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BOSON RANDOM POINT PROCESSES AND CONDENSATION

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This is a short survey of the Boson Random Point Processes method and its application to the mean-field interacting boson gas trapped by a *weak* harmonic potential.

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1. INTRODUCTION: RANDOM POINT PROCESSES

We start by recall of some notations and definitions that we need to formulate our results. For details the reader may consult, for example, the book [1].

(a) Let E be a locally compact metric space serving as a *state space* of ***points***, $\mathfrak{B}$ the Borel σ-algebra, $\mathfrak{B}_\circ \subseteq \mathfrak{B}$ (relatively) compact Borel sets. Let ν be a (*diffusive*) locally finite reference measure on $(E, \mathfrak{B})$. The standard example: ν is the Lebesgue measure and $E = \mathbb{R}^d$.

(b) The space of the locally finite configurations of ***points*** in E is

$$Q(E) := \{\xi \subset E : \mathrm{card}(\xi \cap \Lambda) < \infty \text{ for all } \Lambda \in \mathfrak{B}_\circ\}.$$

Then $Q(\Lambda) := \{\xi \in Q : \xi \subset \Lambda\}$ and the function: $N_\Lambda : \xi \mapsto \mathrm{card}(\xi \cap \Lambda)$.

(c) Each $\xi \in Q$ can be ***identified*** with integer-valued nonnegative *Radon measure*: $\lambda_\xi := \sum\limits_{x\in\xi} \delta_x$ on $\mathfrak{B}$, i.e., $\lambda_\xi(D) := N_D$ is the *number* of ***points*** that fall into the set D for the *locally finite* point configuration $\xi \in Q(D)$.

(c) ***Definition:*** A *random point process* (RPP) in a *locally compact* space E is a random probability Radon measure μ on the *configuration* space $Q(E)$, with expectation that for any *measurable* function is defined by

$$\mathbb{E}_\mu(F) := \int\limits_{Q(E)} \mu(d\xi) F(\xi).$$

• For a *simple* random point process the measure μ assigns a.-s.: $\mu(x) \leqslant 1$, for any single point $x \in Q(E)$.

• By $K(x,y)$ we denote a kernel of nonnegative, self-adjoint, *locally* Tr-class operator $K \geqslant 0$ on $L^2(\Lambda)$.

(d) **Example:** (The *Poisson* RPP π_η with intensity $\eta \geqslant 0$)

(1) For any set $D \subset E$ with finite Lebesgue measure $\nu(D)$, one puts

$$\mathbb{P}\{N_D = n\} = \int\limits_{Q(E)} \pi_\eta(d\xi)\delta_{n,N_D(\xi)} = \frac{(\eta\,\nu(D))^n}{n!}\mathrm{e}^{-\eta\nu(D)}.$$

(2) For *mutually disjoint* subsets $\{D_n \subset \Lambda\}_{n\geqslant 1}$ the *Poisson* RPP π_η is supposed to be ***uncorrelated***:

$$\mathbb{E}_{\pi_\eta}(\delta_{n_1,N_{D_1}(\xi)} \dots \delta_{n_k,N_{D_k}(\xi)}) = \mathbb{E}_{\pi_\eta}(\delta_{n_1,N_{D_1}(\xi)}) \dots \mathbb{E}_{\pi_\eta}(\delta_{n_k,N_{D_k}(\xi)}) = \\ = \frac{(\eta\ \nu(D_1))^{n_1}}{n_1!}\ \mathrm{e}^{-\eta\nu(D_1)} \dots \frac{(\eta\ \nu(D_k))^{n_k}}{n_k!}\mathrm{e}^{-\eta\nu(D_k)}.$$

(e) **Definition:** For any family of *mutually disjoint* subsets $\{D_n \subset \Lambda\}_{n\geqslant 1}$ the ***correlation functions*** (*joint intensities)* of the RPP μ are defined by the densities $\{\rho_n : \Lambda^n \mapsto \mathbb{R}^1_+\}_{n\geqslant 1}$ with respect to the measure ν:

$$\mathbb{E}_\mu\left(\prod_{1\leqslant j\leqslant n} \mathbb{I}_{|\xi\cap D_j|=1}\right) = \int\limits_{D_1\times\ldots\times D_n} \nu(dx_1)\cdots\nu(dx_n)\rho_n(x_1,\ldots,x_n).$$

(f) **Definition:** An RPP is called ***determinantal/permanental*** with (a *locally* Tr-class) kernel K, if it is simple and its correlation functions are

$$\rho_n(x_1,\ldots,x_n) = \mathbf{det}\|K(x_i,x_j)\|_{1\leqslant i,j\leqslant n},$$
$$\rho_n(x_1,\ldots,x_n) = \mathbf{per}\|K(x_i,x_j)\|_{1\leqslant i,j\leqslant n}.$$

For any $n \geqslant 1$ and $x_1,\ldots,x_n \in \Lambda$, $\det_\alpha A := \sum\limits_{\sigma\in\mathfrak{S}_n} \alpha^{n-c(\sigma)} \prod\limits_{1\leqslant i\leqslant n} a_{i\sigma(i)}$ $\alpha = \pm 1 \Leftrightarrow \mathrm{per}/\det$ and $c(\sigma)$ is the number of *cycles* in the permutation σ.

2. FERMION/BOSON RANDOM POINT PROCESSES

2.1. Quantum Statistical Mechanics: Fermions. Let $\mathfrak{H}_L := L^2(\Lambda_L)$, where $\Lambda_L = [-L/2, L/2]^d$ and $\Delta_{L,p}$ be Laplacian with *periodic* boundary conditions on $\partial\Lambda_L$, i.e.,

$$\mathrm{spec}\,(-\Delta_{L,p}) = \{\varepsilon(k) = (2\pi/L)^2\|k\|^2 : k \in \mathbb{Z}^d\}.$$

Then the Gibbs semigroup kernel for the inverse temperature β has the form

$$(G_{\beta,L})(x,y) := (\mathrm{e}^{\beta\Delta_L})(x,y) = \\ = \sum_{k\in\mathbb{Z}^d} \mathrm{e}^{-\beta\varepsilon(k)}\phi_{k,L}(x)\overline{\phi_{k,L}(y)} = \sum_{k\in\mathbb{Z}^d}(G_\beta)(x,y+kL),$$

where the «heat» semigroup kernel is

$$(G_\beta)(x,y) := \lim_{L\to\infty}(G_{\beta,L})(x,y) = (4\pi\beta)^{-d/2}\exp -\left(\frac{\|x-y\|^2}{4\beta}\right).$$

Remark: It is known that any n-particle free-fermion wave function is the *Slater* determinant:

$$\Psi_{k_1,\ldots,k_n}(x_1,\ldots,x_n) = \frac{1}{\sqrt{n!}}\det\|\phi_{k_i,L}(x_j)\|_{1\leqslant i,j\leqslant n}.$$

The corresponding n-point free-fermion joint *probability distribution* density: $p_{n,L}(x_1,\ldots,x_n) := |\Psi_{k_1,\ldots,k_n}(x_1,\ldots,x_n)|^2$, or

$$p_{n,L}(x_1,\ldots,x_n) = \frac{1}{n!}\det\|\phi_{k_i,L}(x_j)\|_{1\leqslant i,j\leqslant n}\overline{\det\|\phi_{k_i,L}(x_j)\|_{1\leqslant i,j\leqslant n}}.$$

Since $\det A \det B = \det A\, B$, one gets

$$p_{n,L}(x_1,\ldots,x_n) = \frac{1}{n!}\det\|K_{n,L}(x_i,x_j)\|_{1\leqslant i,j\leqslant n},$$

where $K_{n,L}(x,y) = \sum_{1\leqslant i\leqslant n}\phi_{k_i,L}(x)\overline{\phi_{k_i,L}(y)}$ is the kernel of orthogonal projection on the $\mathrm{Env}\{\phi_{k_1,L},\ldots,\phi_{k_n,L}\}$.

Since the k-point *marginal* correlation functions are

$$p^{(k)}_{n,L}(x_1,\ldots,x_n) := \frac{n!}{(n-k)!}\int p_{n,L}(x_1,\ldots,x_n)dx_{k+1},\ldots,dx_n = \\ = \det\|K_{n,L}(x_i,x_j)\|_{1\leqslant i,j\leqslant k},$$

the *determinantal* RPP $\mu^F_{n,L}$ generated by the joint probability distribution density $p_{n,L}$ is correctly defined for n free fermions in the cube Λ_L.

Canonical Ensemble: Probability density distribution of n free-fermion positions in the cube Λ_L

$$p_{n,L}(x_1,\ldots,x_n;\beta) := Z^{-1}_{\Lambda,F}(\beta,n)\times \\ \times \sum_{(k_1,\ldots,k_n)\in(\mathbb{Z}^n)}\overline{\Psi_{k_1,\ldots,k_n}(x_1,\ldots,x_n)}\left(\bigotimes^n G_{\beta,L}\Psi_{k_1,\ldots,k_n}\right)(x_1,\ldots,x_n).$$

Proposition [2]: Let $(x_1,\ldots,x_n) \mapsto \xi := \sum_{1\leqslant j\leqslant n} \delta_{x_j} \in Q(\Lambda_L)$. Then $p_{n,L}(x_1,\ldots,x_n;\beta)$ induces a *determinantal* RPP $\mu^F_{\beta,n,L}$ with matrix

$$K_{\beta,n,L}(x_i,x_j) := (G_{\beta,L})(x_i,x_j),$$

i.e., a probability measure $d\mu^F_{\beta,n,L}(\xi)$ on the configuration space $Q(\Lambda_L)$.

Laplace Transformation: Let $\langle \xi, f\rangle := \sum_{1\leqslant j\leqslant n} f(x_j)$, where *nonnegative* $f \in C_0(\Lambda_L)$. Then for $\widetilde{G}_{\beta,L} := \sqrt{G_{\beta,L}}\ \mathrm{e}^{-f}\ \sqrt{G_{\beta,L}}$,

$$\mathbb{E}_{\beta,n,L}(\mathrm{e}^{-\langle\xi,f\rangle}) := \int\limits_{Q(\Lambda_L)} d\mu^F_{\beta,n,L}(\xi)\ \mathrm{e}^{-\langle\xi,f\rangle} =$$

$$= \int\limits_{\Lambda_L^n} dx_1\ldots dx_n\ p_{n,L}(x_1,\ldots,x_n;\beta)\exp\left\{-\sum_{1\leqslant j\leqslant n} f(x_j)\right\} =$$

$$= \int\limits_{\Lambda_L^n} dx_1\ldots dx_n \det\|(\widetilde{G}_{\beta,L})(x_i,x_j)\| / \int\limits_{\Lambda_L^n} dx_1\ldots dx_n \det\|(G_{\beta,L})(x_i,x_j)\|.$$

Example: For the Poisson RPP, one obtains

$$\int\limits_{Q(\Lambda)} d\pi_\eta(\xi)\,\mathrm{e}^{-\langle\xi,f\rangle} = \int\limits_{Q(\Lambda)} d\pi_\eta(\xi)\exp\left[-\sum_{x\in\xi} f(x)\right] =$$

$$= \sum_{n=0}^{\infty} \mathbb{E}_{\pi_\eta}\left(\prod_{1\leqslant j\leqslant n} \mathbb{I}_{|\xi\cap dx_j|=1}\right)\exp\left[-\sum_{x_j} f(x_j)\right] =$$

$$= \sum_{n=0}^{\infty} \frac{1}{n!}\int\limits_{\Lambda^n} \nu(dx_1)\ldots\nu(dx_n)\eta^n \exp\left[-\sum_{1\leqslant j\leqslant n} f(x_j)\right] =$$

$$= \exp\left[-\int\limits_{\Lambda} dx\,\eta(1-\mathrm{e}^{-f(x)})\right].$$

Thermodynamic Limit [2]: For $n/L^d \to \rho$ a weak limit of the RPP: $w - \lim_{L\to\infty} \mu^F_{\beta,n,L} = \mu^F_{\beta,\rho}$, exists and

$$\int\limits_{Q(\mathbb{R}^d)} d\mu^F_{\beta,\rho}(\xi)\,\mathrm{e}^{-\langle\xi,f\rangle} = \mathrm{Det}\left[I - \sqrt{1-\mathrm{e}^{-f}}\,z_* G_\beta (I + z_* G_\beta)^{-1}\sqrt{1-\mathrm{e}^{-f}}\right],$$

$$\rho = \int\limits_{\mathbb{R}^d} \frac{d^d q}{(2\pi)^d}\,\frac{z_*\,\mathrm{e}^{-\beta\|q\|^2}}{1 + z_*\,\mathrm{e}^{-\beta\|q\|^2}} = (z_* G_\beta (I + z_* G_\beta)^{-1})(x,x).$$

For a Tr-class integral operator J on $L^2(\Lambda,\nu)$, the ***Fredholm determinant/permanent*** is defined by the Vere-Jones formula [1]:

$$\mathrm{Det}([I-\alpha J]^{-1/\alpha}) = \sum_{s=0}^{\infty}\int_{\Lambda^s} \nu^{\otimes s}(dx_1\ldots dx_n)\,\mathrm{det}_\alpha \|J(x_i,x_j)\|_{1\leqslant i,j\leqslant n},$$

where $\mathrm{det}_{\alpha=\pm 1} = \mathrm{per}/\det$.

2.2. Quantum Statistical Mechanics: Bosons

Grand-Canonical Ensemble: Probability density distribution of n free-boson positions in the cube Λ_L is defined by

$$p_{n,L}(x_1,\ldots,x_n;\beta) := Z^{-1}_{\Lambda,B}(\beta,n)\times$$
$$\times \sum_{(k_1,\ldots,k_n)\in(\mathbb{Z}^n)} \overline{\Psi_{k_1,\ldots,k_n}(x_1,\ldots,x_n)}\left(\bigotimes^n G_{\beta,L}\Psi_{k_1,\ldots,k_n}\right)(x_1,\ldots,x_n),$$

$$\Psi_{k_1,\ldots,k_n}(x_1,\ldots,x_n) = \frac{1}{\sqrt{n!\prod_l n(k_l)!}}\,\mathrm{per}\|\phi_{k_i,L}(x_j)\|_{1\leqslant i,j\leqslant n}.$$

The boson RPP $d\mu^B_{\beta,n,L}(\xi)$ on the configuration space $Q(\Lambda_L)$ is implied by $p_{n,L}$. In the (*grand-*)canonical thermodynamic limit for particle densities $\rho<\rho_c(\beta)$ (or solutions $z_*(\beta,\rho)<1$), where

$$\rho = \int_{\mathbb{R}^d} \frac{d^d q}{(2\pi)^d}\,\frac{z_*\,\mathrm{e}^{-\beta\|q\|^2}}{1-z_*\,\mathrm{e}^{-\beta\|q\|^2}} = (z_*G_\beta(I-z_*G_\beta)^{-1})(x,x) < \rho_c(\beta),$$

one obtains [3]

$$\int_{Q(\mathbb{R}^d)} d\mu^B_{\beta,\rho}(\xi)\,\mathrm{e}^{-\langle\xi,f\rangle} = \mathrm{Det}\left[I+\sqrt{1-\,\mathrm{e}^{-f}}z_*G_\beta(I-z_*G_\beta)^{-1}\sqrt{1-\,\mathrm{e}^{-f}}\right]^{-1}.$$

Proposition [4]: For densities $\rho>\rho_c(\beta)$ we have $z_*=1$ and

$$\int_{Q(\mathbb{R}^d)} d\mu^B_{\beta,\rho}(\xi)\,\mathrm{e}^{-\langle\xi,f\rangle} = \frac{\exp\left[-(\rho-\rho_c(\beta))(\sqrt{1-\,\mathrm{e}^{-f}},[I+K_f]^{-1}\sqrt{1-\mathrm{e}^{-f}})\right]}{\mathrm{Det}\,[I+K_f]},$$

where $K_f := \sqrt{1-\mathrm{e}^{-f}}G_\beta(I-G_\beta)^{-1}\sqrt{1-\mathrm{e}^{-f}}$ is from the Tr-class. Therefore, the free boson RPP for $\rho>\rho_c(\beta)$ is a convolution of the boson RPP at $z_*=1$ and a boson process (see *numerator*) proportional to the condensate density: $\rho-\rho_c(\beta)$.

2.3. Grand-Canonical (β, μ) Free Bose Gas. (a) Consider *independent* random variables $k \mapsto N_k \in \mathbb{N} \cup \{0\}$, $k \in \Lambda^*{}_L$, in the probability space $\Omega := \times_{k \in \Lambda^*{}_L} \Omega_k$.

(b) For *bosons* the one-mode random *occupation* numbers are $N_k \geqslant 0$, but for *fermions* they are $N_k = 0, 1$.

(c) *Probabilities* (N. B. for *bosons*: $\mu < 0$, since $\varepsilon_k = \|k\|^2 \geqslant 0$) are

$$\Pr_{\beta,\mu}(N_k) := \frac{e^{-\beta(\varepsilon_k - \mu) N_k}}{\Xi_k(\beta, \mu)}, \quad k \in \Lambda^*{}_L.$$

(d) *Expectations* are: $\mathbb{E}_{\beta,\mu}(N_k) = \{e^{\beta(\varepsilon_k - \mu)} - 1\}^{-1}$, for $k \in \Lambda^*{}_L$ and $z_* := e^{-\beta\,\mu}$.

(e) Expectation value of the *total* density of bosons in $\mathbb{R}^d$ is

$$\lim_{L\to\infty} \rho_{\Lambda_L}(\beta, \mu) := \lim_{L\to\infty} \frac{1}{|\Lambda_L|} \sum_{k \in \Lambda^*} \mathbb{E}_{\beta,\mu}(N_k) = \int_0^\infty \frac{d\tilde{\mathcal{N}}_d(E)}{e^{\beta(E-\mu)} - 1}.$$

3. BOSONS IN A WEAK HARMONIC TRAP

3.1. Weak Harmonic Trap [5]. One-particle Hamiltonian of the harmonic oscillator

$$h_\kappa = \frac{1}{2} \sum_{j=1}^{d} \left(-\frac{\partial^2}{\partial x_j^2} + \frac{x_j^2}{\kappa^2} - \frac{1}{\kappa} \right)$$

is a self-adjoint operator in the Hilbert space $\mathfrak{H} := L^2(\mathbb{R}^d)$, with

$$\mathrm{Spec}\,(h_\kappa) = \{\epsilon_\kappa(s) := |s|_1/\kappa \,|\, s = (s_1, \cdots, s_d) \in \mathbb{N}^d\}, \quad |s|_1 := \sum_{j=1}^{d} s_j.$$

In this setup the «thermodynamic limit» is an «opening» of the trap, i.e., $\kappa \to \infty$, called the Weak Harmonic Trap (WHT) limit.

Perfect Bose-gas expectation value of the ***total*** number of particles is

$$N_\kappa(\beta, \mu) = \frac{1}{\beta} \frac{\partial \ln \Xi_{0,\kappa}(\beta, \mu)}{\partial \mu} = \sum_{s \in \mathbb{N}^d} \frac{1}{e^{\beta(\epsilon_\kappa(s) - \mu)} - 1}.$$

Since $N_\kappa(\beta, \mu)$ diverges for $\kappa \to \infty$ as κ^d, the scaled particle density is defined by

$$\rho_\kappa(\beta, \mu) := \frac{1}{\kappa^d} \sum_{s \in \mathbb{N}^d} \frac{1}{e^{\beta(\epsilon_\kappa(s) - \mu)} - 1},$$

$$\rho(\beta,\mu) = \lim_{\kappa\to\infty} \rho_\kappa(\beta,\mu) = \int\limits_{[0,\infty)^d} \frac{dp}{\mathrm{e}^{\beta(|p|_1-\mu)}-1} = \sum_{s=1}^{\infty} \frac{\mathrm{e}^{\beta\mu s}}{(\beta s)^d}.$$

Notice that the *Integrated Density of States* ($\mathcal{N}_d(E)$) and the *critical density* $\rho_c(\beta)$ are related by the limit of the measure:

$$\mathcal{N}_{d,\kappa}(E) = \frac{1}{\kappa^d} \sum_{s\in\mathbb{N}^d} \theta(E - |s|_1/\kappa).$$

Then we obtain in the $\kappa \to \infty$ limit

$$d\mathcal{N}_d(E) = \frac{E^{d-1}}{\Gamma(d)}\, d\,E \neq \frac{E^{(d-2)/2}}{(2\pi)^{d/2}\Gamma(d/2)}\, dE = d\tilde{\mathcal{N}}_d(E),$$

$$\rho_c(\beta) := \zeta(d)/\beta^d \neq \zeta(d/2)/(2\pi\beta)^{d/2} =: \tilde{\rho}_c(\beta).$$

3.2. Mean-Field Interaction and Main Results. A model of the mean-field interacting bosons trapped by the harmonic potential is defined by the *grand-canonical* partition function

$$\Xi_{\lambda,\kappa}(\beta,\mu) := \sum_{n=0}^{\infty} \mathrm{e}^{\beta(\mu n - \lambda n^2/2\kappa^d)} \mathrm{Tr}_{\mathfrak{H}^{\mathrm{n}}_{\mathrm{symm}}}[\otimes^n G_\kappa(\beta)],$$

where $G_\kappa(\beta) = \mathrm{e}^{-\beta h_\kappa}$ is the Gibbs semigroup for the oscillator process. Here $\beta > 0$, $\lambda > 0$ and $\mu \in \mathbb{R}^1$.

Theorem [5]: *Normal phase*. Let $\mu < \mu_{\lambda,c}(\beta) := \lambda\rho_c(\beta)$. Then the boson RPP $\mu_{\kappa,\beta,\mu}$ converges weakly in the WHT limit $\kappa \to \infty$ to the RPP μ_{β,r_*} with the Laplace transformation:

$$\mathbb{E}_{\beta,r_*}\left[\mathrm{e}^{-\langle f,\xi\rangle}\right] = \mathrm{Det}\left[1 + \sqrt{1-\mathrm{e}^{-f}}\, r_* G_\beta (1 - r_* G_\beta)^{-1} \sqrt{1-\mathrm{e}^{-f}}\right]^{-1},$$

where $r_* = r_*(\beta,\mu,\lambda) \in (0,1)$ is a unique solution of the equation

$$\beta\mu = \ln r + \lambda\beta \int\limits_0^\infty \frac{d\mathcal{N}_d(E)}{r^{-1}\mathrm{e}^{\beta\,E}-1}, \quad r := \mathrm{e}^{\beta(\mu-\lambda\rho)} < 1.$$

Theorem [5]: *Condensed phase*. For $\mu > \mu_{\lambda,c}(\beta) (:= \lambda\rho_c(\beta))$ the Laplace transformation of the boson RPP measure has the following limit:

$$\lim_{\kappa\to\infty} \frac{1}{\kappa^{d/2}} \ln \mathbb{E}_{\beta,\mu}\left[\mathrm{e}^{-\langle f,\xi\rangle}\right] = -\frac{\mu - \mu_{\lambda,c}(\beta)}{\pi^{d/2}\lambda} (\sqrt{1-\mathrm{e}^{-f}}, (1+K_f)^{-1}\sqrt{1-\mathrm{e}^{-f}}),$$

where the operator

$$K_f := \left(G_\beta^{1/2}(1-G_\beta)^{-1/2}\sqrt{1-\mathrm{e}^{-f}}\right)^* \left(G_\beta^{1/2}(1-G_\beta)^{-1/2}\sqrt{1-\mathrm{e}^{-f}}\right)$$

is a positive trace-class operator on $\mathfrak{H} = L^2(\mathbb{R}^d)$ for $d > 2$.

Remark: *Condensed phase*. Similar to the homogeneous free Bose gas the resulting RPP is a *convolution* of ***two*** Bose RPP [5].

Local Particle Density.

Corollary: *Normal phase*. Let $f \in C_0(\mathbb{R}^d)$ and $f \geqslant 0$. For $\mu < \mu_{\lambda,c}(\beta)$

$$\mathbb{E}_{\beta,r_*}\left[\langle f, \xi\rangle\right] = \mathrm{Tr}\left[f\, r_* G(\beta)(1 - r_* G(\beta))^{-1}\right] = \rho_{r_*} \int\limits_{\mathbb{R}^d} dx f(x),$$

where the local density ρ_{r_*} in the neighbourhood of the bottom of the WHT potential is given by

$$\rho_{r_*} = r_* G(\beta)(1 - r_* G(\beta))^{-1}(x, x) = \sum_{n=1}^{\infty} \frac{r_*^n}{(2\pi\beta n)^{d/2}}.$$

Corollary: *Condensed phase*. For $\mu > \mu_{\lambda,c}(\beta)$ one obtains

$$\liminf_{\kappa\to\infty} \frac{\mathbb{E}_{\kappa,\beta,\mu,\lambda}\left[\langle f, \xi\rangle\right]}{\kappa^{d/2}} \geqslant \frac{\mu - \mu_{\lambda,c}(\beta)}{\pi^{d/2}\lambda} \int\limits_{\mathbb{R}^d} dx f(x).$$

3.4. Global Particle Density. The results of the Theorem and Corollary in the *noncondensed* regime has the following interpretation: in the WHT limit the position distribution of the MF interacting bosons in the neighbourhood of the ***origin*** of coordinates (i.e., at the bottom of the WHT potential) is close to that for the free BG corresponding to a substitution of the *unconventional* parameter r_* by the conventional z_*. The information about the particle position distribution in domains distant from the bottom of the WHT is missing in the limit μ_{β,r_*} since the test function f has a finite support.

In order to take this «tail»-particles into account, we have to use for our model the standard definition of the grand-canonical global number of particles:

$$\rho^{(\mathrm{tot})}_{\kappa,\lambda}(\beta,\mu) := \frac{1}{\kappa^d\,\beta}\frac{\partial \ln \Xi_\kappa(\beta,\mu)}{\partial\mu} =$$

$$\frac{1}{\kappa^d\,\Xi_{\kappa,\lambda}(\beta,\mu)} = \sum_{n=0}^{\infty} n\,\mathrm{e}^{\beta(\mu n - \lambda n^2/2\kappa^d)}\mathrm{Tr}_{\mathfrak{H}^n_{\mathrm{symm}}}[\otimes^n G_\kappa(\beta)].$$

Since κ^d is interpreted as the effective volume of the model, the function $\rho_{\kappa,\lambda}^{(\mathrm{tot})}(\beta,\mu)$ represents an effective total space-averaged density of the nonhomogeneous boson gas.

Theorem [5]: *Global density = experiment*. In the WHT limit

$$\rho_{\lambda}^{(\mathrm{tot})}(\beta,\mu) = \lim_{\kappa\to\infty} \rho_{\kappa,\lambda}^{(\mathrm{tot})}(\beta,\mu) = \lim_{\kappa\to\infty} \kappa^{-d}\mathrm{Tr}\,[r_* G_\kappa (1 - r_* G_\kappa)^{-1}]$$

exists and satisfies the following properties:

(i) for $\mu \leqslant \mu_{\lambda,c}(\beta)$ one has

$$\rho_{\lambda}^{(\mathrm{tot})}(\beta,\mu) = \int\limits_0^\infty \frac{d\mathcal{N}_d(E)}{r_*^{-1}\mathrm{e}^{\beta E} - 1} \text{ and } \beta\mu = \log r_* + \lambda\beta\rho_{\lambda}^{(\mathrm{tot})}(\beta,\mu);$$

(ii) for $\mu > \mu_{\lambda,c}(\beta)$: $(\rho_c^{(\mathrm{tot})}(\beta) := \lim_{\mu\to\mu_c(\beta)} \rho_{\lambda}^{(\mathrm{tot})}(\beta,\mu) = \zeta(d)/\beta^d)$

$$\rho_{\lambda}^{(\mathrm{tot})}(\beta,\mu) = \frac{\mu}{\lambda} = \frac{\mu - \mu_{\lambda,c}(\beta)}{\lambda} + \rho_c^{(\mathrm{tot})}(\beta).$$

3.5. Conclusion: Bosons in a Weak Harmonic Trap. Different behaviour of the space distributions of bosons described in the Theorems above has the following explanation:

In the *normal case* the bosons are distributed almost uniformly in the region of radius κ according to the shape of the oscillator process kernel.

On the other hand, in the *condensed phase case* the condensed part of particles $\kappa^d(\rho_{\lambda}^{(\mathrm{tot})}(\beta,\mu) - \rho_{\lambda,c}^{(\mathrm{tot})}(\beta)) = \kappa^d(\mu - \mu_{\lambda,c}(\beta))/\lambda$ is localized in the region of radius $O(\kappa^{1/2})$ according to profile of the square of the ground-state wave function

$$\Omega_\kappa(x) = \frac{1}{(\pi\kappa)^{d/4}}\mathrm{e}^{-\|x\|^2/2\kappa} \equiv \phi_{s=0,\,\kappa}(x).$$

Whereas the particles outside of the condensate are spread out essentially over the region of radius κ.

4. LDP FOR NONINTERACTING BRPP

In this section we consider the limiting theorems: Law of Large Numbers (LLN), Central Limit Theorem (CLT) and Large Deviation Principle (LDP), for the free Bose gas [6].

4.1. Noninteracting BRPP with BEC. Proposition [4]: For continuous $f \geqslant 0$ with a compact support we define two BRPP by generating functionals:

$$\int\limits_{Q(\mathbb{R}^d)} d\mu_{K,z}^{(\mathrm{det})}(\xi)\,\mathrm{e}^{-\langle f,\xi\rangle} = \det[1+K_f(z)]^{-1}, \quad z=\mathrm{e}^{\beta\mu}\leqslant 1,$$

$$\int\limits_{Q(\mathbb{R}^d)} d\mu_{K,\rho}(\xi)\mathrm{e}^{-\langle f,\xi\rangle} = \exp\left[-\rho\Big(\sqrt{1-\mathrm{e}^{-f}}, \frac{1}{1+K_f(1)}\sqrt{1-\mathrm{e}^{-f}}\Big)\right],$$

where $K_f(z) := \sqrt{1-\mathrm{e}^{-f}}zG_\beta(1+zG_\beta)^{-1}\sqrt{1-\mathrm{e}^{-f}}$ and $G_\beta := \mathrm{e}^{\beta\Delta}$. Then the BRPP for the *ideal* gas is $\mu^B_{K,\rho\leqslant\rho_c} = \mu^{(\mathrm{det})}_{K,z\leqslant 1}$, but in the regime of BEC ($\rho > \rho_c$) it is *convolution* of the two Random Point Processes:

$$\mu^B_{K,\rho>\rho_c} := \mu^{(\mathrm{det})}_{K,z=1} * \mu_{K,\rho=\rho-\rho_c} = \text{ (non-condensate)}*\text{(condensate).}$$

Theorem (LLN) [6]: For continuous function $f \geqslant 0$ with a compact support, the limit

$$\lim_{\kappa\to\infty}\frac{1}{\kappa^d}\langle f(\cdot/\kappa),\xi\rangle = \rho\int\limits_{\mathbb{R}^d} dx\, f(x)$$

holds in $L^2(Q(\mathbb{R}^d),\mu^B_{K,\rho})$.

Theorem (CLT) [6]: Let $\rho > \rho_c$. Then for $\kappa \to \infty$ the family of random variables

$$X_\kappa := \frac{\langle f(\,\cdot/\kappa),\xi\rangle - \rho\kappa^d \int\limits_{\mathbb{R}^d} f(x)\,dx}{\sqrt{2(\rho-\rho_c)}\|(-\beta\Delta)^{-1/2}f\|_{HS}\kappa^{(d+2)/2}}$$

converges in distribution to the standard Gaussian random variable:

$$\lim_{\kappa\to\infty}\int\limits_{Q(\mathbb{R}^d)} d\mu^B_{K,\rho>\rho_c}(\xi)\mathrm{e}^{itX_\kappa} = \mathrm{e}^{-t^2/2}.$$

Large Deviation Principle in the BEC Regime.

Theorem (LDP) [6]: For $\rho > \rho_c$ there exists a convex rate function $I(s) := \sup_{s\in\mathbb{R}}\big(st - P(t)\big)$, such that

$$\limsup_{\kappa\to\infty}\frac{1}{\kappa^{d-2}}\log\mu^B_{K,\rho}\Big(\frac{1}{\kappa^d}\big\langle f\big(\,\cdot/\kappa\big),\ \xi\big\rangle\in F\Big) \leqslant -\inf_{s\in F} I(s),\ \text{ for closed } F\subset\mathbb{R},$$

and

$$\liminf_{\kappa\to\infty}\frac{1}{\kappa^{d-2}}\log\mu^B_{K,\rho}\Big(\frac{1}{\kappa^d}\big\langle f\big(\,\cdot/\kappa\big),\ \xi\big\rangle\in G\Big) \geqslant -\inf_{s\in G} I(s),\ \text{ for open } G\subset\mathbb{R}.$$

$$P(t) := \lim_{\kappa\to\infty} \frac{1}{\kappa^{d-2}} \log \int\limits_{Q(\mathbb{R}^d)} d\mu^B_{K,\rho}(\xi) \mathrm{e}^{t\langle f(\,\cdot/\kappa),\xi\rangle/\kappa^2} = P^{(\mathrm{det})}_{K,z=1}(t) + P_{K,\rho-\rho_c}(t) =$$

$$= \begin{cases} t\rho_c \int\limits_{\mathbb{R}^d} f(x)\,dx + (\rho-\rho_c)t^2(f, (-\beta\Delta - tf)^{-1}f), & t < \|\sqrt{f}(-\beta\Delta)^{-1}\sqrt{f}\|^{-1}, \\ +\infty, & t \geqslant \|\sqrt{f}(-\beta\Delta)^{-1}\sqrt{f}\|^{-1}. \end{cases}$$

4.3. Conclusion: BEC versus the Normal Phase. Let $D_\kappa := \langle f(\cdot/\kappa), \xi\rangle/\kappa^d$ be a *random empirical density* of particles localized in the region of the length scale κ.

For the BEC case $\rho > \rho_c$***:***

(i) The random variable D_κ converges for $\kappa \to \infty$ to its expectation value $m := \rho \int\limits_{\mathbb{R}^d} f(x)\,dx$ in *mean*.

(ii) The law of the random variable $\kappa^{(d-2)/2}(D_\kappa - m)$ converges to the *normal* distribution as $\kappa \to \infty$.

(iii) The law of the random variable D_κ manifests a Large Deviation Property with the parameter κ^{d-2}.

For the normal phase $\rho \leqslant \rho_c$***:***

(i) also holds;

(ii) holds but for $\kappa^{d/2}(D_\kappa - m)$, instead of $\kappa^{(d-2)/2}(D_\kappa - m)$;

(iii) holds with the order κ^d, instead of κ^{d-2}.

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