$$X^+_{26}\,X^+_{45}\,|\Lambda\rangle,\ X^+_{26}\,X^+_{45}\,X^+_{35}\,X^+_{46}\,|\Lambda\rangle,\ X^+_{35}\,X^+_{45}\,|\Lambda\rangle,\ X^+_{16}\,X^+_{26}\,X^+_{45}\,X^+_{46}\,|\Lambda\rangle,\ r_1 \geqslant 2,$$
$$X^+_{16}\,X^+_{26}\,|\Lambda\rangle,\ X^+_{16}\,X^+_{26}\,X^+_{35}\,X^+_{46}\,|\Lambda\rangle,\ X^+_{16}\,X^+_{26}\,X^+_{45}\,X^+_{36}\,|\Lambda\rangle, \quad r_1 \geqslant 2,$$
$$X^+_{26}\,X^+_{35}\,X^+_{45}\,|\Lambda\rangle,\ X^+_{16}\,X^+_{26}\,X^+_{45}\,|\Lambda\rangle,\ X^+_{16}\,X^+_{26}\,X^+_{45}\,X^+_{35}\,X^+_{46}\,|\Lambda\rangle, \qquad r_1 \geqslant 3,$$
$$X^+_{16}\,X^+_{26}\,X^+_{35}\,X^+_{45}\,|\Lambda\rangle, \qquad r_1 \geqslant 4.$$

Thus, correspondingly to the cases in (A.22a)–(A.22e) we have now 24, 23, 20, 13, 3 states.

The irreps with $r_1 > 2$ appear as the term $\hat{L}_{\Lambda+\beta_{34}}$ of (2.93), while those with $r_1 > 3$ appear also as the term $\hat{L}_{\Lambda+\alpha_{3,4+N}+\beta_{34}}$ of (2.90) but only when $j_2 = 1/2$ in Λ there. (The irreps with $r_1 = 2$ have appeared in quasi-decompositions (A.64).)

The cases (A.66) and (A.67) share 21 states (for $r_1 \geqslant 4$). The 3 states by which they differ are the last states on the 3rd, 6th, 7th lines of (A.66) and 2nd, 4th, 6th lines of (A.67).

bc) $d = d^2_{21} = d^3_{22} = 1 + j_1 + r_1, j_2 = 0,\ z = 1 + j_1$.

Here there hold character formulae (2.98) when $j_1 r_1 > 0$ and (2.99) when $j_1 = 0, r_1 > 0$ (both these cases without counterterms for $r_1 \geqslant 4$), and finally when $r_1 = 0$, (2.100) holds independently of the value of j_1 — these are the chiral massless UIRs.

The generators X^+_{36}, X^+_{46}, and in addition X^+_{15} for $j_1 > 0$ (resp., the state $X^+_{15}\,X^+_{25}\,|\Lambda\rangle$, and its descendants for $j_1 = 0$) are eliminated. The maximal number of states is 24 = 8(chiral)×3(antichiral), achieved for $r_1 \geqslant 4$. The antichiral sector for $r_1 >$ consists of the two states in (A.39) and the vacuum, while the chiral sector is given by (A.43) for $j_1 > 0$ and by (A.48) for $j_1 = 0$.

The 24 states for $j_1 > 0$ are given explicitly as

$$|\Lambda\rangle,\ X^+_{25}\,|\Lambda\rangle,\ X^+_{26}\,X^+_{25}\,|\Lambda\rangle, \qquad r_1 \geqslant 0,$$
$$X^+_{16}\,X^+_{25}\,|\Lambda\rangle,\ X^+_{45}\,X^+_{25}\,|\Lambda\rangle, \qquad r_1 \geqslant 1,$$
$$X^+_{26}\,|\Lambda\rangle,\ X^+_{26}\,X^+_{16}\,X^+_{25}\,|\Lambda\rangle,\ X^+_{45}\,X^+_{26}\,X^+_{25}\,|\Lambda\rangle,\ ,X^+_{16}\,|\Lambda\rangle, \qquad r_1 \geqslant 1,$$
$$X^+_{45}\,|\Lambda\rangle,\ X^+_{45}\,X^+_{16}\,X^+_{25}\,|\Lambda\rangle,\ X^+_{35}\,X^+_{45}\,X^+_{25}\,|\Lambda\rangle, \qquad r_1 \geqslant 1,$$
$$X^+_{45}\,X^+_{26}\,|\Lambda\rangle,\ X^+_{45}\,X^+_{26}\,X^+_{16}\,X^+_{25}\,|\Lambda\rangle,\ X^+_{16}\,X^+_{26}\,|\Lambda\rangle,\ X^+_{35}\,X^+_{45}\,X^+_{26}\,X^+_{25}\,|\Lambda\rangle,\ r_1 \geqslant 2,$$
$$X^+_{35}\,X^+_{45}\,|\Lambda\rangle,\ X^+_{35}\,X^+_{45}\,X^+_{16}\,X^+_{25}\,|\Lambda\rangle,\ X^+_{45}\,X^+_{16}\,|\Lambda\rangle, \quad r_1 \geqslant 2,$$

$$X^+_{45}\,X^+_{16}\,X^+_{26}\,|\Lambda\rangle,\ X^+_{35}\,X^+_{45}\,X^+_{26}\,|\Lambda\rangle,\ X^+_{35}\,X^+_{45}\,X^+_{26}\,X^+_{16}\,X^+_{25}\,|\Lambda\rangle, X^+_{35}\,X^+_{45}\,X^+_{16}\,|\Lambda\rangle,\quad r_1 \geqslant 3,$$
$$X^+_{35}\,X^+_{45}\,X^+_{16}\,X^+_{26}\,|\Lambda\rangle,\qquad r_1 \geqslant 4. \tag{A.68}$$

Thus, correspondingly to the cases in (A.22a)–(A.22e) we have now 24, 23, 19, 12, 3 states.

The irreps with $r_1 > 2$ appear (up to two times) in decomposition (2.87) as explained in detail in the main text for type bc). (The irreps with $r_1 = 2$ have appeared in quasi-decomposition (A.58).)

The 24 states for $j_1 = 0$ are given explicitly as

$$|\Lambda\rangle,\ X^+_{25}\,|\Lambda\rangle,\ X^+_{26}\,X^+_{25}\,|\Lambda\rangle,\ r_1 \geqslant 0,$$
$$X^+_{16}\,X^+_{25}\,|\Lambda\rangle,\ X^+_{45}\,X^+_{25}\,|\Lambda\rangle,\ X^+_{26}\,X^+_{15}\,|\Lambda\rangle,\quad r_1 \geqslant 1,$$

$$X^+_{26}\,|\Lambda\rangle,\ X^+_{26}\,X^+_{16}\,X^+_{25}\,|\Lambda\rangle,\ X^+_{45}\,X^+_{26}\,X^+_{25}\,|\Lambda\rangle,\quad r_1 \geqslant 1,$$
$$X^+_{45}\,|\Lambda\rangle,\ X^+_{45}\,X^+_{16}\,X^+_{25}\,|\Lambda\rangle,\ X^+_{35}\,X^+_{45}\,X^+_{25}\,|\Lambda\rangle,\ X^+_{45}\,X^+_{26}\,X^+_{15}\,|\Lambda\rangle,\quad r_1 \geqslant 1,$$
$$X^+_{45}\,X^+_{26}\,|\Lambda\rangle,\ X^+_{45}\,X^+_{26}\,X^+_{16}\,X^+_{25}\,|\Lambda\rangle,\ X^+_{16}\,X^+_{26}\,|\Lambda\rangle,\ X^+_{35}\,X^+_{45}\,X^+_{26}\,X^+_{25}\,|\Lambda\rangle,\ r_1 \geqslant 2,$$
$$X^+_{35}\,X^+_{45}\,|\Lambda\rangle,\ X^+_{35}\,X^+_{45}\,X^+_{16}\,X^+_{25}\,|\Lambda\rangle,\ X^+_{35}\,X^+_{45}\,X^+_{26}\,X^+_{15}\,|\Lambda\rangle,\quad r_1 \geqslant 2,$$
$$X^+_{45}\,X^+_{16}\,X^+_{26}\,|\Lambda\rangle,\ X^+_{35}\,X^+_{45}\,X^+_{26}\,|\Lambda\rangle,\ X^+_{35}\,X^+_{45}\,X^+_{26}\,X^+_{16}\,X^+_{25}\,|\Lambda\rangle,\quad r_1 \geqslant 3,$$
$$X^+_{35}\,X^+_{45}\,X^+_{16}\,X^+_{26}\,|\Lambda\rangle,\quad r_1 \geqslant 4. \tag{A.69}$$

Thus, correspondingly to the cases in (A.22a)–(A.22e) we have now 24, 23, 20, 13, 3 states.

The irreps with $r_1 > 2$ appear as the term $\hat{L}_{\Lambda+\beta_{12}}$ of (2.93), while those with $r_1 > 3$ appear also as the term $\hat{L}_{\Lambda+\alpha_{15}+\beta_{12}}$ of (2.87) but only when $j_1 = 1/2$ in Λ there. (The irreps with $r_1 = 2$ have appeared in quasi-decomposition (A.64).)

bd) $d = d^2_{21} = d^4_{22} = r_1,\quad j_1 = j_2 = 0 = z.$

The generators $X^+_{15},\ X^+_{25},\ X^+_{36},\ X^+_{46}$ are eliminated. For $r_1 = 1$ also the generators $X^+_{16},\ X^+_{35}$ are eliminated. For $r_1 = 0$ the remaining two generators $X^+_{26},\ X^+_{45}$ are eliminated and we have the trivial irrep as explained in general.

For $r_1 > 0$ the character formula is (2.101) with $i_0 = i'_0 = 0$. The maximal number of states is nine and the list of states together with the conditions when they exist are

$$|\Lambda\rangle,\qquad r_1 \geqslant 0,$$
$$X^+_{26}\,|\Lambda\rangle,\quad X^+_{45}\,|\Lambda\rangle,\qquad r_1 \geqslant 1,$$
$$X^+_{16}\,X^+_{26}\,|\Lambda\rangle,\quad X^+_{35}\,X^+_{45}\,|\Lambda\rangle,\quad X^+_{26}\,X^+_{45}\,|\Lambda\rangle,\qquad r_1 \geqslant 2, \tag{A.70}$$
$$X^+_{16}\,X^+_{26}\,X^+_{45}\,|\Lambda\rangle,\quad X^+_{26}\,X^+_{35}\,X^+_{45}\,|\Lambda\rangle,\qquad r_1 \geqslant 3,$$
$$X^+_{16}\,X^+_{26}\,X^+_{35}\,X^+_{45}\,|\Lambda\rangle,\qquad r_1 \geqslant 4.$$

Thus, correspondingly to the cases in (A.22a)–(A.22e) we have now $9, 8, 6, 3,$ 1 states. The mixed massless irrep is obtained for $d = r_1 = 1$ and consists of the first three states above — as was shown in general.

The irreps with $r_1 > 4$ have appeared in decomposition (2.93), cf. type ac) above. (The irreps with $r_1 = 4$ have appeared in quasi-decomposition (A.64).)

Appendix B

ODD REFLECTIONS

Below we repeat first the original text of our submission to «Concise Encyclopedia of Supersymmetry» (Eds. S. Duplij, W. Siegel, and J. Bagger. Kluwer Acad. Publ., 2003).

Odd Reflection — action of an odd root α on the dual χ^* of the Cartan subalgebra χ of a basic classical Lie superalgebra $\mathcal{G}$. Let $\Delta = \Delta_{\bar{0}} \cup \Delta_{\bar{1}}$ be the root system of $\mathcal{G} = \mathcal{G}_{\bar{0}} \oplus \mathcal{G}_{\bar{1}}$, where $\Delta_{\bar{0}}$ is the root system of the even subalgebra $\mathcal{G}_{\bar{0}}$ of $\mathcal{G}$, and the set of odd roots $\Delta_{\bar{1}}$ is the weight system of the representation of $\mathcal{G}_{\bar{0}}$ in $\mathcal{G}_{\bar{1}}$. The action of $\alpha \in \Delta_{\bar{1}}$ on $\Lambda \in \chi^*$ is defined by

$$\begin{aligned}
&s_\alpha \Lambda = \Lambda - 2\frac{(\alpha,\Lambda)}{(\alpha,\alpha)}\,\alpha, \qquad (\alpha,\alpha) \neq 0,\\
&s_\alpha \Lambda = \Lambda + \alpha, \qquad (\alpha,\alpha) = 0,\ (\alpha,\Lambda) \neq 0,\\
&s_\alpha \Lambda = \Lambda, \qquad (\alpha,\alpha) = 0,\ (\alpha,\Lambda) = 0, \quad \alpha \neq \Lambda,\\
&s_\alpha \alpha = -\alpha, \qquad (\alpha,\alpha) = 0,
\end{aligned}$$

where $(\cdot,\cdot)$ is the standard bilinear product in χ^*. As in the even case one has: $s_\alpha^{-1} = s_{-\alpha}$, but an odd reflection is not always a reflection since $s_\alpha^2 \neq id_{\chi^*}$ if $(\alpha,\alpha) = 0$. In particular, one has

$$s_\alpha^n \Lambda = \Lambda + n\alpha, \quad (\alpha,\alpha) = 0, \quad (\alpha,\Lambda) \neq 0, n \in \mathbb{Z},$$

i.e., in this situation the odd reflection acts as a translation.

Note that if $\alpha, \beta, \alpha+\beta \in \Delta$ and $(\alpha,\alpha)(\beta,\beta)(\alpha+\beta,\alpha+\beta) = 0$, then $s_{\alpha+\beta}$ cannot be expressed in terms of s_α, s_β.

The odd reflections s_α with $(\alpha,\alpha) = 0$ generate an infinite Abelian group with elements

$$\prod_{\alpha\in\Delta_{\bar{1}},\ (\alpha,\alpha)=0} s_\alpha^{n_\alpha}, \quad n_\alpha \in \mathbb{Z}.$$

This group does not preserve $\Delta, \Delta_{\bar{0}}, \Delta_{\bar{1}}$*.

Appendix C

CHARACTERS OF THE EVEN SUBALGEBRA

For the characters of the even subalgebra we first recall its structure: $\mathcal{G}_0^{\mathbb{C}} = sl(4) \oplus gl(1) \oplus sl(N)$ of $\mathcal{G}^{\mathbb{C}}$. We choose a basis in which the Cartan subalgebra $\mathcal{H}$ of $\mathcal{G}^{\mathbb{C}}$ is also a Cartan subalgebra of $\mathcal{G}_0^{\mathbb{C}}$. Since the subalgebra $\mathcal{G}_0^{\mathbb{C}}$ is reductive, the corresponding character formulae will be given by the products of the character formulae of the two simple factors $sl(4)$ and $sl(N)$.

We start with the $sl(4)$ case. We denoted the six positive roots of $sl(4)$ by α_{ij}, $1 \leqslant i < j \leqslant 4$. For the simplification of the character formulae we use notation for the formal exponents corresponding to the $sl(4)$ simple roots: $t_j \equiv e(\alpha_{j,j+1})$, $j = 1, 2, 3$; then for the three nonsimple roots we have: $e(\alpha_{13}) = t_1 t_2$, $e(\alpha_{24}) = t_2 t_3$, $e(\alpha_{14}) = t_1 t_2 t_3$. In terms of these, the character formula for a Verma module over $sl(4)$ is

$$\mathrm{ch}_0\, V^{\Lambda^s} = \frac{e(\Lambda^s)}{(1-t_1)(1-t_2)(1-t_3)(1-t_1t_2)(1-t_2t_3)(1-t_1t_2t_3)}, \qquad \text{(C.1)}$$

where by Λ^s we denote the $sl(4)$ lowest weight.

The representations of $sl(4)$, which we consider, are infinite-dimensional. When $d > d_{\max}$, then all the numbers: $n_2, n_{13}, n_{24}, n_{14}$ from (1.17) cannot be positive integers. Then the only reducibilities of the $sl(4)$ Verma module are related to the complexification of the Lorentz subalgebra of $su(2,2)$, i.e., with $sl(2) \oplus sl(2)$, and the character formula is given by the product of the two character formulae for finite-dimensional $sl(2)$ irreps. In short, the $sl(4)$ character formula is

$$\begin{aligned}\mathrm{ch}_0\, L_{\Lambda^s} &= \mathrm{ch}_0\, V^{\Lambda^s} - \mathrm{ch}_0\, V^{\Lambda^s + n_1\alpha_{12}} - \mathrm{ch}_0\, V^{\Lambda^s + n_3\alpha_{34}} + \mathrm{ch}_0\, V^{\Lambda^s + n_1\alpha_{12} + n_3\alpha_{34}} = \\ &= \frac{e(\Lambda^s)\ (1-t_1^{n_1})\ (1-t_3^{n_3})}{(1-t_1)(1-t_2)(1-t_3)(1-t_1t_2)(1-t_2t_3)(1-t_1t_2t_3)} = \\ &= e(\Lambda^s)\ \mathcal{Q}^s_{n_1,n_2}, n_1 = 2j_1+1,\ n_3 = 2j_2+1,\ d > d_{\max}, \qquad \text{(C.2)}\end{aligned}$$

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*$Dobrev\ V.\ K., Petkova\ V.\ B.$ On the Group-Theoretical Approach to Extended Conformal Supersymmetry: Function Space Realizations and Invariant Differential Operators // Fortschr. Phys. 1987. Bd. 35. S. 537–572.

Note that this reference in this paper is [101]. The above text was submitted in June 2000. Recently, we were informed that a similar definition was proposed independently in [129], while a different, geometric, version of odd reflections was introduced in [130] (see also [131]).

and we have introduced for later use notation $\mathcal{Q}^s_{n_1,n_2}$ for the character factorized by $e(\Lambda^s)$. The above formula obviously has the form (2.5) replacing $W \mapsto W_2 \times W_2$, where W_2 is the two-element Weyl group of $sl(2)$.

When $d \leqslant d_{\max}$, there are additional even reducibilities, cf. (1.47), (1.51), (1.52), and the discussion in-between.

Thus, we need additional formulae for $\mathrm{ch}_0\, L_{\Lambda^s}$:

$$\mathrm{ch}_0\, L_{\Lambda^s} = $$
$$= e(\Lambda^s)\mathcal{Q}^s_{n_1,n_2}(1 - t_1 t_2 t_3) = \frac{e(\Lambda^s)(1-t_1^{n_1})\ (1-t_3^{n_3})}{(1-t_1)(1-t_2)(1-t_3)(1-t_1t_2)(1-t_2t_3)}, \tag{C.3a}$$

for (1.52a), $\quad d = d^1_{N1} = d^3_{NN} = 2 + j_1 + j_2, \quad j_1 j_2 > 0;$

$$= e(\Lambda^s)\mathcal{Q}^s_{1,2}(1 - t_2 t_3) = \frac{e(\Lambda^s)(1+t_3)}{(1-t_2)(1-t_1t_2)(1-t_1t_2t_3)}, \tag{C.3b}$$

for (1.52b), $\quad d = d^1_{N1} = d^4_{NN} = 3/2, \quad j_1 = 0, \quad j_2 = 1/2;$

$$= e(\Lambda^s)\mathcal{Q}^s_{2,1}(1 - t_1 t_2) = \frac{e(\Lambda^s)\ (1+t_1)}{(1-t_2)(1-t_2t_3)(1-t_1t_2t_3)}, \tag{C.3c}$$

for (1.52d) $d = d^2_{N1} = d^3_{NN} = 3/2, \quad j_1 = 1/2, \quad j_2 = 0;$

$$= e(\Lambda^s)\ \mathcal{Q}^s_{1,1}(1 - t_1 t_2^2 t_3) = \frac{e(\Lambda^s)(1-t_1t_2^2t_3)}{(1-t_2)(1-t_1t_2)(1-t_2t_3)(1-t_1t_2t_3)},$$

for (1.52e), (1.52f), $\quad d = 1, \quad j_1 = j_2 = 0.$ (C.3d)

In the case of $sl(N)$, the representations are finite-dimensional since we induce from UIRs of $su(N)$. The character formula is (2.5), which we repeat in order to introduce the corresponding notation:

$$\mathrm{ch}_0\, L_{\Lambda^u}(r_1, \ldots, r_{N-1}) = \sum_{w \in W_u} (-1)^{\ell(w)} \mathrm{ch}_0\, V^{w \cdot \Lambda^u}, \quad \Lambda^u \in -\Gamma^u_+. \tag{C.4}$$

The index u is to distinguish the quantities pertinent to the case.

REFERENCES

1. *Lee S. et al.* // Adv. Theor. Math. Phys. 1998. V. 2. P. 697; hep-th/9806074.
2. *D'Hoker E. et al.* // Nucl. Phys. B. 2000. V. 589. P. 3–37; hep-th/0003218.
3. *Ferrara S., Fronsdal C.* Conformal Fields in Higher Dimensions // Recent Developments in Theoretical and Experimental General Relativity, Gravitation and Relativistic Field Theories. Pt. A. Rome, 2000. P. 508–527; Quantization, Gauge Theory, and Strings. M., 2000. V. 1. P. 405–426; hep-th/0006009.
4. *D'Hoker E., Pioline B.* Near-Extremal Correlators and Generalized Consistent Truncation for $AdS_{4|7} \times S^{7|4}$ // JHEP. 2000. 0007. 021; hep-th/0006103.

5. *Nelson A., Strassler M. J.* // Phys. Rev. D. 1999. V. 60. P. 015004; hep-ph/9806346; JHEP. 2000. 0009. 030; hep-ph/0006251.
6. *Osborn H.* // Ann. Phys. (N. Y.). 1999. V. 272. P. 243; hep-th/9808041.
7. *Park J.-H.* // Nucl. Phys. B. 1999. V. 559. P. 455; hep-th/9903230.
8. *Freedman D. Z. et al.* // Adv. Theor. Math. Phys. 1999. V. 3. P. 363; hep-th/9904017.
9. *Gremm M., Kapustin A.* // JHEP. 1999. 9907. 005; hep-th/9904050.
10. *Aharony O. et al.* // Phys. Rep. 2000. V. 323. P. 184; hep-th/9905111.
11. *Rehren K.-H.* Algebraic Holography // Ann. Henri Poincare. 2000. V. 1. P. 607–623; hep-th/9905179; QFT Lectures on AdS-CFT // Zlatibor Summer School on Modern Math. Phys. (SFIN XVIII (A1)), Aug. 2004. Belgrade, 2005. P. 95–118; hep-th/0411086.
12. *Ceresole A. et al.* // Phys. Rev. D. 2000. V. 61. P. 066001; hep-th/9905226; Class. Quant. Grav. 2000. V. 17. P. 1017; hep-th/9910066.
13. *Bianchi M. et al.* // JHEP. 1999. 9909. 020; hep-th/9906188; Nucl. Phys. B. 2000. V. 584. P. 216; hep-th/0003203.
14. *D'Hoker E. et al.* The Many Faces of the Superworld. P. 332; hep-th/9908160.
15. *Ferrara S., Zaffaroni A.* // Conf. Moshe Flato. Dijon, 1999. V. I. Math. Phys. Stud. 2000. V. 21. P. 177–188; hep-th/9908163.
16. *Andrianopoli L. et al.* // Adv. Theor. Math. Phys. 1999. V. 3. P. 1149; hep-th/9912007.
17. *Ferrara S., Sokatchev E.* // Lett. Math. Phys. 2000. V. 52. P. 247; hep-th/9912168; J. Math. Phys. 2001. V. 42. P. 3015; hep-th/0010117.
18. *Pelc O.* // JHEP. 2000. 0003. 012; hep-th/0001054.
19. *Ferrara S.* // TMR Conf., Paris, Sept. 1–7, 1999; hep-th/0002141.
20. *Ferrara S., Sokatchev E.* // JHEP. 2000. 0005. 038; hep-th/0003051; Intern. J. Mod. Phys B. 2000. V. 14. P. 2315; hep-th/0007058; New J. Phys. 2002. V. 4. P. 2–22; hep-th/0110174.
21. *Bastianelli F., Zucchini R.* // JHEP. 2000. 0005. 047; hep-th/0003230.
22. *Heslop P. J., Howe P. S.* // Class. Quant. Grav. 2000. V. 17. P. 3743; hep-th/0005135; TMR Conf., Paris, Sept. 2000; hep-th/0009217.
23. *Ferrara S., Sokatchev E.* // Intern. J. Theor. Phys. 2001. V. 40. P. 935; hep-th/0005151.
24. *Duff M. J., Liu J. T., Stelle K. S.* // J. Math. Phys. 2001. V. 42. P. 3027; hep-th/0007120.
25. *Liu J. T., Sati H.* // Nucl. Phys. B. 2001. V. 605. P. 116; hep-th/0009184.
26. *D'Auria R. et al.* Spinor Algebras // J. Geom. Phys. 2001. V. 40. P. 101–128; hep-th/0010124.
27. *Klemm D., Sabra W. A.* // JHEP. 2001. 0102. 031; hep-th/0011016.
28. *Freedman D. Z., Henry-Labordere P.* // TMR Conf., Paris, Sept. 7–13, 2000; hep-th/0011086.
29. *Martin S. P., Wells J. D.* // Phys. Rev. D. 2001. V. 64. P. 036010; hep-ph/0011382.
30. *Argyres P. C.* // Nucl. Phys. B (Proc. Suppl.). 2001. V. 101. P. 175; hep-th/0102006.
31. *Arutyunov G. et al.* // Nucl. Phys. B. 2002. V. 620. P. 380; hep-th/0103230.
32. *Bianchi M. et al.* // JHEP. 2001. 0105. 042; hep-th/0104016.
33. *Arutyunov G., Eden B., Sokatchev E.* // Nucl. Phys. B. 2001. V. 619. P. 359; hep-th/0105254.
34. *Heslop P. J., Howe P. S.* // Phys. Lett. B. 2001. P. 516. P. 367; hep-th/0106238; Nucl. Phys. B. 2002. V. 626. P. 265–286; hep-th/0107212; Aspects of $N = 4$ SYM. hep-th/0307210.
35. *Eden B., Sokatchev E.* // Nucl. Phys. B. 2001. V. 618. P. 259; hep-th/0106249.

36. *Penati S., Santambrogio A.* // Nucl. Phys. B. 2001. V. 614. P. 367; hep-th/0107071.

37. *Eden B., Ferrara S., Sokatchev E.* // JHEP. 2001. 0111. 020; hep-th/0107084.

38. *Kubo J., Suematsu D.* // Phys. Rev. D. 2001. V. 64. P. 115014; hep-ph/0107133.

39. *West P.* // Nucl. Phys. B (Proc. Suppl.). 2001. V. 101. P. 112.

40. *Heslop P. J.* // Class. Quant. Grav. 2002. V. 19. P. 303; hep-th/0108235; Aspects of Superconformal Field Theories in Six Dimensions // JHEP. 2004. 0407. 056; hep-th/0405245.

41. *Ryzhov A. V.* // JHEP. 2001. 0111. 046; hep-th/0109064; Operators in the $\mathrm{D} = 4$, $N = 4$ SYM and the *AdS*/CFT Correspondence. hep-th/0307169; UCLA thesis. 169 p.

42. *D'Hoker E., Ryzhov A. V.* Three-Point Functions of Quarter BPS Operators in $N = 4$ SYM // JHEP. 2002. 0202. 047; hep-th/0109065.

43. *Hoffmann L. et al.* Multitrace Quasi-Primary Fields of $N = 4$ SYM from *AdS* n-Point Functions // Nucl. Phys. B. 2002. V. 641. P. 188–222; hep-th/0112191.

44. *Arutyunov G., Sokatchev E.* Implications of Superconformal Symmetry for Interacting (2,0) Tensor Multiplets // Nucl. Phys. B. 2002. V. 635. P. 3–32; hep-th/0201145; Class. Quant. Grav. 2003. V. 20. P. L123–L131; A note on the perturbative properties of BPS operators. hep-th/0209103.

45. *D'Hoker E., Freedman D. Z.* Strings, Branes and Extra Dimensions. Boulder, 2001. P. 3–158; Supersymmetric Gauge Theories and the *AdS*/CFT Correspondence. hep-th/0201253.

46. *Takayanagi T., Terashima S.* Strings on Orbifolded PP Waves // JHEP. 2002. 0206. 036; hep-th/0203093.

47. *Sezgin E., Sundell P.* Massless Higher Spins and Holography // Nucl. Phys. B. 2002. V. 644. P. 303–370; hep-th/0205131.

48. *Bianchi M. et al.* // Nucl. Phys. B. 2002. V. 646. P. 69–101; On Operator Mixing in $N = 4$ SYM. hep-th/0205321.

49. *Arutyunov G. et al.* Nonprotected Operators in $N = 4$ SYM and Multiparticle States of AdS_5 SUGRA // Nucl. Phys. B. 2002. V. 643. P. 49–78; hep-th/0206020.

50. *Fernando S., Gunaydin M., Pavlyk O.* // JHEP. 2002. 0210. 007; Spectra of PP-Wave Limits of M-/Superstring Theory on $AdS(p) \times S^q$ Spaces. hep-th/0207175.

51. *Dolan F. A., Osborn H.* // Ann. Phys. 2003. V. 307. P. 41–89; On Short and Semishort Representations for Four-Dimensional Superconformal Symmetry // DAMTP/02-114; hep-th/0209056.

52. *Beisert N.* BMN Operators and Superconformal Symmetry // Nucl. Phys. B. 2003. V. 659. P. 79–118; hep-th/0211032; The Complete One-Loop Dilatation Operator of $N = 4$ Super Yang–Mills Theory // Nucl. Phys. B. 2004. V. 676. P. 3–42; hep-th/0307015; The Dilatation Operator of $N = 4$ Super Yang–Mills Theory and Integrability // Phys. Rep. 2004. V. 405. P. 1–202; hep-th/0407277.

53. *D'Hoker E. et al.* Systematics of Quarter BPS Operators in $N = 4$ SYM // JHEP. 2003. 0304. 038; hep-th/0301104.

54. *Cheng Shun-Jen, Lam Ngau, Zhang R. B.* // J. Algebra. 2004. V. 273. P. 780–805; math.RT/0301183.

55. *Beisert N., Kristjansen C., Staudacher M.* // Nucl. Phys. B. 2003. V. 664. P. 131–184; The Dilatation Operator of $N = 4$ Super Yang–Mills Theory. hep-th/0303060.

56. *Dhar A., Mandal G., Wadia S. R.* String Bits in Small Radius AdS and Weakly Coupled $N = 4$ Super Yang–Mills Theory: I. hep-th/0304062.

57. *Bianchi M., Morales J. F., Samtleben H.* // JHEP. 2003. 0307. 062; On Stringy $AdS_5 \times S^5$ and Higher Spin Holography. hep-th/0305052.

58. *Drummond J. M. et al.* Integral Invariants in $N = 4$ SYM and the Effective Action for Coincident D-Branes // JHEP. 2003. 0308. 016; hep-th/0305202.

59. *Fukuma M., Matsuura S., Sakai T.* Holographic Renormalization Group // Prog. Theor. Phys. 2003. V. 109. P. 489–562.

60. *Leonhardt T., Manvelyan R., Ruehl W.* // Nucl. Phys. B. 2003. V. 667. P. 413–434; The Group Approach to AdS Space Propagators: hep-th/0305235; The Group Approach to AdS Space Propagators: A Fast Algorithm // J. Phys. A. 2004. V. 37. P. 7051; hep-th/0310063.

61. *Terning J.* TASI-2002 Lectures: Nonperturbative Supersymmetry // Part. Phys. and Cosmology. Boulder, 2002. P. 343–443; hep-th/0306119.

62. *Beisert N., Staudacher M.* // Nucl. Phys. B. 2003. V. 670. P. 439–463; The $N = 4$ SYM Integrable Super Spin Chain. hep-th/0307042.

63. *Kovacs S.* // Nucl. Phys. B. 2004. V. 684. P. 3–74; On Instanton Contributions to Anomalous Dimensions in $N = 4$ Supersymmetric Yang–Mills Theory. hep-th/0310193.

64. *Beisert N. et al.* On the Spectrum of AdS/CFT Beyond Supergravity // JHEP. 2004. 0402. 001; hep-th/0310292; Higher Spin Symmetry and $N = 4$ SYM // JHEP. 2004. 0407. 058; hep-th/0405057.

65. *Kujawa J.* Crystal Structures Arising from Representations of $GL(m|n)$. math.RT/0311251.

66. *Metsaev R. R.* Massive Totally Symmetric Fields in $AdS(d)$ // Phys. Lett. B. 2004. V. 590. P. 95–104; hep-th/0312297; Eleven Dimensional Supergravity in Light Cone Gauge // Phys. Rev. D. 2005. V. 71. P. 085017; hep-th/0410239; Mixed Symmetry Massive Fields in $AdS(5)$ // Class. Quant. Grav. 2005. V. 22. P. 2777–2796; hep-th/0412311; AdS Friendly Light-Cone Formulation of Conformal Field Theory. hep-th/0512330.

67. *Schnitzer H. J.* Gauged Vector Models and Higher-Spin Representations in AdS_5 // Nucl. Phys. B. 2004. V. 695. P. 283–300; hep-th/0310210; Confinement/Deconfinement Transition of Large N Gauge Theories with N_f Fundamentals: N_f/N Finite // Nucl. Phys. B. 2004. V. 695. P. 267–282; hep-th/0402219.

68. *Csaki C., Meade P., Terning J.* A Mixed Phase of SUSY Gauge Theories from a-Maximization // JHEP. 2004. 0404. 040; hep-th/0403062.

69. *Ruehl W.* Lifting a Conformal Field Theory from D-Dimensional Flat Space to (D+1)-Dimensional AdS Space // Nucl. Phys. B. 2005. V. 705. P. 437–456; hep-th/0403114; The Masses of Gauge Fields in Higher Spin Fieldc Theory on $AdS(4)$ // Phys. Lett. B. 2005. V. 605. P. 413–418; hep-th/0409252.

70. *Zoubos K.* A Conformally Invariant Holographic Two-Point Function on the Berger Sphere // JHEP. 2005. 0501. 031; hep-th/0403292.

71. *Lee S., Park J. J.* Noncentral Extension of the $AdS(5) \times S$-5 Superalgebra: Supermultiplet of Brane Charges // JHEP. 2004. 0406. 038; hep-th/0404051.

72. *Dolan F. A., Gallot L., Sokatchev E.* On Four-Point Functions of Half-BPS Operators in General Dimensions // JHEP. 2004. 0409. 056; hep-th/0405180.

73. *Arnaudon D. et al.* General Boundary Conditions for the $sl(N)$ and $sl(M/N)$ Open Spin Chains // J. Stat. Mech.: Theor. Exp. (JSTAT). 2004. V. 08. P. 005; math-ph/0406021.

74. *Nirschl M., Osborn H.* Superconformal Ward Identities and Their Solution // Nucl. Phys. B. 2005. V. 711. P. 409–479; hep-th/0407060.

75. *Demir D. A.* Renormalization Group Invariants in the MSSM and Its Extensions // JHEP. 2005. 0511. 003; hep-ph/0408043.

76. *Eden B., Jarczak C., Sokatchev E.* A Three-Loop Test of the Dilatation Operator in $N = 4$ SYM // Nucl. Phys. B. 2005. V. 712. P. 157–195; hep-th/0409009.

77. *Bianchi M.* Higher-Spin Symmetry (Breaking) in $N = 4$ SYM Theory and Holography // Comptes Rendus Phys. 2004. V. 5. P. 1091–1099; hep-th/0409292; Fortsch. Phys. 2005. Bd. 53. S. 665–691; hep-th/0409304.

78. *Morales J. F., Samtleben H.* Higher-Spin Holography for SYM in d Dimensions // Phys. Lett. B. 2005. V. 607. P. 286–293; hep-th/0411246.

79. *Burrington B. A., Liu J. T., Sabra W. A.* AdS_5 Black Holes with Fermionic Hair // Phys. Rev. D. 2005. V. 71. P. 105015; hep-th/0412155.

80. *Manvelyan R., Ruehl W.* The Masses of Gauge Fields in Higher Spin Field Theory on the Bulk of AdS_4 // Phys. Lett. B. 2005. V. 613. P. 197–207; hep-th/0412252; The Off-Shell Behaviour of Propagators and the Goldstone Field in Higher Spin Gauge Theory on AdS_{d+1} Space // Nucl. Phys. B. 2005. V. 717. P. 3–18; hep-th/0502123.

81. *Carmeli C. et al.* Unitary Representations of Super Lie Groups and Applications to the Classification and Multiplet Structure of Super Particles // Commun. Math. Phys. 2006. V. 263. P. 217–258; hep-th/0501061.

82. *Barabanschikov A. et al.* The Spectrum of Yang–Mills on a Sphere // JHEP. 2006. 0601. 160; hep-th/0501063.

83. *Krasnov K., Louko J.* $SO0(1, d + 1)$ Racah Coefficients: Type I Representations // J. Math. Phys. 2006. V. 47. P. 033513; math-ph/0502017.

84. *Bianchi M., Didenko V.* «Massive» Higher Spin Multiplets and Holography. hep-th/0502220.

85. *Genovese L., Stanev Y. S.* Rationality of the Anomalous Dimensions in $N = 4$ SYM Theory // Nucl. Phys. B. 2005. V. 721. P. 212–228; hep-th/0503084.

86. *D'Alessandro M., Genovese L.* A Wide Class of Four-Point Functions of BPS Operators in $N = 4$ SYM at Order g^4 // Nucl. Phys. B. 2006. V. 732. P. 64–88; hep-th/0504061.

87. *Argyres P. C. et al.* Classification of $N = 2$ Superconformal Field Theories with Two-Dimensional Coulomb Branches. hep-th/0504070.

88. *Bianchi M., Heslop P. J., Riccioni F.* More on La Grande Bouffe // JHEP. 2005. 0508. 088; hep-th/0504156.

89. *Arvidsson P.* Superconformal Symmetry in the Interacting Theory of (2,0) Tensor Multiplets and Self-Dual Strings // J. Math. Phys. 2006. V. 47. P. 042301; hep-th/0505197.

90. *Ibe M. et al.* Conformally Sequestered SUSY Breaking in Vector-Like Gauge Theories // Phys. Rev. D. 2006. V. 73. P. 015004; hep-ph/0506023; More on Conformally Sequestered SUSY Breaking // Phys. Rev. D. 2006. V. 73. P. 035012; hep-ph/0509229.

91. *Milanesi G., O'Loughlin M.* Singularities and Closed Time-Like Curves in Type-IIB 1/2 BPS Geometries // JHEP. 2005. 0509. 008; hep-th/0507056.

92. *Henn J., Jarczak C., Sokatchev E.* On Twist-Two Operators in $N = 4$ SYM // Nucl. Phys. B. 2005. V. 730. P. 191–209; hep-th/0507241.

93. *Zaikin V. N., Palchik M. Ya.* Secondary Fields in $\mathrm{D} > 2$ Conformal Theories. hep-th/0509103.

94. *Kinney J. et al.* An Index for 4-Dimensional Super-Conformal Theories. hep-th/0510251.

95. *Nakayama Yu.* Index for Orbifold Quiver Gauge Theories. hep-th/0512280; Index for Supergravity on $AdS_5 \times T^{1,1}$ and Conifold Gauge Theory. hep-th/0602284.

96. *Berkooz M., Reichmann D., Simon J.* A Fermi Surface Model for Large Supersymmetric AdS_5 Black Holes. hep-th/0604023.

97. *Nahm W.* // Nucl. Phys. B. 1978. V. 135. P. 149.

98. *Haag R., Lopuszanski J. T., Sohnius M.* // Nucl. Phys. B. 1975. V. 88. P. 257.

99. *Flato M., Fronsdal C.* // Lett. Math. Phys. 1984. V. 8. P. 159.

100. *Dobrev V. K., Petkova V. B.* On the Group-Theoretical Approach to Extended Conformal Supersymmetry: Classification of Multiplets // Lett. Math. Phys. 1985. V. 9. P. 287.

101. *Dobrev V. K., Petkova V. B.* On the Group-Theoretical Approach to Extended Conformal Supersymmetry: Function Space Realizations and Invariant Differential Operators // Fortschr. Phys. 1987. Bd. 35. S. 537–572; First as Preprint IC/85/29. ICTP Trieste, 1985.

102. *Dobrev V. K., Petkova V. B.* All Positive Energy Unitary Irreducible Representations of Extended Conformal Supersymmetry // Phys. Lett. B. 1985. V. 162. P. 127–132.

103. *Dobrev V. K., Petkova V. B.* // Lecture Notes in Physics. Berlin, 1986. V. 261. P. 291; 300.

104. *Minwalla S.* // Adv. Theor. Math. Phys. 1998. V. 2. P. 781–846.

105. *Dobrev V. K.* Positive Energy Unitary Irreducible Representations of D $= 6$ Conformal Supersymmetry // J. Phys. A. 2002. V. 35. P. 7079–7100; hep-th/0201076.

106. *Siegel W.* // Nucl. Phys. B. 1981. V. 177. P. 325.

107. *Howe P. S., Stelle K. S., Townsend P. K.* // Ibid. V. 192. P. 332.

108. *Gunaydin M., Marcus N.* // Class. Quant. Grav. 1985. V. 2. P. L11.

109. *Shapovalov N. N.* // Funkts. Anal. Prilozh. 1972. V. 6(4). P. 65; English transl. Funct. Anal. Appl. 1972. V. 6. P. 307.

110. *Dixmier J.* Enveloping Algebras. N. Y.: North Holland, 1977.

111. *Kac V. G.* Infinite-Dimensional Lie Algebras. Boston: Birkhäuser, 1983.

112. *Kac V. G.* // Lect. Notes in Math. 1978. V. 676. P. 597–626.

113. *Dobrev V. K.* Multiplet Classification of the Reducible Elementary Representations of Real Semisimple Lie Groups: the $SO_e(p,q)$ Example // Lett. Math. Phys. 1985. V. 9. P. 205–211; Talk at the I National Congress of Bulg. Physicists, Sofia, 1983; INRNE Sofia preprint, 1983.

114. *Dobrev V. K.* // Rep. Math. Phys. 1988. V. 25. P. 159–181.

115. *Dobrev V. K., Petkova V. B.* Odd Reflection, Contribution to «Concise Encyclopedia of Supersymmetry». Kluwer Acad. Publ., 2003. P. 282–283. (However, this is not the original text of our contribution — that is given here in Appendix B.)

116. *Kac V. G.* // Adv. Math. 1977. V. 26. P. 8–96; Commun. Math. Phys. 1977. V. 53. P. 31–64 (the second paper is an adaptation for physicists of the first paper).

117. *Mack G.* // Ibid. V. 55. P. 1.

118. *Dobrev V. K.* Subsingular Vectors and Conditionally Invariant (q-deformed) Equations // J. Phys. A: Math. Gen. 1995. V. 28. P. 7135–7155.

119. *Kazhdan D., Lusztig G.* // Inv. Math. 1979. V. 53. P. 165.

120. *Kac V. G.* Characters of Typical Representations of Classical Lie Superalgebras // Comm. Algebra. 1977. V. 5. P. 889–897.

121. *Bernstein I. N., Leites D. A.* // C. R. Acad. Bulg. Sci. 1980. V. 33. P. 1049.

122. *Van der Jeugt J. et al.* // Comm. Algebra. 1990. V. 18. P. 3453; J. Math. Phys. 1990. V. 31. P. 2278–2304.

123. *Van der Jeugt J.* // Comm. Algebra. 1991. V. 19. P. 199.

124. *Serganova V.* Kazhdan–Lusztig Polynomials and Character Formula for the Lie Superalgebra $gl(m|n)$ // Selecta Math. 1996. V. 2. P. 607–654.

125. *van der Jeugt J., Zhang R. B.* Characters and Composition Factor Multiplicities for the Lie Superalgebra $gl(m|n)$ // Lett. Math. Phys. 1999. V. 47. P. 49–61.

126. *Brundan J.* Kazhdan–Lusztig Polynomials and Character Formulae for the Lie Superalgebra $gl(m|n)$ // J. Am. Math. Soc. 2002. V. 16. P. 185–231; Kazhdan–Lusztig Polynomials and Character Formulae for the Lie Superalgebra $q(n)$ // Adv. Math. 2004. V. 182. P. 28–77.

127. *Su Yucai, Zhang R. B.* Character and Dimension Formulae for General Linear Superalgebra. math. QA/0403315.

128. *Dobrev V. K.* Character of the Unitarizable Highest Weight Modules Over the $N = 2$ Superconformal Algebras // Phys. Lett. B. 1987. V. 186. P. 43–51.

129. *Serganova V. V.* Appendix to the paper: *Leites D. A., Saveliev M. V., Serganova V. V.* // Proc. of Group Theoretical Methods in Physics. Yurmala, 1985 (in Russian: M., 1985. P. 377–394; in English: VNU. Dordrecht: Sci. Press, 1987).

130. *Penkov I., Skornyakov I.* // C. R. Acad. Sci. (Paris). 1984. V. 299, Serie I. P. 1005–1008.

131. *Penkov I.* // J. Sov. Math. 1990. V. 51. P. 2108.