ФИЗИКА ЭЛЕМЕНТАРНЫХ ЧАСТИЦ И АТОМНОГО ЯДРА
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PERIPHERAL PROCESSES IN QED AT HIGH ENERGIES

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We consider the high-energy processes in QED frames in peripheral kinematics. The key feature of this kinematics is that processes have large cross sections which do not decrease with the increasing of the initial center-of-mass energy. Two purposes to study peripheral processes are: the background processes with large total cross sections and the structure of jets in the fragmentation region.

We describe various QED peripheral processes in terms of Impact Factors (IF) and give the explicit expressions for the differential distributions and spin correlation effects, as well as estimates of the total cross section of peripheral processes in photon–photon, photon–lepton, and lepton–lepton collisions.

A special attention is paid to the small-angle Bhabha scattering process which is relevant for beam monitoring at LEP I, LEP II.

Based on analytical properties of the amplitudes, some relations (QED sum rules) between the high-energy asymptotics of the cross sections of inelastic processes in e^+e^- collisions and higher-order perturbative contributions to the electron Dirac and Pauli form factors are derived.

For practical using we present some loop momentum integrals.

Рассмотрены различные процессы КЭД в периферической кинематике. Важнейшим свойством этих процессов являются довольно большие сечения и их независимость от энергии начальных частиц, поэтому их изучение представляется интересной задачей как для учета фоновых процессов, так и для исследования структуры струйных событий.

Приводятся сечения различных процессов КЭД в периферической кинематике в рамках импакт-факторов, исследованы спиновые эффекты, приведены дифференциальные сечения для фотон-лептонных и лептон-лептонных соударений.

Отдельная секция посвящена баба-рассеянию на малые углы, и рассмотрены конкретные приложения для современных установок LEP I и LEP II.

На основе аналитических свойств амплитуды приведены соотношения (правила сумм), связывающие асимптотику сечения электрон-позитронного рассеяния и радиационные поправки к формфакторам электрона.

В конце обзора для удобства приведены явные значения некоторых петлевых интегралов Фейнмана для процесса аннигиляции двух фотонов в лептонную пару.

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To the blessed memory of Teachers Alexandr Ilich Akhiezer, Sergej Semenovich Sannikov, and Vladimir Naumovich Gribov

INTRODUCTION

The increasing accuracy of modern experiments which investigate the manifestations of weak and strong interactions require an adequate knowledge of the background reactions. The origin of the background is mainly due to electrodynamics interaction. As a rule, it is sizable and can imitate the characteristic features of the predictions from the SM (for the complete list of abbreviations see the end of Introduction) and the strong interaction of hadrons.

Modern methods developed in the frames of QCD, when applied to the problems of QED, allow one to take into account the contributions of QED nature with sufficient accuracy to be used in experiments.

In the study of processes at high energies, two expansion parameters are relevant; firstly, the power corrections in the variable $(m^2/s)^n$, where m is the electron mass. Here $\sqrt{s}$ is the total energy in the center of mass of initial colliding particles, which is supposed to be much larger than the characteristic masses of particles taking part in the process.

The second important parameter which enters in the description of processes at high energies is the so-called «large logarithm» $L = \ln(s/m^2)$, where m is the smallest mass of charged particle (lepton or pion) taking part in the process. This logarithm for modern experimental facilities can reach a value of 20. In high-energy QED, the quantity $(\alpha/\pi)L$ plays the role of expansion parameter in a perturbative expansion on the fine structure constant $\alpha = 1/137$, therefore it appears unavoidable to take into account higher powers of the expansion.

A wide class of processes at high energy of initial colliding particles has a cross section which does not decrease with $\sqrt{s}$. These processes (called *peripheral processes*) correspond to large values of orbital momentum of the initial state. Main purpose of this investigation is rigorous estimation of background processes of QED origin at colliders in experiments on «new physics» searching. The other possibility is to investigate the jet content in QED as a realistic model for similar problems in QCD.

These processes have a large cross section and have a practical interest for precision measurements as monitoring the intensity of the beam and its polarization properties.

In 1966, the picture of these processes was built by H. Cheng and T. T. Wu, describing high-energy small transfer momentum (peripheral) kinematics of the elastic scattering amplitudes in terms of Impact Factors (IF). These universal quantities describe the fragmentation property of each individual particle. For the subprocesses of $\gamma\gamma^* \to e^+e^-$, $\pi^+\pi^-$, electron–photon scattering with additional lepton and pion pair creation the relevant cross sections and matrix element in terms of chiral amplitudes are calculated. In Sec. 1 the radiative corrections to the IF for photon, electron with taking into account additional soft and hard-photon emission are presented. All leading logarithm contributions are taken into account in the frames of structure function approach.

For small-angle Bhabha scattering the estimation of the cross section in two-loop approximation is done.

At the end of Introduction we put the sum rules for processes of electron–positron and closely related electron–proton scattering. Using the analytical properties of the amplitudes we put the relation between nucleon form factors and a difference of proton and neutron differential electroproduction cross sections. In particular, for the case of small transferred momenta, one finally derives a sum rule, relating the Dirac proton mean square radius and anomalous magnetic moments of proton and neutron to the integral over a difference of the total proton and neutron photoproduction cross sections

In Sec. 2 we give the tables for one-loop Feynman integrals of scalar, vector, and tensor types, with two, three, four, and five denominators. All formulae are presented with the accuracy up to the terms of the order of the ratio of the electron to the muon masses squared, and the kinematic invariants are assumed to be large compared to the electron mass squared.

Throughout our paper we use the next designations:

FD — Feynman diagram
IF — impact factor
LBL — light-by-light
LC — light-cone
LLA — leading logarithmic approximation
QCD — quantum chromodynamics
QED — quantum electrodynamics
RC — radiative corrections
SM — Standard Model.

1. PERIPHERAL PROCESSES

1.1. QED Processes in Peripheral Kinematics at Polarized $\gamma\gamma$, $e^\pm\gamma$, $ee^\pm$ Colliders. QED processes with production of two up to six particles by high-energy colliding beams have attracted both theoretical and experimental attention during the last four decades. Accelerators with high-energy colliding e^+e^-, γe, $\gamma\gamma$, and $\mu^+\mu^-$ beams are now widely used or designed to study fundamental interactions [1]. Some processes of QED might play an important role at these colliders, especially those inelastic processes which cross section does not decrease with increasing energy. Polarization is also included in future plans, therefore these QED processes have to be described in more detail, including the calculation of cross sections with definite helicities of the initial particles — leptons ($\ell = e$ or μ) and photons γ. These reactions have the form of a two-jet process with exchange of a virtual photon γ^* in the t channel (see Fig. 1).

A lot of attention was paid in the literature to the calculation of helicity amplitudes of QED processes at high-energy colliders (see [2] and references therein). Keeping in mind the possible physical programme at $\gamma\gamma$ and lepton-γ colliders, a precise knowledge of calibration and monitoring processes is needed, as suggested, for example, in [3]. The calibration processes are known QED processes with sufficiently large cross sections, which have clear signatures for the detection. A wide domain of physics can be investigated in peripheral processes such as heavy leptons and mesons (scalar and pseudoscalar) production, where the relevant QED monitoring processes have to be measured.

The general feature of peripheral processes is the important property of their nondecreasing cross sections in the limit of high total energy $\sqrt{s}$ in CMS of the initial particles. It is then possible to produce and measure jets containing two or three particles.

The helicity amplitudes for subprocesses of type $2 \to 3$ have in general a complicated form. We do not give explicit expressions for the corresponding cross sections, indicating only the general procedure for deriving them.

Keeping in mind the increasing accuracy of the experiments, RC must be taken into account. For this aim the subprocess of pair creation by real and virtual photon with one-loop RC is considered, as well as the crossing subprocess — the real photon emission in virtual photon–electron collisions.

1.1.1. Kinematics in Quasi-Peripheral Region. Throughout the Section it is implied that the energy fractions of the jet components are positive quantities with values of the order of unity (the sum of the energy fractions of each jet is unity) and that the values of the components of the three-momentum transversal to the beam direction are much larger compared to their rest masses. Therefore the mass of the jet particles can be neglected.

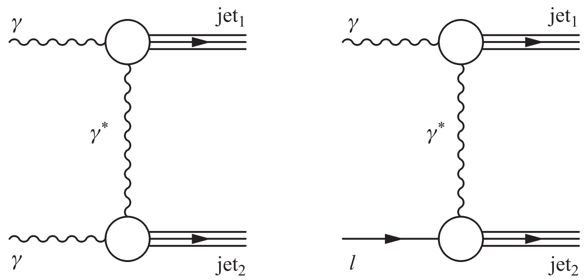


Fig. 1. Exchange of a virtual photon γ^* in the t channel of the processes $\gamma\gamma$, and γl ($l = e, \mu$)

The corresponding amplitudes include several FD. Fortunately, in the high-energy limit the number of essential FD contributing to the «leading» approximation greatly reduces. The method that we use here, has the advantage to estimate the uncertainty due to «nonleading» contributions which are of the order of

$$\frac{m^2}{s_1}, \quad \frac{s_1}{s}, \quad \frac{s_2}{s}, \quad \frac{\alpha}{\pi}\ln\frac{s}{m^2}, \tag{1.1}$$

where $s_{1,2}$ are the invariant masses squared of the jets $1, 2$. The last term in (1.1) corresponds to the lowest RC, which will not be considered in this Section. The angles θ_i between the emitted particles and the corresponding projectile directions (see Fig. 2) are assumed to be of the order:

$$\frac{m_i}{\sqrt{s}} \ll \theta_i \sim \frac{\sqrt{s_i}}{\sqrt{s}} \ll 1, \tag{1.2}$$

where m_i is the typical mass of the jet particle.

In this approach we can consider the initial particles (with four-momenta p_1, p_2) as massless and use the Sudakov parameterization of four-momenta of the particles as [4]:

$$q_i = \alpha_i p_2 + \beta_i p_1 + q_{i\perp}, \quad q_{i\perp} p_{1,2} = 0, \quad q_{i\perp}^2 = -\mathbf{q}_i{}^2 < 0. \tag{1.3}$$

The Sudakov parameters β_i are quantities of the order of unity for the momenta of the particles belonging to the $\mathrm{jet1}$ and obeying the conservation law $\sum_{\mathrm{jet1}} \beta_i = 1$. The components of the particle momenta of $\mathrm{jet1}$ along the four-momentum p_2 are small positive numbers which can be determined from the on-mass-shell conditions of the $\mathrm{jet1}$ particles:

$$q_i^2 = s\alpha_i\beta_i - \mathbf{q}_i^2 \approx 0, \quad \alpha_i = \frac{\mathbf{q}_i^2}{s\beta_i} \ll 1. \tag{1.4}$$

Here $\mathbf{q}$ is the transverse part of 4-momentum vector q, and in this Section we will imply this designation for all 2-fold vectors.

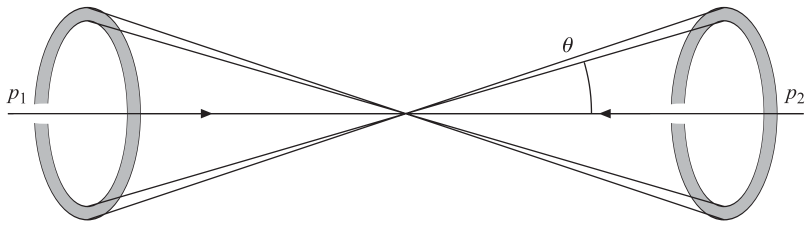


Fig. 2. Scheme of collision of initial beams with detection of two jets moving in cones within the angle θ

Similar considerations hold for the 4-momenta of the particles belonging to the jet2, namely, $\alpha_j \sim 1$, $\sum_{\text{jet2}} \alpha_j = 1$, $\beta_j = \mathbf{q}_j^2/(s\alpha_j) \ll 1$.

Among all possible FD, describing the process in the lowest (Born) order of PT, only one should be considered, the one which gives a contribution to the cross section which does not decrease with increasing s. It corresponds to a photon t-channel one-particle state.

It is known [5, 6] that the matrix elements of the peripheral processes have a factorized form and the cross section can be written in terms of the so-called IF, which describe the subprocess where the interaction of the internal virtual photon with one of the initial particles produces a jet along a direction close to the projectile momentum. Therefore the problem can be formulated in terms of computation of IF. For processes with initial photons in a definite state of polarization (described in terms of Stoke's parameters) we construct the relevant chiral matrices from bilinear combinations of chiral amplitudes. The last step consists in the calculation of the differential cross sections.

The matrix element, which corresponds to the main («leading») contribution to the cross section, has the form

$$M = iJ_1^\mu \frac{g_{\mu\nu}}{q^2} J_2^\nu, \tag{1.5}$$

where J_1^μ and J_2^ν are the currents of the upper (associated with jet1) and of the lower blocks of the relevant FD, respectively, and $g_{\mu\nu}$ is the metric tensor. The current J_1^μ describes the scattering of an incoming particle of momentum p_1 with a virtual photon and the subsequent transition to the first jet (similar to J_2^ν). The matrix elements (1.5) can be written in the form (see the Appendices in [4])

$$M = 2i\frac{s}{q^2} I_1 I_2, \quad I_1 = \frac{1}{s} J_1^\mu p_{2\mu}, \quad I_2 = \frac{1}{s} J_2^\nu p_{1\nu}, \tag{1.6}$$

which follows from the Gribov representation of the metric tensor,

$$g^{\mu\nu} = \frac{2}{s}(p_2^\mu p_1^\nu + p_2^\nu p_1^\mu) + g_\perp^{\mu\nu} \approx \frac{2}{s} p_2^\mu p_1^\nu. \tag{1.7}$$

I_1 and I_2 will be referred to as LC projections. The invariant mass squared of the jets can be also expressed in terms of the Sudakov parameters of the exchanged photon

$$\begin{gathered} q = \alpha p_2 + \beta p_1 + q_\perp, \quad (q + p_1)^2 = s_1 = -\mathbf{q}^2 + s\alpha, \\ (-q + p_2)^2 = s_2 = -\mathbf{q}^2 - s\beta, \quad q^2 = s\alpha\beta - \mathbf{q}^2 \approx -\mathbf{q}^2, \end{gathered} \tag{1.8}$$

q is the momentum of virtual photon (photon exchange between particles with impulses p_1 and p_2). Here and below we use the symbol $\approx$ for the approximation where we neglect the terms which do not contribute in the limit $s \to \infty$.

The singularity of the matrix element (1.6) at $\mathbf{q} = 0$ is fictitious (excluding the elastic scattering case). Indeed, one can see that it cancels due to the current conservation

$$\begin{aligned} q_\mu J_1^\mu &\approx (\alpha p_2 + q_\perp)_\mu J_1^\mu = 0, \qquad & p_{2\mu} J_1^\mu &= \frac{s}{s\alpha}\mathbf{q}\mathbf{J}_1, \\ q_\nu J_2^\nu &\approx (\beta p_1 + q_\perp)_\nu J_2^\nu = 0, \qquad & p_{1\nu} J_2^\nu &= \frac{s}{s\beta}\mathbf{q}\mathbf{J}_2. \end{aligned} \tag{1.9}$$

We obtain the modified form of the matrix element for peripheral processes:

$$\begin{gathered} M[a(p_1,\eta_1) + b(p_2,\eta_2) \to \mathrm{jet}_{1\lambda_1} + \mathrm{jet}_{2\lambda_2}] = i(4\pi\alpha)^{\frac{n_1+n_2}{2}} \frac{2s}{\mathbf{q}^2} m_{1\lambda_1}^{\eta_1} m_{2\lambda_2}^{\eta_2}, \\ n_{1,2} \geqslant 2, \quad \eta, \lambda = \pm 1, \end{gathered} \tag{1.10}$$

where η_i describe the polarization states of the projectile $i = a, b$; λ_i describes the polarization states of participants of the corresponding jet. The numbers of the QED vertices in the upper and lower blocks of FD (see Fig. 1) are denoted by $n_{1,2}$.

For the matrix elements $m_{1,2}$ of the subprocesses $\gamma^*(q) + a(p_1,\eta_1) \to \mathrm{jet}_1(\lambda_1)$ and $\gamma^*(q) + b(p_2,\eta_2) \to \mathrm{jet}_2(\lambda_2)$ we give here two alternative forms:

$$m_{1\lambda_1}^{\eta_1} = \frac{1}{s} p_{2\mu} J_{1\lambda_1}^{\eta_1\mu}, \quad m_{1\lambda_1}^{\eta_1} = \frac{\mathbf{q}\mathbf{J}_{1\lambda_1}^{\eta_1}}{s_1 + \mathbf{q}^2}. \tag{1.11}$$

Similar expressions hold for the lower block. The second representation is used below. Equation (1.11) can be used as a check of validity of gauge invariance, verifying that the matrix elements vanish in the limit $\mathbf{q} \to 0$.

Let us verify that the differential cross section of peripheral processes does not depend on the total CMS energy $\sqrt{s}$. For that we rewrite the phase volume of the final two-jet kinematical state in a more convenient form:

$$\begin{aligned} dF = (2\pi)^4 \delta^4 \left(p_1 + p_2 - \sum_i p_i^{(1)} - \sum_j p_j^{(2)} \right) dF^{(1)} dF^{(2)} = \\ = (2\pi)^4 d^4 q \delta^4_{(1)} \delta^4_{(2)} dF^{(1)} dF^{(2)}, \end{aligned}$$

$$\begin{gathered} \delta^4_{(1)} = \delta^4 \left(p_1 + q - \sum_i p_i^{(1)} \right), \quad \delta^4_{(2)} = \delta^4 \left(p_2 - q - \sum_j p_j^{(2)} \right), \\ dF^{(1,2)} = \Pi_i \frac{d^3 p_i^{(1,2)}}{2\varepsilon_i^{(1,2)} (2\pi)^3}. \end{gathered} \tag{1.12}$$

Using Sudakov's parameterization for the transferred four-momentum q

$$d^4q = \frac{s}{2} d\alpha\, d\beta\, d^2q_\perp = \frac{1}{2s} ds_1\, ds_2\, d^2q_\perp, \tag{1.13}$$

where $s_{1,2}$ is the invariant mass squared of the jets, the phase volume can be written in a factorized form:

$$dF = \frac{(2\pi)^4}{2s} d^2q_\perp\, ds_1\, dF^{(1)}\delta^4_{(1)}\, ds_2\, dF^{(2)}\delta^4_{(2)}. \tag{1.14}$$

Using Eq. (1.14) for the matrix element and the phase volume for the cross section of the peripheral process in the case of polarized initial particles (photons or electrons), the cross section is

$$d\sigma^{\eta_1\eta_2} = \frac{\alpha^{n_1+n_2}\pi^2(4\pi)^{2+n_1+n_2}\, d^2q_\perp}{(\mathbf{q}^2)^2}\, d\Phi_1^{\eta_1}(\mathbf{q})\, d\Phi_2^{\eta_2}(\mathbf{q}), \tag{1.15}$$

where the differential IF $d\Phi_i^{\eta_i}$ are defined as:

$$d\Phi_i^{\eta_i}(\mathbf{q}) = \int ds_i \sum_{\lambda_j} |m^{\eta_i}_{i\lambda_j}|^2\, dF_i \delta^4_{(i)}, \quad i = 1, 2. \tag{1.16}$$

Below we use the term «impact factor» or «chiral amplitudes of subprocess» instead of «differential impact factor» for simplicity. The «impact factor» as defined in the paper of Cheng–Wu [5] is introduced to describe elastic amplitudes for the process $a + b \to a + b$ in the case of small angle high-energy scattering. This will be considered in Subsec. 1.3.

The matrix elements for definite chiral states of all particles $m^{\eta_i}_{i(\lambda)}$, where the subscript (λ) denotes the set of chiral parameters of the final state, are calculated and listed below.

In the case of initial polarized photons, a description in terms of Stoke's parameters $\xi_{1,2,3}$, $\xi_1^2 + \xi_2^2 + \xi_3^2 \leqslant 1$ is commonly used. The matrix element squared in the right-hand side (r.h.s.) of Eq. (1.16) should be replaced by [7, 8]

$$T_\gamma = \mathrm{Tr}(m\rho) = \frac{1}{2}\,\mathrm{Tr}\begin{pmatrix} m^{++} & m^{+-} \\ m^{-+} & m^{--} \end{pmatrix}\begin{pmatrix} 1+\xi_2 & i\xi_1 - \xi_3 \\ -i\xi_1 - \xi_3 & 1-\xi_2 \end{pmatrix}, \tag{1.17}$$

where the elements of the spin matrix m are:

$$\begin{aligned} m^{++} &= \sum_\lambda |m^+_{(\lambda)}|^2, & m^{+-} &= \sum_\lambda m^+_{(\lambda)}(m^-_{(\lambda)})^*, \\ m^{--} &= \sum_\lambda |m^-_{(\lambda)}|^2, & m^{-+} &= (m^{+-})^*. \end{aligned} \tag{1.18}$$

In the case of initial fermion we choose its polarization equal to $\eta_i = +1$. The matrix element squared for electron scattering must be summed over the polarization of the final particles:

$$T_e = \sum_\lambda |m^+_{(\lambda)}|^2. \tag{1.19}$$

The cross section $d\sigma_{n_1,n_2}$ of the process of type $2 \to n_1 + n_2$ with production of two jets

$$\begin{aligned} &a(p_1,\eta_1) + b(p_2,\eta_2) \to \\ &\to a_1(r_1,\lambda_1) + \ldots + a_{n_1}(r_{n_1},\lambda_{n_1}) + b_1(q_1,\sigma_1) + \ldots + b_{n_2}(q_{n_2},\sigma_{n_2}), \end{aligned} \tag{1.20}$$

where the energy fractions $x_1, \ldots, x_{n_1}, \sum x_i = 1$, the transversal components of momenta $\mathbf{r}_1, \ldots, \mathbf{r}_{n_1}$, $\sum \mathbf{r}_i = \mathbf{q}$ of jet a and similar quantities $y_i, \mathbf{q}_i, \sum y_i = 1, \sum \mathbf{q}_i = -\mathbf{q}$ for the other jet b, are:

$$d\sigma_{22} = \frac{\alpha^4}{2^2\pi^4} T^{(1)}_{a\to 2} T^{(2)}_{b\to 2} \frac{d^2q}{(\mathbf{q}^2)^2} d^2r_1\, d^2q_1 \frac{dx_1\, dy_1}{x_1x_2y_1y_2}, \tag{1.21}$$

$$d\sigma_{23} = \frac{\alpha^5}{2^4\pi^6} T^{(1)}_{a\to 2} T^{(2)}_{b\to 3} \frac{d^2q}{(\mathbf{q}^2)^2} d^2r_1\, d^2q_1\, d^2q_2 \frac{dx_1 dy_1\, dy_2}{x_1x_2y_1y_2y_3}, \tag{1.22}$$

$$d\sigma_{33} = \frac{\alpha^6}{2^6\pi^8} T^{(1)}_{a\to 3} T^{(2)}_{b\to 3} \frac{d^2q}{(\mathbf{q}^2)^2} d^2q_1\, d^2q_2 d^2r_1\, d^2r_2 \frac{dx_1\, dx_2\, dy_1\, dy_2}{x_1x_2x_3y_1y_2y_3}, \tag{1.23}$$

$$a, b = e, \gamma.$$

1.1.2. The Subprocesses $\gamma\gamma^* \to e^+e^-$, $\pi^+\pi^-$. Let us consider first the contribution of the lepton pair production subprocess to the photon IF:

$$\gamma(k_1,\eta) + \gamma^*(q) \to e^-(q_-,\lambda) + e^+(q_+,-\lambda). \tag{1.24}$$

The matrix element of the subprocess has the form (we omit the factor $4\pi\alpha$)

$$\begin{gathered} m^{\eta\mu}_{1\lambda} = -\bar{u}_\lambda(q_-)\Big[\hat{\varepsilon}^\eta \frac{\hat{q}_- - \hat{k}_1}{\kappa_{1-}}\gamma^\mu + \gamma^\mu \frac{-\hat{q}_+ + \hat{k}_1}{\kappa_{1+}}\hat{\varepsilon}^\eta\Big] v_{-\lambda}(q_+), \\ \bar{u}_\lambda = \bar{u}\,\omega_{-\lambda}, \qquad v_\lambda = \omega_{-\lambda} v, \qquad \kappa_{1\pm} = 2k_1q_\pm, \qquad \omega_\lambda = \frac{1}{2}(1+\lambda\gamma_5). \end{gathered} \tag{1.25}$$

We imply that all the particles are massless. A definite chiral state of the initial photon polarization vector has the form [9, 10]:

$$\hat{\varepsilon}^\lambda_1 = N_1[\hat{q}_-\hat{q}_+\hat{k}_1\omega_{-\lambda} - \hat{k}_1\hat{q}_-\hat{q}_+\omega_\lambda], \quad N_1^2 = \frac{2}{s_1\kappa_{1+}\kappa_{1-}}, \quad s_1 = 2q_+q_-. \tag{1.26}$$

The chiral amplitudes $m^\eta_\lambda = (1/s) m^{\eta\mu}_{1(\lambda)} p_{2\mu}$ have the form:

$$\begin{aligned} m^+_{1+} &= -\frac{N_1}{s}\bar{u}\hat{q}_+\hat{q}\hat{p}_2\omega_+ v, & m^+_{1-} &= -\frac{N_1}{s}\bar{u}\hat{p}_2\hat{q}\hat{q}_-\omega_- v, \\ m^-_{1-} &= -\frac{N_1}{s}\bar{u}\hat{q}_+\hat{q}\hat{p}_2\omega_- v, & m^-_{1+} &= -\frac{N_1}{s}\bar{u}\hat{p}_2\hat{q}\hat{q}_-\omega_+ v. \end{aligned} \tag{1.27}$$

The elements of spin matrix m (see Eq. (1.18)) in the case of lepton pair production are:

$$\begin{aligned} m^{++}_{e^+e^-} &= m^{--}_{e^+e^-} = \frac{2\mathbf{q}^2}{\mathbf{q}_+^2\mathbf{q}_-^2} x_+ x_- (x_+^2 + x_-^2), \\ m^{+-}_{e^+e^-} &= (m^{-+}_{e^+e^-})^* = -\frac{4\mathbf{q}^2}{\mathbf{q}_+^2\mathbf{q}_-^2}(x_+x_-)^2 \mathrm{e}^{2i\theta}, \end{aligned} \tag{1.28}$$

where $x_\pm$ are the energy fractions carried out by the components of the pair, with $x_+ + x_- = 1$, θ is the angle between two Euclidean vectors $\mathbf{q} = \mathbf{q}_- + \mathbf{q}_+$ and $\mathbf{Q} = x_+\mathbf{q}_- - x_-\mathbf{q}_+$.

In the case of charged pion pair production

$$\gamma(p_1, \eta) + \gamma^*(q) \to \pi^+(q_+) + \pi^-(q_-) \tag{1.29}$$

we have

$$m^\eta = \frac{1}{s}\varepsilon^\eta_{1\nu} p_2^\mu m^\nu_\mu = \frac{x_+}{p_1q_-}\varepsilon^\eta_1 q_- + \frac{x_-}{p_1q_+}\varepsilon^\eta_1 q_+ - \frac{2}{s}(\varepsilon^\eta_1 p_2)\,. \tag{1.30}$$

Using the photon polarization vector written as

$$\varepsilon^\eta_{1\mu} = N_1[(q_+p_1)q_{-\mu} - (q_-p_1)q_{+\mu} + i\eta\varepsilon_{\mu\alpha\beta\gamma}q_-^\alpha q_+^\beta p_1^\gamma]\,, \tag{1.31}$$

we obtain the chiral amplitude of the pion pair production process (here we use $\epsilon_{\alpha\beta\gamma\delta}p_1^\alpha p_2^\beta q_-^\gamma q_+^\delta = (s/2)[\mathbf{q}_-\mathbf{q}_+]_z$, with $\epsilon_{\alpha\beta\gamma\delta}$ being the antisymmetric tensor, with $\epsilon_{0123} = 1$)

$$m^\eta = -N_1(\mathbf{Q}\mathbf{q} + i\eta[\mathbf{Q}, \mathbf{q}]_z) = -N_1|\mathbf{q}|\,|\mathbf{Q}|\,\mathrm{e}^{i\eta\theta}, \quad \theta = \widehat{\mathbf{q}\mathbf{Q}}, \tag{1.32}$$

where we imply that the direction of the z axis is along the photon three-momentum and use the relation $[\mathbf{q}_-, \mathbf{q}_+]_z = [\mathbf{Q}, \mathbf{q}]_z$. The pion chiral matrix can be written as

$$\begin{aligned} m^{++}_{\pi^+\pi^-} &= m^{--}_{\pi^+\pi^-} = \frac{2\mathbf{q}^2}{\mathbf{q}_+^2\mathbf{q}_-^2}(x_+x_-)^2, \\ m^{+-}_{\pi^+\pi^-} &= (m^{-+}_{\pi^+\pi^-})^* = \frac{2\mathbf{q}^2}{\mathbf{q}_+^2\mathbf{q}_-^2}(x_+x_-)^2\,\mathrm{e}^{2i\theta}. \end{aligned} \tag{1.33}$$

For the two-pair production process

$$\begin{aligned} &\gamma_1(p_1, \boldsymbol{\xi}_1) + \gamma_2(p_2, \boldsymbol{\xi}_2) \to a(q_-) + \bar{a}(q_+) + b(p_-) + \bar{b}(p_+), \\ &q_\pm = \alpha_\pm p_2 + x_\pm p_1 + q_{\pm\perp}, \quad p_\pm = y_\pm p_2 + \beta_\pm p_1 + p_{\pm\perp}, \end{aligned} \tag{1.34}$$

the differential cross section (assuming that the pair $a\bar{a}$ moves along the direction of the photon 1 and the pair $b\bar{b}$ moves along the direction of the photon 2) has the form (1.21), (1.22) with:

$$T^{(1)}_{\gamma\to 2} = \frac{\mathbf{q}^2}{\mathbf{q}_+^2 \mathbf{q}_-^2}(x_+x_-)^2[1 - \xi_3\cos(2\theta) + \xi_1\sin(2\theta)] \text{ for } \pi^+, \pi^-, \tag{1.35}$$

$$T^{(1)}_{\gamma\to 2} = \frac{\mathbf{q}^2}{\mathbf{q}_+^2 \mathbf{q}_-^2}(x_+x_-)\{x_+^2 + x_-^2 + 2x_+x_-[\xi_3\cos(2\theta) + \xi_1\sin(2\theta)]\} \text{ for } e^+, e^- \tag{1.36}$$

with a similar expression for $T^{(2)*}$. We remind that the formulae obtained here are valid at large transverse component of the jet particle momentum, compared to the masses of the particles,

$$\mathbf{q}_-^2 \sim \mathbf{q}_+^2 \sim \mathbf{p}_+^2 \sim \mathbf{p}_-^2 \gg m^2, \quad \mathbf{q}_+ = \mathbf{q} - \mathbf{q}_-, \quad \mathbf{p}_+ = -\mathbf{q} - \mathbf{p}_-, \tag{1.37}$$

and for finite energy fractions $x_\pm \sim y_\pm \sim 1$, which correspond to emission angles of jet particles $\theta_i = |\mathbf{q}_i|/(x_i\varepsilon) \gg m/\varepsilon$ which are considerably larger than the mass-to-energy ratio (ε is the energy of the initial particle in CMS).

1.1.3. Subprocesses $\gamma\gamma^* \to e^+e^-\gamma$, $\pi^+\pi^-\gamma$. Here and below for subprocesses of type $2 \to 3$ we restrict ourselves to the calculation of the chiral amplitudes and to the check of their gauge invariance properties.

The subprocess

$$\gamma(k, \lambda) + \gamma^*(q) \to e^+(q_+, -\lambda_-) + e^-(q_-, \lambda_-) + \gamma(k_1, \lambda_1) \tag{1.38}$$

is described by six FD. A standard calculation of chiral amplitudes $m^\lambda_{\lambda_1\lambda_-}$ leads to

$$\begin{aligned} m^+_{++} &= -\frac{s_1 N N_1}{s}\bar{u}(q_-)\hat{q}_+\hat{q}\hat{p}_2\omega_+ v(q_+) = (m^-_{--})^*, \\ m^+_{+-} &= -\frac{s_1 N N_1}{s}\bar{u}(q_-)\hat{p}_2\hat{q}\hat{q}_-\omega_- v(q_+) = (m^-_{-+})^*, \\ m^+_{-+} &= \frac{N N_1}{s}\bar{u}(q_-)A^+_{-+}\omega_+ v(q_+) = (m^-_{+-})^*, \\ m^+_{--} &= \frac{N N_1}{s}\bar{u}(q_-)A^+_{--}\omega_- v(q_+) = (m^-_{++})^*, \end{aligned} \tag{1.39}$$

*In paper [12] Eq. (2.36) contains a misprint in the sign of $\xi_3^{(1,2)}$.

with $A^{+}_{--}(k,k_1) = A^{+}_{-+}(-k_1,-k)$,

$$\begin{gathered} N^2 = \frac{2}{s_1\kappa_-\kappa_+}, \quad N_1^2 = \frac{2}{s_1\kappa_{1+}\kappa_{1-}}, \\ s_1 = 2q_+q_-, \quad \kappa_\pm = 2kq_\pm, \quad \kappa_{1\pm} = 2k_1q_\pm \end{gathered} \tag{1.40}$$

and the following expression for A^{+}_{-+}

$$A^{+}_{-+} = \frac{s_1}{(q_+-q)^2}\hat{k}\hat{q}_+\hat{k}_1(-\hat{q}_+ + \hat{q})\hat{p}_2 - \hat{q}_+(\hat{q}_- - \hat{k})\hat{p}_2(\hat{q}_+ + \hat{k}_1)\hat{q}_- \\ - \frac{s_1}{(q_- - q)^2}\hat{p}_2(\hat{q}_- - \hat{q})\hat{k}\hat{q}_-\hat{k}_1. \tag{1.41}$$

Substituting

$$\hat{p}_2 \approx \frac{1}{\alpha}(\hat{q} - \hat{q}_\perp) = \frac{s}{s\alpha}[\hat{q}_+ + \hat{k_1} + (\hat{q}_- - \hat{k}) - \hat{q}_\perp]$$

in the second term of r.h.s. of Eq. (1.41), we have

$$A^{+}_{-+} = -ss_1\kappa_{1+}\left[\frac{x_+}{(q_+-q)^2} + \frac{1}{s\alpha}\right]\hat{k} - ss_1\kappa_-\left[\frac{x_-}{(q_--q)^2} + \frac{1}{s\alpha}\right]\hat{k}_1 + \\ + \frac{s_1}{(q_+-q)^2}\hat{k}\hat{q}_+\hat{k}_1\hat{q}_\perp\hat{p}_2 + \frac{s_1}{(q_--q)^2}\hat{p}_2\hat{q}_\perp\hat{k}\hat{q}_-\hat{k}_1 + \\ + \frac{s}{s\alpha}\hat{q}_+(\hat{q}_- - \hat{k})\hat{q}_\perp(\hat{q}_+ + \hat{k}_1)\hat{q}_-, \tag{1.42}$$

with

$$\begin{gathered} (q_\pm - q)^2 = -\mathbf{q}^2 + 2\mathbf{q}\mathbf{q}_\pm - s\alpha x_\pm, \quad s\alpha = \frac{\mathbf{k}_1^2}{x_1} + \frac{\mathbf{q}_-^2}{x_-} + \frac{\mathbf{q}_+^2}{x_+}, \\ x_1 + x_- + x_+ = 1, \quad \kappa_\pm = \frac{\mathbf{q}_\pm^2}{x_\pm}, \quad \kappa_{1\pm} = \frac{1}{x_1x_\pm}(x_1\mathbf{q}_\pm - x_\pm\mathbf{k}_1)^2, \end{gathered} \tag{1.43}$$

and $x_\pm = 2p_2q_\pm/s$, $x_1 = 2p_2k_1/s$. The gauge invariance property (the chiral amplitudes must vanish as $\mathbf{q} \to 0$) can be seen explicitly.

The procedure of constructing the chiral matrix is straightforward and can be performed in terms of simple traces. We will not explicate it here.

Let us consider the subprocess

$$\gamma(k,\lambda) + \gamma^*(q) \to \pi^+(q_+) + \pi^-(q_-) + \gamma(k_1,\lambda_1). \tag{1.44}$$

The matrix element is described by 12 FD. Its expression can be considerably simplified following the formalism of [15] for the photon polarization vectors (compare with (1.31)):

$$\varepsilon_\mu^\lambda(k) = \frac{N}{2}\mathrm{Sp}\,(\gamma_\mu \hat{q}_- \hat{q}_+ \hat{k}\omega_\lambda), \quad \varepsilon_\mu^{\lambda_1}(k_1) = \frac{N_1}{2}\mathrm{Sp}\,(\gamma_\mu \hat{q}_- \hat{q}_+ \hat{k}_1\omega_\lambda), \tag{1.45}$$

with similar expressions for N, N_1 as in the case of the $\gamma\gamma^* \to e^+e^-\gamma$ subprocess. The polarization vectors are chosen in such a form, that to satisfy the Lorentz conditions $\varepsilon(k)k = 0$, $\varepsilon(k_1)k_1 = 0$ and the gauge condition $\varepsilon(k)q_- = = \varepsilon(k_1)q_- = 0$.

The matrix element has the form (at this stage the Bose symmetry is lost):

$$\begin{aligned} m_{\lambda_1}^\lambda &= \frac{1}{s}p_2^\rho \varepsilon^\mu(k)\varepsilon_1^{*\sigma}(k_1)O_{\rho\mu\sigma} = \\ &= \frac{4x_-}{(q_- - q)^2}\Big[\frac{(\varepsilon_1 q_+)(\varepsilon q)}{\kappa_{1+}} - \frac{(\varepsilon_1 q)(\varepsilon q_+)}{\kappa_+}\Big] + \frac{4(\varepsilon p_2)(\varepsilon_1 q_+)}{s\kappa_{1+}} - \\ &\qquad - \frac{4(\varepsilon_1 p_2)(\varepsilon q_+)}{s\kappa_+} + 2(\varepsilon\varepsilon_1)\Big[\frac{x_+}{(q_+ - q)^2} - \frac{x_-}{(q_- - q)^2}\Big], \end{aligned} \tag{1.46}$$

where we imply $\varepsilon = \varepsilon^\lambda$, $\varepsilon_1 = \varepsilon_1^{\lambda_1}$ and $x_\pm = 2p_2 q_\pm/s$, $x_1 = 2p_2 k_1/s$, where $x_+ + x_- + x_1 = 1$.

For $\lambda_1 = \lambda$ we have

$$\begin{aligned} m_\lambda^\lambda &= s_1 N N_1 [A_1 + i\lambda B_1], \quad A_1 = -\mathbf{Q}\mathbf{q}, \\ B_1 &= [\mathbf{Q}\times\mathbf{q}]_z, \quad \mathbf{Q} = \mathbf{q}_- x_+ - \mathbf{q}_+ x_-. \end{aligned} \tag{1.47}$$

For the case of opposite chiralities we have

$$m_{-\lambda}^\lambda = s_1 N N_1 [A + i\lambda B],$$

$$\begin{aligned} A = \frac{1}{2x_1 x_- x_+}&\left[\mathbf{Q}^2\mathbf{k}_1^2 - \mathbf{q}_-^2(x_1\mathbf{q}_+ - x_+\mathbf{k}_1)^2 - \mathbf{q}_+^2(x_1\mathbf{q}_- - x_-\mathbf{k}_1)^2\right] \times \\ &\times \left(\frac{x_+}{(q_+ - q)^2} - \frac{x_-}{(q_- - q)^2}\right) - \mathbf{Q}\mathbf{q}, \end{aligned} \tag{1.48}$$

$$\begin{aligned} B &= \left(\frac{x_+}{(q_+ - q)^2} + \frac{x_-}{(q_- - q)^2}\right) \times \\ &\times (s\alpha[\mathbf{q}_- \times \mathbf{q}_+]_z - s\alpha_-[\mathbf{q}\times\mathbf{q}_+]_z + s\alpha_+[\mathbf{q}\times\mathbf{q}_-]_z) + 2[\mathbf{q}_-\times\mathbf{q}_+]_z - [\mathbf{Q}\times\mathbf{q}]_z, \end{aligned}$$

$$s\alpha_\pm = \frac{\mathbf{q}_\pm^2}{x_\pm}, \quad s\alpha = \frac{\mathbf{k}_1^2}{x_1} + s\alpha_+ + s\alpha_-.$$

We can see that the Bose symmetry is restored.

1.1.4. Subprocesses $e\gamma^* \to e\gamma$, $e+\gamma+\gamma$. Consider first the Compton subprocess*

$$e(p,\lambda_1)+\gamma^*(q)\to\gamma(k,\lambda)+e(p',\lambda_1). \tag{1.49}$$

For the chiral matrix elements we have (we choose $\lambda_1=+1$)

$$\begin{aligned}
m_\lambda^+ &= \frac{N}{s}\bar{u}(p')[-\hat{p}\omega_\lambda(\hat{p}'+\hat{k})\hat{p}_2-\hat{p}_2(\hat{p}-\hat{k})\hat{p}'\omega_{-\lambda}]\omega_+u(p),\\
m_+^+ &= -\frac{N}{s}\bar{u}(p')\hat{p}\hat{q}\hat{p}_2\omega_+u(p),\\
m_-^+ &= -\frac{N}{s}\bar{u}(p')\hat{p}_2\hat{q}\hat{p}'\omega_+u(p), \quad N^2=\frac{1}{(pp')\kappa\kappa'},
\end{aligned} \tag{1.50}$$

and κ, κ' are defined in (1.52). The sum of module squared of the matrix elements is

$$T_e = 2\frac{\mathbf{q}^2}{\kappa\kappa'}[1+x'^2], \tag{1.51}$$

with

$$\kappa=2kp=\frac{\mathbf{k}^2}{x}, \quad \kappa'=2kp'=\frac{1}{xx'}(\mathbf{p}\,'x-\mathbf{k}x')^2, \tag{1.52}$$

and $x=2kp_2/2pp_2$, $x'=1-x$ are the energy fractions of the photon and the electron in the final state.

Consider now the double Compton subprocess (see Fig. 3, a):

$$e(p,\eta)+\gamma^*(q)\to e(p',\eta)+\gamma(k_1,\lambda_1)+\gamma(k_2,\lambda_2). \tag{1.53}$$

The chiral matrix elements $m^\eta_{\lambda_1\lambda_2}$ are

$$\begin{aligned}
m_{++}^+ &= (m_{--}^-)^* = -\frac{s_1N_1N_2}{s}\bar{u}(p')\hat{p}\hat{q}\hat{p}_2\omega_+u(p),\\
m_{--}^+ &= (m_{++}^-)^* = -\frac{s_1N_1N_2}{s}\bar{u}(p')\hat{p}_2\hat{q}\hat{p}'\omega_+u(p),\\
m_{+-}^+ &= (m_{-+}^-)^* = \frac{N_1N_2}{s}\bar{u}(p')A_{+-}^+\omega_+u(p),\\
m_{-+}^+ &= (m_{+-}^-)^* = \frac{N_1N_2}{s}\bar{u}(p')A_{-+}^+\omega_+u(p),
\end{aligned} \tag{1.54}$$

*The case of real initial photons was considered in paper [12].

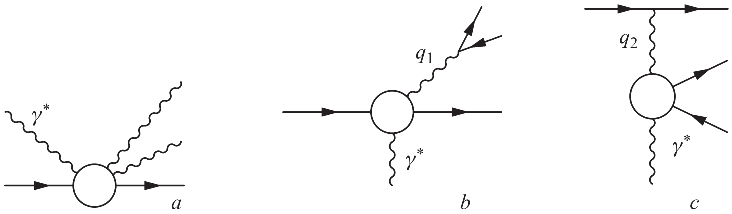


Fig. 3. Feynman diagrams describing: the subprocess $\gamma^* e^- \to \gamma\gamma e^-$ (a); the subprocesses of pair production $\gamma^* e \to e a\bar{a}$ by bremsstrahlung (b) and double photon mechanisms (c)

with $A^{+}_{-+}(k_1,k_2) = A^{+}_{+-}(k_2,k_1)$ and

$$A^{+}_{+-}(k_1,k_2) = \frac{s_1}{(p'-q)^2}\hat{p}_2(\hat{p}'-\hat{q})\hat{k}_1\hat{p}'\hat{k}_2 + \hat{p}(\hat{p}'+\hat{k}_1)\hat{p}_2(\hat{p}-\hat{k}_2)\hat{p}' + \\ + \frac{s_1}{(p+q)^2}\hat{k}_1\hat{p}\hat{k}_2(\hat{p}+\hat{q})\hat{p}_2, \quad (1.55)$$

with

$$s_1 = 2pp', \quad N_i^2 = \frac{2}{s_1\kappa_i\kappa_i'}, \quad \kappa_i = 2pk_i, \quad \kappa_i' = 2p'k_i. \quad (1.56)$$

To check the gauge invariance property of two last amplitudes, let us substitute $p_2 = (q - q_\perp)/\alpha_q$ in the r.h.s. of Eq. (1.55) and obtain:

$$A^{+}_{+-}(k_1,k_2) = ss_1\kappa_1'\left(\frac{x'}{(p'-q)^2} + \frac{1}{s\alpha_q}\right)\hat{k}_2 + ss_1\kappa_2\left(\frac{1}{(p+q)^2} - \frac{1}{s\alpha_q}\right)\hat{k}_1 + \\ + \frac{s_1}{(p+q)^2}\hat{k}_1\hat{p}\hat{k}_2\hat{q}_\perp\hat{p}_2 - \frac{s_1}{(p'-q)^2}\hat{p}_2\hat{q}_\perp\hat{k}_1\hat{p}'\hat{k}_2 - \hat{p}(\hat{p}'+\hat{k}_1)\hat{q}_\perp(\hat{p}-\hat{k}_2)\hat{p}'\frac{s}{s\alpha_q}. \quad (1.57)$$

We can verify that this expression vanishes at $\mathbf{q} = 0$, using the following relations:

$$(p'-q)^2 = -\mathbf{q}^{\,2} + 2\mathbf{p}'\mathbf{q} - sx'\alpha_q, \quad (p+q)^2 = -\mathbf{q}^{\,2} + s\alpha_q,$$
$$\alpha_q = \alpha' + \alpha_1 + \alpha_2, \quad x' + x_1 + x_2 = 1, \quad s\alpha' = \frac{(\mathbf{p}')^2}{x'}, \quad s\alpha_i = \frac{\mathbf{k}_i^2}{x_i}, \quad (1.58)$$
$$\kappa_i = s\alpha_i, \quad \kappa_i' = \frac{1}{x'x_i}(\mathbf{k}_i x' - \mathbf{p}' x_i)^2,$$

here we use Sudakov decomposition (1.3), and $x_{1,2} = 2k_{1,2}p/2pp_2$, $x' = 1 - x_1 - x_2$ are the energy fractions of photons and scattered electron.

One can proceed further in the calculation of the chiral matrix, with a similar procedure as detailed in the previous Section.

1.1.5. Subprocesses $e\gamma^* \to e\pi^+\pi^-$, $e\mu^+\mu^-$. The matrix element of the pion pair production subprocess

$$e(p,\eta) + \gamma^*(q) \to \pi^+(q_+) + \pi^-(q_-) + e(p',\eta) \tag{1.59}$$

can be written in the form

$$m^\eta = \bar{u}(p')[\hat{B} + \hat{D}]\omega_\eta u(p), \tag{1.60}$$

where $\hat{B}$ and $\hat{D}$ arise respectively from the bremsstrahlung and the double photon mechanisms (see Fig. 3, b, c):

$$\begin{aligned}\hat{B} &= \frac{1}{q_1^2}\left[B\hat{q}_1 + \frac{1}{s(p+q)^2}\hat{q}_1\hat{q}\hat{p}_2 - \frac{1}{s(p'-q)^2}\hat{p}_2\hat{q}\hat{q}_1\right],\\ \hat{D} &= \frac{1}{q_2^2}\left[D(2\hat{q}_- + \hat{q}_2) - 2\frac{x_-}{(q-q_-)^2}\hat{q}_\perp + \frac{2(\mathbf{q}^2 - 2\mathbf{q}\,\mathbf{q}_-)}{s(q-q_-)^2}\hat{p}_2\right],\end{aligned} \tag{1.61}$$

$$q_1 = q_+ + q_-, \quad q_2 = p' - p,$$

with B and D given by

$$\begin{gathered}B = \frac{x'}{(p'-q)^2} + \frac{1}{(p+q)^2}, \quad D = \frac{x_-}{(q_- - q)^2} - \frac{x_+}{(q-q_+)^2},\\ x_\pm = \frac{2p_2q_\pm}{s}, \quad x' = \frac{2p_2p'}{s}, \quad x_+ + x_- + x' = 1.\end{gathered} \tag{1.62}$$

For the squares of modulo of the chiral amplitudes, which enter in (1.22), (1.23), we have for $e\pi^+\pi^-$:

$$T^{(1)}_{e\to 3} = |m^+|^2 = \mathrm{Sp}\,(\hat{p}'(\hat{B} + \hat{D})\hat{p}(\hat{B} + \hat{D})\omega_+). \tag{1.63}$$

For the subprocess of the muon pair production

$$e(p,\eta) + \gamma^*(q) \to \mu^+(q_+) + \mu^-(q_-) + e(p',\eta), \tag{1.64}$$

the bremsstrahlung and the two-photon mechanisms as well must be taken into account (see Fig. 3, b, c)

$$\begin{aligned}m_\lambda^+ = \frac{1}{q_1^2}\bar{u}(p')B_\mu\omega_+u(p) \times \bar{u}(q_-)\gamma^\mu\omega_\lambda v(q_+) +\\ + \frac{1}{q_2^2}\bar{u}(p')\gamma_\nu\omega_+u(p) \times \bar{u}(q_-)D_\nu\omega_\lambda v(q_+),\end{aligned} \tag{1.65}$$

with the double photon mechanism contribution

$$D_\nu = D\gamma_\nu + \frac{1}{s(q-q_+)^2}\gamma_\nu \hat{q}\hat{p}_2 - \frac{1}{s(q-q_-)^2}\hat{p}_2\hat{q}\gamma_\nu, \tag{1.66}$$

and the bremsstrahlung mechanism

$$B_\mu = B\gamma_\mu - \frac{1}{s(p'-q)^2}\hat{p}_2\hat{q}\gamma_\mu + \frac{1}{s(p+q)^2}\gamma_\mu\hat{q}\hat{p}_2, \tag{1.67}$$

B and D are defined in Eq. (1.62).

To perform the conversion to the Lorentz indices μ, ν in Eq. (1.65), one can use the projection operators. For the case of equal chiralities $\eta = \lambda = +1$, we choose the projection operator as

$$P_+ = \frac{\bar{u}(p)\hat{q}_+\omega_+ u(q_-)}{\bar{u}(p)\hat{q}_+\omega_+ u(q_-)}. \tag{1.68}$$

Inserting it in Eq. (1.65) and using the relation $\omega_+ u(p)\bar{u}(p) = \omega_+\hat{p}$, we obtain

$$\begin{aligned} m_+^+ = \frac{-2}{\bar{u}(p)\hat{q}_+\omega_+ u(q_-)}\bar{u}(p')\Bigg\{ \left(\frac{D}{q_2^2} + \frac{B}{q_1^2}\right)\hat{q}_-\hat{q}_+\hat{p} + \\ + \frac{\hat{q}_-\hat{q}_+\hat{p}\hat{q}_\perp\hat{p}_2}{s}\left[\frac{1}{q_2^2(q-q_+)^2} - \frac{1}{q_1^2(p+q)^2}\right] + \\ + \frac{\hat{p}_2\hat{q}_\perp\hat{q}_-\hat{q}_+\hat{p}}{s}\left[\frac{1}{q_2^2(q_- - q)^2} - \frac{1}{q_1^2(p'-q)^2}\right]\Bigg\}\omega_+ v(q_+) = \\ = \frac{-2}{\bar{u}(p)\hat{q}_+\omega_+ u(q)_-}\bar{u}(p')A_+^+\omega_+ v(q_+). \end{aligned} \tag{1.69}$$

For the case of opposite chiralities $\eta = -\lambda = +1$, we use the projection operator

$$P_- = \frac{\bar{u}(p)\omega_- u(q_-)}{\bar{u}(p)\omega_- u(q_-)}.$$

Similar calculations lead to the result

$$\begin{aligned} m_-^+ = \frac{2}{\bar{u}(p)\omega_- u(q_-)}\bar{u}(p')\Bigg\{ \left(\frac{D}{q_2^2} + \frac{B}{q_1^2}\right)2(pq_-) + \\ + 2\frac{\hat{p}\hat{q}_-\hat{q}_\perp\hat{p}_2}{s}\left[\frac{1}{q_2^2(q-q_+)^2} + \frac{1}{q_1^2(p_1-q_-)^2}\right] - \\ - \frac{\hat{p}\hat{q}_\perp\hat{p}_2\hat{q}_-}{s}\left[\frac{1}{q_2^2(q-q_-)^2} + \frac{1}{q_1^2(p+q)^2}\right] - \\ - \frac{\hat{q}_-\hat{p}_2\hat{q}_\perp\hat{p}}{s}\left[\frac{1}{q_2^2(q-q_-)^2} + \frac{1}{q_1^2(p+q)^2}\right]\Bigg\}\omega_- v(q_+) = \\ = \frac{2}{\bar{u}(p)\omega_- u(q_-)}\bar{u}(p')A_-^+\omega_- v(q_+). \end{aligned} \tag{1.70}$$

The property, following from gauge invariance, that A_+^+, A_-^+ vanish when $|\mathbf{q}| \to 0$ can be explicitly seen from (1.69), (1.70).

For the sum of chiral amplitudes squared, one finds for $e\mu^+\mu^-$ subsystem

$$T^{(1)}_{e\to 3} = \frac{1}{(pq_+)(q_-q_+)}\mathrm{Sp}\,(\hat{p}' A_+^+ \hat{q}_+ \tilde{A}_+^+ \omega_+) + \frac{2}{pq_-} Sp(\hat{p}' A_-^+ \hat{q}_+ \tilde{A}_-^+ \omega_+)\ . \quad (1.71)$$

The magnitude of the cross sections of Eqs. (1.21)–(1.23) is of the order of $\alpha^n/\mu^2 \gg \alpha^n/s$, $n = 4, 5, 6$ where $\mu^2 = \max(s_1, s_2)$. It is large enough to be measured, and does not depend on s. The strategy of calculation of cross section, using the helicity amplitudes of subprocesses $2 \to 3$, described above, can be implemented to numerical programmes for a realistic simulation of experiments.

1.1.6. Subprocess $e\gamma^* \to ee\bar{e}$. The particle momenta for the subprocess $e\gamma^* \to ee\bar{e}$ are defined as

$$e(p, l_p) + \gamma^*(q) \to e(p_1, l_1) + e(p_2, l_2) + \bar{e}(p_+, t),$$

where $l_i, t = \pm$ are the chiralities of initial and final fermions. Without loss of generality we will consider below $l_p = +$. For the sum on the chiral states of the module squared of the relevant matrix elements we obtain

$$\sum |M^{l_p}_{l_1 l_2 t}|^2 = 2[|M^+_{++-}|^2 + |M^+_{+-+}|^2 + |M^+_{-++}|^2]. \quad (1.72)$$

Eight Feynman diagrams enter in the description of this subprocess, which form four gauge-invariant sets of amplitudes. The general form of the chiral amplitudes is

$$\begin{aligned} M^+_{l_1 l_2 t} = -\frac{1}{s_1}(4\pi\alpha)^{3/2}\,\{\ &\delta_{l_1,+}\delta_{t,-l_2}[\bar{u}^{l_2}(p_2)\gamma_\lambda v^t(p_+)\bar{u}^{l_1}(p_1)A_\lambda u^+(p) + \\ &+ \bar{u}^{l_1}(p_1)\gamma_\sigma u^+(p)\bar{u}^{l_2}(p_2)B_\sigma v^t(p_+)] - \\ &- \delta_{l_2,+}\delta_{t,-l_1}[\bar{u}^{l_1}(p_1)\gamma_\eta v^t(p_+)\bar{u}^{l_2}(p_2)D_\eta u^+(p) + \\ &+ \bar{u}^{l_2}(p_2)\gamma_\delta \bar{u}^+(p)\bar{u}^{l_1}(p_1)C_\delta v^t(p_+)]\ \}\,. \end{aligned} \quad (1.73)$$

Applying projection operators to provide the contraction on vector indices we have

$$\begin{aligned} |M^+_{++-}|^2 = \frac{(4\pi\alpha)^3}{2s_1^2 pp_+}\Bigg[&\frac{1}{p_2p_+}\frac{1}{4}Sp(\hat{p}_1 m^{(1)}_{++-}\hat{p}_+(m^{(1)}_{++-})^+) + \\ &+ \frac{1}{p_+p_1}\frac{1}{4}Sp(\hat{p}_2 m^{(2)}_{++-}\hat{p}_+(m^{(2)}_{++-})^+) - \\ &- \frac{2}{p_1p_+p_2p_+}\frac{1}{4}Sp(\hat{p}_1 m^{(1)}_{++-}\hat{p}_+(m^{(2)}_{++-})^+\hat{p}_2\hat{p}_+)\Bigg], \end{aligned}$$

$$|M^+_{+-+}|^2 = \frac{(4\pi\alpha)^3}{2s_1^2 pp_2}\frac{1}{4}Sp(\hat{p}_1 m_{+-+}\hat{p}_+(m_{+-+})^+),$$

$$|M^+_{-++}|^2 = \frac{(4\pi\alpha)^3}{2s_1^2 pp_1}\frac{1}{4}Sp(\hat{p}_2 m_{-++}\hat{p}_+(m_{-++})^+),$$

with

$$m_{+-+} = \gamma_\sigma \hat{p}\hat{p}_2 B_\sigma + A_\lambda \hat{p}\hat{p}_2\gamma_\lambda, \quad m_{-++} = \gamma_\delta \hat{p}\hat{p}_1 C_\delta + D_\eta \hat{p}\hat{p}_1\gamma_\eta,$$

$$m^{(1)}_{++-} = A_\lambda \hat{p}\hat{p}_+\hat{p}_2\gamma_\lambda + \gamma_\sigma \hat{p}\hat{p}_+\hat{p}_2 B_\sigma, \quad m^{(2)}_{++-} = \gamma_\sigma \hat{p}\hat{p}_+\hat{p}_1 C_\sigma + D_\eta \hat{p}\hat{p}_+\hat{p}_1\gamma_\eta,$$

$$q_1^2 A_\lambda = \frac{\hat{q}_\perp(\hat{p}_1-\hat{q})\gamma_\lambda}{(p_1-q)^2} + \frac{\gamma_\lambda(\hat{p}+\hat{q})\hat{q}_\perp}{(p+q)^2}, \quad q_2^2 B_\sigma = \frac{\hat{q}_\perp(\hat{p}_2-\hat{q})\gamma_\sigma}{(p_2-q)^2} + \frac{\gamma_\sigma(\hat{q}-\hat{p}_+)\hat{q}_\perp}{(p_+-q)^2},$$

$$q_4^2 C_\sigma = \frac{\hat{q}_\perp(\hat{p}_1-\hat{q})\gamma_\sigma}{(p_1-q)^2} + \frac{\gamma_\sigma(\hat{q}-\hat{p}_+)\hat{q}_\perp}{(q-p_+)^2}, \quad q_3^2 D_\eta = \frac{\hat{q}_\perp(\hat{p}_2-\hat{q})\gamma_\eta}{(p_2-q)^2} + \frac{\gamma_\eta(\hat{p}+\hat{q})\hat{q}_\perp}{(p+q)^2},$$

$$q_1^2 = (p_2+q_+)^2, \quad q_2^2 = (p-p_1)^2, \quad q_3^2 = (p_1+q_+)^2, \quad q_4^2 = (p-p_2)^2.$$

1.2. Radiative Corrections to Chiral Amplitudes. *1.2.1. Photon Impact Factor: Virtual and Real Soft-Photon Emission Contribution.* In the Born approximation there present two Feynman diagrams describing the subprocess

$$\gamma(p_1) + \gamma^*(q) \to e^-(q_-) + e^+(q_+).$$

The one-loop level radiative correction to the corresponding amplitude is described in terms of eight Feynman diagrams. The whole set of them can be separated in two classes: one corresponds to the case when the virtual photon is absorbed by electron line, and the second when the photon is absorbed by the positron line. One can restrict oneself to consideration of one of them, as well as the other can be obtained by exchanges of chirality and four-momenta of the particles:

$$\Phi_-^{\gamma,+-} = \Phi_+^{\gamma,++}(q_- \leftrightarrow q_+). \tag{1.74}$$

Here the subscript describes the absorbtion of the virtual photon by the electron $(-)$ or the positron $(+)$ line, superscript denotes polarizations of initial photon and final electron.

One class of RC to the electron IF consists of the renormalized electron mass operator and the vertex function with only one off-mass-shell electron. Its contribution can be written in the form [11]:

$$\Phi^{\gamma,\lambda\sigma}_{-,V\Sigma} = \frac{(4\pi\alpha)^{3/2}}{16\pi^2}\bar{u}(q_-)\frac{\hat{p}_2}{s}\frac{\hat{p}_1-\hat{q}_+}{-\chi_+}\Bigg[\hat{e}^\lambda(3-2l_l+l_+) + \\ + \int\frac{d^4k}{i\pi^2}\frac{\gamma^\mu(-\hat{q}_+ + \hat{p}_1 - \hat{k})\hat{e}^\lambda(-\hat{q}_+ - \hat{k})\gamma^\mu}{(0)(\tilde{2})(q)}\Bigg]\omega^\sigma v(q_+), \tag{1.75}$$

where the χ_+ denominators (0), $(\tilde{2})$, (q) are defined in Subsec. 2.1 and $l_+ = \ln(\chi_+/m^2)$, $l_l = \ln(m^2/\lambda^2)$.

After multiplying with the corresponding Born amplitude and integrating over the four-momentum k we obtain

$$2\Phi^{\gamma,+-}_{-,V\Sigma}(\Phi^{\gamma,+-}_B)^* = \frac{8\alpha^2}{\chi_+\chi_-}(\mathbf{q}\mathbf{q}_-)\left(l_+ - \frac{1}{2}\right), \tag{1.76}$$

when the polarization of the final electron is positive. Here we used the Sudakov parameterization:

$$\begin{gathered} q = \alpha p_1 + \beta p_2 + q_\perp, \quad q_\pm = x_\pm p_1 + \beta_\pm p_2 + q_{\pm\perp}, \\ q_\perp^2 = -\mathbf{q}^2, \quad q_{\pm\perp}^2 = -\mathbf{q}_\pm^2. \end{gathered} \tag{1.77}$$

The contribution of the other polarization (when the virtual photon is absorbed by the electron line) is equal to zero. All other contributions with different polarization and absorbtion of virtual photon by positron line can be obtained with the substitution (1.74).

The contribution of the vertex function with a virtual photon can be written in the form:

$$\begin{aligned} \Phi^{\gamma,\lambda\sigma}_{V,-} = \frac{(4\pi\alpha)^{3/2}}{-16\pi^2\chi_+}\int\frac{d^4k}{i\pi^2}\frac{1}{(0)(2)(q)}\times \\ \times\,\bar{u}(q_-)\gamma_\mu(\hat{q}_- - \hat{k})\frac{\hat{p}_2}{s}(\hat{q}_- - \hat{q} - \hat{k})\gamma_\mu(\hat{p}_1 - \hat{q}_+)\hat{e}^\lambda\omega^\sigma v(q_+). \end{aligned} \tag{1.78}$$

Using the tables of integrals (see Subsec. 2.1) we obtain

$$\begin{gathered} 2\Phi^{\gamma_1+-}_{V,-}(\Phi^{\gamma_1+-}_B)^* = -2A_-|\Phi^{\gamma_1+-}_B|^2, \\ A_- = \frac{\alpha}{2\pi}\left[-\frac{1}{2}L - \frac{1}{4} - \frac{\chi_+ + 2\mathbf{q}^2}{2a}l_+ + \frac{3\mathbf{q}^2}{2a}l_q - \mathbf{q}^2 J_{02q}\right], \\ a = \chi_+ - \mathbf{q}^2, \quad l_q = \ln\frac{\mathbf{q}^2}{m^2}, \quad L = \ln\frac{\Lambda^2}{m^2}, \end{gathered} \tag{1.79}$$

with Λ-ultraviolet cut-off parameter. We remind that we work in the frame of unrenormalized field theory. The regularization procedure consists in the replacement $L \to 2l_l - 9/2$ [7, 8].

The most complicated case is the calculation of box-type contribution. It can be written in the form:

$$\begin{aligned} \Phi^{\gamma,\lambda\sigma}_{-,box} = \frac{(4\pi\alpha)^{3/2}}{16\pi^2}\int\frac{d^4k}{i\pi^2}\frac{1}{(0)(1)(2)(q)}\times \\ \times\,\bar{u}(q_-)(\hat{q}_- - \hat{k})\frac{\hat{p}_2}{s}(\hat{q}_- - \hat{q} - \hat{k})\hat{e}^\lambda(-\hat{q}_+ - \hat{k})\omega^\sigma v(q_+). \end{aligned} \tag{1.80}$$

All the details about loop calculation and relevant integrals can be found in Subsec. 2.1. It is worth to mention that in the case of box-type contribution all polarizations have nonzero value.

The contribution of additional real soft-photon emission to the light cone projector has a standard form

$$\Phi^{\gamma}_{\rm soft} = \Phi^{\gamma}_{B}\sqrt{4\pi\alpha}\left(\frac{q_-}{q_-k} - \frac{q_+}{q_+k}\right)e(k). \tag{1.81}$$

The corresponding contribution to the differential IF $d\tau^{\gamma}$ is

$$\int \frac{d^3k}{16\omega\pi^3}\sum|\Phi^{\gamma}_{\rm soft}|^2 \tag{1.82}$$

for $\omega < \Delta\varepsilon \ll \varepsilon$. The result can be expressed in terms of the Born differential IF $d\tau^{\gamma}_{B}$ as

$$d\tau^{\gamma}_{\rm soft} = \frac{\alpha}{\pi}d\tau^{\gamma}_{B}\left\{(l_s-1)\left[l_l + \ln\frac{\Delta^2}{x_+x_-}\right] + \frac{1}{2}l_s^2 - \frac{1}{2}\ln^2\frac{x_+}{x_-} - \frac{\pi^2}{6}\right\}, \tag{1.83}$$

$$\Delta = \frac{\Delta\varepsilon}{\varepsilon}, \quad l_s = \ln\frac{s_1}{m^2}.$$

We took into account the fact that the emission angle between the three-momenta of the pair components in the center-of-mass frame of the colliding particles is small.

After summing all contributions (1.75), (1.78), (1.80), including the soft-photon contribution (1.83), we explicitly see the cancellation of the auxiliary parameter λ and of the large logarithm squared:

$$2\left(d\tau^{\gamma,+\pm}_{\pm,{\rm box}} + d\tau^{\gamma,+\pm}_{\mp,V+\Sigma} + d\tau^{\gamma,+\pm}_{\mp,V}\right) + d\tau^{\gamma\pm}_{\rm soft} = \\ = \frac{\alpha}{\pi}d\tau^{\gamma\pm}_{B}\left\{(l_s-1)\left[2\ln\Delta + \frac{3}{2} - \ln(x_+x_-)\right] + K^{\gamma,\pm}_{SV}\right\}.$$

The analytic form of $K^{\gamma,\pm}_{SV}$ is rather complicated (see [11]). The leading logarithm contribution is proportional to the Born cross section, which is in agreement with the predictions of the structure function approach, namely the leading logarithm contribution is exactly the Δ part of the evolution equation kernel. All nonleading terms are gathered in the so-called K-factor, a smooth function of the order of magnitude of about unity.

The contribution from the emission of hard photon can be presented as the sum of two parts, corresponding to collinear and noncollinear kinematics. It will be considered below.

1.2.2. Electron Impact Factor: Virtual and Real Soft-Photon Contribution. Subprocess of the Compton scattering in the Born approximation

$$e(p_1) + \gamma^*(q) \to e(p_1') + \gamma(k') \tag{1.84}$$

is described by two Feynman diagrams, whereas in 1-loop approximation one must take into account eight Feynman diagrams. All the relevant amplitudes can be classified into six sets, the contribution of three of them to the light cone projector can be obtained by the substitution:

$$\Phi_i^{e,+-} = \Phi_f^{e,++}(p_1 \leftrightarrow -p_1'), \tag{1.85}$$

where the subscript corresponds to the interaction of the virtual photon with the initial i or the scattered f electron.

We take into account the contributions from self-energy, vertex, and box-type FD:

$$\begin{aligned}
\Phi_{i,V}^{e,+\lambda} &= \frac{(4\pi\alpha)^{3/2}}{-16\pi^2\kappa}\int\frac{d^4k}{i\pi^2}\frac{\bar{u}(p_1')\dfrac{\hat{p}_2}{s}(\hat{p}_1 - \hat{k}'\gamma_\mu(\hat{p}_1' - \hat{k} - \hat{k}'\hat{e}^\lambda)(\hat{p}_1 - \hat{k})\gamma_\mu\omega_+ u(p_1)}{(0)(2)(q)},\\
\Phi_{i,\Sigma}^{e,+\lambda} &= \frac{(4\pi\alpha)^{3/2}}{-16\pi^2\kappa}(l_t - 2l_\lambda + 3)\bar{u}(p_1')\omega_- \frac{\hat{p}_2}{s}(\hat{p}_1' - \hat{q})\hat{e}^\lambda\omega_+ u(p_1),\\
\Phi_{i,H}^{e,+\lambda} &= \frac{(4\pi\alpha)^{3/2}}{-16\pi^2\kappa}\int\frac{d^4k}{i\pi^2}\frac{\bar{u}(p_1')\gamma_\mu(\hat{p}_1' - \hat{k})\dfrac{\hat{p}_2}{s}(\hat{p}_1' - \hat{k} - \hat{q})\gamma_\mu(\hat{p}_1 - \hat{k}_1)e^\lambda\omega_+ u(p_1)}{(0)(1)(q)},\\
\Phi_{i,\mathrm{box}}^{e,+\lambda} &= \frac{(4\pi\alpha)^{3/2}}{16\pi^2}\int\frac{d^4k}{i\pi^2}\frac{\bar{u}(p_1')\gamma_\mu(\hat{p}_1' - \hat{k})\dfrac{\hat{p}_2}{s}(\hat{p}_1' - \hat{k} - \hat{q})\hat{e}^\lambda\gamma_\mu\omega_+ u(p_1)}{(0)(1)(2)(q)},\\
&\kappa = 2p_1k', \quad \kappa' = 2p_1'k'.
\end{aligned} \tag{1.86}$$

The first three contributions are (see details in Subsec. 2.1)

$$\begin{aligned}
&2\Phi_{vi}^{++}(\Phi_B^{++})^* = 2A|\Phi_B^{++}|^2,\\
&A = \frac{\alpha}{2\pi}\left[-\frac{1}{2}L - \frac{1}{4} - \mathbf{q}^2 I_{01\bar{q}} + \frac{3\mathbf{q}^2}{2d}l_q + \frac{\kappa' - 2\mathbf{q}^2}{2d}l_{\kappa'}\right],\\
&d = \kappa' - \mathbf{q}^2, \quad l_{\kappa'} = \ln\frac{\kappa'}{m^2} - i\pi,\\
&2\Phi_{\Sigma vi}^{+-}(\Phi_B^{+-})^* = \frac{8\alpha^2}{\kappa\kappa'}(\frac{1}{2} - l_\kappa)x'[x'\mathbf{k}' - x\mathbf{p}_1]\mathbf{q}, \quad \mathbf{k}' + \mathbf{p} = \mathbf{q}.
\end{aligned} \tag{1.87}$$

The soft photon contribution has a standard form

$$d\tau_{\mathrm{soft}}^{+\pm} = d\tau_B^{+\pm}\frac{\alpha}{\pi}\left\{(l_u - 1)[l_l + 2\ln\Delta - \ln(x')] + \frac{1}{2}l_u^2 - \frac{1}{2}\ln^2(x') - \frac{\pi^2}{6}\right\}, \tag{1.88}$$

where $l_u = \ln u/m^2$, $u = 2p_1'p_1$, E, $E' = x'E$ are the energies of the initial and the scattered electrons. We can write the contribution to the electron IFs with definite chiral state:

$$2\left[d\tau_{i,V}^{e,+\pm} + d\tau_{i,\Sigma}^{e,+\pm} + d\tau_{i,V}^{e,+\pm} + d\tau_{i,\mathrm{box}}^{e,+\pm}\right] + d\tau_{\mathrm{soft}}^{+\pm} = \\ = d\tau_B^{+\pm}\frac{\alpha}{\pi}\left[(l_u - 1)\left(2\ln\Delta + \frac{3}{2} - \ln x'\right) + K_{e,SV}^{+\pm}\right]. \quad (1.89)$$

Again here we can see the cancellation of auxiliary «photon mass» parameter λ and the agreement with the predictions of the structure functions approach.

The analytic expressions of $K_{e,SV}^{+\pm}, K_{\gamma,SV}^{+\pm}$ can be found in [11].

1.2.3. Hard-Photon Emission in Collinear Kinematics. For appropriate consideration of RC to chiral amplitudes, we have to consider additional hard collinear photon emission. It is convenient to distinguish the collinear and noncollinear kinematics of emission of a hard photon. For this aim we introduce an auxiliary small parameter $\theta_0 \ll 1$. Collinear kinematics corresponds to the case when the photon is emitted by the charged particle at an angle of $\theta \leqslant \theta_0$ with respect to the direction of motion of the (initial or final) charged particle. Noncollinear kinematics corresponds to larger emission angles: $\theta > \theta_0$. The chiral amplitudes in noncollinear kinematics can be calculated using the methods developed by the CALCUL Collaboration [9]. The contribution from collinear kinematics can be obtained using the quasi-real electron method developed in [13, 14]. The total sum does not depend on the parameter θ_0. The cancellation of the θ_0 dependence constitutes also a check of the calculations. The nonleading contributions from additional hard-photon emission essentially depend on the experimental setup. These are included as K factors in the structure function picture of IFs.

The contribution to the photon IF in collinear kinematics is

$$d\tau_{\gamma,\mathrm{coll}}^{\lambda} = \frac{\alpha}{2\pi}\int\limits_{x_-(1+\Delta)}^{1}\frac{dz_-}{z_-}\left[\frac{1+\tilde{x}_-^2}{1-\tilde{x}_-}(l_s + r_- + \ln\theta_0^2 - 1) + 1 - \tilde{x}_-\right]d\tau_B^{\lambda\gamma\gamma*}\left(\frac{q_-}{z_-}, q_+\right) + \\ + \frac{\alpha}{2\pi}\int\limits_{x_+(1+\Delta)}^{1}\frac{dz_+}{z_+}\left[\frac{1+\tilde{x}_+^2}{1-\tilde{x}_+}\left(l_s + r_+ + \ln\theta_0^2 - 1\right) + 1 - \tilde{x}_+\right]d\tau_B^{\lambda\gamma\gamma^*}\left(q_-, \frac{q_+}{z_+}\right), \quad (1.90)$$

where the first term in square brackets corresponds to the emission of the hard photon along the electron and the second one along the positron (from the created pair). Moreover, we use the notations

$$\tilde{x}_\pm = \frac{x_\pm}{z_\pm},\ l_s = \ln\frac{2q_+q_-}{m^2} = \ln\frac{2E^2x_+x_-(1-c_\pm)}{m^2},\ r_\pm = \ln\frac{x_\pm}{2x_\mp(1-c_\pm)}, \tag{1.91}$$

where the quantity $c_\pm$ is the cosine of the angle between the pair momenta in CMS of the colliding beams. The «shifted» photon IF (the conservation law reads as $p_1 + q = q_-/z_- + q_+/z_+$) is

$$\begin{gathered} d\tau_B^{\lambda\gamma\gamma^*}\left(\frac{q_+}{z_+}, \frac{q_-}{z_-}, x_\pm\right) = \frac{4\alpha^2}{\mathbf{q}_+^2\mathbf{q}_-^2}\tilde{x}_\pm^2 d^2q_- d\tilde{x}_-, \quad \tilde{x}_+ + \tilde{x}_- = 1, \\ \mathbf{q} = \frac{1}{z_-}\mathbf{q}_- + \frac{1}{z_+}\mathbf{q}_+. \end{gathered} \tag{1.92}$$

A similar method can be applied to the problem of the calculation of the contribution to the IF of electron of collinear photon emission with the result

$$\begin{aligned} d\tau^\lambda_{e,\mathrm{coll}} = {} & \frac{\alpha}{2\pi}\int\limits_0^{1-\Delta} dz_1\left[\frac{1+z_1^2}{1-z_1}(l_u + l_1 + \ln\theta_0^2 - 1) + 1 - z_1\right] d\tau_B^{\lambda e\gamma^*}(p_1z_1, p_1') + \\ & + \frac{\alpha}{2\pi}\int\limits_{x'(1+\Delta)}^{1}\frac{dz_2}{z_2}\left[\frac{1+(x'/z_2)^2}{1-(x'/z_2)}(l_u + l_2 + \ln\theta_0^2 - 1) d\tau_B^{\lambda e\gamma^*}\left(p_1, \frac{1}{z_2}p_1'\right)\right], \end{aligned} \tag{1.93}$$

where the first term in the square brackets describes the emission from the initial electron and the second term describes the emission from the scattered electron. Here we use the next definitions:

$$l_u = \ln\frac{2p_1p_1'}{m^2} = \ln\frac{2E^2x'(1-c)}{m^2},\ l_1 = \ln\frac{z_1^2}{2x'(1-c)},\ l_2 = \ln\frac{x'}{2z_2^2(1-c)}, \tag{1.94}$$

where c is the cosine of the angle between the initial and the scattered electrons momenta in CMS of the initial particles.

The «shifted» electron IF in the Born approximation (the four-momentum conservation law reads as $z_1p_1 + q = p_1'/z_2 + k_1$) is

$$\begin{gathered} d\tau_{e\gamma}^\pm\left(z_1p_1, \frac{p_1'}{z_2}\right) = \frac{4\alpha^2\mathbf{q}^2}{\kappa\kappa'}\eta^\pm\frac{z_2d^2kdx_1}{x_1x'}, \quad \eta^+ = z_1^2, \quad \eta^- = \left(\frac{x'}{z_2}\right)^2, \\ \kappa = \frac{z_1}{x}\mathbf{k}_1^2 \quad \kappa' = \frac{z_2\left(\mathbf{p}_1'x_1 - \mathbf{k}_1\dfrac{x'}{z_2}\right)^2}{x_1x'}, \quad x_1 + \frac{x'}{z_2} = 1, \quad \mathbf{q} = \mathbf{k}_1 + \frac{1}{z_2}\mathbf{p}_1'. \end{gathered} \tag{1.95}$$

The terms containing «large» logarithms $l_s - 1$, $l_u - 1$ will be included in the lepton nonsinglet structure functions in the form of the Drell–Yan IFs, whereas the remaining terms contribute to the relevant K factors. The results for the electron IF can therefore be rewritten in the frame of the structure function approach (the chiral indices are suppressed)

$$d\tau^{e,\mathrm{coll}} = \int\limits_0^1 dz_1 \left[P_\theta(z_1)\frac{\alpha}{2\pi}(l_u - 1) + \frac{\alpha}{\pi}K^i_{\mathrm{coll}}\right] d\tau^{i,e}(z_1 p_1, p_1') + d\tau^i_{\mathrm{comp}} + \\ + \int\limits_0^1 \frac{dz_2}{z_2}\left[P_\theta\left(\frac{x'}{z_2}\right)\frac{\alpha}{2\pi}(l_u - 1) + \frac{\alpha}{\pi}K^f_{\mathrm{coll}}\right] d\tau^{f,e}\left(p_1, \frac{p_1'}{z_2}\right) + d\tau^f_{\mathrm{comp}}. \quad (1.96)$$

For the photon IF we have

$$d\tau^{\gamma,\mathrm{coll}} = \left[\int\limits_0^1 \frac{dz_-}{z_-}P_\theta\left(\frac{x_-}{z_-}\right)\frac{\alpha}{2\pi}(l_s - 1) + \frac{\alpha}{\pi}K^f_{\mathrm{coll}}\right] d\tau\left(\frac{q_-}{z_-}, q_+\right) + d\tau^{-,\gamma}_{\mathrm{comp}} + \\ + \left[\int\limits_0^1 \frac{dz_+}{z_+}P_\theta\left(\frac{x_+}{z_+}\right)\frac{\alpha}{2\pi}(l_s - 1) + \frac{\alpha}{\pi}K^f_{\mathrm{coll}}\right] d\tau\left(q_-, \frac{q_+}{z_+}\right) + d\tau^{+,\gamma}_{\mathrm{comp}}, \quad (1.97)$$

where

$$P_\theta(z) = \frac{1+z^2}{1-z}\theta(1 - z - \Delta), \quad (1.98)$$

and

$$\begin{aligned} d\tau^{-,\gamma}_{\mathrm{comp(coll)}} &= \frac{\alpha}{2\pi}\int\limits_0^1 \frac{dz_-}{z_-}P_\theta\left(\frac{x_-}{z_-}\right) d\tau^\gamma_B\left(\frac{q_-}{z_-}, q_+\right)\ln\theta_0^2(r_- + 1 - \tilde{x}_-), \\ d\tau^{+,\gamma}_{\mathrm{comp(coll)}} &= \frac{\alpha}{2\pi}\int\limits_0^1 \frac{dz_+}{z_+}P_\theta\left(\frac{x_+}{z_+}\right) d\tau^\gamma_B\left(q_-, \frac{q_+}{z_+}\right)\ln\theta_0^2(r_+ - 1 - \tilde{x}_+), \\ d\tau^{i,e}_{\mathrm{comp(coll)}} &= \frac{\alpha}{2\pi}\int\limits_0^1 dz_1 P_\theta(z_1) d\tau^e_B(z_1 p_1, p_1')\ln\theta_0^2(l_1 + 1 - z_1), \\ d\tau^{f,e}_{\mathrm{comp(coll)}} &= \frac{\alpha}{2\pi}\int\limits_0^1 \frac{dz_2}{z_2}P_\theta\left(\frac{x'}{z_2}\right) d\tau^e_B\left(p_1, \frac{p_1'}{z_2}\right)\ln\theta_0^2(l_2 + 1 - \frac{x'}{x_2}). \end{aligned} \quad (1.99)$$

1.2.4. Noncollinear Hard-Photon Emission Contribution to the Electron and Photon Impact Factor. Contribution to electron IF from the channel of double Compton scattering process

$$\begin{gathered} e(p_1,\lambda_1)+\gamma^*(q)\to\gamma(k_1,\lambda_1)+\gamma(k_2\lambda_2)+e(p_1',\lambda_e), \\ u=2p_1p_1', \quad \kappa_i=2k_ip_1, \quad \kappa_i'=2k_ip_1', \end{gathered} \tag{1.100}$$

with emission of both the final electrons outside a narrow cone defined by $\theta>\theta_0$, can be calculated using the chiral amplitudes technique [9]. The result is

$$\begin{gathered} d\tau^{e\gamma\gamma}_{\lambda_e\lambda_1\lambda_2}=\frac{\alpha^3}{2\pi^2}|m^{\lambda_e}_{\lambda_1\lambda_2}|^2\frac{d^2k_1d^2k_2dx_1dx_2}{x_1x_2x'}|_{\theta_{1,2}>\theta_0}, \\ x'=1-x_1-x_2, \quad \mathbf{q}=\mathbf{k}_1+\mathbf{k}_2+\mathbf{p}_1', \end{gathered} \tag{1.101}$$

with

$$\begin{aligned} |m^+_{++}|^2&=\frac{4\mathbf{q}^2u}{\kappa_1\kappa_2\kappa_1'\kappa_2'}, \\ |m^+_{--}|^2&=\frac{4(x')^2\mathbf{q}^2u}{\kappa_1\kappa_2\kappa_1'\kappa_2'}, \\ |m^+_{-+}(k_1,k_2)|^2&=|m^+_{+-}(k_2,k_1)|^2, \\ |m^+_{+-}(k_2,k_1)|^2&=\frac{4}{u^2\kappa_1\kappa_2\kappa_1'\kappa_2'}\mathrm{Sp}\big[\hat p_1'B^+_{+-}\omega_+\hat p_1\tilde B^+_{+-}\big], \end{aligned} \tag{1.102}$$

and

$$\begin{aligned} B^+_{+-}=&-\frac{1}{(p_1+q)^2}\hat p_1\hat k_1\hat p_1'\hat p_1\hat k_2(\hat p_1+\hat q)\frac{\hat p_2}{s}- \\ &-\frac{1}{(p_1'-q)^2}\frac{\hat p_2}{s}(\hat p_1'-\hat q)\hat k_1\hat p_1'\hat p_1\hat k_2\hat p_1'\hat p_1(\hat p_1'+\hat k_1)\frac{\hat p_2}{s}(\hat p_1-\hat k_2)\hat p_1'. \end{aligned} \tag{1.103}$$

It was explicitly shown [11] that the quantity $B^+_{+-}(k_1,k_2,q)$ vanishes at $|\mathbf{q}|\to 0$. This property is the consequence of gauge invariance implement for the virtual photon with momentum q.

For the aim of checking the cancellation of the θ_0 dependence of the sum of the contributions of collinear and noncollinear kinematics, we write the limiting expressions for $|m^+_{ij}|^2$ for the emission of real photons kinematics

$$\theta_1>\theta_0, \quad \theta_1\to\theta_0, \quad \theta_2\gg\theta_0, \tag{1.104}$$

where θ_1 is the angle of emission of a photon with momentum k_1 with respect to the initial or final electron momentum. These limiting values are

$$\begin{aligned}
(|m^+_{+-}|^2 + |m^+_{++}|^2)_{\kappa_1\to 0} &= \frac{4\mathbf{q}^2}{\kappa_1}\frac{[(x')^2 + (1-x_1)^2]}{x_1(1-x_1)^2\kappa_2\kappa_2'},\\
(|m^+_{+-}|^2 + |m^+_{++}|^2)_{\kappa_1'\to 0} &= \frac{4\mathbf{q}^2}{\kappa_1'}\frac{x'}{x_1\kappa_2\kappa_2'}[1 + (1-x_2)^2],\\
(|m^+_{-+}|^2 + |m^+_{--}|^2)_{\kappa_1\to 0} &= \frac{4\mathbf{q}^2}{\kappa_1}\frac{1}{x_1\kappa_2\kappa_2'}[(x')^2 + (1-x_1)^2],\\
(|m^+_{-+}|^2 + |m^+_{--}|^2)_{\kappa_1'\to 0} &= \frac{4\mathbf{q}^2}{\kappa_1'}\frac{(x')^3}{x_1(1-x_2)^2\kappa_2\kappa_2'}[1 + (1-x_2)^2].
\end{aligned} \tag{1.105}$$

For small emission angles we can express all the invariants in terms of angular two-dimensional vectors in the plane transversal to the beam axis:

$$\begin{aligned}
&\mathbf{k}_1 = Ex_1\boldsymbol{\theta}_1, \quad \mathbf{p}_1{}' = Ex'\theta', \quad \int\limits_{\theta_1>\theta_0} \frac{d^2k_1}{\kappa_1} = \pi x_1 \ln\frac{1}{\theta_0^2} + \ldots,\\
&\int \frac{d^2k_1}{\kappa_1'} = \frac{x_1}{x'} \int\limits_{|\boldsymbol{\theta}_1-\boldsymbol{\theta}'|>\theta_0} \frac{d^2\theta_1}{(\boldsymbol{\theta}_1-\boldsymbol{\theta}')^2} = \frac{\pi x_1}{x'}\ln\frac{1}{\theta_0^2} + \ldots
\end{aligned} \tag{1.106}$$

One can be convinced that explicitly the sum of collinear and noncollinear contributions to the electron IF, summed over the chiral states of the final hard photon, is independent of θ_0:

$$d\tau^e_{\text{hard, nc}} = \sum_{\lambda_1\lambda_2} (d\tau^{e\gamma\gamma}_{+\lambda_1,\lambda_2} + d\tau^{i,e}_{\text{comp}} + d\tau^{f,e}_{\text{comp}}). \tag{1.107}$$

Its value, however, depends essentially on the experimental photon detection setup.

Similar calculations for the hard photon emitted in noncollinear kinematics for photon impact factor:

$$\begin{aligned}
\gamma(k,\lambda) + \gamma^*(q) &\to e^-(q_-,\lambda_-) + e^+(q_+,-\lambda_-) + \gamma(k_1,\lambda_1),\\
s_1 = 2q_-q_+, &\quad \kappa_\pm = 2kq_\pm, \quad \kappa_{1\pm} = 2k_1q_\pm,
\end{aligned} \tag{1.108}$$

with chiral amplitudes defined as $m^\lambda_{\lambda_1\lambda_-}$, gives

$$\begin{aligned}
d\tau^{e^+e^-\gamma}_{\lambda\lambda_1\lambda_-} &= \frac{\alpha^3}{2\pi^2}|m^\lambda_{\lambda_1\lambda_-}|^2\frac{d^2q_-d^2q_+dx_+dx_-}{x_1x_+x_-},\\
x_1 = 1 - x_+ - x_-, &\quad \mathbf{q} = \mathbf{q}_- + \mathbf{q}_+ + \mathbf{k}_1,
\end{aligned} \tag{1.109}$$

with

$$
\begin{aligned}
|m^+_{++}|^2 &= \frac{4\mathbf{q}^2 s_1 x_+^2}{\kappa_-\kappa_{1-}\kappa_+\kappa_{1+}}, \\
|m^+_{+-}|^2 &= \frac{4\mathbf{q}^2 s_1 x_-^2}{\kappa_-\kappa_{1-}\kappa_+\kappa_{1+}}, \\
|m^+_{-+}|^2 &= \frac{4}{s_1^2\kappa_-\kappa_{1-}\kappa_+\kappa_{1+}} \mathrm{Sp}\left[\hat{q}_- A^+_{-+}\omega_+\hat{q}_+\tilde{A}^+_{-+}\right], \\
|m^+_{--}(k,k_1)|^2 &= |m^+_{-+}(-k_1,-k)|^2,
\end{aligned}
\tag{1.110}
$$

and

$$
A^+_{-+} = \frac{s_1}{(q_+-q)^2}\hat{k}\hat{q}_+\hat{k}_1(-\hat{q}_+ + \hat{q})\frac{\hat{p}_2}{s} - \frac{s_1}{(q_- - q)^2}\frac{\hat{p}_2}{s}(\hat{q}_- - \hat{q})\hat{k}\hat{q}_-\hat{k}_1 - \\
- \hat{q}_+(\hat{q}_- - \hat{k})\frac{\hat{p}_2}{s}(\hat{q}_+ + \hat{k}_1)\hat{q}_-. \tag{1.111}
$$

Again it was demonstrated in [11] the proportionality $A^+_{-+} \sim |\mathbf{q}|$ at small $|\mathbf{q}|$.

To check the cancellation of θ_0 dependence let us consider the limiting values of $|m^+_{\lambda-\lambda_1}|^2$ for emission angles close to the direction of the momentum of one of charged particles.

$$
\begin{aligned}
(|m^+_{++}|^2 + |m^+_{+-}|^2)_{\kappa_{1-}\to 0} &= \frac{4\mathbf{q}^2}{\kappa_{1-}}\frac{(x_+)^2 x_-}{x_1(1-x_+)^2\kappa_+\kappa_-}[x_-^2 + (1-x_+)^2], \\
(|m^+_{-+}|^2 + |m^+_{--}|^2)_{\kappa_{1-}\to 0} &= \frac{4\mathbf{q}^2}{\kappa_{1-}}\frac{x_-}{x_1\kappa_+\kappa_-}[x_-^2 + (1-x_+)^2], \\
(|m^+_{-+}|^2 + |m^+_{--}|^2)_{\kappa_{1+}\to 0} &= \frac{4\mathbf{q}^2}{\kappa_{1-}}\frac{x_+ x_-}{x_1(1-x_-)^2\kappa_+\kappa_-}[x_+^2 + (1-x_-)^2], \\
(|m^+_{++}|^2 + |m^+_{+-}|^2)_{\kappa_{1+}\to 0} &= \frac{4\mathbf{q}^2}{\kappa_{1-}}\frac{x_-}{x_1\kappa_+\kappa_-}[x_+^2 + (1-x_-)^2].
\end{aligned}
\tag{1.112}
$$

One can verify the cancellation of the θ_0 dependence in the sum of the collinear kinematics of the noncollinear contributions to the photon IF, summed on hard-photon chiral states:

$$
d\tau^\gamma_{\mathrm{hard,\,nc}} = d\tau^{e^+e^-\gamma}_{+\lambda,\lambda} + d\tau^{-,\gamma}_{\mathrm{comp}} + d\tau^{+,\gamma}_{\mathrm{comp}}. \tag{1.113}
$$

The numerical value of $d\tau^\gamma_{\mathrm{hard,\,nc}}$ depends also on the experimental setup and will not be considered here.

1.2.5. Drell–Yan Picture of Process. We have obtained that the IFs of both electron and photon in LLA can be written in the partonic form of the Drell–Yan process, in terms of structure functions for any chiral states of initial and final particles (the chiral indices are suppressed):

$$(d\tau_B + d\tau_{SV} + \sum d\tau_{\mathrm{hard}})^{\gamma}(q_-, q_+) = \\ = \int\limits_{x_-}^{1} \frac{dz_-}{z_-} \int\limits_{x_+}^{1} \frac{dz_+}{z_+} D\left(\frac{x_-}{z_-}\right) D\left(\frac{x_+}{z_+}\right) d\tau_B^{\gamma}\left(\frac{q_-}{z_-}, \frac{q_+}{z_+}\right) \times \\ \times \left(1 + \frac{\alpha}{\pi}[K_{SV}^{\gamma} + K_{\mathrm{coll}}^{\gamma} + K_{\mathrm{ncol}}^{\gamma}]\right), \quad (1.114)$$

$$(d\tau_B + d\tau_{SV} + d\tau_{\mathrm{hard}})^{e}(p_1, p_1') = \\ = \int\limits_{0}^{1} dz_1 \int\limits_{x'}^{1} \frac{dz_2}{z_2} D(z_1) D\left(\frac{x'}{z_2}\right) d\tau_B^{e}\left(z_1 p_1, \frac{p_1'}{z_2}\right) \times \\ \times \left(1 + \frac{\alpha}{\pi}[K_{SV}^{e} + K_{\mathrm{coll}}^{e} + K_{\mathrm{hard}}^{e}]\right), \quad (1.115)$$

where $D(z) = D(z, l_s)$ is the nonsinglet structure function of a fermion (the defintion could be find in [58]). K_{coll} can be derived from Eq. (1.96), where the terms containing $\ln\theta_0$ are eliminated. The form of K_{ncol} (after proper regularization, compensating the divergent terms in the limit $\theta_0 \to 0$) strongly depends on the details of experiment — tagging the additional hard photon.

The contributions from pairs production channels to the electron IF are not discussed here.

1.3. Radiative Corrections to the Cheng–Wu Electron Impact Factor. It is well known (see [4–6]) that the QED peripheral scattering amplitude for the process $A + B \to A' + B'$ (with internal quantum numbers a, b, a', b') at high energy

$$A(p_A, a) + B(p_B, b) \to A(p_A', a') + B(p_B', b'), \\ s = (p_A + p_B)^2 \gg -t = -(p_A - p_A')^2 \sim m^2 \quad (1.116)$$

can be written in the form

$$A(s,t) = \frac{is}{(2\pi)^2} \int \frac{d^2k\ \tau^A(\mathbf{k},\mathbf{r})\,\tau^B(\mathbf{k},\mathbf{r})}{[(\mathbf{k}+\mathbf{r})^2 + \lambda^2][(\mathbf{k}-\mathbf{r})^2 + \lambda^2]} \\ \left(1 + \mathcal{O}\left(\frac{t}{s}\right)\right), \quad 4\mathbf{r}^2 = -t > 0, \quad (1.117)$$

where τ^i is the so-called Cheng–Wu IF, associated with the particle i. This form was found to be valid in the first nontrivial order of the perturbation theory. Here λ is the photon mass and the two-dimensional vectors $\mathbf{r}$ and $\mathbf{k}$ are orthogonal to the momenta p_A, p_B of the initial particles. The IF τ describes the inner structure of the colliding particles. For the electron $\tau^e = 4\pi\alpha\delta_{ij}$, where indices i, j define its polarization states. The expression for the impact factor of an on-mass-shell photon can be written in the form [5]:

$$\tau_{ij}^{\gamma} = 8\alpha^2 \int\limits_0^1 dy \int\limits_0^1 dx_+\, dx_- \delta(x_+ + x_- - 1)(A_{ij} - B_{ij}), \tag{1.118}$$

with

$$A_{ij} = \frac{1}{4\mathbf{r}^2 x_+^2 y(1-y) + m^2} \left\{ 8x_+^3 x_- y(1-y) r_i r_j - x_+^2 \mathbf{r}^2 \left[1 - 8x_+ x_- \left(y - \frac{1}{2} \right)^2 \right] \delta_{ij} \right\},$$

$$B_{ij} = \frac{1}{4\mathbf{Q}^2 y(1-y) + m^2} \left\{ 8x_+ x_- y(1-y) Q_i Q_j - \mathbf{Q}^2 \left[1 - 8x_+ x_- \left(y - \frac{1}{2} \right)^2 \right] \delta_{ij} \right\},$$

$$\mathbf{Q} = \frac{1}{2}(\mathbf{k} + \mathbf{r}) - x_+ \mathbf{r},$$

where i, j describe the photon polarization states.

In the case of small angle e^+e^- scattering, the amplitude for the diagrams with the multiphoton exchange has the eikonal representation

$$\begin{aligned} A(s,t) &= A_0(s,t)\, \mathrm{e}^{i\delta(t)}, \\ A_0(s,t) &= 4\pi\alpha \frac{2}{st} \bar{u}(p_1') \hat{p}_2 u(p_1) \bar{v}(p_2) \hat{p}_1 v(p_2') = 4\pi\alpha \frac{2s}{t} N_1 N_2, \\ \sum_{\text{pol}} |N_i|^2 &= 2, \quad \delta(t) = -i\alpha \ln \frac{-t}{\lambda^2}. \end{aligned} \tag{1.119}$$

Here we use the fact, that at high energies only the longitudinal polarizations of the t-channel virtual photons are important:

$$\bar{u}(p_1') \gamma_\mu u(p_1) \bar{v}(p_2) \gamma_\nu v(p_2') G^{\mu\nu}(q), \quad G^{\mu\nu}(q) = \frac{1}{q^2} \frac{2\, p_2^\mu p_1^\nu}{s}, \quad q^2 = t. \tag{1.120}$$

Radiative corrections to A_0 due to the so-called «decorated boxes» diagrams are assumed to lead to a generalized eikonal representation:

$$A = A_0(s,t) [\Gamma_1(t)]^2\, \mathrm{e}^{i\delta(t)}, \tag{1.121}$$

where $\Gamma_1(t)$ is the Dirac form factor of electron

$$V^\mu(t) = \gamma^\mu \Gamma_1(t) + \frac{\sigma^{\mu\nu} q_\nu}{2m} \Gamma_2(t), \quad \Gamma_1(t) = 1 + \frac{\alpha}{\pi} \Gamma_1^{(2)}(t) + \ldots \tag{1.122}$$

Note, that one should include in $\delta(t)$ also corrections to the virtual photon Green function, leading in particular to the electric charge renormalization.

In the next Subsection we verify the generalized eikonal representation for the so-called «decorated boxes» amplitudes [16].

1.3.1. One-Loop Correction to the Electron IF. Keeping in mind that the amplitude for the near forward scattering with the two-photon exchange is pure imaginary (we omit the corrections of the order of m^2/s), we can calculate its s-channel discontinuity. There are RC to this discontinuity from the virtual photons and from the emission of the real photon in the intermediate state. We will separate the last contribution in two parts corresponding to the emission of soft and hard photons.

The virtual photon contribution contains the electron vertex function for the case when the initial and final electrons are on mass shell:

$$\begin{gathered}
\Delta\tau_e^{\rm virt} = \frac{\alpha}{\pi}\tau_e^{(0)}[F_1^{(2)}(k^2) + F_1^{(2)}(k'^2)], \\
F_1^{(2)}(t) = -G(t)\,\ln\frac{m}{\lambda} - G_1(t) - T(t), \\
G(t) = \frac{1+a^2}{2a}\ln b - 1, \quad G_1(t) = 1 - \frac{1+2a^2}{4a}\ln b, \\
T(t) = \frac{1+a^2}{2a}\left[-\frac{1}{4}\ln^2 b + \ln b\ln(1+b) - \int\limits_1^b \frac{dx}{x}\ln(1+x)\right], \\
a = \sqrt{1 - \frac{4m^2}{t}}, \quad b = \frac{a+1}{a-1}, \quad t < 0, \\
k = p - p_1, \quad k' = p_1 - p_1',
\end{gathered} \tag{1.123}$$

where the momenta of initial and final electrons are p, p_1' and the momentum of electron in the intermediate state is p_1.

The contribution from the emission of a soft photon has the classical form:

$$\begin{gathered}
-\frac{\alpha}{4\pi^2}\int\left(\frac{p_1}{p_1k_1} - \frac{p}{pk_1}\right)\left(\frac{p_1}{p_1k_1} - \frac{p_1'}{p_1'k_1}\right)\tau_e^{(0)}\frac{d^3k_1}{\omega_1}\Big|_{\omega_1<\delta E}, \\
\delta E \ll E = \sqrt{s}/2.
\end{gathered} \tag{1.124}$$

The energies of initial, intermediate, and final particles are approximately equal (but large in comparison with the electron mass), and for small scattering angles we can use the relations:

$$\frac{1}{2\pi}\int\frac{d^3k_1}{\omega_1}\frac{m^2}{(p_ik_1)^2}=2L_e,$$

$$\frac{1}{2\pi}\int\frac{d^3k_1}{\omega_1}\frac{p_ip_j}{(p_ik_1)(p_jk_1)}=\frac{1+a^2}{a}\left[L_e\ln b-\frac{1}{4}\ln^2 b+\ln b\ln(1+b)-\int\limits_1^b\frac{dx}{x}\ln(1+x)\right], \tag{1.125}$$

$$L_e=\ln\Delta+\ln\frac{m}{\lambda},\quad t=(p_i-p_j)^2,\quad \Delta=\frac{\delta E}{E}\ll 1,$$

with the quantities a, b defined above. Thus, we obtain

$$\Delta\tau_e^{\rm soft}=\frac{\alpha}{\pi}\left[\left(G(k^2)+G(k'^2)-G(t)\right)\ L_e+T(k^2)+T(k'^2)-T(t)\right],$$

where $T(t)$, $G(t)$ are defined above.

At last we consider the hard-photon emission. Its contribution to the imaginary part of the electron–electron scattering amplitude can be presented in the form

$$\mathrm{Im}_sA(s,t)=-s\frac{\alpha^3}{2\pi^2}\int\frac{d^2k}{k^2k'^2}N_1N_2\frac{d^2k_1dx}{x(1-x)}I(x,k_1,k),\quad \Delta<x<1, \tag{1.126}$$

where x is the energy fraction of the hard photon. We obtain

$$I(x,k_1,k)=\frac{1}{d_1d_2}(-4m^2+2t_1z)+\frac{1}{d_1d_1'}(-4m^2x^2(1-x)+2tz(1-x))+$$
$$+\frac{1}{d_2d_1'}(-4m^2+2t_2z)-2z\frac{1}{d_1}-4z\frac{1}{d_2}+\frac{8m^2}{d_2^2}-2z\frac{1}{d_1'},$$
$$z=1+(1-x)^2, \tag{1.127}$$

where

$$d_1=(p-k_1)^2-m^2=-\frac{1}{x}[m^2x^2+\mathbf{k}_1^2],$$
$$d_2=(p_1+k_1)^2-m^2=\frac{1}{x(1-x)}[m^2x^2+(x\mathbf{k}-\mathbf{k}_1)^2],$$
$$d_1'=(p_1'-k_1)^2-m^2=-\frac{1}{x}[m^2x^2+(x\mathbf{q}-\mathbf{k}_1)^2], \tag{1.128}$$
$$2pp_1=t_1=\frac{1}{1-x}[m^2z+(\mathbf{k}-\mathbf{k}_1)^2],$$
$$2p_1p_1'=t_2=\frac{1}{1-x}[m^2z+(x\mathbf{q}+\mathbf{k}_1-\mathbf{k})^2].$$

The subsequent integration is straightforward and gives the result:

$$\Delta\tau_e^{\mathrm{hard}} = \tau_e^{(0)}\frac{\alpha}{\pi}\left[\Big(G(k^2)+G(k^{'2})-G(t)\Big)\ln\frac{1}{\Delta}+G_1(k^2)+G_1(k^{'2})-G_1(t)\right].$$

The interference of two amplitudes with the photon emitted by two initial particles is small $\sim\mathcal{O}(t/s)$. This fact is known in literature as the up-down cancellation. The contribution of the diagrams with the two-photon exchange is pure imaginary and, consequently, does not interfere with the real Born amplitude. Adding together all contributions we obtain the final result for one-loop RC to the electron IF

$$\Delta\tau_e = \frac{\alpha}{\pi}\tau_e^{(0)}F_1^{(2)}(t), \quad \tau_e^{(0)} = 4\pi\alpha. \tag{1.129}$$

This result agrees with the generalized eikonal form of the small-angle scattering amplitude. But in the upper orders the eikonal representation is violated, as it will be shown below.

1.3.2. Generalized Eikonal Representation. The above result for RC to the electron IF can be obtained in a simple way. Let us consider again the case of the decorated box, when the positron block corresponds to the Born diagram whereas the electron one contains the set of four Feynman's graphs with a virtual photon. We express the components of the exchanged photon momentum in terms of the squared invariant energies s_1, s_2 for electron and positron blocks using the Sudakov parameters

$$k = \alpha p_2 + \beta p_1 + k_\perp, \quad d^4k = \frac{1}{2s}ds_1 ds_2 d^2k_\perp, \quad k_\perp^2 = -\mathbf{k}^2,$$
$$s_1 = (k-p_1)^2 = -s\alpha - \mathbf{k}^2 + m^2, \quad s_2 = (k+p_2)^2 = s\beta - \mathbf{k}^2 + m^2.$$

Performing the s_2 integration by residue from the propagator of the intermediate positron (it takes into account also the diagram with the crossed photon lines), we obtain the following expression for the total RC:

$$\frac{4\,\alpha}{s\,(2\pi)^2}\int\frac{d^2\mathbf{k}}{(\mathbf{k}^2+\lambda^2)((\mathbf{q}-\mathbf{k})^2+\lambda^2)}\int\limits_C ds_1 p_2^\mu p_2^\nu \bar{u}(p_1')A_{\mu\nu}u(p_1), \tag{1.130}$$

where $\bar{u}(p_1')A_{\mu\nu}u(p_1)$ is the Compton scattering amplitude, corresponding to the Feynman diagrams having only s-channel singularities and the contour C is situated above these singularities. The amplitude has the pole at $s_1 = m^2$, which corresponds to the electron intermediate state, and the right-hand cut starting from $s_1 = (m+\lambda)^2$, which corresponds to one-electron and one-photon intermediate state.

Using the Sudakov parameterization for the photon momentum k we can present p_2^μ in the form

$$p_2^\mu = \frac{1}{\alpha}(k - k_\perp - \beta p_1) \approx -\frac{s}{s_1 + \mathbf{k}^2}(k - k_\perp)^\mu \tag{1.131}$$

omitting the small contribution $\sim 1/s$ proportional to βp_1.

Let us consider the product of two terms in the right-hand side of this equation with the Compton amplitude $A_{\mu\nu}$. The contribution of the term $\sim k_\perp$ is zero:

$$|\mathbf{k}| s p_2^\nu \int\limits_C \frac{ds_1 k_\perp^\mu}{(s_1 + \mathbf{k}^2)|\mathbf{k}|} \bar{u}(p_1') A_{\mu\nu}(s_1, k, k') u(p_1) = 0. \tag{1.132}$$

This conclusion follows from the convergence of the integral over the large circle in the s_1 plane and the absence of the left cut. This property is valid for the planar Feynman graphs. The convergence of the integral is a consequence of the fact that for the physical (transverse) polarizations of the virtual photon the quantity $e^\mu p_2^\nu A_{\mu\nu}, \mathbf{e} = \mathbf{k}_\perp / |\mathbf{k}|$ behaves at large s_1 as m^2/s_1.

Applying the Ward identity for the first contribution $\sim k^\mu$ we obtain:

$$p_2^\mu p_2^\nu \bar{u}(p_1') A_{\mu\nu}(s_1) u(p_1) = -\frac{s e^2}{s_1} p_2^\nu \bar{u}(p_1') \Gamma^\nu(q) u(p_1), \;\; s_1 \gg m^2. \tag{1.133}$$

Now the integral over the large semicircle gives the generalized eikonal result $\sim \Gamma_\nu$, which means, in particular, that for physical $t < 0$ the total contribution of the various intermediate states is not zero. In particular, we see that RC to IF of electron containing infrared divergences cancel only in the total cross section with the contribution of the inelastic process — the photon emission.

In the case of the n-photon exchange, the eikonal result for the scattering amplitude corresponds to the classical picture in which all intermediate fermions are on their mass shell. It is a consequence of the fact, that the Born amplitude for the t-channel photon interactions with external particles tends to zero as $(p_A\, k_i)^{-2}$ when $(p_A\, k_i) \to \infty$, which gives a possibility of calculating all integrals over $(p_A\, k_i)$ by residues. For RC corresponding to the decorated diagrams with one additional virtual photon we can use the arguments similar to the two-photon case. The physical reason for the generalized eikonal result for their total contribution is that the integration over the invariant s_i, corresponding to the virtuality of the inner fermion line, to which the virtual gluon line is attached, gives zero because after the cancellation of the renormalization effects due to the Ward identity the amplitude at large s_i behaves as $1/s_i^2$. The nonvanishing result is obtained only from the diagrams in which the virtual gluon line is attached to the external fermion lines but in this case we obtain the generalized eikonal result. This argument is not valid for nonplanar diagrams because they have the left and right singularities in s_i planes.

1.3.3. Impact Factors in the Two-Loop Approximation. In RC to the photon IF the infrared divergencies are cancelled in the sum of contributions from the $e^+e^-\gamma$ and e^+e^- intermediate states. With the use of the crossing relations for $t = 0$ [17] one can express the contribution to τ^γ from the $e^+e^-\gamma$ intermediate state in terms of the contribution to τ^e from the $e\gamma\gamma$ intermediate state which was investigated in papers [18–20]. We estimate here RC for $t = 0$ only at small virtualities of the exchanged photon $\mathbf{k}^2$. Their value can be extracted from the results of paper [21], where the one-loop correction to the cross section of pair production by photon on the coulomb field of nuclei was calculated:

$$\sum_{i=1}^{2}[\tau + \Delta\tau]_{ii}^{\gamma}(k, 0) = \frac{28\mathbf{k}^2\alpha^2}{9m^2}[1 + \delta_p], \quad \mathbf{k}^2 \ll m^2,$$
$$\delta_p = \frac{\alpha}{\pi}\frac{9}{14}\left(\frac{1128}{35}\zeta(3) - \frac{6971}{210}\right) = 0.009. \tag{1.134}$$

RC to the photon IF can be found easily also in the region $k^2 \gg m^2$ where one can use the DGLAP evolution equations [22, 23].

Let us consider now RC to the electron IF. The generalized eikonal (GE) hypothesis is violated in the 2-loop approximation. (This fact was verified explicitly for $t = 0$.) Indeed, if the GE hypothesis would be valid, the complete compensation of contributions from the transition of the initial electron to the intermediate states e, $e\gamma$ and $e\gamma\gamma$ would take place. However, it was shown, that the total contribution is not zero and is equal to an interference term for the e^+e^--pair production amplitudes.

To clarify this result, let us write down the impact factor in the form

$$\tau^A = \int\limits_C \frac{ds_1}{2\pi\, i}\frac{1}{s^2}J_{\mu\nu}^{(A)}p_B^\mu p_B^\nu, \tag{1.135}$$

where the quantity $(1/s^2)J_{\mu\nu}^{(A)}p_B^\mu p_B^\nu$ is expressed in terms of amplitudes $J^{(A)}$ for the scattering of the virtual photon on the initial particles and does not depend on s at $s \to \infty$. $J_{\mu\nu}^e$ corresponds to contributions of all possible diagrams contrary to the planar amplitude $A_{\mu\nu}$ discussed in the previous section. The integration contour C is displaced in correspondence with the Feynman prescription between the right- and left-hand side singularities of the amplitude. The right singularities are the poles at $s_1 = m^2$ and the cuts at $s_1 > (m+\lambda)^2$, $s_1 > (m+2\lambda)^2$, and $s_1 > 9m^2$. There are also left singularities at the same points for the crossing variable $u_1 = -s_1 - t - 2m^2 + \mathbf{k}^2 + (\mathbf{q} - \mathbf{k})^2$. The additional e^+e^- pair can be produced according to the Bethe–Heitler or bremsstrahlung mechanisms. There are also interference terms taking into account the identity of the final electrons. The most important contribution is from the Bethe–Heitler mechanism, corresponding to the

e^+e^- pair production by two virtual photons. The corresponding impact factor contains the divergency in s_1 related to the presence of two-photon intermediate states in the crossing channel. For the case of $t = 0$ this contribution was calculated in [26]. We write down it here only in the Weizsacker–Williams approximation, where it has the form of the sum rule for the Borselino formulae for the total cross section $\sigma(s_1)$ for the e^+e^- pair production in the electron–photon collisions through the Bethe–Heitler mechanism:

$$\tau^e_{\mathrm{BH}_e} = \mathbf{k}^2 \int\limits_{s_{\mathrm{th}}}^{s} \frac{d\,s_1}{\pi} \frac{\sigma(s_1)^{\gamma e \to e e \bar{e}}}{s_1} = \frac{\alpha^3 \mathbf{k}^2}{\pi m^2} \left(a \, \ln^2 \frac{s}{m^2} + b \, \ln \frac{s}{m^2} + c \right), \qquad a = \frac{14}{9}, \quad b = -\frac{218}{27}, \quad c = \frac{418}{27} - \frac{13}{2}\zeta(2). \tag{1.136}$$

As it was discussed above, the logarithmic dependence on the upper limit s in the integral over s_1 should be subtracted in a self-consistent way to avoid the double counting, because the logarithmic contributions are summed by the Bethe–Saltpeter equation (cf. the analogous procedure for the BFKL Pomeron in the next-to-leading approximation [24] (and references therein)).

For the muon production we have

$$\tau^e_{\mathrm{BH}_\mu} = \frac{\alpha^3 \mathbf{k}^2}{\pi M^2} \left(a_1 \, \ln^2 \frac{s}{M^2} + b_1 \, \ln \frac{s}{M^2} + c_1 \right), \qquad a_1 = \frac{14}{9}, \quad b_1 = -\frac{218}{27} + \frac{28}{9} \ln \left(\frac{M}{m} \right), \quad c_1 = \frac{3011}{324} - \frac{28}{9}\zeta(2) - \frac{107}{9} \ln \left(\frac{M}{m} \right). \tag{1.137}$$

Here m and M are masses of electron and muon, correspondingly.

In the case of the bremsstrahlung mechanism of the e^+e^- pair production its contribution should be added with the corresponding two-loop RC to the electron form factor for the elastic intermediate state and leads to the result corresponding to the generalized eikonal approximation due to the fact, that the corresponding diagrams are planar.

Among many Feynman's graphs obtained from the interference between the various amplitudes for the pair production there are only four nonplanar diagrams corresponding to the identity of electrons in the final state in the Bethe–Heitler mechanism. Only they give nonvanishing result for τ^e at $t = 0$. The corresponding contribution in the Weizsacker–Williams approximation was calculated in [17]:

$$\tau^{(e)}_{\mathrm{int}} \approx \frac{\mathbf{k}^2}{m^2} \frac{\alpha^3}{\pi} \left(\frac{221}{315} + \frac{41549}{6300}\zeta(2) - \frac{216}{105}\zeta(3) - \frac{792}{105}\xi(2) \ln 2 \right) \approx \approx \frac{\mathbf{k}^2}{m^2} \frac{\alpha^3}{\pi} (-3.57). \tag{1.138}$$

It leads to the sum rules for the integrals from the one- and two-photon bremsstrahlung cross sections and the slope of the Dirac form factor at $t = 0$ [20].

Finally, the total two-loop contribution to the electron impact factor can be written as follows:

$$\tau^e = \frac{\alpha^2}{\pi^2} \tau_e^0 F_1^{(4)} + \tau_{\rm BH}^e. \tag{1.139}$$

Here $F_1^{(4)}$ is the full two-loop correction to the Dirac form factor (including nonplanar diagrams and the diagrams with the inner fermion loop). The term $\tau_{\rm BH}^e$ is the total contribution from the imaginary part corresponding to the Bethe–Heitler mechanism of the pair production including the interference effects, related with the identity of the produced electrons ($\tau_{\rm BH}^e = \tau_{{\rm BH}_e}^e + \tau_{{\rm BH}_\mu}^e + \tau_{\rm int}^{(e)}$ for $t = \mathbf{k}^2 = 0$).

The physical meaning of this formula is obvious: the nontrivial corrections to IF are related only with the charged particle production in the intermediate states.

1.3.4. Higher Order Radiative Corrections to Impact Factor. In the three-loop approximation to the photon impact factor the most important contribution corresponds to the diagram with two fermion loops connected in the t channel by two photons. It contains the logarithmic divergency $\sim \ln(s/m^2)$ because of the imaginary part of the corresponding amplitude proportional to s_1 for large s_1. In particular, for $t = \mathbf{k}^2 = 0$ the impact factor can be expressed in terms of the integral from the cross section for the transition of two real photons into two e^+e^- pairs.

The growth of the impact factor $\sim \ln(s/m^2)$ is related to the logarithmic increase of the number of the dipoles (lepton pairs) at large energies. The effect of fermion's identity in the intermediate state does not have any influence on this growth. Also the contribution from the diagrams with one e^+e^- pair and several photons gives a finite contribution to the photon impact factor.

Let us consider now three-loop corrections to the electron impact factor. The most important contribution $\sim \ln^2(s/m^2)$ appears from one-loop RC to the Bethe–Heitler mechanism of the e^+e^- production. Other diagrams lead to finite terms. The generalized eikonal representation is violated due to the nonplanar diagrams related to the e^+e^- pair production, but there is another reason for its violation. It is related with the charge parity conservation in QED. Indeed, two external photons with their momenta k and $q - k$ cannot pass through the fermion loop to the three-photon intermediate state in the t channel. Therefore the generalized eikonal representation, containing in particular the form factor corresponding to the transition of the external photon through the fermion loop into the three-photon state, cannot be valid in three-loop approximation.

The methods, which were developed above for QED, can be used also for QCD, where we urgently need to calculate the radiative corrections to impact factors of the virtual photon and other particles to find the energy region of applicability of the BFKL theory in the next-to-leading approximation [24].

1.4. Small Angle Processes. *1.4.1. Bremsstrahlung at Bhabha Scattering and Crossed Process.* For the case when one of projectile is elastically scattered, differential cross sections are logarithmically enhanced. It is well-known phenomenon called Weizsacker–Williams approximation. Using the infinite momentum technique one can calculate the cross section with power accuracy-improving the logarithmical ones.

Let us demonstrate this method considering the bremsstrahlung in electron–positron (electron) scattering (see [4, 7, 8, 33] and references therein)

$$e^-(p_1) + e^+(p_2) \to (e^-(p_1') + \gamma(k_1)) + e^+(p_2'),$$
$$s = (p_1 + p_2)^2 \gg m^2, \quad -t = -k^2 = -(p_2 - p_2')^2 \sim m^2, \quad m = m_e,$$
$$k = p_2' - p_2, \quad q = p_1 - p_1'.$$

The main contribution, nondecreasing with energy, arises from the kinematical region when the scattered electron with photon form a jet moving close to the direction of motion of the initial electron (the center of mass of initial electron and positron is implied). The typical scattering angles are of the order of m/E, $4E^2 = s$. The spectator-positron scattered in backward direction as well moves close to the direction of initial positron.

Let us write down the 4-momenta of particles and the momentum transferred between fermions k in terms of Sudakov's variables (closely related to infinite momentum approach):

$$k = \alpha\tilde{p}_2 + \beta\tilde{p}_1 + k_\perp, \quad p_1' = \alpha'\tilde{p}_2 + \beta'\tilde{p}_1 + p'_{1_\perp}, \quad a_\perp^2 = -\mathbf{a}^2 < 0, \tag{1.140}$$
$$k_1 = \alpha_1\tilde{p}_2 + \beta_1\tilde{p}_1 + k_{1\perp}, \quad a_\perp p_1 = a_\perp p_2 = 0, \quad \tilde{p}_{1,2} = p_{1,2} - p_{2,1}\frac{m^2}{s}.$$

Although the fine structure constant and part of momentum k have the same designation α, we do not think they can be mixed.

Four-vectors $\tilde{p}_{1,2}$ are almost light-like ones $\tilde{p}_{1,2}^2 = O(m^6/s^2)$. Using the on-mass-shell condition of the scattered positron

$$p_2'^2 - m^2 = (p_2 + k)^2 - m^2 = m^2\alpha + s\beta(1+\alpha) - \mathbf{k}^2 = 0,$$

one can see that the square of momentum transferred

$$k^2 = -\frac{\mathbf{k}^2 + \alpha^2 m^2}{1+\alpha} < 0$$

is nonzero negative quantity. Components of jet particles momenta along p_1 (quantities of the order of unity) as well as transversal ones obey the conservation low

$$\mathbf{k} + \mathbf{p}_1' + \mathbf{k}_1 = 0, \quad \beta_1 + \beta' = 1, \quad x = 1 - \beta_1. \tag{1.141}$$

Small components of jet particles momenta (ones along p_2) can be found using the on-mass-shell conditions $p_1'^2 = m^2, k_1^2 = 0$:

$$s\alpha_1 = \frac{\mathbf{k}_1^2}{1-x}, \quad s(\alpha + \alpha_1) = -\frac{(1-x)m^2 + \mathbf{q}^2}{x}, \quad \mathbf{q} = \mathbf{k} + \mathbf{k}_1. \tag{1.142}$$

The value of Sudakov parameter α can be related with invariant mass square of jet $s_1 = (p_1 - k)^2 = (p_1' + k_1)^2 = m^2 - s\alpha - \mathbf{k}^2$.

We note that the square of momentum transfer is negative and is restricted by the magnitude below $|k^2| > m^2\gamma^{-4}, \gamma = m/E$, as well as the jet invariant mass in the region of maximal contribution to the cross section is of the order of electron mass.

The phase volume of the final particles can be expressed in terms of Sudakov's variables in full analogy with the previous section and has the form:

$$d\Gamma = \frac{d^3p_1'}{E_1'}\frac{d^3k_1}{\omega_1}\frac{d^3p_2'}{E_2'}\frac{(2\pi)^4}{(2\pi)^9}\delta^4(p_1 + p_2 - k_1 - p_1' - p_2') = \\ = \frac{1}{4s}\frac{1}{(2\pi)^5}\frac{dxd^2\mathbf{k}d^2\mathbf{q}}{x(1-x)}. \tag{1.143}$$

Matrix element of the process of radiative Bhabha scattering has a contribution from eight Feynman diagrams. Contribution of only two of them, of scattering type survive in the region of high energies. Using Gribov's prescription for the Green function of the virtual photon we can write it in the form:

$$\begin{aligned} M^{e^+e^-\to e^+(e^-\gamma)} &= \frac{2s(4\pi\alpha)^{3/2}}{k^2}\varepsilon_\rho(k_1)N_2\bar{u}(p_1')V^\rho u(p_1), \\ N_2 &= \frac{1}{s}\bar{v}(p_2)\hat{p}_1 v(p_2'), \\ V^\rho &= \gamma_\rho\frac{\hat{p}_1 - \hat{k} + m}{(p_1-k)^2 - m^2}\frac{\hat{p}_2}{s} + \frac{\hat{p}_2}{s}\frac{\hat{p}_1 - \hat{k}_1 + m}{(p_1-k_1)^2 - m^2}\gamma_\rho, \end{aligned} \tag{1.144}$$

with $\epsilon(k_1)$ — the polarization vector of photon. Denominators of the electron Green functions can be expressed in terms of Sudakov's variables:

$$\begin{aligned} (p_1 - k)^2 - m^2 &= -\mathbf{k}^2 + \frac{(1-x)m^2 + \mathbf{q}^2}{x} + \frac{(\mathbf{q}-\mathbf{k})^2}{1-x} = \frac{d}{x(1-x)}, \\ (p_1 - k_1)^2 - m^2 &= -(\mathbf{q}-\mathbf{k})^2 - (1-x)m^2 - \frac{x}{1-x}(\mathbf{q}-\mathbf{k})^2 = -\frac{d_1}{1-x}. \end{aligned} \tag{1.145}$$

Further manipulations to obtain the differential cross section are standard. Differential cross section of radiative Bhabha scattering with emission of hard photon along the initial electron is

$$d\sigma^{e^+e^-\to e^+(e^-\gamma)} = \frac{2\alpha^3}{\pi^2}\frac{1-x}{d^2d_1^2}[\mathbf{k}^2(1+x^2)dd_1 - 2x(d-d_1)^2m^2]\frac{d^2\mathbf{q}\,d^2\mathbf{k}\,dx}{(\mathbf{k}^2+m^2\alpha^2)^2},$$
$$d = (1-x)^2m^2 + (\mathbf{q}-x\mathbf{k})^2, \qquad d_1 = (1-x)^2m^2 + (\mathbf{q}-\mathbf{k})^2, \qquad (1.146)$$
$$\alpha = -\frac{\mathbf{q}^2 + (1-x)^2m^2}{sx(1-x)}.$$

In the case of radiative scattering of electron on a heavy nuclei, $\mathbf{k}$ is the momentum transferred to it in the rest frame of nuclei and the quantity α must be replaced by

$$m\alpha = -\frac{(1-x)^2m^2+\mathbf{q}^2}{2Ex(1-x)}, \qquad (1.147)$$

with E — the energy of the initial electron in the nuclei rest frame.

The differential cross section in this form is convenient both for further numerical or analytical investigation. For instance, we can obtain, by integrating it on momentum transferred to the nuclei, the distribution on photon transversal momentum and its energy fraction (in agreement with the result of paper [27]):

$$\sigma^{e^+e^-\to e^+(e^-\gamma)} = \frac{2\alpha^3}{\pi m^4}\frac{1-x}{c^2}\times$$
$$\times\left[2[1+x^2-\frac{4x(1-x)^2}{c}+\frac{4x(1-x)^4}{c^2}]\ln\frac{sx}{(1-x)m^2} - \right.$$
$$\left. -(1+x)^2 + \frac{16x(1-x)^2}{c} - \frac{16x(1-x)^4}{c^2}\right]dxd^2k_1, \quad (1.148)$$

$$x = 1-\frac{\omega}{E}, \quad c = (1-x)^2 + \frac{\mathbf{k}_1^2}{m^2}.$$

The same form has the distribution on photon parameters emitted along the initial positron. The total spectrum can be obtained by integration on transversal momentum of photon and further multiplying on factor 2:

$$\frac{d\sigma^{e^+e^-\to e^+e^-\gamma}}{dx} =$$
$$= \frac{4\alpha^3}{m^2(1-x)}\left[\frac{4}{3}x + (1-x)^2\right]\left[2\ln\frac{s}{m^2} + 2\ln\frac{x}{1-x} - 1\right]. \quad (1.149)$$

In the same way one can obtain the inclusive distribution on the scattered electron (in agreement with [28]):

$$\frac{d\sigma^{e^+e^-\to e^+(e^-\gamma)}}{dx\,d^2\mathbf{q}} = \frac{4\alpha^3(1-x)}{\pi m^2a^2}\left[\left[1+x^2-\frac{4x(1-x)^2\mathbf{q}^2}{m^2a^2}\right]\left[\ln\frac{s}{m^2}+\ln x\right]+\right.$$
$$+\frac{x\ln x}{1-x}\left[1+x^2+\frac{4x^2(3-x)\mathbf{q}^2}{m^2a^2}\right]-\frac{(1+x^2)^2}{2(1-x)^2}+\frac{8x(1-x+x^2)\mathbf{q}^2}{m^2a^2}-$$
$$\left.-\frac{xa(1+x^2+4x^2/a)}{(1-x)^2r}\ln\frac{x[2(1-x)-a+r]}{2x(1-x)+a+r}\right], \quad (1.150)$$

$$r=\sqrt{a(a+4x)}, \quad a=(1-x)^2+\frac{\mathbf{q}^2}{m^2}.$$

For the case of scattering on a heavy nuclei Y the logarithmical term $\ln s/m^2$ must be replaced by $\ln 2E/m$. This formula can be considerably simplified for sufficiently large electron scattering angle $\theta = |\mathbf{q}|/xE \gg m/E = 1/\gamma$ and $1-x\sim 1$:

$$\frac{d\sigma^{ee\pm\to(e\gamma)e^\pm}}{dxd\theta^2}=\frac{4\alpha^3(1+x^2)}{x^2\theta^4E^2(1-x)}\left[(1-x+x^2)\ln\frac{4E^2}{m^2}+\ln x-\frac{1}{2}(1+x^2)+x\ln\frac{\theta^2}{4}\right], \quad (1.151)$$
$$\frac{d\sigma^{eY\to(e\gamma)Y}}{dxd\theta^2}=\frac{4\alpha^3(1+x^2)}{x^2\theta^4E^2(1-x)}\left[(1+x^2)\left(\ln\frac{2E}{m}-\frac{1}{2}\right)+\ln x+x\ln\frac{\theta^2}{4}\right].$$

We put for completeness the distribution on angles of emission of both the scattered electron and the positron assuming them to be large compared with $1/\gamma$:

$$\frac{d\sigma}{d\theta_+^2\,d\theta_-^2\,d\varphi}=\frac{4\alpha^3}{\pi s}\frac{F(\rho,c)+F(1/\rho,c)}{\theta_+^2\theta_-^2(\theta_+^2+\theta_-^2+2\theta_+\theta_-c)}, \quad (1.152)$$

where $\rho=\theta_+/\theta_-$, $c=\cos\varphi$, φ is the azimuthal angle between the planes containing the beam axes and the scattered electron and positron, $\theta_\pm$ is the polar angle of scattered positron and electron. The function F has the form:

$$F(\rho,c)=\frac{1}{2}+2c\rho+\frac{1}{2}(-1+\rho^2-2c\rho-4c^2\rho^2)\ln\frac{\rho^2+2c\rho+1}{\rho^2}+$$
$$+(1+c\rho+2(c\rho)^2+4(c\rho)^3-\rho^2-3c\rho^3)\frac{\arctan\left(\dfrac{\sqrt{1-c^2}}{\rho+c}\right)}{\rho\sqrt{1-c^2}}, \quad (1.153)$$

we suppose here $\theta_\pm\gg 1/\gamma$.

Quite similar calculation can be applied to the problem of the double bremsstrahlung process for the case of emission by electron and positive(negative) charged muon (photons are emitted in opposite directions in the center-of-mass reference frame). The relevant cross section has the form:

$$\begin{gathered}\frac{d\sigma^{e^+e^-\to(e^+\gamma)(e^-\gamma)}}{d^2\mathbf{k}\,d^2\mathbf{q}_1\,d^2\mathbf{q}_2\,dx\,dy}=\frac{\alpha^4(1-x)(1-y)}{\pi^4(\mathbf{k}^2)^2}\frac{R_1R_2}{(d_1d_2d_3d_4)^2},\\ \begin{aligned}R_1&=\mathbf{k}^2d_1d_2(1+x^2)-2xm_1^2(d_1-d_2)^2,\\ R_2&=\mathbf{k}^2d_3d_4(1+y^2)-2ym_2^2(d_3-d_4)^2,\\ d_1&=m_1^2(1-x)^2+(\mathbf{q}_1-\mathbf{k}x)^2,\\ d_2&=m_1^2(1-x)^2+(\mathbf{q}_1-\mathbf{k})^2,\\ d_3&=m_2^2(1-y)^2+(\mathbf{q}_2+\mathbf{k}y)^2,\\ d_4&=m_2^2(1-y)^2+(\mathbf{q}_2+\mathbf{k})^2.\end{aligned}\end{gathered}\tag{1.154}$$

Here $x, -\mathbf{q}_1$ are the energy fraction and the transversal momentum of the scattered electron, $y, -\mathbf{q}_2$ — the corresponding values for muon, $m_{1,2}$ — their masses.

Differential cross section of double photon emission in the same direction has much more complicated form [29]. It will be considered in kinematics of large angles emission later.

In the similar way the mechanisms of electroproduction of electron or muon pairs at colliding $e^+e^\pm$ were investigated.

The two-photon mechanism of electroproduction was considered first in [30, 31], and later in more detail in [26, 32, 33]; bremsstrahlung ones — in [29]. The effect of identity of fermions in the final state was considered in [17]. Spectra in fragmentation region including the charge-odd contributions as well as fermion statistics were investigated in [34]. We refer for details to the cited papers.

Differential cross sections of different processes in fragmentation region with the same content of initial and final particles are connected with each other due to crossing relation. Cross sections of processes one of which can be obtained from another by rearrangement of initial particle with one of final ones from its jet with the subsequent replacing them by antiparticles, turns out to be related by some algebraical transformation [35]. Namely, the summed on spin states of all particles matrix element squared are connected by relation

$$\begin{aligned}&\sum|M_{a\to c}(k,\beta;k_i,\beta_i;s)|^2=\\ &\qquad=\eta_{ca}\sum\left|M_{\bar{c}\to\bar{a}}\left(\frac{k}{\beta},1/\beta;k_i-k\frac{\beta_i}{\beta},-\frac{\beta_i}{\beta};-s\beta\right)\right|^2,\end{aligned}\tag{1.155}$$

with k, β — transversal momentum and the energy fraction of particle c; k_i, β_i — the similar quantities for other particles from that jet; η_{ac} equals $+1$, if both touched particles are of the same statistic, and -1 in the case when one of them is fermion and the other is boson. In particular the spectra related as

$$\frac{d\sigma^{b+a\to c+\ldots}}{d\beta} = \varphi_{a\to c}(\beta, s) = -\beta\eta_{ac}\varphi_{\bar{c}\to\bar{a}}\left(\frac{1}{\beta}, -s\beta\right). \tag{1.156}$$

For instance, starting from the spectral distribution on the photon energy fraction in the process of photon emission along electron in electron–positron scattering inferred above we have

$$\frac{d\sigma^{ea\to(e\gamma)a}}{dy} = \varphi(y,s) = \frac{2\alpha^3}{y}\left[\frac{4}{3}(1-y)+y^2\right]\left[2\ln\frac{s(1-y)}{m^2 y}-1\right], \quad y=\frac{\omega}{E},$$
$$\frac{d\sigma^{\gamma a\to e^-e^+a}}{dz} = \psi(z) = \frac{2\alpha^3}{m^2}\left[1-\frac{4}{3}z(1-z)\right]\left[2\ln\frac{sz(1-z)}{m^2}-1\right], \quad z=\frac{E_-}{\omega}. \tag{1.157}$$

One can be convinced in the validity of the relation $\psi(z,s) = z\varphi(1/z, -sz)$. Let us consider now the spectrum of emission of two photons by electron and positron [36]:

$$\frac{d\sigma^{e_-e^+\to(e^-\gamma)(e^+\gamma)}}{dy_1\,dy_2} = \\ = \frac{8\alpha^4}{\pi m^2}\frac{(1-y_1)(1-y_2)\eta_1+\eta_2[y_1^2(1-y_2)+y_2^2(1-y_1)]+\eta_3 y_1^2 y_2^2}{y_1 y_2},$$
$$\eta_1 = \frac{5}{4}+\frac{7}{8}\xi_3, \quad \eta_2 = \frac{1}{2}+\frac{7}{8}\xi_3 \quad \eta_3 = \frac{7}{8}\xi_3, \quad \xi_3 = \sum_1^\infty \frac{1}{n^3} \approx 1.202\ldots,$$

with $y_{1,2} = \omega_{1,2}/E$ — the energy fractions of photons. Multiplying the right-hand part of this equation on x_1x_2 and performing the substitution $y_{1,2} \to 1/x_{1,2}$, we obtain the spectral distribution on the energy fractions x_1, x_2 of electrons in the process of two-pair production at photons collision:

$$\frac{d\sigma^{\gamma\gamma\to(e^-(x_1)e^+)(e^-(x_2)e^+)}}{dx_1dx_2} = \frac{8\alpha^4}{\pi m^2}[x_1(1-x_1)x_2(1-x_2)\eta_1 - \\ -\eta_2[x_1(1-x_2)+x_2(1-x_1)]+\eta_3]. \tag{1.158}$$

Integration of this spectrum on the energy fractions gives the known result for the total cross section of the process of two electron–positron pairs production in two-photon high energy collisions [37]:

$$\sigma^{\gamma\gamma\to 2e^+2e^-} = \frac{\alpha^4}{\pi m^2}\left[\frac{175}{36}\xi_3 - \frac{19}{18}\right]. \tag{1.159}$$

Starting from the double emission in opposite directions spectrum, one can obtain the spectrum of bremsstrahlung at pair creation in the high-energy photon electron collisions:

$$\frac{d\sigma^{\gamma e\to(e^-e^+)(e\gamma)}}{dy_1\,dy}=\frac{8\alpha^4}{\pi m^2}\left[[\eta_2-\eta_1 y(1-y)]\left(\frac{1}{y_1}-1\right)+\eta_3 y_1-\eta_2 y_1 y(1-y)\right], \tag{1.160}$$

with $y_1=\omega/E$, $y=E_-/E$ — the energy fractions of the emitted photon and the electron from the pair created.

It must be noted that the crossing relation discussed above is violated beyond Born approximation since the amplitudes become in general complex.

1.4.2. Inclusive Distributions for Two-Pair Production Processes at Photon Collisions. Starting from early 70th the processes with colliding photon beams become close interest. In papers [37] the total cross sections of processes $\gamma\gamma\to 2e^+2e^-; e^+e^-\mu^+\mu^-$ were calculated. These QED processes can be used as a calibration ones for the planned photon colliding beams in full analogy with the double-photon emission process at electron collisions. Besides, such a type of processes is an essential background in any physical processes to be studied. More possibilities can be obtained using the polarized beams.

For calibration purposes the differential inclusive cross sections for the case of large scattered angles of the final particles become relevant. In paper [38] it was shown that the inclusive cross sections weakly depend on the circular polarization of initial photon beams, but the effects of linear polarization can be rather big ($\sim 19\%$). The polarization effects weakly manifest themselves in the total cross sections.

Let us consider the process of two-pair production at photon collisions

$$\gamma(k_2)+\gamma(k_1)\to(\mu^+(p_+)+\mu^-(p_-))+(e_-(q_-)+e^+(q_+)). \tag{1.161}$$

There present many Feynman diagrammes contributing to matrix element (about 40). In high-energy limit only eight of them survive (i.e, give the nonvanishing contribution in high-energy limit); restricting ourselves with the case when muon pair moves along photon with momentum k_1 and electron pair moves in opposite direction, only four Feynman diagrams become relevant. Using Sudakov's parameterization of 4-momenta of the problem

$$\begin{gathered}
p_-=\alpha_-k_2+xk_1+p_{-\perp},\quad q_-=yk_2+\beta_-k_1+q_{-\perp},\\
k=\alpha k_2+\beta k_1+k_\perp,\quad k_1^2=k_2^2=0,\\
d^4k=\frac{ds_1\,ds_2}{2s}d^2k_\perp,\quad d^4p_-=\frac{s}{2}d\alpha_-\,dx\,d^2p_{-\perp},\\
d^4q_-=\frac{s}{2}d\beta_-\,dy\,d^2q_{-\perp},
\end{gathered} \tag{1.162}$$

where k is the momentum transferred between pairs, energy fractions x,y are the positive quantities of the order of unity, $s_1=(k_1+k)^2=-\mathbf{k}^2+s\alpha$,

$s_2 = (k_2 - k)^2 = -\mathbf{k}^2 - s\beta$ are the invariant masses squared of jets which are supposed to be much smaller compared with the total CMS energy square $s = (k_1 + k_2)^2$. Using the arguments given above the cross section can be written in the frames of impact picture form of H. Cheng and T. T. Wu as

$$d\sigma^{\gamma\gamma\to e\bar{e}\mu\bar{\mu}} = \frac{\alpha^4}{4\pi}\int \frac{d^2k}{\pi(\mathbf{k}^2)^2} d\Phi^e(\mathbf{k}, x, \epsilon_1, p_{-\perp})\, d\Phi^\mu(\mathbf{k}, y, \epsilon_2, q_{-\perp}), \quad (1.163)$$

with $\epsilon_{1,2}$ — polarization vectors of photons and the impact factors defined as

$$d\Phi^e = \frac{dx\, d^2p_{-\perp}}{\pi x(1-x)} \sum \left|\frac{1}{s} T_{\mu\nu}^{\gamma\gamma^*\to e\bar{e}} e_1^\mu k_2^\nu\right|^2,$$
$$d\Phi^\mu = \frac{dy\, d^2q_{-\perp}}{\pi y(1-y)} \sum \left|\frac{1}{s} T_{\alpha\beta}^{\gamma\gamma^*\to\mu^-\mu^+} e_1^\alpha k_1^\beta\right|^2. \quad (1.164)$$

In the lowest order of PT we have

$$\sum \left|\frac{1}{s} T_{\mu\nu}^{\gamma\gamma^*\to e\bar{e}} e_1^\mu k_2^\nu\right|^2 = 8x^2(1-x)^2 \times$$
$$\times \left[\frac{\mathbf{k}^2(\mathbf{e}_1\mathbf{e}_1^*)}{4x(1-x)a_+a_-} - \frac{(\mathbf{e}_1\mathbf{p}_+)(\mathbf{e}_1^*\mathbf{p}_+)}{a_+^2} - \frac{(\mathbf{e}_1\mathbf{p}_-)(\mathbf{e}_1^*\mathbf{p}_-)}{a_-^2} - \right.$$
$$\left. - \frac{1}{a_+a_-}((\mathbf{e}_1\mathbf{p}_+)(\mathbf{e}_1^*\mathbf{p}_-) + (\mathbf{e}_1\mathbf{p}_-)(\mathbf{e}_1^*\mathbf{p}_+))\right],$$

$$a_\pm = \mathbf{p}_\pm^2 + m^2, \qquad \mathbf{p}_+ + \mathbf{p}_- = \mathbf{k}.$$

The similar expression can be obtained for the muon impact factor. Performing the integration on the transversal momenta of pair component at fixed momentum transfer $\mathbf{k}$, we obtain the spectral distribution:

$$d\sigma^{\gamma\gamma\to e^+e^-\mu^+\mu^-} = \frac{16\alpha^4}{\pi}\int \frac{d^2k}{\pi((\mathbf{k})^2)^2} \times$$
$$\times \left[(\mathbf{e}_1\mathbf{e}_1^*(-\psi + 2x(1-x)(\varphi-\psi)) + 2\frac{(\mathbf{e}_1\mathbf{k})(\mathbf{e}_1^*\mathbf{k})}{\mathbf{k}^2} x(1-x)(2\psi-\varphi))\right] \times$$
$$\times [e_1 \to e_2; x \to y; \psi \to \psi'; \varphi \to \varphi', m \to M], \quad (1.165)$$

with $\varphi = -1 + (1+2z)L_z$, $\psi = zL_z$, $z = \mathbf{k}^2/(4m^2)$; m, M are the masses of electron and muon, and $L_z = \dfrac{\ln(\sqrt{z} + \sqrt{z+1})}{\sqrt{z(z+1)}}$. Functions φ', ψ' are defined as

$$\varphi' = \varphi(\tilde{z}), \quad \psi' = \psi(\tilde{z}), \quad \tilde{z} = \frac{\mathbf{k}^2}{4M^2}. \quad (1.166)$$

Introducing the spin density matrix of photons

$$\sum_\lambda \mathbf{e}_\alpha^\lambda \mathbf{e}_\beta^{\lambda *} = \frac{1}{2}(1+\boldsymbol{\sigma}\boldsymbol{\xi})_{\alpha\beta}, \qquad \mathbf{e}_\alpha^* \mathbf{e}_\alpha = 1, \tag{1.167}$$

with ξ_1, ξ_2, ξ_3 — Stokes parameters, using the relations

$$\begin{aligned}
&4\int \frac{d\mathbf{k}^2}{(\mathbf{k}^2)^2}\psi(z)\psi(\tilde z) = \frac{1}{4}l^2 + l + 2,\\
&4\int \frac{d\mathbf{k}^2}{(\mathbf{k}^2)^2}\varphi(z)\varphi(\tilde z) = \frac{2}{3}l^2 + \frac{14}{9}l + \frac{154}{27},\\
&4\int \frac{d\mathbf{k}^2}{(\mathbf{k}^2)^2}\psi(z)\varphi(\tilde z) = \frac{1}{3}l^2 + \frac{13}{9}l + \frac{80}{27},\\
&4\int \frac{d\mathbf{k}^2}{(\mathbf{k}^2)^2}\varphi(z)\psi(\tilde z) = \frac{1}{2}l^2 + l + 4, \quad l = \ln\frac{M^2}{m^2}
\end{aligned} \tag{1.168}$$

and performing the integration on the energy fractions, we obtain for the total cross section

$$\sigma^{\gamma\gamma\to e^+e^-\mu^+\mu^-} = \frac{8\alpha^4}{\pi M^2}\left[\frac{7}{54}l^2 + \frac{103}{162}l + \frac{485}{486}\right] + \frac{2\alpha^4}{27\pi M^2}\left(l - \frac{1}{3}\right)a + \mathcal{O}\left(\frac{m^2}{M^2}\right),$$

$$a = l_1 l_2 \cos(2\gamma),$$

with $l_1 = [\xi_1^2 + \xi_3^2]^{1/2}$ — the degree of polarization of the photon with momentum k_1; l_2 — the similar quantity for the photon with momentum k_2; γ is the angle between the directions of their maximal polarization.

The similar results can be obtained for the process of electron–positron pair creation with a pair of charged pions (we suggest pions to be point-like or structureless)

$$\pi\frac{d\Phi^\pi}{d^2\mathbf{q}_{-\perp}dy} = -4y(1-y)\Bigg(\frac{(\mathbf{e}_2\mathbf{q}_-)(\mathbf{e}_2^*\mathbf{q}_-)}{a_-^2} + \frac{(\mathbf{e}_2\mathbf{q}_+)(\mathbf{e}_2^*\mathbf{q}_+)}{a_+^2} + \\ + \frac{1}{a_+a_-}((\mathbf{e}_2\mathbf{q}_+)(\mathbf{e}_2^*\mathbf{q}_-) + (\mathbf{e}_2\mathbf{q}_-)(\mathbf{e}_2^*\mathbf{q}_+))\Bigg),$$

$$a_\pm = \mathbf{q}_\pm^2 + M_\pi^2, \qquad \mathbf{q}_+ + \mathbf{q}_- = -\mathbf{k}.$$

For the total cross section we have

$$\sigma^{\gamma\gamma\to e\bar e\pi\bar\pi} = \frac{4\alpha^4}{27\pi M_\pi^2}\left[L_\pi^2 + \frac{16}{3}L_\pi + \frac{163}{18} - \frac{3L_\pi - 1}{12}a\right], \quad L_\pi = \ln\frac{M_\pi^2}{m^2}. \tag{1.169}$$

The integrated over muon 4-impulses (moving along photon with momentum k_1) distribution has the form [38]

$$\frac{d\sigma}{d^2q_\perp\,dx} = \frac{2\alpha^4}{3\pi^2 M_\mu^4}[F_0 + F_3\Sigma_3 + F_-\Sigma_- + F_+\Sigma_+ + F_3'\Sigma_3'], \qquad (1.170)$$

with

$$\Sigma_3 = l_1\cos(2\gamma_1), \quad \Sigma_\pm = l_1 l_2 \cos(2(\gamma_1 \pm \gamma_2)), \quad \Sigma_3' = l_2\cos(2\gamma_2), \qquad (1.171)$$

$\gamma_1(\gamma_2)$ is the azimuthal angle between transversal vectors of meson momentum $\mathbf{q}$ and the direction of maximal polarization of photon with momentum $k_1(k_2)$. The coefficient functions are

$$\begin{aligned}
F_0 &= \frac{R}{(1+\rho)^2} - \frac{2x(1-x)}{(1+\rho)^4}[(1+\rho^2)R - (2L+1)(\rho^2-4\rho+1)- \\
&\qquad\qquad - 2l(\rho^2 - 8\rho + 3) - 4\rho], \\
F_3 &= \frac{4\rho x(1-x)}{(1+\rho)^4}\left[R - 6L - 3 + \frac{1}{\rho^2}(-9\rho^2+4\rho+1)l - \frac{1}{\rho} - \rho\right], \\
F_- &= \frac{x(1-x)}{(1+\rho)^4}\left[L + 2l - 3 + 3\rho + \frac{1}{2}\rho^2\right], \\
F_+ &= \frac{x(1-x)}{(1+\rho)^4}\left[\rho^2(L-3) + \frac{17}{2} + 8\rho + \frac{3}{\rho} - l\left(\frac{3}{\rho^2} + \frac{10}{\rho} + 12 + 6\rho - \rho^2\right)\right], \\
F_3' &= -\frac{(\rho+1)l - \rho}{\rho(\rho+1)^2} - \frac{x(1-x)}{(1+\rho)^4}\left[2\rho L + 4\rho^2 - 5\rho + 2 - 2l\left(\frac{1}{\rho} + 3 + \rho + \rho^2\right)\right],
\end{aligned}$$

with

$$\begin{gathered}
\rho = \frac{\mathbf{q}^2}{M^2}, \quad l = \ln(1+\rho), \quad L = \ln\frac{M^2}{m^2}, \\
R = L^2 + 2(2L+1)l + L + 4l^2 + \frac{\pi^2}{3} - 2 + 2\mathrm{Li}_2(-\rho).
\end{gathered} \qquad (1.172)$$

The dependence of the inclusive cross section on the parameters of the «alien» photon with momentum k_2 is defined by the ratio $F_\pm/F_0$, F_3'/F_0 and is weak (of the order of 3%). The dependence on the polarization of «his» photon (k_1) is defined by the ratio F_3/F_0. At $\rho = 1.2$, $x = 0.5$ this quantity equals 0.19.

1.4.3. Small-Angle Bhabha Scattering. An accurate verification of the SM was one of the primary aims of LEP facility [39]. The small-angle Bhabha scattering process was used to measure the luminosity of electron–positron colliders. At LEP, an experimental accuracy on the luminosity of the order of $|\delta\sigma|/\sigma < 0.001$ has been reached. However, to obtain the total accuracy, a systematic theoretical error must also be added. This precision calls for an equally accurate theoretical expression for the Bhabha scattering cross section in order to extract the SM parameters from the observed distributions. The knowledge of Bhabha cross section in the Born approximation

$$\frac{d\sigma_B}{dO} = \frac{\alpha^2}{4s}\left(\frac{3+c^2}{1-c}\right)^2, \qquad c = \cos\theta \tag{1.173}$$

becomes insufficient. For small scattering angle θ (scattering angle of electron), taking into account the electroweak corrections, we have

$$\frac{d\sigma_B}{\theta\, d\theta} = \frac{8\pi\alpha^2}{E^2\theta^4}\left[1 - \frac{1}{2}\theta^2 + \frac{9}{40}\theta^4 + \delta_{\mathrm{weak}}\right]. \tag{1.174}$$

For the pure QED case the lowest order RC was calculated a long time ago [40, 41]. Taking into account a contribution from soft-photon emission with photon energy not exceeding (CMS implied) some small quantity $\omega/E < \Delta \ll 1$, we have $d\sigma^{(1)}/dc = d\sigma_B/dc(1+\delta_V+\delta_S)$, with [40, 41, 43]:

$$\begin{aligned}\delta_V + \delta_S = \frac{2\alpha}{\pi}\Bigg[2\left(1 - \ln\frac{s}{m^2} + 2\ln\left(\cot\frac{\theta}{2}\right)\right)\ln\frac{1}{\Delta} + \\ + \mathrm{Li}_2\left(\frac{1+c}{2}\right) - \mathrm{Li}_2\left(\frac{1-c}{2}\right) - \frac{23}{9} + \frac{11}{6}\ln\frac{s}{m^2}\Bigg] + \\ + \frac{\alpha}{\pi}\frac{1}{(3+c^2)^2}\Bigg[\frac{\pi^2}{3}(2c^4 - 3c^3 - 15c) + 2(2c^4 - 3c^3 + 9c^2 + 3c + 21)\ln^2\left(\sin\frac{\theta}{2}\right) - \\ - 4(c^4 + c^2 - 2c)\ln^2\left(\cos\frac{\theta}{2}\right) - 4(c^3 + 4c^2 + 5c + 6)\ln^2(\tan\frac{\theta}{2}) + \\ + \frac{2}{3}(11c^3 + 33c^2 + 21c + 111)\ln(\sin\frac{\theta}{2}) + \\ + 2(c^3 - 3c^2 + 7c - 5)\ln\left(\cos\frac{\theta}{2}\right) + 2(c^3 + 3c^2 + 3c + 9)\delta_t - \\ - 2(c^3 + 3c)(1-c)\delta_s\Bigg],\end{aligned} \tag{1.175}$$

where $\delta_t(\delta_s)$ is defined by contributions to vacuum polarization $\Pi(t), \Pi(s)$

$$\begin{aligned}\Pi(t) = \frac{\alpha}{\pi}[\delta_t + \frac{1}{3}L_t - \frac{5}{9}] + \frac{\alpha^2}{4\pi^2}L_t, \quad L_t = \ln\frac{-t}{m^2}, \quad Q^2 = -t = 2E^2(1-c), \\ Re\Pi(s) = \frac{\alpha}{\pi}[\delta_s + \frac{1}{3}L_s - \frac{5}{9}] + \frac{\alpha^2}{4\pi^2}L_s, \quad L_s = \ln\frac{4E^2}{m^2}.\end{aligned} \tag{1.176}$$

In the Standard Model δ_t contains contributions of muons, tau-leptons, W bosons, and hadrons:

$$\delta_t = \delta_t^\mu + \delta_t^\tau + \delta_t^W + \delta_t^H, \quad \delta_s = \mathrm{Re}\,\delta_t(Q^2 \to -s), \tag{1.177}$$

and the first three contributions are theoretically calculable:

$$\delta_t^\mu = \frac{1}{3}\ln\frac{Q^2}{m_\mu^2} - \frac{5}{9}, \quad \delta_t^\tau = \frac{1}{2}v(1-\frac{1}{3}v^2)\ln\frac{v+1}{v-1} + \frac{1}{3}v^2 - \frac{8}{9}, \tag{1.178}$$

$$v=\sqrt{1+\frac{4m_\tau^2}{Q^2}}, \quad \delta_t^w = \frac{1}{4}w(w^2-4)\ln\frac{w+1}{w-1} - \frac{1}{2}w^2 + \frac{11}{6}, \quad w=\sqrt{1+\frac{4m_w^2}{Q^2}}.$$

The contribution of the hadrons δ_t^H can be expressed in terms of the experimentally measurable cross section of the e^+e^- annihilation

$$\delta_t^H = \frac{Q^2}{4\pi\alpha^2}\int\limits_{4m_\pi^2}^{\infty}\frac{\sigma^{e^+e^-\to h(x)}}{x+Q^2}dx. \tag{1.179}$$

In the small scattering angle limit we obtain from (1.176):

$$\frac{d\sigma^{(1)}}{dc} = \frac{d\sigma_B}{dc}\frac{1}{(1-\Pi(t))^2}(1+\delta),$$

$$\delta = 2\frac{\alpha}{\pi}\left[2(1-L_t)\ln\frac{1}{\Delta} + \frac{3}{2}L_t - 2\right] + \\ + \frac{\alpha}{\pi}\theta^2\left[\ln\Delta + \frac{3}{16}l^2 + \frac{7}{12}l - \frac{19}{18} + \frac{1}{4}(\delta_t - \delta_s)\right]. \tag{1.180}$$

$$l = \ln\frac{\theta^2}{4}.$$

This representation gives a possibility of verifying explicitly that the terms of relative order θ^2 in RC are really small. Large contribution proportional to $\ln\Delta$ will disappear when adding the cross section of hard-photon emission.

Further simplification follows from the generalized eikonal representation of small-angle scattering amplitude

$$A(s,t) = A_0(s,t)F_1(t)^2\frac{1}{1-\Pi(t)}\,\mathrm{e}^{i\varphi(t)}\left[1 + O\left(\frac{\alpha}{\pi}\frac{Q^2}{s}\right)\right], \quad s \gg Q^2, \tag{1.181}$$

where $F_1(t) = 1 + \frac{\alpha}{\pi}F_1^{(1)}(t) + \left(\frac{\alpha}{\pi}\right)^2 F_1^{(2)}(t) + \ldots$ is the Dirac form factor of the electron, $\varphi(t) = -\alpha\ln(-t/\lambda^2)$ is the Coulomb phase. These arguments permit one to omit the contributions from annihilation channel as well as multiple-photon exchange ones in the scattering channel.

In the lowest order of RC we also have to consider additional hard-photon emission. Single hard-photon emission contribution strongly depends on the experimental setup. The differential cross section has the form [43]

$$\frac{d\sigma_B^{e^+e^-\to e^+e^-\gamma}}{dx_1\, d^2q_1\, dx_2\, d^2q_2} = \frac{2\alpha^3}{\pi^2}\left[\frac{R(x_1;\mathbf{q}_1,\mathbf{q}_2)\delta(1-x_2)}{(\mathbf{q}_2^2)^2(1-\Pi(-\mathbf{q}_2^2))^2}+\right.$$
$$\left.+\frac{R(x_2;\mathbf{q}_2,\mathbf{q}_1)\delta(1-x_1)}{(\mathbf{q}_1^2)^2(1-\Pi(-\mathbf{q}_1^2))^2}\right](1+O(\theta^2)), \quad (1.182)$$

with

$$R(x,\mathbf{q}_1,\mathbf{q}_2) = \frac{1+x^2}{1-x}\left[\frac{\mathbf{q}_2^2(1-x)^2}{d_1 d_2} - \frac{2m^2(1-x)^2 x}{1+x^2}\frac{(d_1-d_2)^2}{(d_1 d_2)^2}\right], \quad (1.183)$$

and

$$d_1 = m^2(1-x)^2 + (\mathbf{q}_1-\mathbf{q}_2)^2 \quad d_2 = m^2(1-x)^2 + (\mathbf{q}_1 - x\mathbf{q}_2)^2, \quad (1.184)$$

here $x_{1,2}, q_{1,2}$ are the energy fractions and the transverse components of the momenta of electron and positron.

The two-loop level RC consists of virtual two-loop vertex function contribution, RC to a single bremsstrahlung amplitude (which we put in the form of sum soft and hard real photon emission parts) and two real photons contribution which as well we will separate as soft and hard parts. The first one can be expressed in the form

$$\frac{d\sigma_{VV}}{dc} = \frac{d\sigma_B}{dc}\left(\frac{\alpha}{\pi}\right)^2 \frac{1}{(1-\Pi(t))^2}[6F_1^{(1)}(t)^2 + 4F_1^{(2)}], \quad (1.185)$$

with

$$F_1^{(1)}(t) = (L_t-1)\ln\frac{\lambda}{m} + \frac{3}{4}L_t - \frac{1}{4}L_t^2 - 1 + \frac{\pi^2}{12}. \quad (1.186)$$

It is convenient to separate the photons and virtual pair intermediate state contributions as $F_1^{(2)} = F^{\gamma\gamma} + F^{e^+e^-}$,

$$F^{\gamma\gamma} = \frac{1}{32}L_t^4 - \frac{3}{16}L_t^3 + (\frac{17}{32} - \frac{1}{8}\xi_2)L_t^2 + (-\frac{21}{32} - \frac{3}{8}\xi_2 + \frac{3}{2}\xi_3)L_t +$$
$$+ \frac{1}{2}(L_t-1)^2\ln^2\frac{m}{\lambda} + (L_t-1)[-\frac{1}{4}L_t^2 + \frac{3}{4}L_t - 1 + \frac{1}{2}\xi_2]\ln\frac{m}{\lambda} + O(1),$$
$$F^{e^+e^-} = -\frac{1}{36}L_t^3 + \frac{19}{72}L_t^2 - (\frac{265}{216} + \frac{1}{6}\xi_2)L_t + \mathcal{O}(1); \quad (1.187)$$
$$\xi_2 = \frac{\pi^2}{6}, \quad \xi_3 = \sum_1^\infty \frac{1}{n^3} \approx 1.202.$$

Cross section of emission of two soft photons, the energy of which does not exceed ΔE, is:

$$d\sigma_{SS} = d\sigma_B \frac{8}{(1-\Pi(t))^2}\left(\frac{\alpha}{\pi}\right)^2 \left[(L_t - 1)\left(\ln\frac{m}{\lambda} + \ln\Delta\right) + \frac{1}{4}L_t^2 - \frac{1}{2}\xi_2\right]^2. \quad (1.188)$$

The contribution of virtual correction to the single soft-photon emission is

$$d\sigma_{SV} = d\sigma_B \frac{16}{(1-\Pi(t))^2}\left(\frac{\alpha}{\pi}\right)^2 F_1^{(1)}(t)\left[(L_t - 1)\left(\ln\frac{m}{\lambda} + \ln\Delta\right) + \frac{1}{4}L_t^2 - \frac{1}{2}\xi_2\right]. \quad (1.189)$$

The contribution from $F_1^{e^+e^-}$ contains cubic on large logarithm term L_t^3 which is cancelled when we take into account the soft-pair production contribution [42]

$$d\sigma_S^{e^+e^-} = d\sigma_B \frac{1}{(1-\Pi(t))^2}\left(\frac{\alpha}{\pi}\right)^2 R_S^{e^+e^-},$$
$$R_S^{e^+e^-} = \frac{1}{18}L_t^3 + \left(\frac{1}{3}\ln\Delta - \frac{5}{18}\right)L_t^2 + \left(\frac{2}{3}\ln^2\Delta - \frac{10}{9}\ln\Delta + \frac{28}{27} - \frac{1}{3}\xi_2\right)L_t + \mathcal{O}(1),$$

here $\Delta = (\epsilon_+ + \epsilon_-)/E \ll 1$, $\epsilon_\pm$ are c.m.s. energies of pair components. As a result, we obtain $R_{S+V}^{e^+e^-} = R_S^{e^+e^-} + 2F_1^{e^+e^-}$:

$$R_{S+V}^{e^+e^-} = L_t^2\left(\frac{1}{3}\ln\Delta + \frac{1}{4}\right) + L_t\left(\frac{2}{3}\ln^2\Delta - \frac{10}{9}\ln\Delta - \frac{2}{3}\xi_2 - \frac{17}{12}\right) + \mathcal{O}(1). \quad (1.190)$$

When evaluating the corrections arising from virtual and real soft photon accompanied by emission of hard ones, we consider two cases. The first one corresponds to emission of the photons by the same fermion, the second one occurs when the hard photon is emitted by another fermion:

$$d\sigma|_{H(S+V)} = d\sigma^{H(S+V)} + d\sigma_{H(S+V)} + d\sigma^H_{(S+V)} + d\sigma_H^{(S+V)}. \quad (1.191)$$

In the case when both photons are emitting we have

$$d\sigma^H_{(S+V)} + d\sigma_H^{(S+V)} = d\sigma_B^{e^+e^-\to e^+e^-\gamma}\frac{2\alpha}{\pi}\left[(L_t - 1)\ln\Delta + \frac{3}{4}L_t - 1\right], \quad (1.192)$$

with $d\sigma_B^{e^+e^-\to e^+e^-\gamma}$ given above. A more complicated expression arises when the same fermion emits virtual, soft, and hard photons. In this case the cross section can be expressed in terms of Compton tensor with heavy photon [18], which describe the subprocess $\gamma^*(q) + e(p_1) \to \gamma(k) + e(q_1)$. In the limit of small-angle photon emission kinematics we have:

$$
\begin{aligned}
& d\sigma^{H(S+V)} = \frac{\alpha^4 dx\, d^2q_1\, d^2q_2}{4x(1-x)\pi^3(\mathbf{q}_2^2)^2}[(B_{11}(s_1,t_1) + 2xB_{12} + x^2B_{11}(t_1,s_1))\rho + T], \\
& T = T_{11}(s_1,t_1) + x^2T_{11}(t_1,s_1) + x(T_{12}(s_1,t_1) + T_{12}(t_1,s_1)), \qquad (1.193) \\
& \rho = 2\left(L_t - \ln\frac{\mathbf{q}_2^2}{-u_1} - 1\right)(2\ln\Delta - \ln x) + 3L_t - \ln^2 x - \frac{9}{2},
\end{aligned}
$$

where $B_{11}(s_1,t_1)$ is the Born tensor component

$$
\begin{gathered}
B_{11}(s_1,t_1) = -4\frac{\mathbf{q}_2^2}{s_1t_1} - \frac{8m^2}{s_1^2}, \quad B_{12} = -\frac{8m^2}{s_1t_1}, \\
s_1 = 2q_1k, \quad t_1 = -2p_1k, \quad u_1 = (p_1-q_1)^2, \quad s_1+t_1+u_1 = -\mathbf{q}_2^2.
\end{gathered} \qquad (1.194)
$$

Quantities T_{11}, T_{12} are finite in zero electron mass limit. Their explicit form can be found in [18].

The double hard-photon bremsstrahlung in opposite directions gives the contribution (see (1.182)):

$$
\frac{d\sigma^{e^+e^-\to(e^+\gamma)(e^-\gamma)}}{dx_1\, d^2q_1\, dx_2\, d^2q_2} = \frac{\alpha^4}{\pi^3}\int \frac{d^2k}{\pi(\mathbf{k}^2)^2}\frac{R(x_1,\mathbf{q}_1,\mathbf{k})R(x_2,\mathbf{q}_2,-\mathbf{k})}{(1-\Pi(-\mathbf{k}^2))^2}. \qquad (1.195)
$$

Now we consider small-angle Bhabha scattering in the frames of the Drell–Yan picture.

Let us introduce the dimensionless quantity $\Sigma = \sigma_{\rm exp}/\sigma_0$, $\sigma_0 = 4\pi\alpha^2/Q_1^2$, $Q_1^2 = E^2\theta_1^2$ with θ_1 — the scattering angle of electron and $\sigma_{\rm exp}$ representing the experimentally observable cross section:

$$
\begin{aligned}
\Sigma = \frac{1}{\sigma_0}\int dx_1 \int dx_2\theta(x_1x_2 - x_c)\times \\
\times \int d^2q_1\theta_1^c \int d^2q_2\theta_2^c \frac{d\sigma^{e^+e^-\to e^+(q_2,x_2)e^-(q_1,x_1)+X}}{dx_1\, d^2q_1\, dx_2\, d^2q_2},
\end{aligned} \qquad (1.196)
$$

where $x_{1,2}, q_{1,2}$ are defined after Eq. (1.184); sx_c is the experimental cut-off on their invariant mass squared and angular cuts are

$$
\theta_1^c = \theta\left(\theta_3 - \frac{|\mathbf{q}_1|}{x_1E}\right)\theta\left(\frac{|\mathbf{q}_1|}{x_1E} - \theta_1\right), \quad \theta_2^c = \theta\left(\theta_4 - \frac{|\mathbf{q}_2|}{x_2E}\right)\theta\left(\frac{|\mathbf{q}_2|}{x_2E} - \theta_2\right), \qquad (1.197)
$$

$\theta(x) = 1$, $x > 0$, $\theta(x) = 0$, $x < 0$ is the step function. In the case of symmetrical angular acceptance (our case below)

$$
\theta_2 = \theta_1, \quad \theta_3 = \theta_4, \quad \rho = \frac{\theta_3}{\theta_1} > 1. \qquad (1.198)
$$

We will present Σ as the sum of various contributions:

$$\Sigma = \Sigma_0 + \Sigma^{\gamma} + \Sigma^{2\gamma} + \Sigma^{e^+e^-} + \Sigma^{3\gamma} + \Sigma^{e^+e^-\gamma}, \qquad (1.199)$$

where $\Sigma_0 = \theta_1^2 \int\limits_{\theta_1^2}^{\theta_3^2} \frac{d\theta^2}{\theta^4}(1-\Pi(t))^{-2} + \Sigma_W + \Sigma_\theta$ stands for the modified Born contribution,

$$\Sigma_W = \theta_1^2 \int\limits_{\theta_1^2}^{\theta_2^2} \frac{dz}{z^2}\left(-\frac{1}{2} + \frac{9z\theta_1^2}{40}\right)(1-\Pi(-zQ_1^2))^{-2},$$

$$\Sigma_W = \theta_1^2 \int\limits_{\theta_1^2}^{\theta_2^2} \frac{dz}{z^2}\delta_{\mathrm{weak}}. \qquad (1.200)$$

In the Born level the contribution from Z boson exchange does not exceed 0.3; Σ^{γ} is one-photon contribution (real and virtual) and so on. Explicit calculations (see details in [44]) give:

$$\Sigma^{\gamma} = \frac{\alpha}{\pi} \int\limits_{1}^{\rho^2} \frac{dz}{z^2} \int\limits_{x_c}^{1} dx (1-\Pi(-zQ_1^2))^{-2} \times$$
$$\times \left[(L-1)P(x)[1+\theta((x\rho)^2 - z)] + \frac{1+x^2}{1-x}k(x,z) - \delta(x-1)\right], \qquad (1.201)$$

$$k(x,z) = \frac{(1-x)^2}{1+x^2}[1+\theta((x\rho)^2-z)] + L_1 + \theta((x\rho)^2 - z)L_2 + \theta(z-(x\rho)^2)L_3,$$

with

$$L_1 = \ln\left|\frac{x^2(z-1)(\rho^2-z)}{(x-z)(x\rho^2-z)}\right|,$$
$$L_2 = \ln\left|\frac{(z-x^2)(x^2\rho^2-z)}{x^2(x-z)(x\rho^2-z)}\right|, \qquad (1.202)$$
$$L_3 = \ln\left|\frac{(z-x^2)(x\rho^2-z)}{(x-z)(x^2\rho^2-z)}\right|,$$

and $P(x)$ is the kernel of evolution equation for nonsinglet structure function:

$$P(x) = \left(\frac{1+x^2}{1-x}\right)_+ =$$
$$= \lim_{\Delta\to 0}\left[\frac{1+x^2}{1-x}\theta(1-x-\Delta) + \left(\frac{3}{2} + 2\ln\Delta\right)\delta(1-x)\right]. \qquad (1.203)$$

The quantity $\Sigma^{2\gamma}$ collects both virtual and real two-photon emission contributions. It can be put in the form:

$$\Sigma^{2\gamma} = \Sigma^{\gamma\gamma} + \Sigma_{\gamma}^{\gamma} + \left(\frac{\alpha}{\pi}\right)^2 \varphi^{\gamma\gamma} \ln\frac{Q_1^2}{m^2}, \tag{1.204}$$

where

$$\Sigma^{\gamma\gamma} = \frac{1}{2}\left(\frac{\alpha}{\pi}\right)^2 \int\limits_1^{\rho^2} \frac{dz}{z^2(1-\Pi(-zQ_1^2))^2} L_t^2 \int\limits_{x_c}^{1} dx \times$$
$$\times \left[\frac{1}{2}P^{(2)}(x)[\theta((x\rho)^2 - z) + 1] + \int\limits_x^1 \frac{dt}{t} P(t) P\left(\frac{x}{t}\right) \theta((t\rho)^2 - z)\right], \tag{1.205}$$

with

$$P^{(2)}(x) = \int\limits_x^1 \frac{dt}{t} P(t) P(\frac{x}{t}) = \lim_{\Delta\to 0}\left[\left[\left(2\ln\Delta + \frac{3}{2}\right)^2 - 4\xi_2\right]\delta(1-x) + \right.$$
$$\left. +2\left[\frac{1+x^2}{1-x}\left(2\ln(1-x) - \ln x + \frac{3}{2}\right) + \frac{1}{2}(1+x)\ln x - 1 + x\right] \times \theta(1-x-\Delta)\right]; \tag{1.206}$$

and

$$\Sigma_{\gamma}^{\gamma} = \frac{1}{4}\left(\frac{\alpha}{\pi}\right)^2 \int\limits_0^{\infty} \frac{dz}{z^2} L_t^2 (1-\Pi(-zQ_1^2))^{-2} \int\limits_{x_c}^{1} dx_1 \int\limits_{x_c/x_1}^{1} dx_2 \times \tag{1.207}$$

$$\times P(x_1)P(x_2)[\theta(z-1)\theta(\rho^2 - z) + \theta(z - x_1^2)\theta(x_1^2\rho^2 - z)] \times$$
$$\times [\theta(z-1)\theta(\rho^2 - z) + \theta(z - x_2^2)\theta(x_2^2\rho^2 - z)]. \tag{1.208}$$

We see that the leading contributions to $\Sigma^{2\gamma}$ can be expressed in terms of kernels of evolution equation for structure functions.

The function $\varphi^{\gamma\gamma}$ collects the next-to-leading contributions which cannot be obtained by the structure functions method (see [58]). Its form can be deduced from comparison of the calculation with logarithmical accuracy performed above with the structure functions approach.

Contribution from pairs production (we restrict ourselves by consideration of only e^+e^- pairs as well muon or pion pairs contribute too small) consists of the mentioned above virtual, soft pairs, and the hard pair creation. Restricting only by leading terms we have [45]:

$$\Sigma^{e^+e^-} = \frac{1}{2}\left(\frac{\alpha}{\pi}\right)^2 \int\limits_1^{\rho^2} \frac{dz}{z^2} L_t^2 \left[1 + \frac{4}{3}\ln(1-x_c) - \frac{2}{3}\int\limits_{x_c}^{1} \frac{dx}{1-x}[1-\theta((x\rho)^2 - z)] + \right.$$
$$\left. + \int\limits_{x_c}^{1} dx \left[\frac{1-x}{6x}(4 + 7x + 4x^2) + (1+x)\left(-\frac{1}{3} + \ln x\right)\right] + O\left(\ln\frac{Q_1^2}{m^2}\right)\right].$$

Let us consider the contributions of the order $(\alpha L)^3$. The relevant iteration of master equations leads to

$$\Sigma^{3\gamma} = \frac{1}{4}\left(\frac{\alpha}{\pi}\ln\frac{Q_1^2}{m^2}\right)^3 \int\limits_1^{\rho^2}\frac{dz}{z^2}\int\limits_{x_c}^1 dx_1 \int\limits_{x_c}^1 dx_2 \theta(x_1x_2 - x_c)\left[\frac{1}{6}\delta(1-x_2)\times\right.$$

$$\left.\times P^{(3)}(x_1)\theta(x_1^2\rho^2 - z) + \frac{1}{2x_1^2}P^{(2)}(x_1)P(x_2)\theta\left(z - \frac{x_2^2}{x_1^2}\right)\theta\left(\frac{x_2^2}{x_1^2}\rho^2 - z\right)\right](1+O(x_c^3)),$$

with $P^{(3)}(x) = \int\limits_x^1 \frac{dt}{t}P^{(2)}(t)P\left(\frac{x}{t}\right)$. In the similar way we obtain:

$$\Sigma^{e^+e^-\gamma} = \frac{1}{4}\left(\frac{\alpha}{\pi}\ln\frac{Q_1^2}{m^2}\right)^3 \int\limits_1^{\rho^2}\frac{dz}{z^2}\int\limits_{x_c}^1 dx_1 \int\limits_{x_c}^1 dx_2 \times \theta(x_1x_2 - x_c)\times$$

$$\times\left[\frac{1}{3}\left[\left(R^P(x_1) + \frac{1}{3}P^{(2)}(x_1) + \frac{2}{3}R(x)\right)\delta(1-x_2)\theta((x_1\rho)^2 - z) + \right.\right.$$

$$\left.\left. + \frac{1}{2x_1^2}P(x_2)R(x_1)\theta\left(z - \frac{x_2^2}{x_1^2}\right)\theta\left(\frac{x_2^2}{x_1^2}\rho^2 - z\right)\right]\right], \quad (1.209)$$

with

$$\begin{aligned}
R(x) &= \frac{2}{3}P(x) + R^s(x),\\
R^s(x) &= \frac{1-x}{3x}(4 + 7x + 4x^2) + 2(1+x)\ln x,\\
R^P(x) &= \left(\frac{3}{2} + 2\ln(1-x)\right)R^s(x) + \frac{2}{3}P^{(2)}(x) + \\
&\quad + (1+x)(-\ln^2 x + 4\mathrm{Li}_2(1-x)) + \frac{1}{3}(-9 - 3x + 8x^2) + \\
&\quad + \frac{2}{3}\left(-\frac{3}{x} - 8 + 8x + 3x^2\right). \qquad (1.210)
\end{aligned}$$

1.5. QED Sum Rules. Due to analyticity of the amplitudes some relations between the high-energy asymptotics of the cross sections of inelastic processes in e^+e^- collisions and higher order perturbative contributions to the electron Dirac and Pauli form factors, can be derived. In particular, the total cross sections turn out to be related to the slope of the Dirac form factor at zero momentum transfer [20]. Applying the similar reasons to electron–proton scattering the photoproduction cross sections on nucleons and deuteron can be expressed in terms of radii and their anomalous magnetic momentum [12, 46].

Let us consider forward Compton scattering of virtual photon on some charged particle. For definiteness we believe it will be an electron on the mass shell:

$$\gamma^*(q,\mu) + e(p_1) \to \gamma^*(q,\nu) + e(p_1), \quad q_0 > 0, \quad q^2 < 0, \quad p_1^2 = m^2. \tag{1.211}$$

Among Feynman diagrams let us choose only those (nonsinglet ones) where the world line of charged particle contains both vertices of absorption and emission of virtual photon, besides, the absorption vertex is situated before the emission one when moving along charged particle world line. It contains in general arbitrary number of vertices of emission of virtual photons and fermion closed loops which corresponds to higher orders of QED PT contributions.

The relevant part of Compton tensor $R_{\mu\nu}$ is not in general gauge invariant $R_{\mu\nu}q_\mu \neq 0$; $R_{\mu\nu}q_\nu \neq 0$, but it indeed does not depend on the choice of virtual photon Green functions as well as the complete set of all possible Feynman diagrams in the given order of PT is implied to be considered.

We introduce further light-cone vector $P, P_0 > 0$, $P^2 = 0$, and build another light-cone vector $p = p_1 - P(m^2/s)$, $p^2 = 0, s = 2Pp_1$. Consider now the light-cone projection of tensor R in the limit of large values of scalar product of these light-cone vectors:

$$J(s_1, \mathbf{q}) = \lim_{s\to\infty} \frac{1}{s^2} R_{\mu\nu} P_\mu P_\nu. \tag{1.212}$$

Consider now the contour integral [20]

$$I(\mathbf{q}) = \int\limits_C ds_1 J(s_1, \mathbf{q}), \tag{1.213}$$

where the integration contour in s_1 is situated below the real axes at negative values of s_1, intersects the real axes in the region $s_1 = 0$ and then lies along real axes having small imaginary part. The singularities of light-cone projection $J(s_1, \mathbf{q})$ are situated on the real axes of physical sheet: the part of Riemann surface restricted by left and right cuts in s_1 plane.

Let us discuss the kinds of singularities of $J(s_1, \mathbf{q})$. Set of Feynman (connected) diagrams with one-particle intermediate state in s_1 channel $(p_1+q)^2 = m^2$ produces a pole situated at $s_1 = -\mathbf{q}^2$. Cuts corresponding to inelastic intermediate states such as electron and photon, electron and an additional pair particle and antiparticle produce the cuts situated at positive s_1, corresponding to threshold of inelastic processes $(p_1 + q)^2 = (m + \lambda)^2; (m + 2M)^2$ with λ, M — fictitious photon mass and mass of a particle from a pair.

The general arguments used in proving the dispersion relation is that all singularities are situated on the real axes. They have a physical interpretation as a poles and thresholds of concrete reactions. In particular for Compton scattering on hadrons such singularities as baryon and meson resonances correspond to the poles situated on the second physical sheets beyond the real axes, so they do not contribute to sum rules considered here.

Left cut singularities of $J(s_1, \mathbf{q})$ as well can be associated with physical processes, but with rather nontrivial interpretation, which will be touched later.

Sum rule appears when we calculate the value $I(\mathbf{q})$ for the case, when the contour is closed to left cut singularities and to the right one including pole contribution, and imply the zero contribution of large circle and convergence of s_1 integral.

First, we note that cuts contributions can be expressed in terms of cross sections of physical processes. So the relevant contribution to the Compton tensor is gauge invariant

$$R_{\mu\nu}^{\rm cut} q_\mu = R_{\mu\nu}^{\rm cut} q_\nu = 0. \tag{1.214}$$

Using Sudakov parameterization $q = \alpha_q P + \beta_q p + q_\perp, q_\perp P = q_\perp p = 0$ and the fact that main components of tensor $R^{\rm cut}$ are ones along 4-vector p, we obtain from gauge condition

$$q_\mu R_{\mu\nu}^{\rm cut} = (\alpha_q P + q_\perp)_\mu R_{\mu\nu}^{\rm cut} = 0.$$

For light-cone projection $J(\mathbf{q})$ we have

$$\frac{1}{s^2} R_{\mu\nu} P_\mu P_\nu = \frac{1}{s_1^2} \mathbf{q}^2 e_i e_j R_{ij}.$$

Here we use expression of invariant mass square s_1 in terms of Sudakov parameters $s_1 = 2qp_1 = s\alpha_q$, $q^2 \approx q_\perp^2 = -\mathbf{q}^2 < 0$ and introduce two-dimensional polarization vectors of virtual photon $\mathbf{e} = \mathbf{q}/|\mathbf{q}|$. Discontinuity of amplitude on the right cut, i.e., the difference of its values on opposite sides of right cut equals to the double imaginary part of amplitude, which can be expressed in terms of total cross section with the help of the optical theorem.

1.5.1. Electron Target. Consider now the zero angle elastic scattering of some charged particle on electron with large total energy $\sqrt{s}$ in the center-of-mass frame:

$$e^\pm(p_2) + e(p_1) \to e^\pm(p_2) + e(p_1), \quad s = (p_2 + p_1)^2 \gg m_i^2, \tag{1.215}$$

where m_i is the typical mass of particles in the process. The nontrivial consequences can be extracted from amplitude corresponding to two virtual photon exchange between projectile and target particle. Amplitude turns out to be almost completely imaginary and proportional to s. It can be expressed in terms of conversion of projectile tensor P, which we will suppose to be written in the Born approximation with the target ones T which is supposed to be a series in QED coupling constant α.

Main contribution to amplitude $A^{\mathrm{elast}}(s) \sim \int d^4q/(q^2)^2 P_{\mu\nu}(p_2) T_{\mu_1\nu_1}(p_1) \times$ $\times g_{\mu\mu_1} g_{\nu\nu_1}$ arises from the so-called «nonsense» components of metrics tensors

$$g_{\mu\mu_1} g_{\nu\nu_1} \sim \frac{2}{s} p_{2\nu_1} p_{1\nu} \frac{2}{s} p_{2\mu_1} p_{1\mu}. \tag{1.216}$$

Expressing loop momentum phase volume as $d^4q = (s/2) d\alpha_q d\beta_q d^2q_\perp =$ $= ds_1 ds_2 d^2\mathbf{q}/(2s)$ and performing the integration on projectile $e^\pm$ invariant mass squared $s_2 = s\beta_q$ and applying optical theorem, we can express the imaginary part of zero angles scattering amplitude in terms of differential cross section of inelastic scattering projectile on target:

$$\frac{d\sigma}{d^2q} = \frac{\alpha}{\pi^2(\mathbf{q}^2)^2} \int ds_1 \frac{1}{s^2} p_{2\mu} p_{2\nu} T_{\mu\nu}. \tag{1.217}$$

In the third order of PT considering projectile and target to be positron and electron, respectively, and taking into account the lowest order radiative corrections to electron part of amplitude, we can work with flat Feynman diagrams. It is known that the corresponding amplitudes do not have left-cut singularities. In this case sum rule has the form:

$$\frac{d\sigma_B - d\sigma^{\mathrm{el}}}{d\mathbf{q}^2} = \frac{d\sigma^\gamma}{d\mathbf{q}^2}, \tag{1.218}$$

where $d\sigma_B$ is the electron–positron elastic scattering differential cross section calculated with structureless leptons; $d\sigma^{\mathrm{el}}$ — cross section with electron form factors taken into account, and $d\sigma^\gamma$ is the electron–positron scattering cross section with additional photon emission. The relevant vertex function has the form:

$$\begin{gathered} \Gamma_\mu = F_1(q^2)\gamma_\mu + \frac{1}{4m} F_2(q^2)[\gamma_\mu, \hat{q}], \\ F_1 = 1 + \frac{\alpha}{\pi} F_1^{(2)} + \ldots, \quad F_2 = \frac{\alpha}{\pi} F_2^{(2)} + \ldots \end{gathered} \tag{1.219}$$

The left-hand side of the sum rule can be expressed in terms of Dirac form factor:

$$\frac{d\sigma_B - d\sigma^{\mathrm{el}}}{d^2q} = -8\alpha^3 F_1^{(2)}(t) \frac{1}{t^2}, \quad t = -\mathbf{q}^2. \tag{1.220}$$

Contribution of Pauli form factor is absent in this order of PT. The lowest order contribution to the Dirac form factor is [7, 8]:

$$F_1^{(2)}(t) = \left(1 + \frac{1+\theta^2}{1-\theta^2}\ln\theta\right)\ln\frac{m}{\lambda} - \frac{3+2\theta+3\theta^2}{4(1-\theta^2)}\ln\theta + \\ + \frac{1+\theta^2}{1-\theta^2}\left(\frac{1}{12}\pi^2 - \frac{1}{4}\ln\theta^2 + \ln\theta\ln(1-\theta) + \mathrm{Li}_2(-\theta)\right), \quad (1.221)$$

$$\theta = -\frac{1-\sqrt{1-4m^2/t}}{1+\sqrt{1-4m^2/t}}.$$

Right-hand part of Eq. (1.218) can be described in terms of real photon emission by electron cross section. It is convenient to distinguish that the emission of soft photon with energy fraction does not exceed some small quantity $\omega/E < \eta \ll 1$ and hard-photon $\eta \ll \omega/E < 1$ emission: $d\sigma^\gamma = d\sigma^\gamma_{\mathrm{soft}} + d\sigma^\gamma_{\mathrm{hard}}$.

For soft-photon contribution one has:

$$\frac{d\sigma^\gamma_{\mathrm{soft}}}{d\mathbf{q}^2} = -\frac{4\alpha^3}{t^2}I(y), \quad \sinh y = \sqrt{\frac{-t}{4m^2}},$$

$$I(y) = 2(1-2y\coth y)\ln\frac{m\eta}{\lambda} + \coth(2y)\Bigg[-y\ln(4\cosh^2 y) + \\ + \mathrm{Li}_2\left(\frac{1+\tanh y}{2}\right) - \mathrm{Li}_2\left(\frac{1-\tanh y}{2}\right)\Bigg]. \quad (1.222)$$

Hard-photon emission contribution has the form [32]:

$$\frac{d\sigma^\gamma_{\mathrm{hard}}}{d\mathbf{q}^2} = \frac{8\alpha^3}{t^2}\int_\eta^1 dx f(x,z), \quad z = \sqrt{\frac{\mathbf{q}^2}{4m^2}},$$

$$f(x,z) = \left(\frac{1}{x}-1\right)\left[-1 + \frac{1+2z^2}{2\sqrt{1+z}}\ln(z+\sqrt{1+z})\right] + x\frac{z}{\sqrt{1+z}}\ln(z+\sqrt{1+z}). \quad (1.223)$$

Performing algebraic transformations one can be convinced in validity of the lowest order sum rule given above.

The similar reasons (absence of the left-hand cut) can be applied to the sum rule connecting the cross sections of $e\bar{e} \to e\mu\bar{\mu}\bar{e}$. Muon-pair production with the relevant contribution (which takes into account the vacuum polarization due to muon–photon self-energy insertion) is connected to the slope of Dirac form factor of electron. Really in the Weizsacker–Williams approximation we have [37]:

$$d\sigma^{e\bar{e}\to e\mu\bar{\mu}\bar{e}} = \frac{2\alpha^2 r_0^2}{\pi}\ln\left(\frac{s}{m_e^2}\right)\left(\frac{77}{18}\xi_2 - \frac{1099}{162}\right), \quad \xi_2 = \frac{\pi^2}{6}. \quad (1.224)$$

This result must be compared with the one obtained in [47, 48]:

$$m_e^2 F_1^{p'}(0) = \lim_{s\to\infty} \frac{\sigma(s)}{16 r_0^2 \pi \ln(s/m_e^2)} = \frac{\alpha^2}{8\pi^2}\left(\frac{77}{18}\xi_2 - \frac{1099}{162}\right). \qquad (1.225)$$

Several similar QED sum rules connecting the inelastic cross sections in electron–positron scattering with the higher contributions to electron form factors were considered in [4]. Considering the processes in the fourth order of PT such as production of lepton–antilepton pairs at electron–positron periphery collisions and double bremsstrahlung processes, the left-cut contributions to the sum rule become important. For the case of pair productions it can be interpreted as contribution to the cross section arising from taking into account identity of produced lepton with the initial one.

Contribution to the sum rules from large circle in s_1 integration as a rule is zero due to convergence of integral $\int (ds_1/s_1)\sigma(s_1)$, which is valid for decreasing cross sections $\sigma(s_1)$. In the case of nondecreasing cross sections, a linear combination of different sum rules in which the corresponding divergent terms are cancelled can be constructed.

1.5.2. Nuclon Target. Starting from very high energy inelastic electron–nucleon scattering with a production of a hadronic state X moving closely to the direction of the initial nucleon, then utilizing analytic properties of parts of forward virtual Compton scattering amplitudes on proton and neutron, one obtains the relation between nucleon form factors and a difference of proton and neutron differential electroproduction cross sections. In particular, for the case of small transferred momenta, one finally derives sum rule, relating Dirac proton mean square radius and anomalous magnetic moments of proton and neutron to the integral over a difference of the total proton and neutron photoproduction cross sections [53–55].

At the end of sixties of the last century, Kurt Gottfried, by consideration of the very high-energy electron–proton scattering and the nonrelativistic quark model of hadrons, has found [49] a sum rule relating to the proton mean square charge radius $\langle r_{Ep}^2\rangle$ and the proton magnetic moment $\mu_p = 1 + \kappa_p$ to the integral over the total proton photoproduction cross section $\sigma_{\rm tot}^{\gamma \rm p}(\nu)$ in the form

$$\int\limits_0^\infty \frac{d\nu}{\nu}\sigma_{\rm tot}^{\gamma \rm p}(\nu) = \frac{\pi^2\alpha}{m_p^2}\left[\frac{4}{3} m_p^2 \langle r_{Ep}^2\rangle + 1 - \mu_p^2\right], \qquad (1.226)$$

where ν is the photon energy in the laboratory frame; α is the fine structure constant, and m_p is the proton mass.

Nowadays it is well known, that the Gottfried sum rule cannot be satisfied since the corresponding integral diverges due to the known rise of the total proton photoproduction cross section at high energies.

In this Subsection by means of a distinct way from the Gottfried approach and considering the nucleon isodoublet (proton and neutron) simultaneously, a new sum rule is derived, which relates Dirac proton mean square radius and anomalous magnetic moments of proton and neutron to the integral over a difference of the total proton and neutron photoproduction cross sections. Thus the rise of both photoproduction cross sections at high energies mutually cancels and the corresponding integral converges.

In derivation of the new sum rule we start also with consideration of the very high-energy electron–nucleon scattering

$$e^-(p_1) + N(p) \to e^-(p_1') + X, \tag{1.227}$$

with production of a hadronic state X moving closely to the direction of initial nucleon. The corresponding one-photon exchange approximation matrix element takes the form

$$M = i\frac{\sqrt{4\pi\alpha}}{q^2}\bar{u}(p_1')\gamma_\mu u(p_1)\langle X \mid J_\nu^{\rm QED} \mid N^{(r)}\rangle g^{\mu\nu}, \tag{1.228}$$

where (r) means a spin state of the nucleon.

The Gribov representation of the metric tensor in the photon propagator of (1.228) is

$$g_{\mu\nu} = g_{\mu\nu}^{\perp} + \frac{2}{s}\left(\tilde{p}_\mu \tilde{p}_{1\nu} + \tilde{p}_\nu \tilde{p}_{1\mu}\right) \approx \frac{2}{s}\tilde{p}_\mu \tilde{p}_{1\nu}, \quad s = (p_1 + p)^2,$$

where

$$\tilde{p}_1 = p_1 - \frac{m_e^2 p}{2p_1 p}, \quad \tilde{p} = p - \frac{m_N^2 p_1}{2p_1 p}$$

are almost light-like vectors. According to the Sudakov expansion of the virtual photon transferred four-momentum $q = p_1 - p_1'$

$$q = \beta_q \tilde{p}_1 + \alpha_q \tilde{p} + q_\perp, \quad q_\perp = (0, 0, \mathbf{q}), \quad \tilde{p}q_\perp = \tilde{p}_1 q_\perp = 0, \quad q_\perp^2 = -\mathbf{q}^2, \tag{1.229}$$

for the corresponding cross section one obtains:

$$d\sigma = \frac{4\pi\alpha}{s(q^2)^2} p_1^\mu p_1^\nu \sum_{X \neq N} \sum_{r=-1/2}^{1/2} \langle N^{(r)} \mid J_\mu^{+\rm QED} \mid X\rangle \langle X \mid J_\nu^{\rm QED} \mid N^{(r)}\rangle d\Gamma, \tag{1.230}$$

where summations through the created hadronic states X and the spin states of the initial nucleon are carried out and $d\Gamma$ denotes the final state phase space volume. Further, approximating square momentum of virtual photon

$$q^2 \approx -\left[\mathbf{q}^2 + \left(\frac{m_e s_1}{s}\right)^2\right]$$

with

$$s_1 = 2(qp) = s\beta_q,$$

i.e., it is related to the invariant square mass of the final hadronic state by the relation

$$m_X^2 = s_1 + q^2 + m_N^2, \tag{1.231}$$

and transforming the phase space volume of the final electron into the form

$$\frac{1}{(2\pi)^3}\frac{d^3p_1'}{2\epsilon_1'} = \frac{1}{(2\pi)^3}d^4q\delta[(p_1-q)^2] = \frac{1}{(2\pi)^3}\frac{ds_1}{2s}d^2q_\perp,$$

one gets the final state phase-space volume in the form

$$d\Gamma = \frac{ds_1}{2s(2\pi)^3}d^2q_\perp d\Gamma_X, \quad d\Gamma_X = (2\pi)^4\delta^4\left(q+p-\sum_j p_j\right)\prod_j \frac{d^3p_j}{2\varepsilon_j(2\pi)^3}. \tag{1.232}$$

Besides, using the current conservation condition

$$q^\mu\langle X \mid J_\mu^{\mathrm{QED}} \mid N^{(r)}\rangle \approx (\beta_q p_1 + q_\perp)^\mu \langle X \mid J_\mu^{\mathrm{QED}} \mid N^{(r)}\rangle = 0 \tag{1.233}$$

($\alpha_q\tilde{p}$ gives a negligible contribution) one can write

$$\int p_1^\mu p_1^\nu \sum_{X\neq N}\sum_{r=-1/2}^{1/2}\langle N^{(r)} \mid J_\mu^{+\mathrm{QED}} \mid X\rangle\langle X \mid J_\nu^{\mathrm{QED}} \mid N^{(r)}\rangle d\Gamma_X = p_1^\mu p_1^\nu \Delta\tilde{A}_{\mu\nu}^{(N)} =$$
$$= \frac{s^2}{s_1^2}\mathbf{q}^2 e^i e^j \Delta\tilde{A}_{ij}^{(N)} = 2i\frac{s^2}{s_1^2}\mathbf{q}^2 \mathrm{Im}\,\tilde{A}^{(N)}(s_1,\mathbf{q}), \quad e_i = \frac{\mathbf{q}_i}{|\mathbf{q}|}, \tag{1.234}$$

where just Cutkosky rule for s-channel discontinuity $\Delta\tilde{A}^{(N)} = 2i\mathrm{Im}\,\tilde{A}^{(N)}$ of the corresponding Feynman amplitude was applied.

Here we would like to note that the amplitude $\tilde{A}(s_1,\mathbf{q})$ by a construction is only a part of the total forward virtual Compton scattering amplitude $A(s_1,\mathbf{q})$, which doesn't contain (unlike the amplitude $A(s_1,\mathbf{q})$) any crossing Feynman diagram contributions. As a result there is no u-channel pole in $\tilde{A}(s_1,\mathbf{q})$, which is a crucial point in a derivation of the new sum rule by using the analytic properties of the latter amplitude.

Since the imaginary part of the crossing Feynman diagrams is starting to be different from zero only above

$$s_1^{(3N)} = 8m_N^2 + \mathbf{q}^2,$$

one can write down an equality relation

$$\mathrm{Im}\,\tilde{A}(s_1,\mathbf{q}) = \mathrm{Im}\,A(s_1,\mathbf{q}) = 4s_1\sigma_{\mathrm{tot}}^{\gamma^*\mathrm{p}\to\mathrm{X}}(s_1,\mathbf{q}) \tag{1.235}$$

for $2m_N m_\pi + m_\pi^2 + \mathbf{q}^2 \leqslant s_1 \leqslant 8m_N^2 + \mathbf{q}^2$. Fortunately, above the threshold, total proton and neutron photoproduction cross sections are almost equal and at the new sum rule one can integrate over them up to infinity.

Using expression (1.230) and integrating over the phase-space volume of the final hadronic state X, as well as over the invariant mass squared m_X^2, i.e., over the variable s_1 (see (1.231)), for a difference of corresponding differential proton and neutron electroproduction cross sections one finds

$$\left(\frac{d\sigma^{\mathrm{e^- p\to e^- X}}(s,\mathbf{q})}{d^2\mathbf{q}} - \frac{d\sigma^{\mathrm{e^- n\to e^- X}}(s,\mathbf{q})}{d^2\mathbf{q}}\right) = \frac{\alpha\mathbf{q}^2}{4\pi^2}\times$$
$$\times \int\limits_{2m_N m_\pi + m_\pi^2 + \mathbf{q}^2}^{\infty} \frac{ds_1}{s_1^2[\mathbf{q}^2 + (m_e s_1/s)^2]^2} \times \left(\mathrm{Im}\,\tilde{A}^{(p)}(s_1,\mathbf{q}) - \mathrm{Im}\,\tilde{A}^{(n)}(s_1,\mathbf{q})\right). \tag{1.236}$$

If one neglects the second term in square brackets of the denominator of the integral (1.236) (due to the small value of m_e and high s in comparison with s_1) and takes into account (1.235) for $\mathbf{q}^2 \to 0$ together with the relation $d^2\mathbf{q} = \pi d\mathbf{q}^2$, one comes to the expression

$$\mathbf{q}^2 \left(\frac{d\sigma^{\mathrm{e^- p\to e^- X}}}{d\mathbf{q}^2} - \frac{d\sigma^{\mathrm{e^- n\to e^- X}}}{d\mathbf{q}^2}\right)\Bigg|_{\mathbf{q}^2\to 0} =$$
$$= \frac{\alpha}{\pi} \int\limits_{2m_N m_\pi + m_\pi^2}^{\infty} \frac{ds_1}{s_1}\left(\sigma_{\mathrm{tot}}^{\gamma\mathrm{p\to X}}(s_1) - \sigma_{\mathrm{tot}}^{\gamma\mathrm{n\to X}}(s_1)\right) \tag{1.237}$$

similar to the difference of the total cross sections of the electroproduction processes on proton and neutron in the Weizsacker–Williams approximation

$$\sigma_{\mathrm{tot}}^{\mathrm{e^- p\to e^- X}}(s) - \sigma_{\mathrm{tot}}^{\mathrm{e^- n\to e^- X}}(s) = 2\frac{\alpha}{\pi}\ln\left(\frac{s}{m_e m_\pi}\right)\times$$
$$\times \int\limits_{2m_N m_\pi + m_\pi^2}^{\infty} \frac{ds_1}{s_1}\left(\sigma_{\mathrm{tot}}^{\gamma\mathrm{p\to X}}(s_1) - \sigma_{\mathrm{tot}}^{\gamma\mathrm{n\to X}}(s_1)\right). \tag{1.238}$$

The analytic properties of the amplitude $\tilde{A}(s_1,\mathbf{q})$ in s_1 plane are consisting of the one-nucleon intermediate state pole at $s_1 = \mathbf{q}^2$, the right-hand cut starting at the pion–nucleon threshold $s_1 = 2m_N m_\pi + m_\pi^2 + \mathbf{q}^2$ and the left-hand cut starting from $s_1 = -\mathbf{q}^2 - 8m_N^2$.

If one defines the path integral I in s_1 plane as presented in Fig. 4, a

$$I = \int\limits_C ds_1 \frac{p_1^\mu p_1^\nu}{s^2} \left(\tilde{A}_{\mu\nu}^{(p)}(s_1, \mathbf{q}) - \tilde{A}_{\mu\nu}^{(n)}(s_1, \mathbf{q}) \right), \tag{1.239}$$

then once the contour C is closed to upper half-plane and the other one to lower half-plane (Fig. 4, b), the following sum rule

$$\pi \left(\mathrm{Res}^{(n)} - \mathrm{Res}^{(p)} \right) = \mathbf{q}^2 \int\limits_{\mathrm{r.h}}^{\infty} \frac{ds_1}{s_1^2} \left(\mathrm{Im}\, \tilde{A}^{(p)}(s_1, \mathbf{q}) - \mathrm{Im}\, \tilde{A}^{(n)}(s_1, \mathbf{q}) \right) \tag{1.240}$$

appears with (an averaging through the initial nucleon and photon spins is performed)

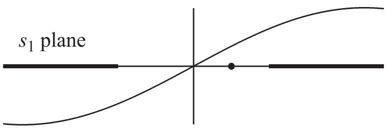
$$\mathrm{Res}^{(N)} = 2\pi\alpha \left(F_{1N}^2 + \frac{\mathbf{q}^2}{4m_N^2} F_{2N}^2 \right) \tag{1.241}$$

to be the one-nucleon intermediate state pole contribution and the left-hand (l.h.) cut contributions from the difference $\left(\mathrm{Im}\, \tilde{A}^{(p)} - \mathrm{Im}\, \tilde{A}^{(n)} \right)$ are mutually annulated.

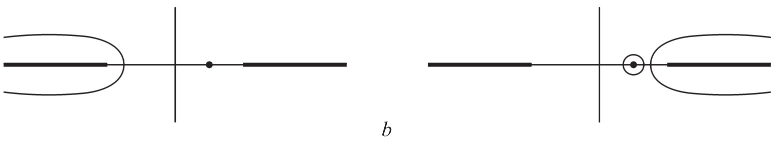
Substituting (1.241) into (1.240) and taking into account (1.236) one comes to the relation

$$F_{1n}^2(-\mathbf{q}^2) + \frac{\mathbf{q}^2}{4m_n^2} F_{2n}^2(-\mathbf{q}^2) F_{1p}^2(-\mathbf{q}^2) - \frac{\mathbf{q}^2}{4m_p^2} F_{2p}^2(-\mathbf{q}^2) =$$
$$= 2\frac{(\mathbf{q}^2)^2}{\pi\alpha^2} \left(\frac{d\sigma^{\mathrm{e^- p \to e^- X}}}{d\mathbf{q}^2} - \frac{d\sigma^{\mathrm{e^- n \to e^- X}}}{d\mathbf{q}^2} \right). \tag{1.242}$$

For $\mathbf{q}^2 = 0$ the right-hand side is equal to zero, but the left-hand side is -1. This is caused by the following reasons. On the right-hand side we take into



a



b

Fig. 4. Sum rule interpretation in s_1 plane. On plot a is drawn the contour C, on the plot b is the contour C closed to the upper and lower half-plane

account only strong interaction effects and on the left-hand side the -1 is given by nonzero proton charge, i.e., pure electromagnetic effect. In order to separate the pure strong interactions from electromagnetic ones on the left-hand side of the sum rule, one has to regularize the latter by adding $+1$ in order to achieve the zero also there. As a result the sum rule takes the final form:

$$1 + F_{1n}^2(-\mathbf{q}^2) + \frac{\mathbf{q}^2}{4m_n^2} F_{2n}^2(-\mathbf{q}^2) - F_{1p}^2(-\mathbf{q}^2) - \frac{\mathbf{q}^2}{4m_p^2} F_{2p}^2(-\mathbf{q}^2) = \\ = 2\frac{(\mathbf{q}^2)^2}{\pi\alpha^2} \left(\frac{d\sigma^{\mathrm{e^- p \to e^- X}}}{d\mathbf{q}^2} - \frac{d\sigma^{\mathrm{e^- n \to e^- X}}}{d\mathbf{q}^2} \right), \quad (1.243)$$

giving into a relation nucleon electromagnetic form factors with a difference of the differential cross sections of deep inelastic electron–proton scattering. There is a challenge to specialized experimental groups to verify the sum rule (1.243).

Now, taking a derivative of both sides in (1.243) according to $\mathbf{q}^2$ for $\mathbf{q}^2 \to 0$ and employing relation (1.237), one comes to the new sum rule relating Dirac proton mean square radius and anomalous magnetic moments of proton and neutron

$$\langle r_{1p}^2 \rangle = 6 \frac{d}{dq^2} F_{1p}(q^2) \Big|_{q^2=0}, \quad \kappa_N = F_{2N}(q^2) \Big|_{q^2=0} \quad (1.244)$$

to the integral over a difference of the total proton and neutron photoproduction cross sections (we used for laboratory frame $s_1 = 2M_N\omega$)

$$\frac{1}{3}\langle r_{1p}^2 \rangle - \frac{\kappa_p^2}{4m_p^2} + \frac{\kappa_n^2}{4m_n^2} = \frac{2}{\pi^2\alpha} \int\limits_{\omega_N}^{\infty} \frac{d\omega}{\omega} \left[\sigma_{\mathrm{tot}}^{\gamma \mathrm{p} \to \mathrm{X}}(\omega) - \sigma_{\mathrm{tot}}^{\gamma \mathrm{n} \to \mathrm{X}}(\omega) \right] \quad (1.245)$$

with $\omega_N = m_\pi + m_\pi^2/2M_N$, in which just a mutual cancellation of the rise of these total proton and neutron photoproduction cross sections for $\omega \to \infty$, created by the Pomeron exchanges, is achieved. It is straightforward to see that this final sum rule is not influenced at all by a renormalization of the left-hand side in (1.243) by $+1$.

Using the Dirac proton mean square radius from [50] and proton and neutron anomalous magnetic moments from [51], the evaluation of the left-hand side in (1.245) gives (1.93 ± 0.18) mb [54].

On the other hand, the data on photoproduction cross section on the neutron are not so known as for the proton case up to now. Nevertheless, taking compilation of both of them from [52], and assuming that both total cross sections are starting at the pion mass and are equal above the last neutron experimental point at $\omega = 17.84$ GeV, one gets on the right-hand side the value (1.92 ± 0.32) mb. So, the sum rule (1.245) can be considered to be satisfied.

Radiative corrections to virtual photon impact factor in the frames of QCD were considered in paper [59].

1.5.3. Photon Target. Sum rules connecting the high-energy asymptotic cross sections of peripheral processes of QED type with the fermion form factors (in particular, with the slope of Dirac form factor) have been found in 1973 [20]. Applications to more complicated two-loop level QED processes have been investigated (see Appendix of the review paper [4]). In a set of the papers [12,46,53,54] some applications to baryon, deuteron, and meson form factors were considered, where connection of these electromagnetic form factors with Q^2-dependent differential cross section of deep inelastic electron–hadron scattering in the peripheral kinematics was investigated.

In this Subsubsection we consider the scattering of electron on photon target with creation of 2 jets in the fragmentation region of the photon [55]. Such kind of problems can be searched at photon–electron colliders constructed on the base of linear electron–positron colliders.

Let us consider the two-photon exchange electron–photon zero-angle scattering amplitude in the process

$$e(p,\lambda)+\gamma(k,\varepsilon)\to e(p,\lambda)+\gamma(k,\varepsilon), \qquad (1.246)$$

in two-loop (α^3) approximation as presented in Fig. 5, with $p^2=m_e^2$, $k^2=0$ and assuming that $s=2p,\ k\gg m_e^2$. Averaging over the initial electron and photon spin states (initial and final spin states are supposed to coincide), one can write down the amplitude of the process (1.246) in the following form:

$$\bar{A}^{e\gamma\to e\gamma}(s,t=0)=s\frac{\alpha}{4\pi^2}\int\frac{d^2\mathbf{q}}{(q^2)^2}ds_1\sum_{\varepsilon}A^{\gamma\gamma\to\gamma\gamma}_{\mu\nu\alpha\beta}\frac{p^\mu p^\nu\varepsilon^\alpha\varepsilon^{*\beta}}{s^2}, \qquad (1.247)$$

where the light-cone projection of the LBL scattering tensor takes the form

$$A^{\gamma\gamma\to\gamma\gamma}_{\mu\nu\alpha\beta}\frac{p^\mu p^\nu\varepsilon^\alpha\varepsilon^{*\beta}}{s^2}=-\frac{8\alpha^2}{\pi^2}\int d^4q_-\Big[\frac{S_1}{D_1}+\frac{S_2}{D_2}+\frac{S_3}{D_3}\Big] \qquad (1.248)$$

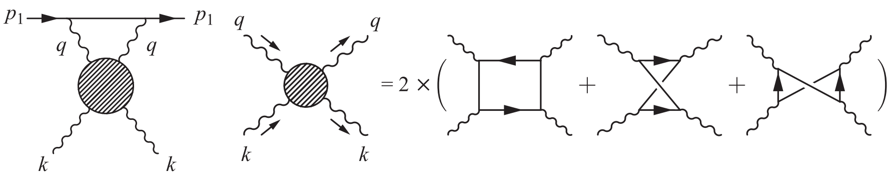


Fig. 5. Feynman diagram of $e\gamma\to e\gamma$ scattering with LBL mechanism to be realized by quark-loops

with (see Fig. 5)

$$\frac{S_1}{D_1} = \frac{1}{4}\frac{\mathrm{Sp}\big[\hat{p}(\hat{q}_- + m_q)\hat{p}(\hat{q}_- - \hat{q} + m_q)\hat{\varepsilon}^*(\hat{q}_- - \hat{q} + \hat{k} + m_q)\hat{\varepsilon}(\hat{q}_- - \hat{q} + m_q)\big]}{(q_-^2 - m_q^2)((q_- - q)^2 - m_q^2)^2((q_- - q + k)^2 - m_q^2)},$$

$$\frac{S_2}{D_2} = \frac{1}{4}\frac{\mathrm{Sp}\big[\hat{p}(\hat{q}_- + m_q)\hat{p}(\hat{q}_- - \hat{q} + m_q)\hat{\varepsilon}^(\hat{q}_- - \hat{q} - \hat{k} + m_q)\hat{\varepsilon}^*(\hat{q}_- - \hat{q} + m_q)\big]}{(q_-^2 - m_q^2)((q_- - q)^2 - m_q^2)^2((q_- - q - k)^2 - m_q^2)},$$

$$\frac{S_3}{D_3} = \frac{1}{4}\frac{\mathrm{Sp}\big[\hat{p}(\hat{q}_- + m_q)\hat{\varepsilon}(\hat{q}_- - \hat{k} + m_q)\hat{p}(\hat{q}_- + \hat{q} - \hat{k} + m_q)\hat{\varepsilon}^*(\hat{q}_- + \hat{q} + m_q)\big]}{(q_-^2 - m_q^2)((q_- + q)^2 - m_q^2)((q_- + q + k)^2 - m_q^2)((q_- - k)^2 - m_q^2)},$$

where q_- means the quark four-momentum in the quark loop of the process $\gamma\gamma \to \gamma\gamma$.

Regularization of LBL tensor is implied to provide the gauge invariance, which consists in removing some constant symmetrical tensor and the latter has no influence on the final results.

Now, taking a derivative of relation (1.247) according to $d^2\mathbf{q}$, one gets rid of the corresponding integral. The analytic properties of the obtained expression in $s_1 = s\alpha_2$ plane are presented in Fig. 6, where also the path C of the integral expression

$$I = \int\limits_C ds_1 \frac{dA^{e\gamma\to e\gamma}(s_1, \mathbf{q})}{d^2\mathbf{q}} \tag{1.249}$$

is drawn. When the integration contour is closed to the right (on s-channel cut), and to the left (on the u-channel cut), one comes to the relation

$$\Delta_u \frac{d\bar{A}^{e\gamma\to e\gamma}(s_1, \mathbf{q})}{d^2\mathbf{q}}|_{\mathrm{left}} = \Delta_s \frac{d\bar{A}^{e\gamma\to e\gamma}(s_1, \mathbf{q})}{d^2\mathbf{q}}|_{\mathrm{right}}, \tag{1.250}$$

where the right s-cannel discontinuity is related, due to optical theorem in a differential form

$$\Delta_s \frac{d\bar{A}^{e\gamma\to e\gamma}(s_1, \mathbf{q})}{d^2\mathbf{q}} = 2s\frac{d\sigma^{e\gamma\to eq\bar{q}}}{d^2\mathbf{q}}, \tag{1.251}$$

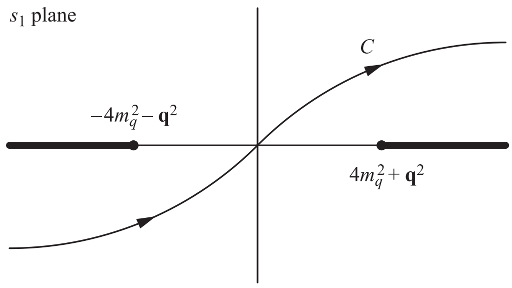


Fig. 6. The path C of an integration in (1.249)

with the $\sigma^{e\gamma\to eq\bar{q}}$ differential cross section of $q\bar{q}$ pair creation by electron on photon to be well known in the framework of QED [56] for l^+l^- pair creation:

$$\frac{d\sigma^{e\gamma\to eq\bar{q}}}{d\mathbf{q}^2} = \frac{4\alpha^3}{3(q^2)^2} f(\frac{\mathbf{q}^2}{m_q^2}), \quad f(\frac{\mathbf{q}^2}{m_q^2}) = (\mathbf{q}^2 - m_q^2)J + 1,$$
$$J = \frac{4}{\sqrt{\mathbf{q}^2(\mathbf{q}^2 + 4m_q^2)}} \ln\left[\sqrt{\frac{\mathbf{q}^2}{4m_q^2}} + \sqrt{1 + \frac{\mathbf{q}^2}{4m_q^2}}\right].$$

But the right-hand cut concerns real two-quark production for $s_1 > 4m_q^2$, which is associated wit 2 jets production.

The left-hand cut contribution has the same form as in QED case with constituent quark masses.

As a result, one obtains

$$\frac{4\alpha^3}{3(\mathbf{q}^2)^2} N_c \sum_q Q_q^4 f\left(\frac{\mathbf{q}^2}{m_q^2}\right) = \frac{d\sigma^{e\gamma\to e2\mathrm{jets}}}{d\mathbf{q}^2}, \tag{1.252}$$

where N_c is the number of colors in QCD, and Q_q is the charge of the quark q in electron charge units. Finally, for the case of small $\mathbf{q}^2$ and applying the Weizsacker–Williams relation, one comes to the sum rule for photon target as follows:

$$\frac{14}{3} \sum_q \frac{Q_q^4}{m_q^2} = \frac{1}{\pi\alpha^2} \int\limits_{4m_q^2}^{\infty} \frac{ds_1}{s_1} \sigma_{\mathrm{tot}}^{\gamma\gamma\to 2\mathrm{jet}}(s_1). \tag{1.253}$$

The quantity $\sigma_{\mathrm{tot}}^{\gamma\gamma\to 2jets}(s_1)$ is assumed to decrease for large values of s_1. It corresponds to the events in $\gamma\gamma$ collisions with creation of two jets, which are not separated by rapidity gaps and for which till the present days there is no experimental information. The latter complicates a verification of the sum rule (1.253).

An evaluation of the left-hand side with the constituent quark masses $m_u = m_d = 280$ MeV and $m_s = 405$ MeV (the contribution of heavy quarks c, b, t is negligible and can be disregarded) gives 5 mb.

The saturation of the right-hand side of the photon sum rule (1.253) with the help of the data on $\sigma_{\mathrm{tot}}^{\gamma\gamma\to X}(s_1)$ [51] on the level of 5 mb is achieved with the upper bound of the corresponding integral to be 2–3 GeV^2.

Unfortunately, the used data are charged by rather large uncertainties and in order to achieve more reliable verification of the sum rule (1.253) the data on $\sigma_{\mathrm{tot}}^{\gamma\gamma\to 2\ \mathrm{jets}}(s_1)$ are highly desirable.

2. TABLE OF INTEGRALS. ONE-LOOP FEYNMAN INTEGRALS

2.1. Loop-Momentum Integrals for Subprocess $\gamma^*\gamma \to e^+e^-$. Here we put the asymptotic expressions for a part of scalar, vector, and tensor integrals, corresponding to the absorption of virtual photon by electron from the pair created in subprocess

$$\begin{gathered}\gamma(p_1) + \gamma^*(q) \to e^-(q_-) + e^+(q_+), \quad q_\pm^2 = m^2, \quad p_1^2 = 0, \\ s_1 = 2q_+q_-, \quad \chi_\pm = 2p_1q_\pm \quad s_1 \sim \chi_\pm \gg m^2.\end{gathered} \tag{2.1}$$

We give first the scalar integrals with two, three, and four (different) denominators, defined as

$$\begin{aligned}&(0) = k^2 - \lambda^2, \\ &(2) = (q_- - k)^2 - m^2 + i0 = k^2 - 2q_-k + i0, \\ &(\bar{2}) = (-q_+ - k)^2 - m^2 + i0, \\ &(q) = (p_1 - q_+ - k)^2 - m^2 + i0.\end{aligned} \tag{2.2}$$

The loop-momentum integrals with the denominator $(\bar{q}) = (q_- - p_1 - k)^2 - m^2$ instead of $(q) = (p_1 - q_+ - k)^2 - m^2$, including scalar, vector, and tensor ones, can be obtained from the ones listed below by means of the replacement:

$$q_- \to -q_+, \quad q_+ \to -q_-, \quad p_1 \to -p_1, \quad \chi_\pm \to \chi_\mp, \quad (2) \to (\bar{2}), \quad (q) \to (\bar{q}). \tag{2.3}$$

So we can restrict ourselves by consideration of only the integrals with denominators (0), (2), $(\bar{2})$, (q).

We note the relation:

$$s_1 + \boldsymbol{q}^{\,2} = \chi_+ + \chi_-. \tag{2.4}$$

Two denominator scalar integrals are defined as

$$I_{ij} = \int \frac{d^4k}{i\pi^2} \frac{1}{(i)(j)}.$$

The explicit expressions for them are

$$\begin{aligned}&I_{02} = L + 1, \qquad &&I_{2q} = L - l_q + 1, \qquad &&I_{0q} = L - l_+ + 1, \\ &I_{0\bar{2}} = L + 1, \qquad &&I_{2\bar{2}} = L - L_s + 1, \qquad &&I_{\bar{2}q} = L - 1.\end{aligned}$$

Here and below we use the notation

$$\begin{gathered}L = \ln \frac{\Lambda^2}{m^2}, \quad l_\pm = \ln \frac{\chi_\pm}{m^2}, \quad l_q = \ln \frac{\boldsymbol{q}^{\,2}}{m^2}, \\ L_s = \ln \frac{s_1}{m^2} - i\pi = l_s - i\pi, \quad l_l = \ln \frac{m^2}{\lambda^2}.\end{gathered}$$

Remind once more that we imply all the kinematic invariants to be greater than electron mass squared $s_1 \sim \mathbf{q}^2 \sim \chi_\pm \gg m^2$ and present below the asymptotic expressions systematically omitting the terms of the order of m^2/s_1 and similar ones.

The tree-denominator scalar integrals $I_{ijk} = \int \dfrac{d^4k}{i\pi^2(i)(j)(k)}$ are

$$\begin{aligned}
I_{0\bar{2}q} &= -\frac{1}{2\chi_+}\Big[l_+^2 + \frac{2\pi^2}{3}\Big],\\
I_{02\bar{2}} &= \frac{1}{2s_1}\Big[l_s^2 + 2l_s l_l - \frac{4\pi^2}{3} - i\pi(2l_s + 2l_l)\Big],\\
I_{2\bar{2}q} &= -\frac{1}{2(s_1 + \mathbf{q}^{\,2})}\Big[l_q^2 - l_s^2 + \pi^2 + 2i\pi l_s\Big],\\
I_{02q} &= \frac{1}{\chi_+ - \mathbf{q}^{\,2}}\Big[l_q(l_q - l_+) + \frac{1}{2}(l_q - l_+)^2 + 2\mathrm{Li}_2\Big(1 - \frac{\chi_+}{\mathbf{q}^{\,2}}\Big)\Big].
\end{aligned} \tag{2.5}$$

The four-denominator integral $I_{02\bar{2}q} = \int \dfrac{d^4k}{i\pi^2(0)(2)(\bar{2})(q)}$ has the form

$$\begin{aligned}
I_{02\bar{2}q} = \frac{1}{s_1\chi_+}\Big[l_q^2 - 2l_+ l_s - l_s l_l + 2\mathrm{Li}_2\Big(1 + \frac{\mathbf{q}^{\,2}}{s_1}\Big) + \frac{\pi^2}{6} + \\
+ i\pi\Big(2l_+ + l_l - 2\ln\Big(1 + \frac{\mathbf{q}^{\,2}}{s_1}\Big)\Big)\Big].
\end{aligned} \tag{2.6}$$

Now we describe the vector integrals

$$I_r^\mu = \int \frac{d^4k k^\mu}{r} = a_r^+ q_+^\mu + a_r^- q_-^\mu + a_r^1 p_1^\mu, \tag{2.7}$$

with $r = (ij), (ijk), (ijkl)$, where $i, j, k, l = (0), (2), (\bar{2}), (q)$.

For the vector integrals with two denominators we have (we put only nonzero coefficients)

$$\begin{aligned}
&a_{2q}^- = a_{2q}^1 = -a_{2q}^+ = \frac{1}{2}\Big(L - l_q + \frac{1}{2}\Big), && a_{0q}^1 = -a_{0q}^+ = \frac{1}{2}\Big(L - l_+ + \frac{1}{2}\Big),\\
&a_{2\bar{2}}^- = -a_{2\bar{2}}^+ = \frac{1}{2}\Big(L - L_s + \frac{1}{2}\Big), && a_{\bar{2}q}^1 = -\frac{1}{2}a_{\bar{2}q}^+ = \frac{1}{2}\Big(L - \frac{3}{2}\Big),\\
&a_{02}^- = \frac{1}{2}L - \frac{1}{4}, && a_{0\bar{2}}^+ = -\frac{1}{2}L + \frac{1}{4},
\end{aligned} \tag{2.8}$$

and the coefficients for the vector integrals with three denominators are

$$
\begin{aligned}
a_{02q}^{-} &= \frac{1}{a}\Big(\chi_+ I_{02q} + \frac{2\chi_+}{a} l_+ - \frac{\mathbf{q}^{\,2} + \chi_+}{a} l_q\Big), \\
a_{02q}^{+} &= -a_{02q}^{1} = \frac{1}{a}\Big(l_+ - l_q\Big), \\
a_{0\bar{2}q}^{1} &= \frac{1}{\chi_+}\Big(-l_+ + 2\Big), \quad a = \chi_+ - \mathbf{q}^{\,2}, \\
a_{0\bar{2}q}^{+} &= -I_{0\bar{2}q} - \frac{1}{\chi_+} l_+, \\
a_{02\bar{2}}^{-} &= -a_{02\bar{2}}^{+} = \frac{1}{s_1} L_s, \\
a_{2\bar{2}q}^{-} &= \frac{1}{c}\Big(L_s - l_q\Big), \\
a_{2\bar{2}q}^{+} &= -I_{2\bar{2}q} + \frac{1}{c}\Big(L_s - l_q\Big), \\
a_{2\bar{2}q}^{1} &= \frac{s_1}{c} I_{2\bar{2}q} + \frac{1}{c}\Big(-l_q + 2\Big) - \frac{2s_1}{c^2}\Big(L_s - l_q\Big), \qquad c = s_1 + \mathbf{q}^{\,2} = \chi_+ + \chi_-.
\end{aligned}
\tag{2.9}
$$

Finally, the coefficient of the vector integral with 4 denominators has the form

$$
\begin{aligned}
a^1 &= \frac{s_1}{d}\Big(\chi_+ A + \chi_- B - s_1 C\Big), \\
a^+ &= \frac{\chi_-}{d}\Big(\chi_+ A - \chi_- B + s_1 C\Big), \\
a^- &= \frac{\chi_+}{d}\Big(-\chi_+ A + \chi_- B + s_1 C\Big), \quad d = 2 s_1 \chi_+ \chi_-, \\
A &= I_{2\bar{2}q} - I_{0\bar{2}q}, \\
B &= I_{02q} - I_{2\bar{2}q}, \\
C &= I_{02q} - I_{02\bar{2}} - \chi_+ I_{02\bar{2}q}.
\end{aligned}
\tag{2.10}
$$

We parameterized the second rank tensor integrals in the form

$$
I_r^{\mu\nu} = \int \frac{d^4k}{i\pi^2} \frac{k_\mu k_\nu}{r} = \Big[a_r^g g + a_r^{11} p_1 p_1 + a_r^{++} q_+ q_+ + a_r^{--} q_- q_- +
$$
$$
+ a_r^{1+}(p_1 q_+ + q_+ p_1) + a_r^{1-}(p_1 q_- + q_- p_1) + a_r^{+-}(q_+ q_- + q_- q_+)\Big]_{\mu\nu}. \tag{2.11}
$$

The coefficients for tensor integral with four denominators are (we suppressed the index $02\bar{2}q$)

$$\begin{aligned}
a^{1+} &= \frac{1}{\chi_+}\Big(A_6 + A_7 - A_{10}\Big), & a^{+-} &= \frac{1}{s_1}\Big(A_2 + A_6 - A_{10}\Big),\\
a^{1-} &= \frac{1}{\chi_-}\Big(A_2 + A_7 - A_{10}\Big), & a^{11} &= \frac{1}{\chi_-}\Big(A_1 - s_1 a^{1+}\Big),\\
a^{--} &= \frac{1}{s_1}\Big(A_5 - \chi_+ a^{1-}\Big), & a^{++} &= \frac{1}{s_1}\Big(A_3 - \chi_- a^{1+}\Big),\\
a^{g} &= \frac{1}{2}\Big(A_{10} - A_2 - \chi_+ a^{1+}\Big),
\end{aligned} \tag{2.12}$$

with

$$\begin{aligned}
A_1 &= a^{1}_{2\bar{2}q} - a^{1}_{0\bar{2}q}, & A_6 &= a^{+}_{02q} - a^{+}_{2\bar{2}q},\\
A_2 &= a^{-}_{2\bar{2}q}, & A_7 &= a^{1}_{02q} - \chi_+ a^{1},\\
A_3 &= a^{+}_{2\bar{2}q} - a^{+}_{0\bar{2}q}, & A_8 &= a^{-}_{02q} - a^{-}_{02\bar{2}} - \chi_+ a^{-},\\
A_4 &= a^{1}_{02q} - a^{1}_{2\bar{2}q}, & A_9 &= a^{+}_{02q} - a^{+}_{02\bar{2}} - \chi_+ a^{+},\\
A_5 &= a^{-}_{02q} - a^{-}_{2\bar{2}q}, & A_{10} &= I_{2\bar{2}q}.
\end{aligned} \tag{2.13}$$

One can verify that the checking relations

$$A_4 = \chi_+ a^{11} + s_1 a^{1-}, \quad A_8 = \chi_- a^{--} + \chi_+ a^{+-}, \quad A_9 = \chi_+ a^{++} + \chi_- a^{+-} \tag{2.14}$$

for the above coefficients (2.13) are fulfilled.

The coefficients entering into the tensor integral $I^{\mu\nu}_{02q}$ are

$$\begin{aligned}
a^{g}_{02q} &= \frac{1}{4}L + \frac{3}{8} + \frac{\mathbf{q}^2}{4a} l_q - \frac{\chi_+}{a} l_+,\\
a^{+-}_{02q} &= -a^{1-}_{02q} = \frac{1}{2a}\Big[\frac{\chi_+}{a}(l_+ - l_q) - 1\Big],\\
a^{++}_{02q} &= a^{11}_{02q} = -a^{1+}_{02q} = \frac{1}{2a}(l_q - l_+),\\
a^{--}_{02q} &= \frac{1}{a^2}\left[\chi_+^2 I_{02q} + \frac{3\chi_+^2}{a} l_+ + \frac{(\mathbf{q})^2 - 4\mathbf{q}^2\chi_+ - 3\chi_+^2}{2a} l_q + \frac{\mathbf{q}^2 - 3\chi_+}{2}\right].
\end{aligned} \tag{2.15}$$

The coefficients entering into the tensor integral $I^{\mu\nu}_{02\bar{2}}$ are

$$a^{g}_{02\bar{2}} = \frac{1}{4}(L - L_s) + \frac{3}{8}, \quad a^{++}_{02\bar{2}} = a^{--}_{02\bar{2}} = \frac{1}{2s_1}(L_s - 1), \quad a^{+-}_{02\bar{2}} = -\frac{1}{2s_1}, \tag{2.16}$$

and the coefficients for the tensor integral $I^{\mu\nu}_{0\bar{2}q}$ are

$$a^{g}_{0\bar{2}q} = \frac{1}{4}(L - l_+) + \frac{3}{8}, \qquad a^{1+}_{0\bar{2}q} = \frac{1}{\chi_+}\Big(l_+ - \frac{5}{2}\Big),$$
$$a^{11}_{0\bar{2}q} = \frac{1}{2\chi_+}(-l_+ + 2), \qquad a^{++}_{0\bar{2}q} = I_{0\bar{2}q} + \frac{1}{2\chi_+}(3l_+ - 1). \tag{2.17}$$

In the case of the tensor integral $I^{\mu\nu}_{2\bar{2}q}$ they have the form

$$a^{g}_{2\bar{2}q} = \frac{1}{2}\Big[\frac{1}{2}L + \frac{3}{4} - \frac{s_1}{2c}L_s - \frac{\mathbf{q}^2}{2c}l_q\Big], \qquad a^{--}_{2\bar{2}q} = -\frac{1}{2c}(l_q - L_s), \tag{2.18}$$

$$a^{++}_{2\bar{2}q} = I_{2\bar{2}q} + \frac{3}{2c}(l_q - L_s), \qquad a^{+-}_{2\bar{2}q} = \frac{1}{2c}(l_q - L_s),$$
$$a^{1-}_{2\bar{2}q} = \frac{1}{c}\Big[-\frac{1}{2} + \frac{s_1}{2c}L_s - \frac{s_1}{2c}l_q\Big], \tag{2.19}$$
$$a^{1+}_{2\bar{2}q} = \frac{1}{c}\Big[-\frac{5}{2} - s_1 I_{2\bar{2}q} + \frac{5s_1}{2c}L_s + \frac{2\mathbf{q}^2 - 3s_1}{2c}l_q\Big],$$

$$a^{11}_{2\bar{2}q} = \frac{1}{c^2}\Big[4s_1 + \mathbf{q}^2 + s_1^2 I_{2\bar{2}q} - \frac{3s_1^2}{c}L_s + \frac{3s_1^2 - (\mathbf{q}^2)^2 - 4s_1\mathbf{q}^2}{2c}l_q\Big]. \tag{2.20}$$

The checking equations for the coefficients (2.18) could be obtained after multiplying $I^{\mu\nu}_{2\bar{2}q}$ by $2(q_+ + q_-)^\nu$ or $2p_1^\nu$, using the relations $2k(q_+ + q_-) = (\bar{2}) - (2)$, $2p_1 k = (\bar{2}) - (q) - \chi_+$ and using the vector integrals (2.7). They have the form:

$$\begin{aligned} 2a^{g}_{2\bar{2}q} + s_1 a^{--}_{2\bar{2}q} + c a^{1-}_{2\bar{2}q} + s_1 a^{+-}_{2\bar{2}q} &= a^{-}_{2q} - a^{-}_{\bar{2}q}, \\ 2a^{g}_{2\bar{2}q} + s_1 a^{++}_{2\bar{2}q} + c a^{1+}_{2\bar{2}q} + s_1 a^{+-} &= a^{+}_{2q} - a^{+}_{\bar{2}q}, \\ c a^{11}_{2\bar{2}q} + s_1 a^{1+}_{2\bar{2}q} + s_1 a^{1-}_{2\bar{2}q} &= a^{1}_{2q} - a^{1}_{\bar{2}q}. \end{aligned} \tag{2.21}$$

Integrals for calculation of the electron impact factor with the denominators

$$\begin{aligned} (0)_e &= k^2 - \lambda^2, \\ (1)_e &= (p_1 - k)^2 - m^2 + i0, \\ (2)_e &= (p_1' - k)^2 - m^2 + i0, \\ (q)_e &= (p_1 - k_1 - k)^2 - m^2 + i0 \end{aligned} \tag{2.22}$$

can be obtained from the cited above by the substitution

$$\int \frac{d^4k}{i\pi^2}\frac{1,\, k,\, kk}{(0)_e(1)_e(2)_e(q)_e} = \mathcal{P}(q_- \to p_1', q_+ \to -p_1, p_1 \to -k_1, q \to q)\times$$
$$\times \int \frac{d^4k}{i\pi^2}\frac{1,\, k,\, kk}{(0)(2)(\bar{2})(q)}. \tag{2.23}$$

An additional set of relevant integrals for the electron impact factor can be obtained by the relevant substitution.

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REFERENCES

1. *Brau J. et al.* International Linear Collider Reference Design Report. 1. Executive Summary. 2. Physics at the ILC. 3. Accelerator. 4. Detectors. JINR, E9-2007-39. Dubna, 2007.
2. *Carimalo C., Schiller A., Serbo V. G.* New Method for Calculating Helicity Amplitudes of Jet-Like QED Processes for High-Energy Colliders. I. Bremsstrahlung Processes // Eur. Phys. J. C. 2002. V. 23. P. 633.
3. *Kuraev E. A., Galynskii M. V., Levchuk M. I.* Principles and Physics Programme of Gamma–Gamma Colliders // Part. Nucl. 2000. V. 31. P. 76 (Fiz. Elem. Chast. At. Yadra. 2000. V. 31. P. 155).
4. *Baier V. N. et al.* Inelastic Processes in Quantum Electrodynamics at High Energies // Phys. Rep. 1981. V. 78. P. 293.
5. *Cheng H., Wu T. T.* High-Energy Elastic Scattering in Quantum Electrodynamics // Phys. Rev. Lett. 1969. V. 22. P. 666.
6. *Gribov V. N., Lipatov L. N., Frolov G. V.* The Leading Singularity in the J Plane in Quantum Electrodynamics // Sov. J. Nucl. Phys. 1971. V. 12. P. 543 (Yad. Fiz. 1970. V. 12. P. 994).
7. *Akhiezer A. I., Berestetskii V. B.* Quantum Electrodynamics. M.: Nauka, 1959; 1981.
8. *Berestetskii V. B., Lifshitz E. M., Pitaevskii L. B.* Quantum Electrodynamics. M.: Nauka, 1989.
9. *Berends F. A. et al.* Multiple Bremsstrahlung in Gauge Theories at High Energies. 2. Single Bremsstrahlung // Nucl. Phys. B. 1982. V. 206. P. 61.
10. *Berends F. A. et al.* Single Bremsstrahlung Processes in Gauge Theories // Phys. Lett. B. 1981. V. 103. P. 124.
11. *Bytev V. et al.* Radiative Corrections to Chiral Amplitudes in Quasiperipheral Kinematics // JETP. 2006. V. 103. P. 224.
12. *Bartos E. et al.* Calibration Processes for Photon–Photon Colliders // Nucl. Phys. B. 2004. V. 676. P. 481.

13. *Baier V. N., Fadin V. S., Khoze V. A.* Quasireal Electron Method in High-Energy Quantum Electrodynamics // Nucl. Phys. B. 1973. V. 65. P. 381.

14. *Kessler P.* Sur une Methode Simplifiee de Calcul pour les Processus Relativistes en Electrodynamique Quantique // Nuovo Cim. 1960. V. 17. P. 809.

15. *Borodulin V. I., Rogalev R. N., Slabospitsky S. R.* CORE: COmpendium of RElations: Version 2.1. hep-ph/9507456.

16. *Kuraev E. A., Lipatov L. N., Shishkina T. V.* QED Radiative Corrections to Impact Factors // JETP. 2001. V. 92. P. 203 (ZhETF. 2001. V. 92. P. 236).

17. *Kuraev E. A., Lipatov L. N., Strikman M. I.* Identity of Electrons in the Final State in Pair Production in Colliding e^+e^- Beams // Yad. Fiz. 1973. V. 18. P. 1270.

18. *Kuraev E. A., Merenkov N. P., Fadin V. S.* The Compton Effect Tensor with Heavy Photon (in Russian) // Yad. Fiz. 1987. V. 45. P. 782 (Sov. J. Nucl. Phys. 1987. V. 45. P. 486);
Kuraev E. A., Merenkov N. P., Fadin V. S. Compton Tensor with Heavy Photon. Preprint INP 86-39. Novosibirsk, 1986.

19. *Kuraev E. A. et al.* Radiative Corrections to Bremsstrahlung Radiation at High Energies // ZETF. 1973. V. 65. P. 2155.

20. *Kuraev E. A., Lipatov L. N., Merenkov N. P.* High-Energy Behavior of Total Cross Sections for e^+e^- Scattering and the Slope at q-Squared $= 0$ of the Dirac Form Factor // Phys. Lett. B. 1973. V. 47. P. 33;
Chang S. J. Radiative Corrections to the e^+e^- Scattering Amplitude at Infinite Energy // Phys. Rev. D. 1970. V. 1. P. 2977.

21. *Vinokurov E. A., Kuraev E. A., Merenkov N. P.* Radiative Corrections to e^+e^- Pair Production Induced by High-Energy Photons Incident on Electrons or Nuclei // ZhETF. 1974. V. 66. P. 1916.

22. *Gribov V. N., Lipatov L. N.* e^+e^- Pair Annihilation and Deep Inelastic *ep* Scattering in Perturbation Theory // Sov. J. Nucl. Phys. 1972. V. 15. P. 675 (Yad. Fiz. 1972. V. 15. P. 1218);
Dokshitzer Y. L. Calculation of the Structure Functions for Deep Inelastic Scattering and e^+e^- Annihilation by Perturbation Theory in Quantum Chromodynamics (in Russian) // Sov. Phys. JETP. 1977. V. 46. P. 641 (ZhETF. 1977. V. 73. P. 1216).

23. *Altarelli G., Parisi G.* Asymptotic Freedom in Parton Language // Nucl. Phys. B. 1977. V. 126. P. 298.

24. *Fadin V. S., Lipatov L. N.* BFKL Pomeron in the Next-to-Leading Approximation // Phys. Lett. B. 1998. V. 429. P. 127.

25. *Gribov V. N. et al.* On the Properties of Amplitudes with Logarithmically Increasing Interaction Radius // Sov. J. Nucl. Phys. 1971. V. 13. P. 381 (Yad. Fiz. 1971. V. 13. P. 670).

26. *Kuraev E. A., Lipatov L. N.* Electron and Muonic Production in e^-e^- and e^+e^- Colliding Beams // Yad. Fiz. 1972. V. 16. P. 1060.

27. *Altarelli G., Buchella F.* Single-Photon Emission in High-Energy e^+e^- Collisions // Nuovo Cim. 1964. V. 34. P. 1337.

28. *Altarelli G., Stella B.* Electron Angular Distribution in $e^-e^- \to e^+e^-$ Gamma at High Energies // Lett. Nuovo Cim. 1974. V. 9. P. 416.

29. *Kuraev E. A. et al.* Double Bremsstrahlung in the Same Direction in e^+e^- Colliding Beams // Sov. J. Nucl. Phys. 1974. V. 19. P. 164 (Yad. Fiz. 1974. V. 19. P. 331).

30. *Landau L. D., Lifschitz E. M.* Production of Electrons and Positrons by a Collision of Two Particles // Phys. Z. Sow. 1934. V. 6. P. 244.

31. *Racah G.* Sulla Nastica di Coppie per Urti di Particelle Eletrizzate // Nuovo Cim. 1937. V. 14. P. 93.

32. *Baier V., Katkov V., Fadin V.* Radiation of Relativistic Electrons. M.: Atomizdat, 1973.

33. *Budnev V. M. et al.* The Two-Photon Particle Production Mechanism. Physical Problems. Applications. Equivalent Photon Approximation // Phys. Rep. 1974. V. 15. P. 181.

34. *Kuraev E. A. et al.* e^+e^- Electroproduction Spectra in the Fragmentation Region // Yad. Fiz. 1976. V. 23. P. 163.

35. *Kuraev E. A., Lipatov L. N., Strikman M. I.* On the Relation between Various Inclusive Processes in Quantum Electrodynamics // Zh. Eksp. Teor. Fiz. 1974. V. 66. P. 838.

36. *Baier V. N., Galitsky V. M.* Double Bremsstrahlung in Electron Collisions // JETP Lett. 1965. V. 2. P. 165.

37. *Cheng H., Wu T. T.* Photon–Photon Scattering Close to the Forward Direction // Phys. Rev. D. 1970. V. 1. P. 3414;
Lipatov L. N., Erolov G. V. Some Processes in Quantum Electrodynamics at High Energies // Yad. Fiz. 1971. V. 13. P. 588.

38. *Kuraev E. A., Schiller A., Serbo V. G.* The Process $\gamma\gamma \to \Pi^+\Pi^-e^+e^-$ for High-Energy $\gamma\gamma$ Beams // Nucl. Phys. B. 1985. V. 256. P. 189; 211.

39. *Bardin D., Hollik W., Passarino G.* Reports of the Working Group on Precision Calculations for the Z Resonance. CERN 95-03. 1995.

40. *Polovin R. V.* Radiative Corrections to the Electron Scattering on Electron and Positron // JETP. 1956. V. 31. P. 449;
Redhead M. L. C. Radiative Corrections to the Scattering of Electrons and Positrons by Electrons // Proc. Roy. Soc. 1953. V. 220. P. 219;
Berends F. A., Kleiss R. Distributions in the Process $e^+e^- \to e^+e^-$ (γ) // Nucl. Phys. B. 1983. V. 228. P. 537;
Budny R. Detailed W^0 Effects in $e^+e^- \to e^+e^-$ // Phys. Lett. B. 1975. V. 55. P. 227.

41. *Berends F. A., Gaemers K. J. F., Gastmans R.* Hard-Photon Corrections for Bhabha Scattering // Nucl. Phys. B. 1974. V. 68. P. 541.

42. *Arbuzov A. B. et al.* Virtual and Soft Real Pair Production in Large-Angle Bhabha Scattering // Phys. At. Nucl. 1997. V. 60. P. 591 (Yad. Fiz. 1997. V. 60. P. 674).

43. *Arbuzov A. B. et al.* Small-Angle Bhabha Scattering // Nucl. Phys. Proc. Suppl. 1996. V. 51C. P. 154;
Arbuzov A. B. et al. Small-Angle Electron–Positron Scattering // Phys. Lett. B. V. 394. P. 218;
Arbuzov A. B. et al. Small-Angle Bhabha Scattering at LEP1: Analytical Results for Wide-Narrow Angular Acceptance // Phys. Lett. B. 1997. V. 399. P. 312.
44. *Merenkov N. P. et al.* Analytical Calculation of Small-Angle Bhabha Cross Section at LEP-1 // Acta Phys. Polon. B. 1997. V. 28. P. 491.
45. *Arbuzov A. B.* Pair Production in Small-Angle Bhabha Scattering // JETP. 1995. V. 81. P. 638 (ZhETF. 1995. V. 108. P. 1164).
46. *Kuraev E. A., Secansky M., Tomasi-Gustafsson E.* High-Energy Inelastic Electron Hadron Scattering, in Peripheral Kinematics: Sum Rules for Hadron Form Factors // Phys. Rev. D. 2006. V. 73. P. 125016.
47. *Appelquist T., Brodsky S. J.* Fourth Order Electrodynamic Corrections to the Lamb Shift // Phys. Rev. A. 1970. V. 2 P. 2293;
Peterman A. Analytic 4th Order Crossed Ladder Contribution to the Lamb Shift // Phys. Lett. B. 1971. V. 35. P. 325.
48. *Barbieri R., Mignaco J. A., Remiddi E.* Electron Form Factors up to Fourth Order. 1 // Nuovo Cim. A. 1972. V. 11. P. 824.
49. *Gottfried K.* Sum Rule for High-Energy Electron–Proton Scattering // Phys. Rev. Lett. 1967. V. 18. P. 1174.
50. *Dubnicka S., Dubnickova A. Z., Weisenpacher P.* Nucleon Electromagnetic Structure Revisited // J. Phys. G. 2003. V. 29. P. 405.
51. *Hagiwara K. et al. (Particle Data Group).* Review of Particle Physics // Phys. Rev. D. 2002. V. 66. P. 010001.
52. *Baldini A. et al.* Group I Elementary Particles, Nuclei and Atoms / Ed. by H. Schopper, Landolt–Börnstein. New Ser. V. 12. Berlin: Springer-Verlag, 1988.
53. *Dubnicka S., Dubnickova A. Z., Kuraev E. A.* Sum Rules for Total Cross Sections of Hadron Photoproduction on Pseudoscalar Mesons // Phys. Rev. D. 2006. V. 74. P. 034023.
54. *Dubnicka S., Dubnickova A. Z., Kuraev E. A.* Sum Rules for Total Cross Sections of Hadron Photoproduction on Ground State $1/2^+$ Octet Baryons // Phys. Rev. D. 2007. V. 75. P. 057901.
55. *Bartos E. et al.* Sum Rule for Photon Target // Phys. Rev. D. 2007. V. 76 P. 057901.
56. *Kel'ner S. R.* Pair Production in Collision between a Fast Particle and Nuclei // Sov. J. Nucl. Phys. 1967. V. 5. P. 788 (Yad. Fiz. 1967. V. 5. P. 1092).
57. *van Neerven W. L., Vermaseren J. A. M.* Large Loop Integrals // Phys. Lett. B. 1984. V. 137. P. 241.
58. *Arbuzov A. B. et al.* Structure Function Approach in QED for High-Energy Processes // Part. Nucl. 2010. V. 41, No. 3. P. 720.
59. *Fadin V. S., Ivanov D. Yu., Kotski M. I.* Nucl. Phys. B. 2003. V. 658. P. 156–174.