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TOWARDS LAGRANGIAN FORMULATIONS OF MIXED-SYMMETRY HIGHER-SPIN FIELDS ON AdS-SPACE WITHIN BFV-BRST FORMALISM

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The spectrum of superstring theory on the $AdS_5 \times S_5$ Ramond–Ramond background in tensionless limit contains integer and half-integer higher-spin fields subject at most to two-rows Young tableaux $Y(s_1, s_2)$. We review the details of a gauge-invariant Lagrangian description of such massive and massless higher-spin fields in anti-de-Sitter spaces with arbitrary dimensions. The procedure is based on the construction of Verma modules, its oscillator realizations and of a BFV-BRST operator for nonlinear algebras encoding unitary irreducible representations of AdS group.

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1. INTRODUCTION

Launch of LHC on the rated capacity assumes not only the answer to the question of existence of Higgs boson, the proof of supersymmetry display and a new insight on the origin of Dark Matter, but permits one to reconsider the problems of a unique description of variety of elementary particles and all known interactions. In this relation, the development of higher-spin (HS) field theory in view of its close relation to superstring theory on constant curvature spaces, which operates with an infinite set of massive and massless bosonic and fermionic HS fields subject to multirow Young tableaux (YT) $Y(s_1, \ldots, s_k)$, $k \geqslant 1$ (see for a review [1]), seems to be actual one. The paper considers the last results of constructing Lagrangian formulations (LFs) for free integer and half-integer HS fields on AdS_d space with $Y(s_1, s_2)$ in Fronsdal metric-like formalism within BFV-BRST approach [2] as a starting point for an interacting HS field theory in the framework of conventional Quantum Field Theory, and in part based on the results presented in [3–6].

This method of Lorentz-covariant constructing LF for HS fields, developed originally in a way that applies to Hamiltonian quantization of gauge theories with a given LF, consists in a solution of the *problem inverse* to that of the method [2] (as in the case of string field theory [7]) in the sense of constructing a classical gauge LF with respect to a nilpotent BFV–BRST operator Q.

In detail, the solution of *inverse problem* includes 4 steps:

• the realization of initial irrep conditions of AdS group, that extract the fields with a definite mass m and generalized spin $\mathbf{s} = (s_1, \ldots, s_k)$ [8] as operator mixed-class constraints o_I in a special Fock space $\mathcal{H}$;

• the additive conversion (following to [9]) of algebra o_I into one of O_I: $O_I = o_I + o'_I$, $[o_I, o'_J\} = 0$, determined on wider Fock space, $\mathcal{H} \otimes \mathcal{H}'$ with only first-class constraints $O_\alpha \subset O_I$;

• the construction of the Hermitian nilpotent BFV-BRST operator Q' for nonlinear superalgebra of converted operators O_I which contains the BFV-BRST operator Q for only subsystem of O_α;

• the finding of Lagrangian $\mathcal{L}$ for given HS field through corresponding scalar product $\langle\,|\rangle$ like $\mathcal{L} \sim \langle\chi|Q|\chi\rangle$, to be invariant with respect to gauge transformations $\delta|\chi\rangle = Q|\Lambda\rangle$ with $|\chi\rangle$ containing initial HS field.

As compared to application of above algorithm for bosonic [10] and fermionic [4, 11] HS fields on $\mathbf{R}^{1,d-1}$ with standard resolution of the 2nd and 3rd steps due to the same Lie (super)algebra structure for o_I, o'_I, O_I: $[o_I, o_J\} = f^K_{IJ} o_K$, their resolution already for totally-symmetric HS fields on AdS_d space [3, 12] is not so easy. It is revealed on stages of Verma module (VM) construction for o'_I and its nonpolynomial(!) oscillator realization in $\mathcal{H}'$ because of AdS radius $r^{-1/2}$ ($r = R/d(d-1)$ for scalar curvature R) [3, 12]. In turn, a construction of BFV-BRST operator Q' does not have the Lie-algebra form, $Q' = \mathcal{C}^I O_I + (1/2)\mathcal{C}^I\mathcal{C}^J f^K_{JI}\mathcal{P}_K$ for (super)algebra of O_I in transiting to AdS space.

The main goals of the paper are to apply the above strategy to construct LFs for bosonic and fermionic HS fields on AdS_d spaces subject to $Y(s_1, s_2)$.

2. BOSONIC FIELDS IN AdS SPACES

A massive integer spin $\mathbf{s} = (s_1, s_2)$ ($s_1 \geqslant s_2$) representation of the AdS group in an AdS_d space is realized in a space of mixed-symmetry tensors,

$$\Phi_{(\mu)_{s_1},(\nu)_{s_2}} \equiv \Phi_{\mu_1\ldots\mu_{s_1},\nu_1\ldots\nu_{s_2}}(x) \longleftrightarrow \begin{array}{|c|c|c|c|c|c|c|c|c|c|}\hline \mu_1 & \mu_2 & \cdot\;\cdot\;\cdot & \cdot & \cdot & \cdot & \cdot & \mu_{s_1} \\ \hline \nu_1 & \nu_2 & \cdot\;\cdot\;\cdot & \cdot & \cdot & \nu_{s_2} \\ \cline{1-6}\end{array}\,, \tag{1}$$

subject to the Klein–Gordon (2) divergentless, traceless and mixed-symmetry equations (3) (for $\beta = (2;3) \Longleftrightarrow (s_1 > s_2; s_1 = s_2)$):

$$\left[\nabla^2 + r[(s_1 - \beta - 1 + d)(s_1 - \beta) - s_1 - s_2] + m^2\right]\Phi_{(\mu)_{s_1},(\nu)_{s_2}} = 0, \tag{2}$$

$$\left(\nabla^{\mu_1}, \nabla^{\nu_1}, g^{\mu_1\mu_2}, g^{\nu_1\nu_2}, g^{\mu_1\nu_1}\right)\Phi_{(\mu)_{s_1},(\nu)_{s_2}} = \Phi_{\{(\mu)_{s_1},\nu_1\}\nu_2\ldots\nu_{s_2}} = 0. \tag{3}$$

To obtain HS symmetry algebra (of o_I) for a description of all integer HS fields, we introduce a Fock space $\mathcal{H}$, generated by 2 pairs of creation $a^i_\mu(x)$ and annihilation $a^{j+}_\mu(x)$ operators, $i,j=1,2,\mu,\nu=0,1,\ldots,d-1$: $[a^i_\mu, a^{j+}_\nu]=-g_{\mu\nu}\delta_{ij}$, and a set of constraints for an arbitrary string-like vector $|\Phi\rangle \in \mathcal{H}$,

$$|\Phi\rangle = \sum_{s_1=0}^{\infty}\sum_{s_2=0}^{s_1}\Phi_{(\mu)_{s_1},(\nu)_{s_2}}(x)a_1^{+\mu_1}\cdots a_1^{+\mu_{s_1}}a_2^{+\nu_1}\cdots a_2^{+\nu_{s_2}}|0\rangle, \quad (4)$$

$$\tilde{l}_0|\Phi\rangle = (l_0+\tilde{m}_b^2+r((g_0^1-2\beta-2)g_0^1-g_0^2))|\Phi\rangle=0,\ l_0=\left[D^2-r\frac{d(d-6)}{4}\right], \quad (5)$$

$$(l^i,l^{ij},t)|\Phi\rangle = \left(-ia^i_\mu D^\mu, \frac{1}{2}a^i_\mu a^{j\mu}, a^{1+}_\mu a^{2\mu}\right)|\Phi\rangle=0,\ i\leqslant j, \quad (6)$$

with number particles operators, $g_0^i=-(1/2)\{a^{i+}_\mu, a^{\mu i}\}$, central charge $\tilde{m}_b^2=m^2+r\beta(\beta+1)$, operator $D_\mu=\partial_\mu-\omega^{ab}_\mu(x)\big(\sum_i a^+_{ia}a_{ib}\big)$, $a_i^{(+)\mu}(x)=e^\mu_a(x)a_i^{(+)a}$ with vielbein e^μ_a, spin connection ω^{ab}_μ, tangent indices a,b. Operator D_μ is equivalent in its action in $\mathcal{H}$ to the covariant derivative ∇_μ (with d'Alambertian $D^2=(D_a+\omega^b{}_{ba})D^a$). The set of 7 primary constraints (5), (6) with $\{o_\alpha\}=\{\tilde{l}_0,l^i,l^{ij},t\}$ are equivalent to Eqs. (2), (3) for all spins.

For Hermiticity of BFV-BRST operator (reality Lagrangian $\mathcal{L}$), the algebra with o_α must be enlarged by adding the operators $\left(l^+_i,l^+_{ij},t^+\right)$, resulting the HS symmetry algebra in AdS_d space with $Y(s_1,s_2)$, denoted as $A(Y(2),AdS_d))$. The Lie subalgebra of operators $l_{ij},t,g^i_0,l^+_{ij},t^+$ is isomorphic to $sp(4)$, whereas the nontrivial quadratic commutators in $A(Y(2),AdS_d))$ are due to operators with D_μ: $l_i,\tilde{l}_0,l_i^+$. For the aim of LF construction it is enough to have a simpler, (the so-called *modified*) algebra $A_{\mathrm{mod}}(Y(2),AdS_d))$, with operator l_0 (5) instead of $\tilde{l}_0$, so that AdS-mass term, $\tilde{m}_b^2+r\big((g_0^1-2\beta-2)g_0^1-g_0^2\big)$, will be restored later within conversion and proper construction of LF. Algebra $A_{\mathrm{mod}}(Y(2),AdS_d))$ contains one first-class constraint l_0, four differential l_i,l_i^+, eight algebraic t,t^+,l_{ij},l^+_{ij} second-class constraints $\theta_{\mathbf{a}}$, operators g_0^i, composing an invertible matrix: $\|[\theta_{\mathbf{a}},\theta_{\mathbf{b}}\}\|=\|\Delta_{\mathbf{ab}}(g_0^i)\|+(o_I)$, and satisfies the nonlinear relations (additional to ones for $sp(4)$) given by the Table with the quantities $\mathcal{K}_1^{bk},W_b^{ki},X_b^{ki}$ to be quadratic in o_I (see [5] for details).

The nonlinear part of algebra $A_{\mathrm{mod}}(Y(2),AdS_d)$

$[\downarrow,\rightarrow\}$	t	t^+	l_0	l^i	l^{i+}	l^{ij}	l^{ij+}	g_0^i
l_0	0	0	0	$-r\mathcal{K}_1^{bi+}$	$r\mathcal{K}_1^{bi}$	0	0	0
l^k	$-l^2\delta^{k1}$	$-l^1\delta^{k2}$	$r\mathcal{K}_1^{bk+}$	W_b^{ki}	X_b^{ki}	0	$-(1/2)l^{\{i+}\delta^{j\}k}$	$l^i\delta^{ik}$
l^{k+}	$l^{1+}\delta^{k2}$	$l^{2+}\delta^{k1}$	$-r\mathcal{K}_1^{bk}$	$-X_b^{ik}$	$-W_b^{ki+}$	$(1/2)l^{\{i}\delta^{j\}k}$	0	$-l^{i+}\delta^{ik}$

3. VERMA MODULE, FOCK SPACE REALIZATION

To find additional parts o'_I within additive conversion for nonlinear algebra $A(Y(2), AdS_d)$ of o_I into algebra $A_c(Y(2), AdS_d)$ of converted constraints O_I, acting in $\mathcal{H}\otimes\mathcal{H}'$, we, first, determine the multiplication law for the algebra $\mathcal{A}'(Y(2), AdS_d)$, which for compactly written one for o_I reads (see [5, 12])

$$[o'_I, o'_J\} = f^K_{IJ}o'_K - f^{KM}_{IJ}o'_M o'_K \quad \text{if} \quad [o_I, o_J\} = f^K_{IJ}o_K + f^{KM}_{IJ}o_K o_M. \qquad (7)$$

Second, following generalization of Poincare–Birkhoff–Witt theorem, we construct VM, based on Cartan-like decomposition enlarged from one for $sp(4)$

$$\mathcal{A}'(Y(2), AdS_d) = \{l'^+_{ij}, t'^+, l'^+_i\} \oplus \{g'^i_0, l'_0\} \oplus \{l'_{ij}, t', l'_i\} \equiv \mathcal{E}^- \oplus H \oplus \mathcal{E}^+. \quad (8)$$

Note, that in contrast to the case of Lie (super)algebra and totally-symmetric HS fields on AdS space [3, 4, 12], the negative root vectors l'^+_1, t'^+, l'^+_2 are not commuted, making the arbitrary vector $|\mathbf{N}\rangle_V = |n_{11}, n_{12}, n_{22}, n_1, n, n_2\rangle_V$

$$|\mathbf{N}\rangle_V \equiv \left(l'^+_{11}\right)^{n_{11}}\left(l'^+_{12}\right)^{n_{12}}\left(l'^+_{22}\right)^{n_{22}}\left(\frac{l'^+_1}{m_1}\right)^{n_1}\left(t'^+\right)^{n}\left(\frac{l'^+_2}{m_2}\right)^{n_2}|0\rangle_V,\ \mathcal{E}^+|0\rangle_V = 0 \qquad (9)$$

from VM, (for the highest weight vector $|0\rangle_V$, $n_{ij}, n_i, n \in \mathbf{N}_0$, and arbitrary constants m_i with dimension of mass) by not proper one for t'^+, l'^+_2! That *nontrivial entanglement* is resolved within iterative procedure, so that the VM for algebra $\mathcal{A}'(Y(2), AdS_d)$ is constructed (see [6]) as well as its realization as formal power series (due to r) in degrees of creation and annihilation operators (B, B^+), $B = (b_i, b_{ij}, b)$ in $\mathcal{H}'$ whose number coincides to ones of $\theta_{\mathbf{a}}$.

4. BRST OPERATOR FOR NONLINEAR ALGEBRA

The system of O_I forming nonlinear algebra $\mathcal{A}_c(Y(2), AdS_d)$ with multiplication law following from Eqs. (7) (see the Table): $[O_I, O_J\} = F^K_{IJ}(o', O)O_K$ for Weyl ordering of quadratic combinations of O_I in right-hand side of $[O_I, O_J]$ [3, 5], now has no the form of closed algebra, because of presence of nontrivial Jacobi identities for 6 triples (L_1, L_2, L_0), (L^+_1, L^+_2, L_0), (L_i, L^+_j, L_0). Indeed, there exists a set of third-order structural functions in terminology of [2] resolving those identities (see [5]).

Therefore, BFV-BRST operator Q' has the terms in the 3rd degree in C^I [5],

$$Q' = Q'_1+Q'_2+\Bigg[r^2\eta_0\eta_i\eta_j\varepsilon^{ij}\Bigg\{\frac{1}{2}\sum_k\Bigg(G_0^k[\mathcal{P}\mathcal{P}^+_{22}-\mathcal{P}^+\mathcal{P}^+_{11}+i\mathcal{P}^+_{12}\sum_l(-1)^l\mathcal{P}^l_G]-$$

$$-\,i(L^+_{11}\mathcal{P}^+ - L^+_{22}\mathcal{P})\mathcal{P}^k_G + 4L^{kk}\mathcal{P}^+_{k2}\mathcal{P}^+_{1k}\Bigg) - L^+_{12}\mathcal{P}^1_G\mathcal{P}^2_G + 2L^{12}\mathcal{P}^+_{22}\mathcal{P}^+_{11}\Bigg\}+$$

$$+\,r^2\eta_0\eta_i^+\eta_j\Bigg\{\Bigg[\sum_k(-1)^k\frac{i}{2}G_0^k\sum_l\mathcal{P}^l_G + 2(L^+_{22}\mathcal{P}^{22} - L^{11}\mathcal{P}^+_{11})\Bigg]\mathcal{P}\delta^{1j}\delta^{2i}+$$

$$+\,\varepsilon^{\{1j}\delta^{2\}i}\Bigg(\imath\sum_k\Bigg[\frac{1}{2}T\mathcal{P}^+ - 2L^{12}\mathcal{P}^+_{12}(-1)^k\Bigg]\mathcal{P}^k_G + 2(L^{12}\mathcal{P} - T\mathcal{P}^{12})\mathcal{P}^+_{22}+$$

$$+\,2(T^+\mathcal{P}^{12} - L^{12}\mathcal{P}^+)\mathcal{P}^+_{11}\Bigg) - T\big[\mathcal{P}^1_G\mathcal{P}^2_G\delta^{2i}\delta^{1j} + 2\mathcal{P}^{11}\mathcal{P}^+_{22}\delta^{1i}\delta^{2j}\big] - 2\sum_k(-1)^k\times$$

$$\times\,\big[(G_0^k\mathcal{P}^{11} + \imath L^{11}\mathcal{P}^k_G)\delta^{1i}\delta^{2j} - (G_0^k\mathcal{P}^{22} + \imath L^{22}\mathcal{P}^k_G)\delta^{2i}\delta^{1j}\big]\mathcal{P}^+_{12}\Bigg\} + \text{h.c.}\Bigg], \quad (10)$$

with the standard form for linear Q'_1 and quadratic Q'_2 terms in ghosts $\mathcal{C}^{I}$*. The Hermiticity of Q' is defined by the rule: $Q'^+K = KQ'$, for operator $K = \hat{1}\otimes K'\otimes\hat{1}_{\text{gh}}$, with nondegenerate K' providing the Hermiticity of o'_I in $\mathcal{H}'$.

5. UNCONSTRAINED LAGRANGIAN FORMULATION

BFV-BRST operator Q for 1st-class constraints $O_\alpha = \{L_0, L^i, L^{ij}, T\}$ is extracted from Q' (10) by collecting the terms with ghosts η^i_G (see [5] for details)

$$Q' = Q + \eta^i_G(\sigma^i + h^i) + \mathcal{B}^i\mathcal{P}^i_G, \quad \sigma^i + h^i = G_0^i + \text{ghosts}. \quad (11)$$

The same is applied to a physical vector $|\chi\rangle \in \mathcal{H}_{\text{tot}} = \mathcal{H}\otimes\mathcal{H}'\otimes\mathcal{H}_{\text{gh}}$, $|\chi\rangle = |\Phi\rangle + |\Phi_A\rangle$, $|\Phi_A\rangle_{|\mathcal{H}} = 0$, with $|\Phi\rangle$ given in (4). From commutativity, $[Q, \sigma^k\} = 0$, and choice of a representation for Hilbert space (as in SFT [7])) it follows the spectral problem from the equation $Q'|\chi\rangle = 0$ [5],

$$Q|\chi\rangle = 0, \quad (\sigma^i + h^i)|\chi\rangle = 0, \quad \text{gh}\,(|\chi\rangle) = 0, \quad (12)$$

thus determining the spectrum of spin values $h^i(s_i) = -\Big(s_i + \frac{d-5}{2} - 2\delta^{i2}\Big)$ and proper eigenvectors $|\chi\rangle_{(s_1,s_2)}$. After substitution: $h^i \to h^i(s_i)$, operator $Q_{(s_1,s_2)}$

*Here $(\mathcal{C}^I, \mathcal{P}_I) = \big(\eta_0, \mathcal{P}_0\,; \eta^i_G, \mathcal{P}^i_G\,; \eta^+_i, \mathcal{P}_i\,; \eta_i, \mathcal{P}^+_i\,; \eta^+_{ij}, \mathcal{P}_{ij}\,; \eta_{ij}, \mathcal{P}^+_{ij}\,; \eta, \mathcal{P}^+\,; \eta^+, \mathcal{P}\big)$.

is nilpotent on each subspace $H_{\rm tot}^{(s_1,s_2)}$ whose vectors satisfy Eq. (12). Hence, the equations of motion (one to one correspond to Eqs. (2), (3)), a sequence of reducible gauge transformations and Lagrangian action have the form

$$Q_{(s_1,s_2)}|\chi^0\rangle_{(s_1,s_2)} = 0,\ \delta|\chi^l\rangle_{(s_1,s_2)} = Q_{(s_1,s_2)}|\chi^{1+1}\rangle_{(s_1,s_2)},\ l = 0,\ldots,6, \quad (13)$$

$$\mathcal{S}_{(s_1,s_2)} = \int d\eta_0\ {}_{(s_1,s_2)}\langle\chi^0|K_{(s_1,s_2)}Q_{(s_1,s_2)}|\chi^0\rangle_{(s_1,s_2)}, \mathrm{for}|\chi^0\rangle \equiv |\chi\rangle. \quad (14)$$

The corresponding LF for bosonic field with spin s subject to $Y(s_1, s_2)$ is a reducible gauge theory of maximally $L = 6$th stage of reducibility*.

SUMMARY

We have briefly considered the method of constructing the LF for free massive mixed-symmetry HS fields on AdS_d space in the framework of BFV-BRST approach. To do so, we have constructed new auxiliary representation for nonlinear algebra which serves for conversion procedure of initial HS symmetry algebra. Then, we have sketched details of systematic way to find BFV-BRST operator for nonlinear operator algebra and presented a proper construction of gauge LF basically for bosonic HS fields. Equations (13), (14) present the basic results of the paper being the first step to interacting theory.

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*The construction of LF for half-integer HS fields subject to $Y(s_1, s_2)$, $s_i = n_i + (1/2)$ is more complicated due to presence of fermionic constraints, which follows from respective irrep conditions on spin-tensor $\Phi_{(\mu)_{n_1},(\nu)_{n_2}A}$, being by Dirac, gamma-traceless and mixed-symmetry equations. As the consequence, the HS symmetry algebra is the superalgebra, but the VM, its oscillator realization for additional parts o'_I and BFV-BRST operator for converted operators O_I exist (see for details [3,6]), as well as the proper LF.

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