ФИЗИКА ЭЛЕМЕНТАРНЫХ ЧАСТИЦ И АТОМНОГО ЯДРА
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INTERTWINING RELATIONS AND DARBOUX TRANSFORMATIONS FOR THE WAVE EQUATIONS

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Using the intertwining operator technique we construct Darboux transformations for the wave equation with position-dependent effective mass and with linearly energy-dependent potentials. The formally adjoint generators of supersymmetry and two formally self-adjoint superpartner Hamiltonians are constructed and they close a quadratic pseudosuperalgebra. The Darboux transformations are constructed in differential and integral forms and an interrelation is established between them. The approach is applied to generation of isospectral potentials with additional or removal bound states or construction of new partner potentials without changing the spectrum, i.e., fully isospectral potentials. The method is illustrated by some concrete examples. The influence of distance between levels on the form of potentials is investigated. In particular, asymmetric double well potentials are generated.

Используя технику соотношений сплетения, мы конструируем преобразования Дарбу для волнового уравнения с эффективной массой, зависящей от координатной переменной, и дополнительным потенциалом, линейно зависящим от энергии. Построены формально сопряженные генераторы суперсимметрии и формально самосопряженные суперпартнеры гамильтонианов, и показано, что они образуют замкнутую суперсимметричную псевдоалгебру. Преобразования Дарбу получены в дифференциальной и интегральной формах, установлена связь между ними. Подход применен для генерирования изоспектральных потенциалов, в которых добавляется или уничтожается одно связанное состояние, а также для конструирования нового потенциала без изменения спектра, т. е. полностью изоспектрального потенциала. Метод проиллюстрирован несколькими примерами. Исследовано влияние расстояния между энергетическими уровнями на форму потенциалов. В частности, построены асимметричные двойные квантовые потенциалы.

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INTRODUCTION

The problem of exact solvability of the Schrödinger equation has been extensively considered since the beginning of quantum mechanics [1]. A factorization technique was introduced by Schrödinger and was used to solve harmonic oscillator, hydrogen atom, Kepler motion. It should be noted that factorization is closely connected with the Darboux transformation [2] and the supersymmetry approach introduced by Witten in quantum mechanics [3]. Many interesting exactly solvable models have been constructed for Schrödinger equation using the Darboux and Bargmann transformation techniques [4–15]. In the last few years,

the research efforts on the topic of algebraic methods have been considerably intensified [16–28] due to the rapid development of nanoelectronics, the basic elements of which are low-dimensional structures such as quantum wells, wires, dots, and superlattices [29–33]. The Darboux transformations are one of the most powerful methods permitting to extend the class of new exactly solvable models. It has been constructed for a variety of linear and nonlinear equations. Among the linear equations to which the Darboux transformations are applicable, there are Pauli and Dirac equations [4, 7, 8], conventional Schrödinger equations (see the reviews in [8–11]), the class of Schrödinger equations with position-dependent effective mass [22–26], and Schrödinger-type equations with energy-dependent potentials [15, 34]. The Darboux transformations are essentially based on intertwining relations and differ from other transformations, that are mostly based on a change of coordinates like point canonical method.

In the present paper, we apply the intertwining operator method to the Schrödinger-type equations with position-dependent effective mass and with linearly energy-dependent potentials and construct first- and second-order Darboux transformations, chains of transformations, factorization, and «pseudosupersymmetry». Darboux transformations for the Schrödinger equation with position-dependent mass, for the wave equation with linearly energy-dependent potentials, as well as Darboux transformations for the conventional Schrödinger equation can be seen as special cases of Darboux transformations for the generalized equation. The position-dependent mass Schrödinger equation is used in different fields of physics, for example, in atomic and nuclear physics for investigation of spectral properties [35, 36], for description of quantum properties of helium clusters and metal clusters [37, 38], as well as in nanoelectronics and optoelectronics for investigation of semiconductor heterostructures [29–33], in which the carrier effective mass depends on position. Although algebraic methods have been used in obtaining exact solutions for Schrödinger equation with position-dependent mass, however, their applications are very complicated for realization and there remain a lot of problems to study. For instance, much attention has been given to the investigation of double-well potentials for conventional Schrödinger equation [39–41]. Nevertheless, this problem has not been investigated for quantum systems endowed with position-dependent mass.

Wave equations with energy-dependent potentials are widely used in relativistic and nonrelativistic quantum mechanics for a long time [42–53], in particular, for investigation of waveguide devices in such modern fields of physics as optoelectronics [33, 54] and photoelectronics [57]. The efforts of many authors have been concentrated on the inverse eigenvalue problems on the determination of $q(x)$ and $v(x)$ from the eigenvalues of the operator [43–48], One can cite more recent investigations, related to a polynomial form of potential coefficients [49, 52, 53] in the second-order differential operator as well as the intertwining technique for wave equations with linearly energy-dependent potentials [34].

The aim of the present paper is to consider a more general problem of generating exact solutions for one-dimensional wave equation with position-dependent mass and with a linear energy-dependent potential by using the intertwining transformation technique. At the beginning the problem with the second-order differential operator of a general form was considered within the classical string model (see [5] and references therein). Similar problems arise in the physics of Earth [43, 55], in acoustic and electromagnetism [56]. In recent time, the rapid progress has been observed in quantum optics, optoelectronics [33], and photoelectronics [57], where the wave equation of a general form is used for the description of wave guide structures as well as the effective position-dependent mass equation, and the wave equation with energy-dependent potentials without effective mass is applied. We think that the development of Darboux or equivalent algebraic transformations for the second-order differential equations of a general form extends the capabilities for investigations in these fields.

The presence of the energy-dependent potential has a few particularities [44]. The main of them leads to the modification of the scalar product, which in contrast to common Schrödinger equation is most considered with the weight function. Instead of Hermitian Hamiltonians it is necessary to work with formally self-adjoint Hamiltonians [58] or, in another terminology, with crypto-Hermitian Hamiltonians [44] or pseudo-Hermitian Hamiltonians [59]. It is known, the Schrödinger equation and the effective mass Schrödinger equation allow a supersymmetry approach and it connects with their respective Darboux transformations. In the case of the generalized Schrödinger-type equation it is necessary to use formally adjoint operators in construction of close pseudosupersymmetry algebra, and this problem should be studied.

The Darboux transformations can be presented not only in a differential form and in an integral form, too. The technique of iterated integral operators for Schrödinger equation is based on the integral Gelfand–Levitan and Marchenko equations [60–62]. For example, the Darboux transformations in the integral form were considered for the conventional Schrödinger equation in [63, 64], the Darboux transformations in the differential and in the integral form and their relationships were studied for the system of Schrödinger equations in [65], and for the Schrödinger equation with variable energy and angular momentum in [66, 67]. As we know, the first attempts to construct the second-order Darboux transformations in the integral form for the Schrödinger equation with position-dependent mass and weighted energy were made in [68]. A combination of the differential and integral Darboux transformations makes the approach more flexible and powerful. Indeed, the integral transformations are preferable at the reconstruction of totally isospectral Hamiltonians, at the construction of Hamiltonians differing by one bound state from the initial one, when this bound state is removed or added at arbitrary energy. In this case differential transformations, accomplished on the excited states, give singular potentials.

In this paper we will focus on Darboux transformations of an arbitrary order applied to the generalized Schrödinger equation with position-dependent mass and energy-dependent potentials. As a first step, we plan to consider a case of first-order intertwining operators and their supersymmetry interpretation. Although supersymmetry relations coincide in form with the standard supersymmetry ones, the intertwining transformation operators are different. These operators are formally adjoint, have a more complicate form and have to factorize the pair of pseudo-Hermitian Hamiltonians. Here, we shall establish a correspondence between the spaces of solutions of the initial and transformed equations. We have to note that some of the results can be obtained from the previous paper [27], where a similar intertwining technique is used for the generalized Schrödinger equations in $(1+1)$ dimensions. But in this paper considerations do not relate with pseudosupersymmetry and, as it is natural for time-dependent equations, generation of isospectral potentials is not considered. At the second step, we shall consider nth-order transformations by iteration. Then, we shall elaborate Darboux transformations in the integral form and show relations with Darboux transformations in the differential form. The procedure can be used for generating families of Hamiltonians with a predetermined spectrum, removing or adding new bound states. To explain our findings, we consider a few different examples of our approach and show how to generate completely isospectral potentials or to construct potentials with addition or moving of bound states from the spectrum of the initial Hamiltonian. We analyze the influence of the distances between the energy levels on the shape of potentials, which can be very complex. We construct asymmetric double-well and even triple-well potentials for the effective mass Schrödinger equation.

The paper is organized as follows. Section 1 is devoted to the generalized Darboux transformation of the first order and corresponding supersymmetry formalism. In Sec. 2 we construct chains of Darboux transformations by iteration of first-order Darboux transformations. In Sec. 3 we derive transformations in an integral form and establish a relationship with differential Darboux transformations, afterwards we consider totally isospectral potentials. In Sec. 4 we illustrate our generalized transformations by concrete examples.

1. THE INTERTWINING RELATIONS AND SUPERSYMMETRY

As is well known, Darboux transformations (or supersymmetry) in nonrelativistic quantum mechanics allow one to produce Hamiltonians whose spectra can differ in one bound state [8]. We will extend the approach to the one-dimensional Schrödinger-type equation with position-dependent mass and weighted energy

$$\left[\frac{d}{dx}\left(\frac{1}{m(x)}\right)\frac{d}{dx}\right]\phi(x)+v(x)\,\phi(x)=q(x)\,\mathcal{E}\,\phi(x). \tag{1}$$

Here $m(x)$ stands for the particle's effective mass, $q(x)\,\mathcal{E}$ is a linearly energy-dependent potential and $v(x)$ denotes the potential, $\phi(x)$ is the wave function and $\mathcal{E}$ denotes real-valued energy, and we use atomic units. The potential functions v, q, and m are assumed to be real and integrable on $[a,b]$ on which $m(x)$, $q(x)$, and $v(x)$ are defined, and $v(x) \in L_1^1$, i.e., $\int\limits_a^b dx|v(x)|(1+|x|) < \infty$. We assume that $m(x) \neq 0$ and $q(x) \neq 0$ on the interval $[a,b]$. In principle, the conditions can be more weak if it is necessary to consider the physical situation with an abrupt interface. It means that effective mass has discontinuities at some points of $[a,b]$. We assume that potential functions $v(x)$ remain finite, but not necessarily continuous, the function $\phi(x)$ must be continuous $\phi_-(x) = \phi_+(x)$ and its derivative should satisfy the condition $\left(\dfrac{1}{m(x)}\dfrac{d\,\phi(x)}{d\,x}\right)_- = \left(\dfrac{1}{m(x)}\dfrac{d\,\phi(x)}{d\,x}\right)_+$, where subindices $-$ and $+$ denote the left- and right-hand sides of mass discontinuity [16, 22]. Even if the potential function has points of singularities, one can use the Darboux transformation method, reconstructing potential excluding the points of singularities (see, e.g., [4]).

The presence of a linearly energy-dependent potential leads to the modification of the scalar product with the weight of $q(x)$: $(f,\chi)_q = \int f^*(x)q(x)\chi(x)\,dx$. As a consequence, in contrast to conventional Schrödinger equation, the role of the self-adjoint operator plays the operator $\mathcal{H}^\dagger = q^{-1}\mathcal{H}^+q$ instead of $\mathcal{H}^+$. It can be shown that the operator $\mathcal{H}^\dagger = \mathcal{H}$, where $\mathcal{H} = -\dfrac{1}{q(x)}\left[\dfrac{d}{dx}\left(\dfrac{1}{m(x)}\right)\dfrac{d}{dx}\right] + \dfrac{v(x)}{q(x)}$ is the second-order differential operator of our wave equation (1). The corresponding eigenfunctions should satisfy the following conditions: $\int\limits_a^b dx\,q(x)|\phi(x)|^2 < \infty$, which is an analog of the square integrability of the Schrödinger equation eigenfunctions. In Russian literature [58] an operator $D^\dagger$ adjoint to D and defined by $D^\dagger = q^{-1}D^+q$ with $q(x) \neq 0$ is called a formally adjoint operator. If $D^\dagger = D$, then it is called a formally self-adjoint operator. Apparently, such kind of operator is a particular case of pseudo-Hermitian operators. By the definition [59], operators D are said to be pseudo-Hermitian with respect to q if $D^\dagger = q^{-1}D^+q = D$. The operator $\mathcal{H}$ of (1) satisfies this condition.

In this section we will briefly present the method of intertwining the operators of the first order, then we will analyze the supersymmetry and the n-fold iteration of the first-order Darboux transformations. The point canonical transformation method is often used (see, e.g., [27, 28]) to convert a general second-order differential equation into another solvable equation. In particular, within the point canonical transformations the Schrödinger equation with position-dependent mass and a linearly energy-dependent potential can be transformed in the form of the constant mass Schrödinger equation by changing the coordinate and wave

function. Evidently, this transformation generates isospectral potentials for the Schrödinger-type equation of a general form and Schrödinger equation with constant mass (see, e.g., [28]). At the same time, in the recent paper [26] it has been shown, the energy spectrum of the position-dependent mass Hamiltonians, in general, is not isospectral with the constant mass Hamiltonians. Evidently, this problem needs further investigations. In general, $m(x)$ and $q(x)$ could lead to a very complex effective potential, for which the solution of the Schrödinger equation can be more complicated than the solution of the initial equation with $m(x)$ and $q(x)$. Note also, if we are going from the Schrödinger equation with the effective potential and constant mass to the position-dependent mass equation, we can reconstruct different mass functions. The procedure is not uniqueness. It should be noted, the procedure of point canonical transformation demands existence of the second-order derivatives of $m(x)$ and $q(x)$, which are included in the effective potential, and it is impossible to consider the situation with abrupt interfaces.

All these reasons motivate investigations on the development of algebraic methods for solving the Schrödinger-type equation in a general form without transforming this equation into the common Schrödinger equation.

1.1. The First-Order Darboux Transformations. Let us consider two generalized Schrödinger equations

$$\mathcal{H}\phi = \mathcal{E}\phi, \quad \mathcal{H} = -\frac{1}{q(x)}\left[\frac{d}{dx}\left(\frac{1}{m(x)}\right)\frac{d}{dx}\right] + \frac{v(x)}{q(x)}, \tag{2}$$

$$\widetilde{\mathcal{H}}\widetilde{\phi} = \mathcal{E}\widetilde{\phi}, \quad \widetilde{\mathcal{H}} = -\frac{1}{q(x)}\left[\frac{d}{dx}\left(\frac{1}{m(x)}\right)\frac{d}{dx}\right] + \frac{\widetilde{v}(x)}{q(x)}, \tag{3}$$

where Hamiltonians $\mathcal{H}$ and $\widetilde{\mathcal{H}}$ differ only in potentials v and $\widetilde{v}$. In the conventional intertwining technique for the Schrödinger equation two one-dimensional Hamiltonians $\mathcal{H} = -\partial_{xx} + v$ and $\widetilde{\mathcal{H}} = -\partial_{xx} + \widetilde{v}$ and corresponding solutions ϕ and $\widetilde{\phi}$ are related by means of a first-order differential operator $\mathcal{L}$

$$\mathcal{L}\mathcal{H} = \widetilde{\mathcal{H}}\mathcal{L}, \tag{4}$$

$$\widetilde{\phi} = \mathcal{L}\,\phi. \tag{5}$$

As is known, the method of intertwining differential operators provides a universal approach to generating new exactly solvable equations [69, 70]. We expand the intertwining relations on our generalized equation in order to determine the intertwining operator $\mathcal{L}$, a new potential, and solutions of transformed equation (3) assuming that the solutions to the initial equation (2) are known. We seek for the intertwiner $\mathcal{L}$ in the form of a linear, first-order differential operator

$$\mathcal{L} = A(x) + B(x)\frac{d}{dx}, \tag{6}$$

where the coefficients A and B are to be determined. To this end, we insert (6) and the explicit form of the Hamiltonians $\mathcal{H}$ and $\widetilde{\mathcal{H}}$ into the intertwining relation (4) and apply it to the solution ϕ of (2). Assuming the linear independence of derivative operators of different order d^k/dx^k, $k = 0, 1, 2, 3$, we obtain the following system of equations for A, B, and $\widetilde{v}$

$$\frac{2}{qm}B' = B\left(\frac{1}{qm}\right)', \tag{7}$$

$$\frac{2}{m}A' + \frac{1}{m}B'' + \left(\frac{1}{m}\right)'B' - Bq\left[\frac{1}{q}\left(\frac{1}{m}\right)'\right]' = B\left(\widetilde{v} - v\right), \tag{8}$$

$$\frac{1}{m}A'' + \left(\frac{1}{m}\right)'A' + Bq\left(\frac{v}{q}\right)' = A(\widetilde{v} - v), \tag{9}$$

where the prime denotes differentiation with respect to x and arguments have been omitted. From (7) it follows

$$B = \frac{\beta}{\sqrt{qm}}, \tag{10}$$

where β is an arbitrary constant of integration. Equations (8) and (9) allow us to determine the potential $\widetilde{v}$ and the function A. Multiplying (8) by A and (9) by B and subtracting the resulting equations we obtain the equation with respect to A

$$-\frac{A''}{qm} + \left(\frac{2A'}{qmB} + \frac{B''}{qmB} + \frac{1}{q}\left(\frac{1}{m}\right)'\frac{B'}{B} - \left[\frac{1}{q}\left(\frac{1}{m}\right)'\right]'\right)A - \\ - B\left(\frac{v}{q}\right)' - \frac{1}{q}\left(\frac{1}{m}\right)'A' = 0. \tag{11}$$

In order to solve this equation, we introduce a new auxiliary function K defined by $K = A/B$. Taking (10) into account, after simplification we arrive at the following nonlinear equation:

$$\frac{d}{dx}\left[\frac{1}{qm}\left(-K' + K^2\right)\right] - \frac{d}{dx}\left(\frac{v}{q}\right) - \frac{d}{dx}\left[\frac{1}{q}\left(\frac{1}{m}\right)'K\right] = 0,$$

from which the generalized Riccati equation follows

$$\frac{1}{qm}\left(-K' + K^2\right) - \frac{v}{q} - \frac{1}{q}\left(\frac{1}{m}\right)'K = -\lambda, \tag{12}$$

where λ is an arbitrary integration constant. Equation (12) turns into the conventional Riccati equation in particular cases of $m = \mathrm{const}$ and $q = \mathrm{const}$. With B

from (10) and $A = KB$ the potential $\widetilde{v}$ can be expressed from (8) in terms of K and known potentials v, m, and q

$$\widetilde{v} = v + 2\sqrt{\frac{q}{m}}\frac{d}{dx}\frac{K}{\sqrt{qm}} - \sqrt{\frac{q}{m}}\frac{d}{dx}\left[\frac{1}{q}\frac{d}{dx}\left(\sqrt{\frac{q}{m}}\right)\right]. \tag{13}$$

The potential $\widetilde{v}$ will be finally determined after finding the function K from the Riccati equation (12), that can be linearized and integrated by introducing an auxiliary function $\mathcal{U} = \mathcal{U}(x)$

$$K = -\frac{\mathcal{U}'}{\mathcal{U}}. \tag{14}$$

Assuming that $\mathcal{U}$ is twice continuously differentiable and substituting (14) into (12) we get the equation

$$-\frac{1}{m}\mathcal{U}'' - \left(\frac{1}{m}\right)'\mathcal{U}' + v\mathcal{U} = q\lambda\mathcal{U}, \tag{15}$$

which is identical to the initial equation (2) at $\mathcal{E} = \lambda$. Since the solutions of (2) with the given potentials v, q, and m are known at all energies $\mathcal{E}$, hence we know the solution $\mathcal{U}$. Once $\mathcal{U}$ is given, the function K is calculated via (14), which in turn determines A by means of $A = BK$ and (10)

$$A = -\frac{1}{\sqrt{qm}}\frac{\mathcal{U}'}{\mathcal{U}}.$$

The constant of integration β has been taken to one. By insertion of K in (13) we obtain the explicit form of the transformed potential

$$\widetilde{v} = v - 2\sqrt{\frac{q}{m}}\frac{d}{dx}\left[\frac{1}{\sqrt{qm}}\frac{\mathcal{U}'}{\mathcal{U}}\right] - \sqrt{\frac{q}{m}}\frac{d}{dx}\left[\frac{1}{q}\frac{d}{dx}\left(\sqrt{\frac{q}{m}}\right)\right]. \tag{16}$$

Finally, the intertwiner $\mathcal{L}$ and the transformation solution $\widetilde{\phi}$ are determined from (6) and (5), respectively,

$$\mathcal{L} = \frac{1}{\sqrt{qm}}\left(\frac{d}{dx} + K\right) = \frac{1}{\sqrt{qm}}\left[\frac{d}{dx} - \frac{\mathcal{U}'}{\mathcal{U}}\right], \tag{17}$$

$$\widetilde{\phi} = \mathcal{L}\phi = \frac{1}{\sqrt{qm}}\left[\frac{d}{dx} - \frac{\mathcal{U}'}{\mathcal{U}}\right]\phi. \tag{18}$$

From (16)–(18) it follows that the transformed potential $\widetilde{v}$, the intertwiner $\mathcal{L}$, and solutions $\widetilde{\phi}$ depend not only on the potential v, but also on the additional potentials m and q. Quite obviously, in order to avoid the creation of singularities in K, A, and $\widetilde{v}$ one must look for nodeless $\mathcal{U}(x)$. It is valid if the transformation

function $\mathcal{U}$ corresponds to the ground state provided $m(x)$ and $q(x)$ do not lead to the extra singularities, which are not present in the initial potential. It is easy to check that all expressions reduce to the well-known ones for the Schrödinger equation if potential functions m and q are taken to be constant. When $q = \mathrm{const}$ or $m = \mathrm{const}$ we get first-order Darboux transformations for position-dependent mass Schrödinger equation [25] or for Schrödinger equation with weighted energy [34] as particular cases of our approach.

Solutions at energy of transformation. Note that relation (18) connects the solutions of two equations (2) and (3) at arbitrary energy except for solutions at energy of transformation, $\mathcal{E} = \lambda$. Evidently, at $\mathcal{E} = \lambda$ the action of Darboux transformation (17) on the function $\mathcal{U}$ and on functions ϕ linearly-dependent on $\mathcal{U}$ gives us $\mathcal{L}\mathcal{U} = 0$. In order to obtain a solution of the transformed equation (3) at energy λ, we replace $\mathcal{U}$ by a linearly-independent solution $\widehat{\mathcal{U}}$

$$\widehat{\mathcal{U}} = \mathcal{U} \int_{x_0}^{x} dx' \frac{m(x')}{|\mathcal{U}(x')|^2}. \tag{19}$$

The limits of integration depend on the boundary conditions. The direct substitution of $\widehat{\mathcal{U}}$ into (15) shows that $\widehat{\mathcal{U}}$ really solves the generalized Schrödinger equation. The action of $\mathcal{L}$ on the function $\widehat{\mathcal{U}}$ gives us a solution η of the transformed equation (3) at energy λ

$$\eta = \mathcal{L}\widehat{\mathcal{U}} = \sqrt{\frac{m}{q}}\,\frac{1}{\mathcal{U}}. \tag{20}$$

By using the generalized Liouville formula (19) once more, one can get a second solution $\widehat{\eta}$ of (3) at energy λ. For this, $\mathcal{U}$ is replaced by η in (19) and with (20) we get

$$\widehat{\eta} = \eta \int_{x_0}^{x} dx' \frac{m(x')}{|\eta(x')|^2} = \sqrt{\frac{m}{q}}\frac{1}{\mathcal{U}} \int_{x_0}^{x} dx' q(x') |\,\mathcal{U}(x')|^2. \tag{21}$$

Thus, if we know all solutions of the initial equation (2), we can determine all solutions of the transformed equation (3), including the solutions at energy of transformation. Notice that if the transformation function $\mathcal{U}$ replies to the bound state of $\mathcal{H}$, then the function η defined by (20) at the energy of transformation λ cannot be normalized. Such bound state is excluded from the spectrum of $\mathcal{H}$. Therefore, the Hamiltonians $\mathcal{H}$ and $\widetilde{\mathcal{H}}$ are isospectral except for the bound state with energy λ, which is removed from the spectrum of $\mathcal{H}$. In the next section we show how one can construct a Hamiltonian with an additional bound state by using the algebra of supersymmetry (SUSY).

1.2. Supersymmetry. As is well known, the SUSY algebra provides a relation between superpartner Hermitian Hamiltonians, which can be presented in a factorized form in terms of Darboux transformation operators $\mathcal{L}$ and its adjoint $\mathcal{L}^+$. Since instead of Hermitian Hamiltonian we have pseudo-Hermitian Hamiltonian (2) with respect to $q(x)$, $\mathcal{H}^\dagger = q^{-1}\mathcal{H}^+ q = \mathcal{H}$ and $q \neq 0$, in order to construct superalgebra we need in formally adjoint operators. Hence, instead of operator $\mathcal{L}^+$ it is necessary to consider the operator $\mathcal{L}^\dagger = q^{-1}\mathcal{L}^+ q$. Therefore, the operator $\mathcal{L}^\dagger$ adjoint to $\mathcal{L} = A + Bd/dx$ is determined as $\mathcal{L}^\dagger = q^{-1}\left(A - Bd/dx - dB/dx\right)q$. After simplification we arrive at

$$\mathcal{L}^\dagger = \frac{1}{\sqrt{qm}}\left(-\frac{d}{dx} + K\right) - \frac{1}{q}\frac{d}{dx}\sqrt{\frac{q}{m}}. \tag{22}$$

Since the operators $\mathcal{H}$ and $\widetilde{\mathcal{H}}$ are formally self-adjoint (pseudo-Hermitian), $\mathcal{H}^\dagger = \mathcal{H}$ and $\widetilde{\mathcal{H}}^\dagger = \widetilde{\mathcal{H}}$ and with account $\mathcal{L}^\dagger = q^{-1}\mathcal{L}^+ q$ the intertwining relation adjoint to (4) reads

$$\mathcal{H}\mathcal{L}^\dagger = \mathcal{L}^\dagger\widetilde{\mathcal{H}}. \tag{23}$$

The generalized Schrödinger equations (2) and (3) can then be written as one single matrix equation in the form

$$\begin{pmatrix} \mathcal{H} - \lambda & 0 \\ 0 & \widetilde{\mathcal{H}} - \lambda \end{pmatrix}\begin{pmatrix} \phi \\ \tilde{\phi} \end{pmatrix} = 0. \tag{24}$$

On defining $H_s = \operatorname{diag}\left(\mathcal{H}, \widetilde{\mathcal{H}}\right)$ and $\Phi = (\phi, \tilde{\phi})^T$, the above matrix Schrödinger equation (24) can be written as

$$\left[H_s - \lambda I\right]\Phi = 0, \tag{25}$$

where I is the 2×2 unity matrix. Similar to the case of the standard Schrödinger equation, we define two supercharge operators Q, $Q^\dagger$ as follows:

$$Q = \begin{pmatrix} 0 & 0 \\ \mathcal{L} & 0 \end{pmatrix}, \qquad Q^\dagger = \begin{pmatrix} 0 & \mathcal{L}^\dagger \\ 0 & 0 \end{pmatrix}, \tag{26}$$

where $\mathcal{L}$ and $\mathcal{L}^\dagger$ are the operators given by (17) and (22), respectively. Since $Q^\dagger$ can be determined as $Q^\dagger = q^{-1}Q^+ q$ and with account of $\mathcal{L}^\dagger = q^{-1}\mathcal{L}^+ q$ one can conclude that the operator $Q^\dagger$ is pseudoadjoint to Q. One can show that the matrix operators Q, $Q^\dagger$, and H_s satisfy the following anticommutation and commutation relations:

$$\{Q, Q\} = \left\{Q^\dagger, Q^\dagger\right\} = 0, \tag{27}$$

$$[Q, H_s] = \left[H_s, Q^\dagger\right] = 0, \tag{28}$$

where $\{\cdot,\cdot\}$ and $[\cdot,\cdot]$ are the anticommutator and the commutator, respectively. The relations (27) are trivially fulfilled, because the matrices in (26) are nilpotent. It is easily seen that equations (28) are equivalent to the intertwining ones (4) and (23).

Now, let us consider the complementing relations of the supersymmetric algebra, that is, the anticommutators $\{Q, Q^\dagger\}$ and $\{Q^\dagger, Q\}$. For this, we calculate the operators $\mathcal{L}^\dagger\mathcal{L}$ and $\mathcal{L}\mathcal{L}^\dagger$, and consider the connections of them with our Hamiltonians $\mathcal{H}$ and $\widetilde{\mathcal{H}}$. By using (17) and (22), we arrive after some algebraic transformations at

$$\mathcal{L}^\dagger\mathcal{L} = -\frac{1}{qm}\partial_{xx} - \frac{1}{q}\left(\frac{1}{m}\right)'\partial_x + \frac{1}{qm}\left(|K|^2 - K'\right) - \frac{1}{q}\left(\frac{1}{m}\right)'K, \tag{29}$$

$$\mathcal{L}\mathcal{L}^\dagger = -\frac{1}{qm}\partial_{xx} - \frac{1}{q}\left(\frac{1}{m}\right)'\partial_x + \frac{1}{qm}\left(|K|^2 + K'\right) + \frac{1}{m}\left(\frac{1}{q}\right)'K - \\ - \frac{1}{\sqrt{qm}}\left[\frac{1}{q}\left(\sqrt{\frac{q}{m}}\right)'\right]'. \tag{30}$$

We express the potential v from the Riccati equation (12) in the form

$$v = \frac{1}{m}\left(-K' + K^2\right) - \left(\frac{1}{m}\right)'K + q\lambda. \tag{31}$$

Using this in (13), we get the following representation for the transformed potential:

$$\tilde{v} = \frac{1}{m}(K^2 + K') + \frac{q}{m}\left(\frac{1}{q}\right)'K - \frac{\sqrt{q}}{\sqrt{m}}\frac{d}{dx}\left[\frac{1}{q}\left(\sqrt{\frac{q}{m}}\right)'\right] + q\lambda. \tag{32}$$

One can easily see that the potential difference is determined as

$$\widetilde{v} - v = 2\sqrt{\frac{q}{m}}\frac{d}{dx}\frac{K}{\sqrt{qm}} - \sqrt{\frac{q}{m}}\frac{d}{dx}\left[\frac{1}{q}\frac{d}{dx}\left(\sqrt{\frac{q}{m}}\right)\right]. \tag{33}$$

On further employing (31) and (32), the formulae (29) and (30) can be rewritten as

$$\mathcal{L}^\dagger\mathcal{L} = -\frac{1}{q}\left[\frac{\partial}{\partial x}\left(\frac{1}{m}\right)\frac{\partial}{\partial x}\right] + \frac{v}{q} - \lambda = \mathcal{H} - \lambda, \tag{34}$$

$$\mathcal{L}\mathcal{L}^\dagger = -\frac{1}{q}\left[\frac{\partial}{\partial x}\left(\frac{1}{m}\right)\frac{\partial}{\partial x}\right] + \frac{\tilde{v}}{q} - \lambda = \widetilde{\mathcal{H}} - \lambda. \tag{35}$$

As can be easily seen, the anticommutation relation is

$$\{Q, Q^\dagger\} = \begin{pmatrix} \mathcal{L}^\dagger\mathcal{L} & 0 \\ 0 & \mathcal{L}\mathcal{L}^\dagger \end{pmatrix} = \begin{pmatrix} \mathcal{H} - \lambda & 0 \\ 0 & \widetilde{\mathcal{H}} - \lambda \end{pmatrix} = \mathcal{H}_s - \lambda I. \tag{36}$$

In components, the latter equality reads

$$\mathcal{H} = \mathcal{L}^\dagger \mathcal{L} + \lambda, \tag{37}$$

$$\widetilde{\mathcal{H}} = \mathcal{L}\mathcal{L}^\dagger + \lambda. \tag{38}$$

One can conclude that the operators $\mathcal{H}_s$, Q, and $Q^\dagger$ close a superalgebra and one can associate a pseudosupersymmetry with a quantum system described by the Hamiltonian $\mathcal{H}_s$.

Note that as soon as the initial Hamiltonian $\mathcal{H}$ is presented in the factorized form (37), one can get its supersymmetric partner in a factorized form (38), too. Indeed, multiplying equation (37) from the left by $\mathcal{L}$ and taking into account the intertwining relation (23) we get

$$\mathcal{L}\mathcal{H}\phi = \mathcal{L}(\mathcal{L}^\dagger \mathcal{L} + \lambda)\phi = (\mathcal{L}\mathcal{L}^\dagger + \lambda)\,\mathcal{L}\phi = \widetilde{\mathcal{H}}\mathcal{L}\phi. \tag{39}$$

It means, that $\widetilde{\mathcal{H}} = \mathcal{L}\mathcal{L}^\dagger + \lambda$.

In summary, we obtained the explicit forms of the supersymmetric partner Hamiltonians $\mathcal{H}$ and $\widetilde{\mathcal{H}}$. The Hamiltonians (37) and (38) are compatible with their definitions (2) and (3), respectively, if the transformed potentials v and $\tilde{v}$ are given by (31) and (32). Finally, taking the difference of the factorized Hamiltonians (37) and (38) gives the potential difference (13) that we obtained for our Darboux transformation. Hence, the Darboux transformation is equivalent to the supersymmetry formalism.

The intertwining relation (23) means that the operator $\mathcal{L}^\dagger$ is also the transformation operator and realizes the transformation from the solutions of (3) to the solutions of (2). Evidently, one can interchange the role of the initial generalized Schrödinger equation (2) and its transformed counterpart (3). To this end, let us express the operators $\mathcal{L}$ and $\mathcal{L}^\dagger$ in terms of functions η given in (20), which are solutions to the equation (3) at energy of transformation λ. First, we rewrite K by using $\mathcal{U}$ from the relation (20)

$$K = -\frac{\mathcal{U}'}{\mathcal{U}} = \frac{1}{2}\frac{q'}{q} - \frac{1}{2}\frac{m'}{m} + \frac{\eta'}{\eta}.$$

Using this in (17) and (22), we obtain after simplifications

$$\mathcal{L}^\dagger = \frac{1}{\sqrt{qm}}\left(-\frac{d}{dx} + \frac{\eta'}{\eta}\right), \qquad \mathcal{L} = \frac{1}{\sqrt{qm}}\left(\frac{d}{dx} + \frac{\eta'}{\eta}\right) + \frac{1}{q}\frac{d}{dx}\sqrt{\frac{q}{m}}. \tag{40}$$

Obviously, the function η is also a transformation function. Notice, $\mathcal{L}^\dagger \eta = 0$, meaning that η belongs to the kernel of the operator $\mathcal{L}^\dagger$. As one can see from (40) and (21), the application of the operator $\mathcal{L}^\dagger$ to the second linearly-independent

solution $\hat{\eta}$ of equation (3) gives back the solutions $\mathcal{U}$ of the initial problem at energy of transformation. Indeed,

$$\mathcal{L}^{\dagger}\hat{\eta} = \frac{1}{\sqrt{qm}}\left(-\frac{d}{dx} + \frac{\eta'}{\eta}\right)\eta \int\limits_{x_0}^{x} dx' \frac{m(x')}{|\eta(x')|^2} = -\sqrt{\frac{m}{q}}\frac{1}{\eta} = -\mathcal{U}.$$

Hence, the solution at energy of transformation λ takes the form

$$\mathcal{U} = \sqrt{\frac{m}{q}}\ \frac{1}{\eta} \tag{41}$$

and, in principle, can reply to the new bound state. The second linearly-independent solution $\widehat{\mathcal{U}}$ of (2) at energy λ can be written in terms of η as follows:

$$\widehat{\mathcal{U}} = \sqrt{\frac{m}{q}}\frac{1}{\eta}\int dx' q|\eta|^2. \tag{42}$$

Introducing the function $\widetilde{K} = \widetilde{K}(x)$ by $\widetilde{K} = \eta'/\eta$ and taking into account $\frac{1}{\sqrt{qm}}\left(\frac{1}{2}\frac{q'}{q} - \frac{1}{2}\frac{m'}{m}\right) = \frac{1}{q}\frac{d}{dx}\sqrt{\frac{q}{m}}$, the expressions (40) for operators $\mathcal{L}^{\dagger}$ and $\mathcal{L}$ can be rewritten as

$$\mathcal{L}^{\dagger} = \frac{1}{\sqrt{qm}}\left(-\frac{d}{dx} + \widetilde{K}\right), \quad \mathcal{L} = \frac{1}{\sqrt{qm}}\left(\frac{d}{dx} + \widetilde{K}\right) + \frac{1}{q}\frac{d}{dx}\sqrt{\frac{q}{m}}. \tag{43}$$

Using (33), the potential v can be expressed in terms of $\widetilde{v}$ as follows:

$$v = \widetilde{v} - \sqrt{\frac{q}{m}}\left[2\frac{d}{dx}\left(\frac{\widetilde{K}}{\sqrt{qm}}\right) + \frac{d}{dx}\left(\frac{1}{q}\frac{d}{dx}\left(\sqrt{\frac{q}{m}}\right)\right)\right], \quad \widetilde{K} = \frac{\eta'}{\eta} \tag{44}$$

and corresponding solutions ϕ are given by

$$\phi = \mathcal{L}^{\dagger}\ \widetilde{\phi} = \frac{1}{\sqrt{qm}}\left[-\frac{d}{dx} + \frac{\eta'}{\eta}\right]\widetilde{\phi}. \tag{45}$$

Thus, the function η becomes a transformation function for the operator $\mathcal{L}^{\dagger}$, which performs the transformation from the potential $\widetilde{v}$ to the potential v and from the solutions of (3) to the solutions of (2). If within the first procedure (17), (18) we constructed the potential $\widetilde{v}$ with one bound state removed, now we can construct the potential v with an additional bound state. Note that we have established a one-to-one correspondence between the spaces of solutions of equations (2) and (3). The operators $\mathcal{L}$ and $\mathcal{L}^{\dagger}$ realize this correspondence for any $\mathcal{E} \neq \lambda$. If $\mathcal{E} = \lambda$, the correspondence is ensured by mapping $\eta \longleftrightarrow \widehat{\mathcal{U}}$ and $\widehat{\eta} \longleftrightarrow \mathcal{U}$.

In particular cases our generalized Darboux transformations are reduced correctly to the known expressions. In the case with a constant weighted energy potential, e.g., $q(x) = 1$, from our supersymmetry approach we get the supersymmetry for the effective mass Schrödinger equation [23, 25]. In the case with constant mass $m(x) = m_0$ from our approach we obtain the supersymmetry for Schrödinger equation with weighted energy [34]. Finally, if $m(x) = m_0$ and $q(x) = 1$, our expressions of supersymmetric algebra are correctly reduced to the conventional ones for the standard Schrödinger equation (see, e.g., [8]).

2. HIGHER-ORDER DARBOUX TRANSFORMATIONS

In this section by considering iterative applications of the first-order Darboux transformations n times, we obtain the nth-order Darboux transformation and show that the nth-order Darboux transformation can be expressed in terms of solutions of an initial equation, with no use of the solutions to intermediate equations.

2.1. Chain of Darboux Transformations. Now we consider iteration of one-step transformations we have obtained in the previous section, in order to construct the higher-order Darboux transformations. To this end, we show that the intertwining operator $\mathcal{L}$ of the nth-order can be obtained from a sequence of n first-order Darboux transformations, like conventional Schrödinger equation, and create a chain of exactly solvable Hamiltonians $\mathcal{H}_1, \mathcal{H}_2, \ldots, \mathcal{H}_n$. Let us define the second-order Darboux transformation as a sequence of two first-order Darboux transformations $\mathcal{L}_1$ and $\mathcal{L}_2$

$$\mathcal{L} = \mathcal{L}_2 \mathcal{L}_1, \tag{46}$$

where $\mathcal{L}_1$ is actually $\mathcal{L}$ given in (17). Let $\mathcal{U}_1$ be an auxiliary solution of (2) at energy λ_1. Then we have

$$\mathcal{L}_1 = \frac{1}{\sqrt{qm}} \left(\frac{d}{dx} + K_1 \right), \quad K_1 = -\frac{\mathcal{U}_1'}{\mathcal{U}_1}. \tag{47}$$

The operator $\mathcal{L}_2$ is determined as follows:

$$\mathcal{L}_2 = \frac{1}{\sqrt{qm}} \left(\frac{d}{dx} + K_2 \right), \quad K_2 = -\frac{\chi_1'}{\chi_1}. \tag{48}$$

The function χ_1 is obtained by means of the first-order Darboux transformation (47), applied to an auxiliary solution $\mathcal{U}_2$ of equation (2) at energy λ_2

$$\chi_1 = \mathcal{L}_1 \mathcal{U}_2 = \frac{1}{\sqrt{qm}} \left(\frac{d}{dx} + K_1 \right) \mathcal{U}_2. \tag{49}$$

Clearly, χ_1 is the solution of the transformed Eq. (3) with the potential $v_1 = \widetilde{v}$, defined as in (13), and χ_1 can be taken as a new transformation function for the Hamiltonian $\mathcal{H}_1 = \widetilde{\mathcal{H}}$ to generate a new potential

$$v_2 = v_1 + 2\sqrt{\frac{q}{m}}\frac{d}{dx}\frac{K_2}{\sqrt{qm}} - \sqrt{\frac{q}{m}}\frac{d}{dx}\left[\frac{1}{q}\frac{d}{dx}\left(\sqrt{\frac{q}{m}}\right)\right] \tag{50}$$

and corresponding solutions

$$\phi_2 = \mathcal{L}_2\phi_1 = \frac{1}{\sqrt{qm}}\left(\frac{d}{dx} + K_2\right)\phi_1, \quad \phi_1 = \mathcal{L}_1\phi. \tag{51}$$

The function $\phi_1 = \widetilde{\phi}$, defined as in (18), is the solution of Eq. (3) with the Hamiltonian $\mathcal{H}_1 = \widetilde{\mathcal{H}}$. In summary, the action of the 2nd-order operator (46) on solutions ϕ of the generalized equation (2) leads to solutions ϕ_2 of

$$\mathcal{H}_2\phi_2(x) = \mathcal{E}\phi_2(x), \quad \mathcal{H}_2 = -\frac{1}{q}\left[\frac{d}{dx}\left(\frac{1}{m}\right)\frac{d}{dx}\right] + \frac{v_2}{q} \tag{52}$$

given by

$$\phi_2 = \mathcal{L}\phi = \mathcal{L}_2\mathcal{L}_1\phi. \tag{53}$$

Iterating the procedure n times in regard to the given operator $\mathcal{H}$ leads to the operator $\mathcal{H}_n$, which satisfies the intertwining relation:

$$\mathcal{L}\mathcal{H} = \mathcal{H}_n\mathcal{L}. \tag{54}$$

The transformed potentials v_n satisfy the following recursion relation

$$v_n = v_{n-1} + 2\sqrt{\frac{q}{m}}\frac{d}{dx}\frac{K_n}{\sqrt{qm}} - \sqrt{\frac{q}{m}}\frac{d}{dx}\left[\frac{1}{q}\frac{d}{dx}\left(\sqrt{\frac{q}{m}}\right)\right], \tag{55}$$

the corresponding solutions are

$$\phi_n = \mathcal{L}\phi = \mathcal{L}_n\phi_{n-1} = \mathcal{L}_n\mathcal{L}_{n-1}\cdots\mathcal{L}_1\phi, \tag{56}$$

where $\mathcal{L}$ is the nth-order operator:

$$\mathcal{L} = \mathcal{L}_n\mathcal{L}_{n-1}\cdots\mathcal{L}_1, \quad \mathcal{L}_n = \frac{1}{\sqrt{qm}}\left(\frac{d}{dx} + K_n\right), \quad K_n = -\frac{\chi'_{n-1}}{\chi_{n-1}}. \tag{57}$$

Thus, the chain of n first-order Darboux transformations results in a chain of exactly solvable Hamiltonians $\mathcal{H} \to \mathcal{H}_1 \to \ldots \to \mathcal{H}_n$. When $\mathcal{L}$ is the nth-order differential operator and the intertwining relation (54) is valid, the so-called nth-order supersymmetry arises, like for the ordinary Schrödinger equation [12].

2.2. Darboux Transformation of the nth Order. Now we show that iteration procedure of the 1st-order Darboux transformations in (55)–(57) can be removed, and transformed potentials v_n and solutions ϕ_n can be expressed in terms of the initial potentials v, m, and q and the family $\{\mathcal{U}_j\}$, $j = 1, 2, \ldots, n$ of auxiliary solutions of the initial equation (2) at energies λ_j. Consider now the 2nd-order transformation in detail. Using the explicit expression for v_1 which appears in the first-order Darboux transformation (13), we present formula (50) for the potential v_2 as

$$v_2 = v + 2\sqrt{\frac{q}{m}}\frac{d}{dx}\frac{K}{\sqrt{qm}} - 2\sqrt{\frac{q}{m}}\frac{d}{dx}\left[\frac{1}{q}\frac{d}{dx}\left(\sqrt{\frac{q}{m}}\right)\right], \quad K = K_1 + K_2. \tag{58}$$

In order to find K, transform $K_2 = -\chi_1'/\chi_1$, representing χ_1 as

$$\chi_1 = \frac{1}{\sqrt{qm}}\frac{W_{1,2}}{\mathcal{U}_1}, \tag{59}$$

where $W_{1,2} = \mathcal{U}_1\mathcal{U}_2' - \mathcal{U}_1'\mathcal{U}_2$ is the Wronskian of the functions $\mathcal{U}_1$ and $\mathcal{U}_2$. Substituting (59) into formula (48) for K_2, we get

$$K_2 = -\frac{\chi_1'}{\chi_1} = -\frac{d}{dx}\left[\ln\left(\frac{1}{\sqrt{qm}}\frac{W_{1,2}}{\mathcal{U}_1}\right)\right], \tag{60}$$

and with account of $K_1 = -\mathcal{U}_1'/\mathcal{U}_1$ we obtain

$$K = -\frac{d}{dx}\left[\ln\frac{W_{1,2}}{\sqrt{qm}}\right]. \tag{61}$$

With the last expression after some manipulations, the new potential v_2 can be expressed as

$$v_2 = v - 2\frac{\sqrt{q}}{\sqrt{m}}\frac{d}{dx}\left[\frac{1}{W_{1,2}}\frac{d}{dx}\frac{W_{1,2}}{\sqrt{qm}}\right] - 2\sqrt{\frac{q}{m}}\frac{d}{dx}\left[\frac{1}{q}\frac{d}{dx}\left(\sqrt{\frac{q}{m}}\right)\right] \tag{62}$$

and finally in the form

$$v_2 = v - 2\sqrt{\frac{q}{m}}\frac{d}{dx}\left[\frac{\sqrt{m}\dfrac{d}{dx}\left(W_{1,2}/m\right)}{\sqrt{q}\,W_{1,2}}\right]. \tag{63}$$

Find now the corresponding functions ϕ_2. To this end, let us transform the relation (51). By analogy with χ_1 the function ϕ_1 can be written in terms of the Wronskian $W_{1,\mathcal{E}} = \mathcal{U}_1\phi' - \mathcal{U}_1'\phi$:

$$\phi_1 = \frac{1}{\sqrt{qm}}\frac{W_{1,\mathcal{E}}}{\mathcal{U}_1}. \tag{64}$$

Let us now calculate the derivative of ϕ_1

$$\phi_1' = (\mathcal{L}_1\phi)' = \left(\frac{1}{\sqrt{qm}\mathcal{U}_1}\right)' W_{1,\mathcal{E}} + \frac{1}{\sqrt{qm}}\phi'' - \frac{1}{\sqrt{qm}}\frac{\mathcal{U}_1''}{\mathcal{U}_1}\phi.$$

Making use of the last expression and the relation (60) for K_2 in (51), we obtain, after some simplification, the solutions as follows:

$$\phi_2 = \frac{1}{qm}\left(\phi'' - \frac{\mathcal{U}_1''}{\mathcal{U}_1}\phi\right) - \frac{1}{qm}\frac{W_{1,2}'}{W_{1,2}}\frac{W_{1,\mathcal{E}}}{\mathcal{U}_1} = \frac{1}{qm}\frac{W_{1,2,\mathcal{E}}}{W_{1,2}}. \tag{65}$$

It is easily seen from (63) and (65) that due to the 2nd-order Darboux transformations, the potential and solutions are completely expressed in terms of the known potential functions, v, m, and q and the solutions $\mathcal{U}_1, \mathcal{U}_2, \phi(\mathcal{E})$ of the initial equation, with no use of the solutions to the intermediate equation with the potential $v_1(x)$.

Clearly, for the next transformation step to be made, one should take a new transformation function χ_2, that corresponds to the potential v_2 at energy λ_3. The solution χ_2 can be obtained by applying the operator $\mathcal{L} = \mathcal{L}_2\mathcal{L}_1$ to the transformation solution $\mathcal{U}_3$ at energy λ_3, that is,

$$\chi_2 = \mathcal{L}_2\mathcal{L}_1\mathcal{U}_3.$$

According to (65), the solution χ_2 can be written as

$$\chi_2 = \frac{1}{qm}\frac{W_{123}}{W_{12}} \tag{66}$$

and can be used to produce a new transformed operator $\mathcal{L}_3$, given by

$$\mathcal{L}_3 = \frac{1}{\sqrt{qm}}\left(\frac{d}{dx} + K_3\right), \quad K_3 = -\frac{\chi_2'}{\chi_2}$$

for generating a new potential v_3 with corresponding solutions ϕ_3 and so on according to (55)–(57). In this way, we can express the transformed potentials v_n of any order in terms of the initial potentials v, q, effective mass m and the family of auxiliary solutions $\mathcal{U}_j$, $j = 1, 2, \ldots, n$ of the initial equation (2) at energies λ_j, which differ from each other:

$$v_n = v + 2\sqrt{\frac{q}{m}}\frac{d}{dx}\frac{K}{\sqrt{qm}} - n\sqrt{\frac{q}{m}}\frac{d}{dx}\left[\frac{1}{q}\frac{d}{dx}\left(\sqrt{\frac{q}{m}}\right)\right], \quad K = K_1 + K_2 + \ldots + K_n. \tag{67}$$

This construction enables us to generate a family of new Hamiltonians of any order and corresponding solutions directly from the initial Hamiltonian and solutions without generating intermediate Hamiltonians. For the constructing the Darboux

transformations of nth order for the conventional Schrödinger equations one can use not only eigenfunctions of the initial Hamiltonian and its associated functions [14]. It would be also interesting to generate the Darboux transformations for the generalized Schrödinger equations by using associated functions.

3. THE INTEGRAL FORM OF DARBOUX TRANSFORMATIONS

In this section the generalized Darboux transformations are represented in the integral form and applied to the construction of Hamiltonians with the same spectrum as the initial one (totally isospectral Hamiltonians), and differing by one bound state and by two bound states.

3.1. First- and Second-Order Integral Transformations. At the beginning, consider *the first-order transformation*. Multiplying both sides of Eq. (2) for ϕ by $\mathcal{U}_1$ and subtracting from the obtained expression Eq. (2) for $\mathcal{U}_1$ multiplied by ϕ, we arrive at

$$\frac{d}{dx}\left(\frac{W_{1,\mathcal{E}}}{m}\right) = (\lambda_1 - \mathcal{E}) q \mathcal{U}_1 \phi. \tag{68}$$

The last expression can be easily integrated:

$$W_{1,\mathcal{E}} = m\left((\lambda_1 - \mathcal{E})\int^x q(x')\,\mathcal{U}_1(x')\,\phi(x')\,dx' + C\right), \tag{69}$$

where C is a constant of integration. Inserting the integration result into formula (64) for ϕ_1, we arrive at the integral form for the 1st-order transformed solutions:

$$\phi_1 = \sqrt{\frac{m}{q}}\frac{1}{\mathcal{U}_1}\left(C + (\lambda_1 - \mathcal{E})\int^x q(x')\,\mathcal{U}_1(x')\,\phi(x')\,dx'\right). \tag{70}$$

The auxiliary solutions χ_1, taken at the energy $\mathcal{E} = \lambda_2$, can be written as

$$\chi_1 = \sqrt{\frac{m}{q}}\frac{1}{\mathcal{U}_1}\left(C_1 + (\lambda_1 - \lambda_2)\int^x q(x')\,\mathcal{U}_1(x')\,\mathcal{U}_2(x')\,dx'\right), \tag{71}$$

where it was used $W_{1,2} = m\Big((\lambda_1 - \lambda_2)\int^x q\,\mathcal{U}_1\mathcal{U}_2\,dx' + C_1\Big)$. The integration limits depend on the boundary conditions.

Now consider *the second-order transformation*. By analogy, using

$$W_{\chi_1,\phi_1} = m\left((\lambda_2 - \mathcal{E})\int^x q(x')\,\chi_1(x')\,\phi_1(x')\,dx' + C\right)$$

in (51), we shall get the integral form for the 2nd-order transformed solutions presented in terms of the 1st-order solutions ϕ_1 and χ_1

$$\phi_2 = \frac{1}{\sqrt{qm}} \frac{W_{\chi_1,\phi_1}}{\chi_1} = \sqrt{\frac{m}{q}} \frac{1}{\chi_1} \left(C + (\lambda_2 - \mathcal{E}) \int^x q(x')\, \chi_1(x')\, \phi_1(x')\, dx' \right). \tag{72}$$

Evidently, the solutions ϕ_2 can be expressed directly in terms of the solutions to the initial equation (2). For this, transform the expression (65) for ϕ_2 having regard to Eq. (2) for ϕ and $\mathcal{U}_1$; using (69) and $\frac{1}{m}[W_{1,2}]' = \left[\frac{W_{1,2}}{m}\right]' - \left[\frac{1}{m}\right]' W_{1,2}$ we obtain

$$\begin{aligned} \phi_2 &= (\lambda_1 - \mathcal{E})\, \phi - \frac{\mathcal{U}_2(\lambda_1 - \lambda_2) W_{1,\mathcal{E}}}{m\Big[C_1 + (\lambda_1 - \lambda_2) \int^x q(x')\, \mathcal{U}_1(x')\, \mathcal{U}_2(x')\, dx'\Big]} = \\ &= (\lambda_1 - \mathcal{E})\, \phi - \frac{\mathcal{U}_2\Big[C + (\lambda_1 - \mathcal{E}) \int^x q(x')\, \mathcal{U}_1(x')\, \phi(x')\, dx'\Big]}{c_1 + \int^x q(x')\, \mathcal{U}_1(x')\, \mathcal{U}_2(x')\, dx'}, \end{aligned} \tag{73}$$

where we have introduced $c_1 = C_1/(\lambda_1 - \lambda_2)$. Using the integral presentation of Wronskian in (63), the transformed potential v_2 can be written as

$$v_2 = v - 2\sqrt{\frac{q}{m}} \frac{d}{dx} \left(\frac{1}{\sqrt{qm}} \frac{q\mathcal{U}_2\, \mathcal{U}_1}{c_1 + \int^x dx' q(x')\, \mathcal{U}_1(x')\, \mathcal{U}_2(x')} \right). \tag{74}$$

Thus, we get the integral form of the first- and second-order Darboux transformations for the potentials and solutions. Note, the spectrum of Hamiltonian $\mathcal{H}_2$ with the potential v_2 differs from the spectrum of the initial Hamiltonian $\mathcal{H}$ by two bound states λ_1 and λ_2.

The formulae (74) and (73) can be rewritten in a more simple form

$$v_2(x) = v(x) - 2\sqrt{\frac{q(x)}{m(x)}} \frac{d}{dx} \left(\frac{1}{\sqrt{q(x)\, m(x)}}\, K(x,x) \right), \tag{75}$$

$$\phi_2(x) = \phi(x) - \int^x K(x,x')\, \phi(x')\, dx' \tag{76}$$

with the operator kernel $K(x,x')$ determined as

$$K(x,x') = \frac{\mathcal{U}_2(x)\, q(x')\, \mathcal{U}_1(x')}{c_1 + \int^x dx'\, q(x')\, \mathcal{U}_1(x')\, \mathcal{U}_2(x')}. \tag{77}$$

In (76) the constant C, connected with the Wronskian $W_{1,\mathcal{E}}$, is chosen to be zero. It is interesting to note, that at $q(x) = \text{const}$ and $m(x) = \text{const}$ the formulae (75) and (76) look like the inverse-problem ones for potentials and solutions [61, 62] obtained with the degenerate kernels $K(x, x')$ except for the form of transformation operator (77). The operator kernel (77) differs from the inverse-problem kernel $K(x, x')$ not only by $q(x) \neq \text{const}$ but by auxiliary functions $\mathcal{U}_1(x)$ and $\mathcal{U}_2(x)$, which correspond to different energies $\lambda_1 \neq \lambda_2$. In the next section we shall consider the case when the 2nd-order Darboux transformations allow one to change the spectrum of a given Hamiltonian on one bound state that corresponds to $\mathcal{U}_1(x) = \mathcal{U}_2(x)$.

3.2. Hamiltonians $\mathcal{H}_2$ Differing by One Bound State and Completely Isospectral Hamiltonians. The first-order supersymmetry, above considered in the differential and integral forms, gives us opportunities to construct isospectral Hamiltonians differing by one bound state. It is interesting to note, if we use Darboux transformation in its integral form, then we directly from (70) obtain the solution of the partner equation (3) at energy of transformation $\mathcal{E} = \lambda_1$ which with an accuracy of an arbitrary constant coincides with (20)

$$\eta = \sqrt{\frac{m}{q}}\,\frac{C}{\mathcal{U}_1}.$$

Now we show how to use double Darboux transformations to generate Hamiltonians $\mathcal{H}_2$, the spectrum of which differs from the spectrum of the initial Hamiltonian $\mathcal{H}$ by one bound state, and how to construct a family of isospectral Hamiltonians $\mathcal{H}_2$, the spectrum of which completely coincides with the spectrum of $\mathcal{H}$. For the 2nd-order transformation we used the transformation function $\chi_1(x)$ obtained within the first step, $\chi_1 = \mathcal{L}_1\mathcal{U}_2 = \dfrac{1}{\sqrt{qm}}\left(\dfrac{d}{dx} - \dfrac{\mathcal{U}_1{}'}{\mathcal{U}_1}\right)\mathcal{U}_2$ with $\lambda_1 \neq \lambda_2$. But for $\lambda_2 = \lambda_1$, we have $\mathcal{U}_2 = \mathcal{U}_1$ and it leads to $\chi_1 = \mathcal{L}_1\mathcal{U}_2 = 0$. For the second transformation one can use the function χ_1 constructed by means of a linear combination of the solutions η and $\hat{\eta}$: $\chi_1 = c_1\eta + c_2\widehat{\eta}$. For our aim we take a linear combination as follows:

$$\chi_1 = c_1\eta + \widehat{\eta} = \sqrt{\frac{m}{q}}\frac{1}{\mathcal{U}_1}\left(c_1 + \int^x dx' q(x') |\,\mathcal{U}_1(x')|^2\right). \tag{78}$$

By analogy with (61) we calculate $K = K_1 + K_2$ with $K_1 = -\mathcal{U}_1{}'/\mathcal{U}_1$ and $K_2 = -\chi_1'/\chi_1$. After simplification we have

$$K_2 = \frac{\mathcal{U}_1'}{\mathcal{U}_1} - \frac{\dfrac{d}{dx}\left[\sqrt{\dfrac{m}{q}}\left(c_1 + \int^x dx' q\,|\,\mathcal{U}_1|^{\,2}\right)\right]}{\sqrt{m/q}\left(c_1 + \int^x dx' q\,|\,\mathcal{U}_1|^2\right)}$$

and

$$K = -\frac{d}{dx}\left(\ln\sqrt{\frac{m}{q}}\right) - \frac{q\,|\mathcal{U}_1|^2}{\left(c_1 + \int\limits^x dx' q\,|\mathcal{U}_1|^2\right)}. \tag{79}$$

Plugging the last expression into formula (58) which defines the potential, after some transformations we arrive at

$$v_2 = v - 2\sqrt{\frac{q}{m}}\frac{d}{dx}\left(\sqrt{\frac{q}{m}}\frac{|\mathcal{U}_1|^2}{c_1 + \int\limits^x dx'\, q(x')\,|\mathcal{U}_1(x')|^2}\right). \tag{80}$$

With χ_1 defined by (78) and ϕ_1 represented by its integral form (70), the relation (72) leads to

$$\phi_2 = \phi(\lambda_1 - \mathcal{E}) - \frac{\mathcal{U}_1}{c_1 + \int\limits^x q(x')\,\mathcal{U}_1^2(x')\,dx'}\times \\ \times\left(C + (\lambda_1 - \mathcal{E})\int\limits^x q(x')\,\mathcal{U}_1(x')\,\phi(x')\,dx'\right). \tag{81}$$

It should be noted, that in contrast to the differential approach, the relations (80) and (81) for the new potential v_2 and the solution ϕ_2 can be obtained directly from (74) and (73), which replies the transformations with two bound states, if one takes $\mathcal{U}_1 = \mathcal{U}_2$.

It is worth noting that the auxiliary function χ_1 can be determined as follows:

$$\chi_1 = \eta + \Gamma\hat{\eta} = \sqrt{\frac{m}{q}}\frac{1}{\mathcal{U}_1}\left(1 + \Gamma\int\limits^x dx'\, q(x')\,\mathcal{U}_1^2(x')\right). \tag{82}$$

Then the potential v_2 and solutions ϕ_2 are rewritten in the form

$$v_2 = v - 2\sqrt{\frac{q}{m}}\frac{d}{dx}\left(\sqrt{\frac{q}{m}}\frac{\Gamma\mathcal{U}_1^2}{1 + \Gamma\int\limits^x dx'\, q(x')\,\mathcal{U}_1^2(x')}\right), \tag{83}$$

$$\phi_2 = (\lambda_1 - \mathcal{E})\phi - \frac{\Gamma\mathcal{U}_1}{1 + \Gamma\int\limits^x q(x')\,\mathcal{U}_1^2(x')\,dx'}\times \\ \times\left[C + (\lambda_1 - \mathcal{E})\int\limits^x q(x')\,\mathcal{U}_1(x')\,\phi(x')\,dx'\right]. \tag{84}$$

In principle, the formulae (82)–(84) coincide with (78), (80), (81) at $\Gamma = 1/c_1$, but they are more suitable for physical applications. Here, the constant Γ plays a role of a normalization constant of the new bound state λ_1 provided the other spectral characteristics coincide.

By analogy with the differential Darboux transformations, the solution of the generalized equation (1) with the potential (83) at energy of transformation $\mathcal{E} = \lambda_1$ can be achieved by means of operator $\mathcal{L}_2$ acting on the solution η from (20), obtained within the first transformation step

$$\eta_2 = \mathcal{L}_2\eta = \frac{1}{\sqrt{qm}}\left(\frac{d}{dx} - \frac{\chi_1'}{\chi_1}\right)\sqrt{\frac{m}{q}}\frac{1}{\mathcal{U}_1}. \tag{85}$$

After transformations (85) with the account of (82) we get

$$\eta_2 = -\frac{\Gamma\mathcal{U}_1}{1 + \Gamma\int\limits^{x} dx'\, q(x')\,\mathcal{U}_1^2(x')}. \tag{86}$$

Note, the solution η_2 can be directly obtained from (84) at $\mathcal{E} = \lambda_1$. Obtaining solutions at energies of transformation from the general formulae at arbitrary energies is one of advantages of integral transformations. The relations for potential (83) and solutions (84) can be rewritten in the form (75) and (76) with the operator kernel $K(x, x')$ determined as

$$K(x, x') = \frac{\Gamma\mathcal{U}_1(x)\, q(x')\,\mathcal{U}_1(x')}{1 + \Gamma\int\limits^{x} dx'\, q(x')\,\mathcal{U}_1^2(x')}.$$

At $m(x) = \mathrm{const} = m_0$ and $q(x) = \mathrm{const}$ our generalized expressions are correctly reduced to the integral equations of inverse problem for the standard Schrödinger equation with the degenerate kernel of transformation (see, e.g., [6, 61, 71]).

It is worth mentioning that the double Darboux transformation with eliminating or adding one bound state at arbitrary energy allows one to avoid the problems with singularities of the transformation kernel $K(x)$ and, as a consequence, to avoid the problems with singularities of the constructed potentials and solutions. We assume, that $m(x)$ and $q(x)$ do not lead to the additional singularities on the tested interval. Let us compare the formulae for the first-order transformation (17), (13) and for the second-order one (79), (83). One can see, if the first step procedure is based on any arbitrary solution $\mathcal{U}$ of the generalized equation (2), rather than the ground state wave function, the superpotential $K = -\mathcal{U}'/\mathcal{U}$ becomes singular. Singularities are localized at the zeros of excited wave functions. It leads to singularities of constructed potentials and solutions. Unlike the 1st-order transformations, the transformation kernel $K(x)$ of the 2nd-order (79) has no singularities at zeros of excited wave functions. It means

that one can make transformations on an arbitrary bound state (not only on the ground state) and construct the potentials and corresponding solutions without additional singularities.

It is interesting to note that by using the double Darboux transformations (83)–(86) we can construct new potentials v_2 without changing the spectrum of the initial potential v, i.e., *fully isospectral potentials* as for the ordinary Schrödinger equation. Indeed, if the bound state λ_1 belongs to the spectrum of the initial Hamiltonian $\mathcal{H}$ and $\Gamma = N_2^2 - N^2$ is a difference between the normalization constants of the bound state λ_1 for $\mathcal{H}_2$ and $\mathcal{H}$, formulae (83)–(86) give us a family of isospectral potentials and corresponding solutions, since the normalization constants can be chosen arbitrary. In quantum mechanics, potentials whose spectra coincide and differ only in the normalizations factors N_2 and N of bound states are called phase-equivalent potentials. Note, the phase-equivalent potentials have different shapes. They can be more deeper and narrow or more shallow and wider and possess the same spectral data, except for normalization constants.

If we assume, the transformation function $\mathcal{U}_1(x)$ to be taken at energy of the bound state, which we would like to add to the initial spectrum, and $\Gamma = N_2^2$ is the corresponding normalization constant, then formula (83) give us the possibility to construct a family of two-parametric potentials with a new bound state λ_1 and an arbitrary Γ, whereas the other spectral characteristics of the spectra produced by the potentials $v_2(x)$ and $v(x)$, coincide. Note, constructed potentials $v_2(x)$ from this family are isospectral among themselves, since posses the coinciding spectra and differ only by the normalization constants. Remind that our eigenfunctions are normalized with weight of $q(x)$: $\Gamma^{-1} = \int q(x)|\,\phi(x)|^2\,dx$.

4. APPLICATION

4.1. The 1st Example. As an illustrative example we present the transformed potential and solutions corresponding to the first, second, and third-order Darboux transformations. We start with the generalized Schrödinger equation (1) with the repulsive Coulomb potential $v(x) = 1/(4\,x)$

$$-\left[\frac{d}{dx}\left(\frac{1}{m(x)}\right)\frac{d}{dx}\right]\phi(x) + \frac{1}{4x}\phi(x) = q(x)\,\mathcal{E}\phi(x), \tag{87}$$

where we choose effective mass as $m = 1/x$ and $q = x$. The general solution of this equation can be written as

$$\phi(x) = \frac{C_1\,\sin(kx)}{k\sqrt{x}} + \frac{C_2\,\cos(kx)}{k\sqrt{x}}. \tag{88}$$

Now we would like to generate potentials with one bound state at energy $\mathcal{E}_1 = -\kappa_1^2$ and obtain corresponding solutions by the first-order supersymmetry transfor-

mations (44) and (45) applied to a special case $\eta = \eta_1$ of the general solution (88)

$$\eta_1 = \frac{C\,\cosh(\kappa_1 x)}{\kappa_1\sqrt{x}}.$$

We obtain the transformation operator $\widetilde{K}_1 = \eta_1'/\eta_1$ in the form

$$\widetilde{K}_1 = -\frac{1}{2x} + \kappa_1 \tanh \kappa_1 x,$$

the potential v_1 and corresponding solutions ϕ_1 at $\mathcal{E} \neq \mathcal{E}_1$

$$v_1(x) = \frac{1}{4x} - 2x\kappa_1^2\left(1 - \tanh^2(\kappa_1 x)\right) = \frac{1}{4x} - \frac{2x\kappa_1^2}{\cosh^2(\kappa_1 x)}, \tag{89}$$

$$\phi_1 = \left[-\frac{d}{dx} + \widetilde{K}_1\right]\phi = \left[-\frac{d}{dx} - \frac{1}{2x} + \kappa_1 \tanh\,\kappa_1 x\right]\phi. \tag{90}$$

The solution at energy of transformation $\mathcal{E} = \lambda_1 = -\kappa_1^2$ is defined in accordance with (41) as

$$\mathcal{U} = \sqrt{\frac{m}{q}}\frac{1}{\eta_1} = \frac{\sqrt{\kappa_1}}{\sqrt{x}\,\cosh(\kappa_1 x)} \tag{91}$$

and corresponds to the bound state. Note, by varying ϕ in (90) we recover all solutions of (1) with the transformed potential (89) and with $m = 1/x$ and $q = x$. In the case when ϕ is chosen to be $\phi(x) = \dfrac{C\,\sin(kx)}{k\sqrt{x}}$ we obtain

$$\phi_1 = -\frac{C\,\cos(kx)}{\sqrt{x}} + \frac{C\kappa_1\,\tanh(\kappa_1 x)\,\sin(kx)}{k\sqrt{x}}. \tag{92}$$

If ϕ is chosen as $\phi = \dfrac{C\,\exp(\pm ikx)}{k\sqrt{x}}$, we get the following partial solutions:

$$\phi_1^{\pm} = (\mp ik + \kappa_1\,\tanh(\kappa_1 x))\,\frac{C\,\exp(\pm ikx)}{k\sqrt{x}}. \tag{93}$$

Thus, we presented the simplest example of exactly solvable problem for the generalized equation (1) with $q(x) = x$, $m(x) = 1/x$ and with a real potential (89), which is singular at zero. The potentials, obtained at different energies of transformation, are depicted in Fig. 1, *a*.

Now employing Darboux transformations of the second order, we shall construct potentials and solutions of the generalized equation with two bound states. We define the auxiliary functions η_1 and η_2 as follows:

$$\eta_1 = \frac{\cosh(\kappa_1 x)}{\sqrt{\kappa_1 x}}, \quad \eta_2 = \frac{\sinh(\kappa_2 x)}{\sqrt{\kappa_2 x}}.$$

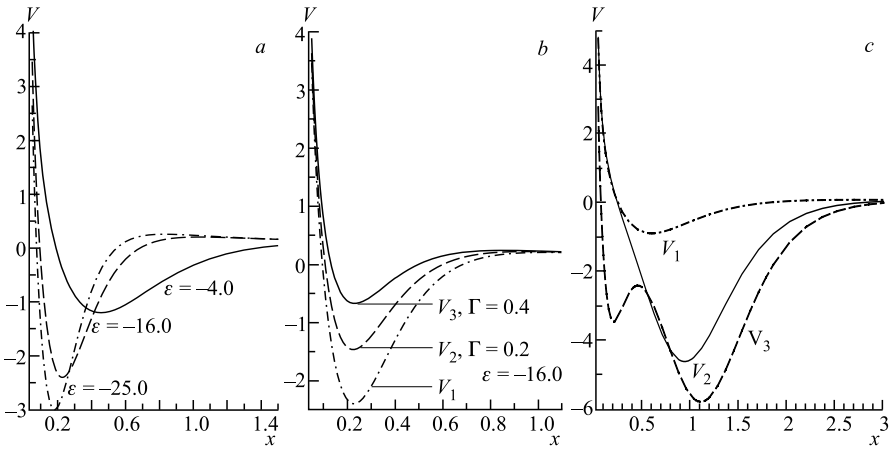

Fig. 1. *a*) Potentials $v_1(x)$ corresponding to one bound state created at different energies. *b*) Completely isospectral potentials with bound state energy $\mathcal{E} = -16.0$, dashed line corresponds to the initial potential v_1, solid line corresponds to its isospectral potentials. *c*) Potentials $v_n,\ n = 1, 2, 3$ having one, two, and three bound states, respectively

For the second step we have $\widetilde{K} = \widetilde{K}_1 + \widetilde{K}_2$, where

$$\widetilde{K}_1 = \frac{\eta_1'}{\eta_1}, \quad \widetilde{K}_2 = \frac{\chi_1'}{\chi_1}, \quad \chi_1 = \frac{1}{\sqrt{qm}}\left(-\eta_2' + \widetilde{K}_1\eta_2\right), \quad \widetilde{K} = -\frac{d}{dx}\left(\ln\frac{W_{1,2}}{\sqrt{qm}}\right).$$

The transformed potential v_2 having two bound states at energies $\lambda_1 = -\kappa_1^2$ and $\lambda_2 = -\kappa_2^2$ can be written as

$$v_2 = v - \frac{2\sqrt{q}}{\sqrt{m}}\frac{d}{dx}\left(\frac{\widetilde{K}}{\sqrt{qm}}\right) - 2\sqrt{\frac{q}{m}}\frac{d}{dx}\left[\frac{1}{q}\frac{d}{dx}\left(\sqrt{\frac{q}{m}}\right)\right]. \tag{94}$$

Finally, for our choice of v, m, and q we obtain

$$v_2 = \frac{9}{4x} - 2x\frac{d^2}{dx^2}\ln W_{1,2}, \tag{95}$$

where $W_{1,2} = W(\eta_1, \eta_2) = \dfrac{1}{x\sqrt{\kappa_2\kappa_1}}(\kappa_2 \cosh(\kappa_1 x)\cosh(\kappa_2 x) - \kappa_1 \sinh(\kappa_2 x)\times \sinh(\kappa_1 x))$ and the corresponding solutions are

$$\phi_2 = \frac{\sqrt{\kappa_1 x}}{\cosh(\kappa_1 x)}\left(\frac{d}{dx}W_{1,\mathcal{E}} - \frac{d(\ln W_{1,2})}{dx}W_{1,\mathcal{E}}\right),$$

where $W_{1,\mathcal{E}} = \dfrac{C}{xk\sqrt{\kappa_1}}(k\,\cosh(\kappa_1 x)\,\cos(kx) - \kappa_1\,\sinh(\kappa_1 x)\,\sin(kx))$ if $\phi(x)$ is chosen as $\phi(x) = \dfrac{C\,\sin(kx)}{k\sqrt{x}}$. We put $\kappa_1 < \kappa_2$. The potentials having two bound states are depicted in Fig. 1, *c*.

By using (55) one can construct the potential v_3 for the generalized Schrödinger equation (3) having three bound states

$$v_3 = \frac{13}{4x} - 2x\frac{d^2}{dx^2}\ln W_{1,2,3}, \tag{96}$$

where the auxiliary functions η_1, η_2, and η_3 determined as

$$\eta_1 = \frac{\cosh(\kappa_1 x)}{\sqrt{\kappa_1\ x}}, \quad \eta_2 = \frac{\sinh(\kappa_2 x)}{\sqrt{\kappa_2\ x}}, \quad \eta_3 = \frac{\cosh(\kappa_3 x)}{\sqrt{\kappa_3\ x}}$$

give us the Wronskian $W_{1,2,3}$

$$\begin{aligned} W_{1,2,3} = \frac{1}{x^{3/2}\sqrt{\kappa_1\kappa_2\kappa_3}}\big[&\cosh(\kappa_1 x)\,\cosh(\kappa_2 x)\kappa_2\,\sinh(\kappa_3 x)\kappa_3^2 - \\ &- \cosh(\kappa_1 x)\,\sinh(\kappa_2 x)\kappa_2^2\,\cosh(\kappa_3 x)\kappa_3 - \\ &- \sinh(\kappa_1 x)\kappa_1\,\sinh(\kappa_2 x)\,\sinh(\kappa_3 x)\kappa_3^2 + \\ &+ \sinh(\kappa_1 x)\kappa_1\,\sinh(\kappa_3 x)\,\sinh(\kappa_2 x)\kappa_2^2 + \\ &+ \cosh(\kappa_1 x)\,\sinh(\kappa_2 x)\kappa_1^2\,\cosh(\kappa_3 x)\kappa_3 - \\ &- \cosh(\kappa_1 x)\kappa_1^2\,\cosh(\kappa_2 x)\kappa_2\,\sinh(\kappa_3 x)\big]. \end{aligned} \tag{97}$$

As an illustrative example we present the potentials v_1, obtained at energy of transformation $\mathcal{E}_1 = -2.25$, v_2, obtained at the energies $\mathcal{E}_1 = -2.25$, $\mathcal{E}_2 = -4.0$, and v_3, calculated at the energies of transformation $\mathcal{E}_1 = -2.25$, $\mathcal{E}_2 = -4.0$, $\mathcal{E}_3 = -25.0$. They are depicted in Fig. 1, *c*.

Now we are going to construct *isospectral potentials*. As an initial potential we take the potential v_1, obtained within the first-order intertwining (89). Using (83) and changing v by v_1 from (89) and $\mathcal{U}_1$ by $\mathcal{U}$ from (91) after simplification we obtain a family of potentials with the same eigenvalue and different values Γ

$$v_3(x) = \frac{1}{4x} - \frac{2x\kappa_1^2}{\cosh^2(\kappa_1 x)} - 2x\frac{d^2}{dx^2}\,\ln\left[1 + \Gamma\,\tanh\,\kappa_1 x\right]. \tag{98}$$

The initial potential v_1 and its isospectral potentials (98) are presented in Fig. 1, *b*.

4.2. The 2nd Example. As the following example let us consider creation of new bound states *for effective mass Schrödinger equation*

$$-\left[\frac{d}{dx}\left(\frac{1}{m(x)}\right)\frac{d}{dx}\right]\phi(x)+v(x)\phi(x)=\mathcal{E}\,\phi(x) \tag{99}$$

that corresponds to the generalized equation (1) with $q(x)=1$. We choose effective mass in the form $m(x)=\alpha^2/x^2$, the initial potential $v(x)=0$ and start with the equation

$$-\left[\frac{d}{dx}\left(\frac{1}{m(x)}\right)\frac{d}{dx}\right]\phi(x)=\mathcal{E}\,\phi(x). \tag{100}$$

The general solution of equation(100) can be written as

$$\phi(x)=\frac{\alpha}{\sqrt{x}}\left[c_1\,\sin\left(\alpha\nu\,\ln\left(x\right)\right)+c_2\,\cos\left(\alpha\nu\,\ln\left(x\right)\right)\right], \tag{101}$$

where $c_1,\ c_2$ are free constants and $\nu^2=(-1+4\alpha^2k^2)/4\alpha^2$. Recently in [25] we have constructed the potentials for effective mass Schrödinger equation (99) with creation of one and two bound states without investigation of potential forms. Here we would like to apply our technique to construction of double-well and even triple-well potentials with creation of two and three bound states, to generation of completely isospectral potentials and to investigation of the influence of position-dependent mass on the form of constructed potentials.

Construction of completely isospectral potentials. As an initial potential we take the modified Pöschl–Teller potential v_1, obtained in [25] within the 1st-order Darboux transformation

$$v_1=-\frac{2\gamma^2}{\cosh^2\left(\alpha\gamma\,\ln\left(x\right)\right)}. \tag{102}$$

It can be easily obtained from (44) at $q(x)=1$ with a particular solution of η_1 given as $\eta_1(x)=\sqrt{\alpha/x}\,\cosh\left(\alpha\gamma_1\,\ln\left(x\right)\right)$. The solution at energy of transformation $\mathcal{E}_1=-\kappa_1^2$ is

$$\mathcal{U}=\frac{\sqrt{m(x)}}{\eta_1(x)}=\frac{\sqrt{\alpha}}{\sqrt{x}\,\cosh\left(\alpha\gamma\,\ln\left(x\right)\right)} \tag{103}$$

and corresponds to the bound state. Using (83) and replacing v by v_1 from (102) and $\mathcal{U}_1$ by $\mathcal{U}$ from (103) after simplification we obtain two-parametric family of isospectral potentials

$$v_3=-\frac{2\gamma^2}{\cosh^2\left(\alpha\gamma\,\ln\left(x\right)\right)}-\frac{2x}{\alpha}\frac{d}{dx}\left[\frac{x}{\alpha}\frac{d}{dx}\,\ln\,P\right], \tag{104}$$

where

$$P = 1 + \Gamma \int^{x} \frac{\alpha}{x \cosh^2(\alpha\gamma \ln(x))}\, dx.$$

All these potentials v_3 posses a single bound state each with the same energy $\mathcal{E}_1 = -\kappa_1^2$ as the initial potential v_1, as well as the normalization constants including in Γ can be chosen arbitrary. In Fig. 2, *a* we have plotted the initial potential calculated by formula (102) and its strictly isospectral potentials, calculated by (104) at different Γ.

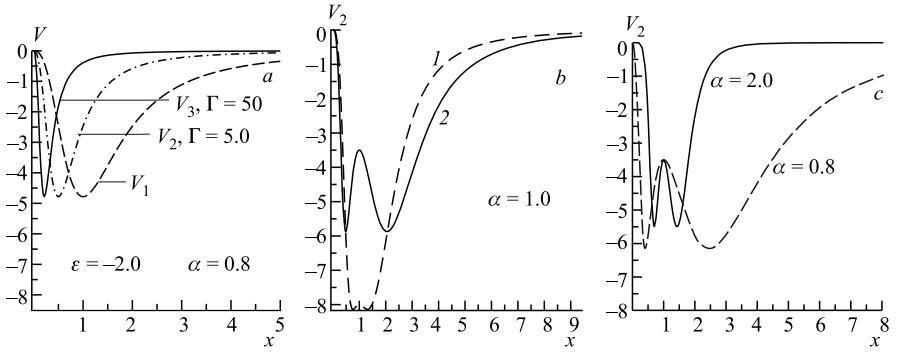

Fig. 2. *a*) Completely isospectral potentials with bound state energy $\mathcal{E} = -2.0$, dashed line corresponds to the initial potential v_1, solid lines correspond to isospectral partner potentials. *b*) The change in form of potentials $v_2(x)$ with two bound states as the levels approach each other: *1* — $\mathcal{E}_1 = -2.0$, $\mathcal{E}_2 = -6.0$; *2* — $\mathcal{E}_1 = -2.0$, $\mathcal{E}_2 = -3.75$. *c*) The influence of $m(x)$ on the behavior of transformed potentials $v_2(x)$ with two bound states $\mathcal{E}_1 = -2.0$, $\mathcal{E}_2 = -3.75$

Influence of distance between levels and effective mass on the form of potentials. By using second-order intertwining let us construct a potential with creation of two bound states. For this, we define the auxiliary transformation functions as follows:

$$\eta_1(x) = \sqrt{\frac{\alpha}{x}}\, \cosh(\alpha\gamma_1 \ln(x)), \quad \eta_2(x) = \sqrt{\frac{\alpha}{x}} \sinh(\alpha\gamma_2 \ln(x)), \qquad (105)$$

where $\gamma_i^2 = -(1 + 4\alpha^2\kappa_i^2)/4\alpha^2$, $i = 1, 2$. We put $\kappa_1 < \kappa_2$. The potential v_2, obtained within the second-order Darboux transformation, can be expressed from (94) with $q(x) = 1$

$$v_2 = v - \frac{2}{\sqrt{m}} \frac{d}{dx} \left[\frac{1}{W_{1,2}} \frac{d}{dx} \frac{W_{1,2}}{\sqrt{m}} \right] - \frac{2}{\sqrt{m}} \frac{d^2}{dx^2} \frac{1}{\sqrt{m}}. \qquad (106)$$

This formula coincides with expression obtained in [25] for effective mass Schrödinger equation. For our choice of $m(x)$ the last term vanishes and the potential is written as

$$v_2 = -\frac{2x}{\alpha}\frac{d}{dx}\left[\frac{x}{\alpha}\frac{d}{dx}\ln W_{1,2}\right], \tag{107}$$

where Wronskian is determined as

$$W_{12} = \frac{\alpha^2}{x^2}\big(\gamma_2\,\cosh(\alpha\gamma_2\,\ln(x))\,\cosh(\alpha\gamma_1\,\ln(x)) - \\ -\gamma_1\,\sinh(\alpha\gamma_1\,\ln x)\,\sinh(\alpha\gamma_2\,\ln(x))\big).$$

The potentials having two bound states are presented in Fig. 2, *b*, *c*. The graphs in (Fig. 2, *b*) depict the forms of constructed potentials in dependence on the distance between energy levels. One can see, if the levels are close to each other, then we construct asymmetric double-well potentials (curve *2* in Fig. 2, *b*) otherwise, we construct asymmetric potentials (curve *1* in Fig. 2, *b*). Note, double-well potentials have attracted some attention over the last years (see, e.g., [10,40,41]). Asymmetric double-well potentials for the ordinary Schrödinger equation were investigated in [39] with introducing a special parameter of asymmetry. In our case asymmetry in forms is a consequence of position-dependent mass $m(x)$ that is singular at zero. The influence of $m(x)$ on the form of constructed potential $v(x)$ is demonstrated in Fig. 2, *c*, *b*. Our analysis shows that the larger α, the shallow and narrow constructed potential (see Fig. 2, *b*, curve *2* and Fig. 2, *c*). In other words, increasing $m(x)$ leads to decreasing potential $v(x)$.

The next considered example illustrates the possibility to construct potentials having three bound states, for all that we can generate double-well potentials and even triple-well potentials. Employing the third-order Darboux transformations (67) with $q(x) = 1$, the potential v_3 can be written as

$$v_3 = -\frac{2x}{\alpha}\frac{d}{dx}\left[\frac{x}{\alpha}\frac{d}{dx}\,\ln\,W_{1,2,3}\right]. \tag{108}$$

The Wronskian $W_{1,2,3}$ is determined by the auxiliary functions $\eta_1(x)$, $\eta_2(x)$, $\eta_3(x)$, where $\eta_1(x)$, $\eta_2(x)$, are defined in (105) and η_3 is given as $\eta_3(x) = \sqrt{\alpha/x}\,\cosh(\alpha\gamma_3\,\ln(x))$. The potentials v_3 calculated by formula (108) are plotted in Fig. 3. As in the previous case with two bound states, the forms of constructed potentials depend on the space between energy levels. We can see if levels are sufficiently distant from one another, we construct simple asymmetric potentials presented in Fig. 3, *a* (curve *1*), if two levels out of three are close to each other, we construct asymmetric double-well potentials (Fig. 3, *a*, curve *2*), if three levels are close to each other, we construct asymmetric triple-well potentials

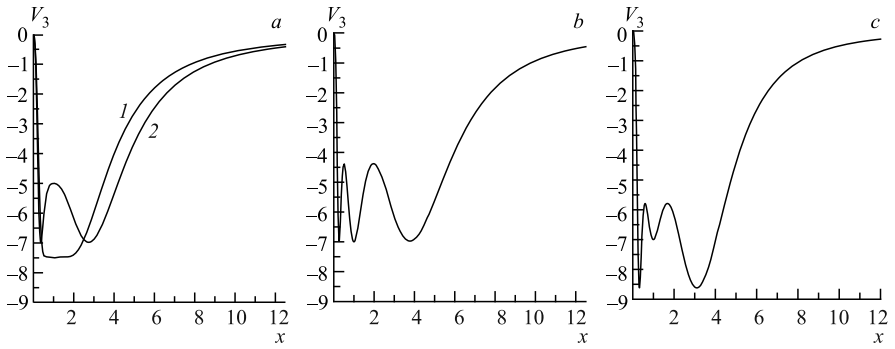

Fig. 3. The change in form of potentials $V_3(x)$ with three bound states as the levels come close to each other. *a*) *1* — $\mathcal{E}_1 = -1.0$, $\mathcal{E}_2 = -3.75$, $\mathcal{E}_3 = -6.25$; *2* — $\mathcal{E}_1 = -1.0$, $\mathcal{E}_2 = -3.75$, $\mathcal{E}_3 = -5.0$; *b*) $\mathcal{E}_1 = -2.0$, $\mathcal{E}_2 = -3.75$, $\mathcal{E}_3 = -5.0$; *c*) $\mathcal{E}_1 = -2.0$, $\mathcal{E}_2 = -4.75$, $\mathcal{E}_3 = -6.0$

(Fig. 3, *b* and *c*). As a final remark, let us note that different distances between levels give us different shapes of potentials. It can be very important for construction and investigation of quantum systems with needed spectral properties, e.g., in nanoelectronics [41].

CONCLUSION

By application of the intertwining operator technique to generalized Schrödinger equation with position-dependent mass and with energy-dependent potentials, Darboux transformations of an arbitrary order have been constructed. The formally adjoint generators of supersymmetry and two formally self-adjoint superpartner Hamiltonians are constructed and they close a quadratic pseudo-superalgebra. The integral Darboux transformation method is elaborated for the generalized Schrödinger equation. An interrelation is found between the differential and integral transformations. The integral Darboux transformations are applied to generation of isospectral Hamiltonians differing by one and by two bound states from the spectrum of the initial one. It is shown how to produce completely isospectral Hamiltonians to a given initial one. Concrete examples demonstrate how to apply the Darboux transformation technique for modeling quantum well potentials with a given spectrum. Hamiltonians with a different number of levels have been produced and the influence of the distance between the levels on the shape of the constructed potentials has been investigated. In particular, asymmetric double-well and triple-well potentials are built. The influence of position-dependent mass on the behavior of the constructed potentials is studied, too.

In concluding remarks, it will be worthwhile to discuss the following questions: A special interest presents the physical situation with a smooth initial potential and discontinuities in the effective mass. Some exactly solvable models with smooth potentials and step mass functions were obtained in [16]. It would be useful to elaborate the extended supersymmetry to generate a nonlinear superalgebra, to reconstruct complex potentials having a real spectrum, and to use not only eigenfunctions of the initial Hamiltonian and its associated functions for constructing Darboux transformations of the nth order as it is developed for the conventional Schrödinger equation [12,14,72,73]. The intertwining technique can be a useful tool to study such kind problems.

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