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# ON SOME CLASSES OF EXACT SOLUTIONS OF SCALAR BORN–INFELD EQUATION

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Some classes of exact solutions of scalar Born–Infeld equation have been found. Certain selected properties of these solutions have been presented.

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## INTRODUCTION

Born–Infeld theory is some generalization of Maxwell theory of electromagnetism. It was created by Max Born and Leopold Infeld, as a continuation of Mie theory of electromagnetism, [3]. Mie wanted to construct such a theory, in which all properties of electron were caused by electromagnetic field. However, this theory was not gauge-invariant. Born and Infeld suggested full relativistic and gauge-invariant theory, which Lagrangian density is given by the square root of the determinant of the sum of the metric tensor and the tensor of the electromagnetic field [2, 3].

In this paper we will investigate scalar Born–Infeld equation [1]:

$$
\begin{aligned}
&(1 - u_{,t}^2 + u_{,x}^2 + u_{,y}^2 + u_{,z}^2)(u_{,xx} + u_{,yy} + u_{,zz} - u_{,tt}) - \\
&\quad - u_{,x}^2 u_{,xx} - u_{,y}^2 u_{,yy} - u_{,z}^2 u_{,zz} - u_{,t}^2 u_{,tt} - 2u_{,x}u_{,y}u_{,xy} - 2u_{,x}u_{,z}u_{,xz} - \\
&\qquad - 2u_{,y}u_{,z}u_{,yz} + 2u_{,t}u_{,x}u_{,tx} + 2u_{,t}u_{,y}u_{,ty} + 2u_{,t}u_{,z}u_{,tz} = 0. \quad (1)
\end{aligned}
$$

Physical context: scalar Born–Infeld equation (1) is connected to Nambu–Goto equation from string theory [5, 7]. Born–Infeld model has been used also by Heisenberg in his model of pion fireball production [7].

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## 1. EXACT SOLUTIONS OF I KIND

In [11], by applying decomposition method, the so-called solutions of I kind have been found:

$$u(x,y,z,t) = \beta_1 + f(a_\mu x^\mu + \beta_2, b_\nu x^\nu + \beta_3, c_\rho x^\rho + \beta_4), \tag{2}$$

where $a_\mu, b_\nu, c_\rho \in R$ — their values are given below, $a_\mu x^\mu = -a_0 x^0 + a_k x^k$, $k = 1,2,3$, $x^0 = t, x^1 = x$, $x^2 = y$, $x^3 = z$, $\mu,\nu,\rho = 0,1,2,3$, $\beta_i$ $(i = 1,2,3,4)$ are arbitrary real constants and $f$ is an *arbitrary*, real function of class $C^2$. Boundary conditions follow from physical application. As we see, the solution (2) is a nonlinear composition of travelling waves.

After inserting (2) into (1), we obtain the so-called determining system of equations, from which we determine the values of the coefficients $a_\mu, b_\nu, c_\rho$. The above ansatz describes the so-called solutions of I kind. Some special case of this solution has been found for static, isotropic Heisenberg ferromagnet in [9, 10], where it is called as instanton solution. So we may say also in the case of the solution (2) of scalar Born–Infeld equation that it is nonlinear composition of instantons. The values of the coefficients $a_\mu, b_\nu, c_\rho$ are:

$$a_0 = \frac{a_3 c_0}{c_3}, \quad a_1 = \frac{-a_3(c_2^2 + c_3^2 - c_0^2)}{c_3\sqrt{c_0^2 - c_2^2 - c_3^2}}, \quad a_2 = \frac{a_3 c_2}{c_3}, \quad b_0 = \frac{b_2 c_0}{c_2}, \tag{3}$$

$$b_1 = -\frac{b_2(c_2^2 + c_3^2 - c_0^2)}{c_2\sqrt{c_0^2 - c_2^2 - c_3^2}}, \quad b_3 = \frac{b_2 c_3}{c_2}, \quad c_1 = \sqrt{c_0^2 - c_2^2 - c_3^2}, \tag{4}$$

$$c_0^2 > c_2^2 + c_3^2, \quad a_3, b_2, c_2, c_3 \neq 0. \tag{5}$$

The solution (2) is a generalization of well-known solution in $(1+1)$ dimensions [13]:

$$u(x,t) = f(-t + x), \tag{6}$$

where $f$ is arbitrary, real function of class $C^2$.

We may obtain also, by applying decomposition method, the solution of Born–Infeld equation (1) in $(2+1)$ dimensions, which has analogical form, as (2), but of course: $u = u(x,y,t)$ and $a_2 = \sqrt{a_0^2 - a_1^2}$, $b_1 = \dfrac{a_1 b_0}{a_0}$, $b_2 = \dfrac{\sqrt{a_0^2 - a_1^2} b_0}{a_0}$, $c_2 = \dfrac{a_1 c_0}{a_0}$, $c_2 = \dfrac{\sqrt{a_0^2 - a_1^2} c_0}{a_0}$ and $a_0^2 > a_1^2, a_1, a_2, b_0, c_0 \in R$.

## 2. GRAUSTEIN-KIND SOLUTIONS

In [4] (and references therein), another exact solution of (1) has been presented:

$$u(x,y,z,t) = f(t - z) - xg(t - z) - yh(t - z), \tag{7}$$

where $f$, $g$, $h$ are arbitrary, real functions of their arguments. This solution describes a domain wall moving along $z$ axis.

We may generalize these solutions and find such solution, which describes a domain wall moving in the plane $x - y$:

$$u(x, y, z, t) = \beta_1 + A_0 f_0(a_\mu x^\mu + \beta_2) + A_1 x^3 f_1(b_\nu x^\nu + \beta_3) + \quad (8)$$

$$+ A_2 x^3 f_2(c_\rho x^\rho + \beta_4), \quad (9)$$

where $\mu, \nu, \rho = 0, 1, 2$, $a_2 = \sqrt{a_0^2 - a_1^2}$, $b_0 = \dfrac{a_0 b_1}{a_1}$, $b_2 = \dfrac{b_1\sqrt{a_0^2 - a_1^2}}{a_1}$, $c_1 = \dfrac{a_1 c_0}{a_0}$, $c_2 = \dfrac{c_0\sqrt{a_0^2 - a_1^2}}{a_0}$ and $f_1$, $f_2$, $f_3$ are *arbitrary*, real functions of class $C^2$, $a_0^2 > a_1^2$, $a_0 \neq 0$, $a_1 \neq 0$, $A_i (i = 0, 1, 2) \in R$, $a_0, a_1, b_1, c_0 \in R$, $\beta_j (j = 1, 2, 3, 4) \in R$.

## 3. SOME ASPECTS OF DYNAMICS OF SOLUTIONS OF SCALAR BORN–INFELD EQUATION

**3.1. Perturbations and Backreaction of the Exact Solutions of Born–Infeld Equation.** Now we will investigate the perturbations of the exact solutions found in the previous section.

1) At the beginning we investigate the perturbations of exact solution in $(1 + 1)$ dimensions. It is well known in the literature [13] and may be found independently by applying the decomposition method.

We search for the solution of (1) in the following form [8]:

$$u(x, t) = \sum_{i=0}^{2} (A)^i u_i(x, t) = u_0(x, t) + A u_1(x, t) + A^2 u_2(x, t), \quad (10)$$

where $u_0(x, t) = f(a_0(x - t))$ ($f$ — arbitrary, real function of class $C^2$, $a_0 \in R$) is the solution of Born–Infeld equation in $(1 + 1)$ dimensions, $u_1(x, t)$ is the perturbation of $u_0(x, t)$, and $u_2(x, t)$ is the backreaction. $A \in R$ is an amplitude of excitations and $A \ll 1$. An interaction of two such solitons as $u_0$ was investigated in [6]. Here we assume *nothing* about the form either of the excitation or of the backreaction. For $u_0 = \tanh(a_0(x - t))$: after inserting (10) into (1) and neglecting the terms by the powers of $A$ more than 1, we obtain:

$$\frac{1}{(\cosh a_0(-t + x + \beta_2))^4} \left[ \frac{\partial^2 u_1}{\partial t^2}(a_0^2 + (\cosh a_0(-t + x + \beta_2))^4) + \right.$$

$$\left. + \frac{\partial^2 u_1}{\partial x^2}(a_0^2 - (\cosh a_0(-t + x + \beta_2))^4) + 2a_0^2 \frac{\partial^2 u_1}{\partial x \partial t} \right] = 0. \quad (11)$$

The general solution of this equation has the form: $u_1(x,t) = F_1(t-x)+G_2(x,t)$, where $F_1$ is some *arbitrary*, real function of its argument and $G_2$ — «a tail» is a linear, unbounded function, and therefore we must reject it. Analogical situation takes place for $u_0 = \mathrm{sech}\,(t-x)$. It turns out that the above situation repeats for the backreaction $u_2$ (we reject the unbounded «tail» function) and finally:

$$u(x,t) = F_0(t-x) + AF_1(t-x) + A^2F_2(t-x), \tag{12}$$

where $F_0(t-x) = \tanh\,(a_0(t-x))$ and $F_i, i = 1, 2$ are *arbitrary*, real functions of class $C^2$.

2) In the case of the solution in $(3+1)$ dimensions, given by (2) and the values of the coefficients (3)–(5), where for the simplicity we put: $a_3 = 0$, $b_2 = 0$, it turned out that the functions of the same form as the unperturbed solution $u_0$, given by (2) and the values of the coefficients (3)–(5) (where $a_3 = 0$, $b_2 = 0$) solve the equations for the perturbation $u_1$ and backreaction $u_2$. However, the search for the «tail» function has not been succeeded. Similar result has taken place in the case of the solution of I kind in $(2+1)$ dimensions.

3) Analogical situation repeats in the case of Graustein-kind solutions.

**3.2. Motion of Perturbed Kink in $(1+1)$ Dimensions.** We must require decaying of perturbation and backreaction, when $x \to \pm\infty$, $t =\to \pm\infty$, we choose: $u_1 = 0.2\,\mathrm{sech}\,(0.5(x-t))$, $u_2 = \dfrac{0.04}{(\exp\,(t-x)+1.7)^2}$. If we define the position of perturbed kink: $u(x,t) = \tanh\,(t-x) + 0.2\,\mathrm{sech}\,(0.5(x-t)) + \dfrac{0.04}{(\exp\,(t-x)+1.7)^2}$, as $u(x,y,z,t) = 0$, then there exists only one real solution of such obtained equation of motion: $x = t - 0.206$. As is well known, the kink may radiate when it is accelerated or it is deformated [12]. Hence we see that our perturbed kink (of Born–Infeld equation (1)) is *not* accelerated and the only source of possible radiation is deformation of the kink.

Because of limited space of this paper, the further physical properties of above-found solutions will be investigated in a separate paper.

**3.3. Nonexistence of Boost Transformation.** If in the case of the solutions (2) (and in the case of $(2+1)$-dimensional analogous solution) we put $a_0 = 0$ in (3)–(5), we will obtain *complex* solutions. It is in a correspondence with the well-known Bernstein theorem [14]. Hence we can establish a general remark: there does not exist boost transformation from *nonlinear*, *real*, space-like and time-independent solutions (because it does not exist) of scalar Born–Infeld equation (1) in Minkowski space-time, to *nonlinear*, *real* and time-dependent solution of this equation.

## CONCLUSIONS

We have obtained generalized Graustein solutions and class of solutions of I kind in $(2+1)$ dimensions. Next we have found the forms of: perturbation and backreaction of kink in $(1 + 1)$ dimensions — they replay the form of unperturbed kink. These last results have been generalized to the solutions in $(2 + 1)$ and $(3 + 1)$ dimensions. We must stress here that in all these cases, the perturbed solutions with backreaction, with the found «tail» functions and without them, *are the solutions* of scalar Born–Infeld equation. We have shown also that the perturbed kink (with backreaction) moves without any acceleration, so the possible source of the radiation of the kink is its deformation and that there does not exist any boost transformation from real, static, nonlinear solution of Born–Infeld equation (1) to real, nonstatic, and nonlinear solution of this equation.

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