

GEOMETRY AND HARMONIC SUPERSPACE: SOME RECENT PROGRESS

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This contribution follows the talk, given by F. Delduc at the conference SQS'2011 in Dubna, Russia (July 18–23, 2011). To a considerable extent it is a summary of known facts about the links between geometry and extended supersymmetry in $d = 1$ mechanics, with emphasis on the harmonic superspace method created in Dubna in the 80s. Some recent developments based on [1] are also presented.

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OUTLINE

This paper is organized as follows:

— *Motion of a particle on a manifold.* An action such that the trajectories are geodesics on some Riemannian manifold is recalled.

— *Supersymmetric extension and torsion.* An extension of this model to $\mathcal{N} = 1$ worldline supersymmetry is introduced. The geometry then naturally contains a torsion.

— *$\mathcal{N} = 2$ supersymmetry.* Constraints on the geometry such that an extended $\mathcal{N} = 2$ supersymmetry emerges are recalled. The general solution to these constraints is given. The objects which parametrize this solution, the prepotentials, are used to construct a superfield action with explicit $\mathcal{N} = 2$ supersymmetry.

— *$\mathcal{N} = 2$ supersymmetry, particular cases via reduction from $d = 2, 4$.* The special cases corresponding to Kähler and Kähler with torsion geometries, which may be obtained by dimensional reduction from higher dimensions, are recalled.

— *$\mathcal{N} = 4$ supersymmetry, constraints.* Geometrical constraints on the geometry such that an extended $\mathcal{N} = 4$ supersymmetry is present are recalled.

— *$\mathcal{N} = 4$ supersymmetry, particular cases via reduction from $d = 2, 4$.* The special cases corresponding to hyper-Kähler (HK) and hyper-Kähler with torsion

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(HKT) geometries, which may be obtained by dimensional reduction from higher dimensions, are recalled.

— *Harmonic superspace.* Basic facts about harmonic superspace are recalled, with the eventual aim to write down the prepotentials of KT and HKT geometries. One has to use $2n$ charge-one superfields, which describe n hypermultiplets with the off-shell content $(4, 4, 0)$.

— *Superfield constraints and action.* A general set of superspace constraints and a general superspace action is proposed for n hypermultiplets in $d = 1$ mechanics, following [1].

— *Components, bridges and metric.* Some basic details of expressing geometrical objects (the metric, in particular) in terms of the initial data in harmonic superspace are described. A thorough study of these geometrical objects leads to the result that the relevant geometry is analogous to the HKT geometry, apart from the fact that the torsion is not closed. Such a geometry is called weak HKT.

— *Beyond weak HKT.* It is conjectured that one might describe more general geometries through the simultaneous use of two different kinds of hypermultiplets. A calculation in $\mathcal{N} = 2$ superspace sustaining this conjecture is outlined.

MOTION OF A PARTICLE IN A RIEMANNIAN MANIFOLD M

We consider a differentiable manifold M and a set of local coordinates x^i , $i = 1, \dots, n$, on M . A particle will follow a trajectory parametrized by coordinates $x^i(t)$ depending on time t . This trajectory may be obtained as a minimum of the action

$$S[x] = \int dt g_{ij}(x) \dot{x}^i \dot{x}^j, \quad \dot{x}^i = \frac{dx^i}{dt}, \quad (1)$$

where $g_{ij}(x)$ are the components of a metric tensor on the manifold M . The equations of motion are given by

$$\ddot{x}^i + \gamma_{jk}^i \dot{x}^j \dot{x}^k = 0, \quad (2)$$

where $\gamma_{jk}^i(x)$ are the Christoffel symbols associated with the metric $g_{ij}(x)$

$$\gamma_{jk}^i = \frac{1}{2} g^{il} (\partial_j g_{lk} + \partial_k g_{lj} - \partial_l g_{jk}). \quad (3)$$

Equations (2) are the equations of geodesics in a particular parametrization, such that the velocity vector has a constant length along the trajectory, $\dot{x}^i \dot{x}^j g_{ij} = \text{const}$.

SUPERSYMMETRIC EXTENSION AND TORSION

We now consider a superspace with coordinates (t, θ) , where θ is a real Grassmann variable. Supersymmetry transformations are realized as particular translations in superspace, with $\delta\theta = \epsilon$, $\delta t = -i\epsilon\theta$, and ϵ is a real Grassmann parameter. The anticommutator of two supersymmetry transformations is a time translation. We introduce superfields $X^i(t, \theta)$ such that their first components $x^i(t) = X^i(t, \theta)|_{\theta=0}$ give back the coordinates of the particle at time t . We shall use the supersymmetric derivative:

$$D = \frac{\partial}{\partial\theta} + i\theta\frac{\partial}{\partial t}, \quad D^2 = i\frac{\partial}{\partial t}. \quad (4)$$

We then write a general supersymmetric action, constrained by the requirement that bosonic component fields have a field equation of second order in time derivatives:

$$S[X] = \int dt d\theta \left(ig_{ij}(X)\dot{X}^i DX^j + \frac{1}{3!}c_{ijk}(X)DX^i DX^j DX^k \right), \quad (5)$$

where $c_{ijk}(x)$ is an antisymmetric tensor, which will play the role of a torsion. In particular, the field equations involve the following covariant derivatives (written for an arbitrary vector field V^j)

$$\nabla_i V^j = \frac{\partial}{\partial x^i} V^j + \Gamma_{ik}^j V^k, \quad (6)$$

where the connexion reads

$$\Gamma_{ik}^j = \gamma_{ik}^j + \frac{1}{2}g^{jl}c_{ikl}. \quad (7)$$

It contains, as a symmetric part, the Christoffel symbols previously introduced and, as an antisymmetric part, the new torsion tensor c_{ikl} . It is still a metric connexion, meaning that the covariant derivative of the metric vanishes.

Thus, given any geometry defined by a metric and a torsion, there is an $\mathcal{N} = 1$ supersymmetric action encoding this geometry.

$\mathcal{N} = 2$ SUPERSYMMETRY

We now look for conditions on the geometry, such that extended worldline $\mathcal{N} = 2$ supersymmetry is in fact present. The way to do that may be found in a 1980 article by L. Alvarez-Gaumé and D. Freedman [2]. We consider a

general form of the transformations under the second supersymmetry, such that it automatically anticommutes with the first supersymmetry

$$\delta X^i = \epsilon' J_j^i(X) D X^j, \quad (8)$$

where ϵ' is an extra Grassmann parameter and $J_j^i(x)$ is a tensor on M . There are now two sources of constraints on the tensor J .

The first one comes from requiring that the new transformations form a supersymmetry algebra. This leads to the equations

$$J_j^i J_k^j = -\delta_k^i, \quad J_i^l \frac{\partial}{\partial x^{[l}} J_{j]}^k - J_j^l \frac{\partial}{\partial x^{[l}} J_{i]}^k = 0, \quad (9)$$

which are summarized by saying that the tensor J is an integrable complex structure.

The second source of constraints comes from requiring that the transformations (8) leave invariant the action (5). This leads to three equations. The first one is

$$g_{ik} J_j^k + J_i^k g_{kj} = 0, \quad (10)$$

and it means that the metric is Hermitian with respect to the complex structure. The second equation is

$$\nabla_i J_j^k + \nabla_j J_i^k = 0, \quad (11)$$

and it means that the symmetrized covariant derivatives of the complex structure vanish. The covariant derivatives are just those introduced in (6) and (7). Finally, the third equation reads

$$\partial_{[i} (J_j^m c_{kl]m}) - 2J_{[i}^m \partial_{[m} c_{jkl]} = 0. \quad (12)$$

It tells us that some 4-form, linear in the torsion c and the complex structure J , vanishes. All these results may be found in a 1990 article by R. Coles and G. Papadopoulos [3].

It turns out that the constraints (9)–(12) may easily be solved. From (9) it follows that there exist local complex coordinates $(z^\alpha, \bar{z}^{\bar{\alpha}})$, such that the complex structure is constant

$$J_\alpha^\beta = i\delta_\alpha^\beta, \quad J_{\bar{\alpha}}^{\bar{\beta}} = -i\delta_{\bar{\alpha}}^{\bar{\beta}}, \quad J_{\bar{\alpha}}^\beta = J_\alpha^{\bar{\beta}} = 0, \quad (13)$$

and the change of coordinates leading from one patch to another has to be holomorphic. In these complex coordinates, the metric has only mixed components $g_{\alpha\bar{\beta}}, g_{\alpha\beta} = g_{\bar{\alpha}\bar{\beta}} = 0$. Finally, from the remaining two constraints (11) and (12), one may show that the torsion is fully specified in terms of the metric and a 2-form $B_{\alpha\beta}, \bar{B}_{\bar{\alpha}\bar{\beta}}$, with the mixed components vanishing.

One may then write an action for this geometry which has explicit $\mathcal{N} = 2$ supersymmetry. One uses an $\mathcal{N} = 2$ superspace with coordinates $(t, \theta, \bar{\theta})$, where θ is now a complex Grassmann variable, and supersymmetric derivatives are

$$D = \frac{\partial}{\partial \theta} + i\bar{\theta} \frac{\partial}{\partial t}, \quad \bar{D} = \frac{\partial}{\partial \bar{\theta}} + i\theta \frac{\partial}{\partial t}, \quad D^2 = \bar{D}^2 = 0, \quad \{D, \bar{D}\} = 2i \frac{\partial}{\partial t}. \quad (14)$$

The coordinates $z^\alpha, \bar{z}^{\bar{\alpha}}$ are defined as the first components of $\mathcal{N} = 2$ superfields $Z^\alpha, \bar{Z}^{\bar{\alpha}}$, which satisfy the chirality constraints $\bar{D}Z^\alpha = 0, D\bar{Z}^{\bar{\alpha}} = 0$. The most general action reads

$$S[Z, \bar{Z}] = \int dt d\theta d\bar{\theta} (g_{\alpha\bar{\beta}} DZ^\alpha \bar{D}\bar{Z}^{\bar{\beta}} + B_{\alpha\beta} DZ^\alpha DZ^\beta + \bar{B}_{\bar{\alpha}\bar{\beta}} \bar{D}\bar{Z}^{\bar{\alpha}} \bar{D}\bar{Z}^{\bar{\beta}}). \quad (15)$$

This action is written in terms of the unconstrained objects $g_{\alpha\bar{\beta}}, B_{\alpha\beta}, \bar{B}_{\bar{\alpha}\bar{\beta}}$, which determine the geometry. It is a general fact that the actions with explicit extended supersymmetry are written in terms of the unconstrained data (called prepotentials) which determine the geometry. This general $\mathcal{N} = 2$ superspace action (15) may be found, together with many other related results, in a 1999 article by C. Hull [4]*.

$\mathcal{N} = 2$ SUPERSYMMETRY, PARTICULAR CASES VIA REDUCTION FROM $d = 2, 4$

Among these $\mathcal{N} = 2$ geometries in supersymmetric mechanics, some special cases originate from theories in dimension $d = 2$ and $d = 4$ through the dimensional reduction to $d = 1$. We recall that, in two dimensions, one separates right-handed and left-handed supersymmetries and uses the symbol $\mathcal{N} = (p, q)$ to denote them.

- $\mathcal{N} = 1$ supersymmetry, $d = 4$ (or $\mathcal{N} = (2, 2)$ supersymmetry, $d = 2$): Torsion vanishes, covariant derivatives of the complex structure vanish. It corresponds to the celebrated Kähler geometry. In this case, the metric may be written as a second derivative

$$g_{\alpha\bar{\beta}} = \partial_\alpha \partial_{\bar{\beta}} K(z, \bar{z}), \quad (16)$$

where the scalar function $K(z, \bar{z})$ is called the Kähler potential. The Kähler potential is not necessarily defined as a function on the whole manifold. It may change, when going from the patch $U_{(a)}$ to another patch $U_{(b)}$, as

$$K_{(b)}(z_\beta, \bar{z}_\beta) = K_{(a)}(z_\alpha, \bar{z}_\alpha) + f(z_\alpha) + \bar{f}(\bar{z}_\alpha). \quad (17)$$

*The $\mathcal{N} = 2$ supersymmetric quantum mechanics associated with the action (15) also exhibits interesting geometric properties, which have recently been analyzed in [5].

In $d = 4$ and $d = 2$, the superspace action is directly determined by the Kähler potential. This was the first example of an action given in terms of the prepotential of the target geometry, and it can be found in a 1979 paper by B. Zumino [6].

• $\mathcal{N} = (2, 0)$, $d = 2$: Torsion is a closed 3-form, covariant derivatives of the complex structure vanish. This geometry is called Kähler with torsion (KT). The prepotential of this geometry has a vector index

$$g_{\alpha\bar{\beta}} = \partial_{\bar{\beta}}V_{\alpha} + \partial_{\alpha}V_{\bar{\beta}}, \quad (18)$$

and the torsion tensor is also determined in terms of the vector prepotential V_{α} , $V_{\bar{\beta}}$, which is not a globally defined vector field, however. The Kähler geometry appears as a special case of the KT geometry, when the vector potential V is expressible as a derivative of the scalar potential K ,

$$V_{\alpha} = \frac{1}{2} \frac{\partial}{\partial z^{\alpha}} K, \quad V_{\bar{\alpha}} = \frac{1}{2} \frac{\partial}{\partial \bar{z}^{\alpha}} K. \quad (19)$$

This geometry was described in a 1985 article by C. Hull and E. Witten [7].

$\mathcal{N} = 4$ SUPERSYMMETRY, CONSTRAINTS

The constraints which ensure the action (5) to possess $\mathcal{N} = 4$ supersymmetry form a natural generalization of the $\mathcal{N} = 2$ constraints [3]. Since we use a formalism with one supersymmetry being explicit, we need three additional supersymmetry transformations anticommuting with the first one. We thus introduce 3 tensors J_a , $a = 1, \dots, 3$, and write the infinitesimal transformations as

$$\delta X^i = \sum_{a=1}^3 \epsilon^a (J_a)^i_j D X^j, \quad (20)$$

where ϵ^a , $a = 1, 2, 3$, are three real Grassmann parameters.

Again, a first set of constraints comes from requiring that the new transformations form the supersymmetry algebra and anticommute with each other. One finds

$$J_a J_b + J_b J_a = -2\delta_{ab} \mathbf{1}, \quad (J_a)^i_j \frac{\partial}{\partial x^i} (J_b)^k_j - (J_a)^i_j \frac{\partial}{\partial x^i} (J_b)^k_i + (a \leftrightarrow b) = 0. \quad (21)$$

Thus, the tensors J_a are three integrable complex structures, which anticommute with each other.

A second set of constraints comes from requiring the invariance of the action (5) under the transformations (20). First, the metric has to be Hermitian with respect to all three complex structures

$$g_{ik} (J_a)^k_j + (J_a)^k_i g_{kj} = 0, \quad (22)$$

second, the symmetrized derivatives of all three complex structures should vanish,

$$D_i(J_a)_j^k + D_j(J_a)_i^k = 0, \quad (23)$$

and, finally, three 4-forms made out of the complex structures and the torsion should vanish

$$\partial_{[i}((J_a)_j^m c_{kl]m}) - 2(J_a)_{[i}^m \partial_{m} c_{jkl]} = 0. \quad (24)$$

The resolution of these constraints is much more difficult than in the $\mathcal{N} = 2$ case. Some particular cases are known, and now we shall recall them.

$\mathcal{N} = 4$ SUPERSYMMETRY, PARTICULAR CASES VIA REDUCTION FROM $d = 2, 4$

Some of the mechanical models with $\mathcal{N} = 4$ supersymmetry may be obtained by dimensional reduction from models in a higher dimension.

- $\mathcal{N} = 2$ supersymmetry, $d = 4$ (or $\mathcal{N} = (4, 4)$ supersymmetry, $d = 2$): Torsion vanishes, all three complex structures are annihilated by covariant derivatives and form the quaternionic algebra

$$J_a J_b = -\delta_{ab} \mathbf{1} + \epsilon_{abc} J_c. \quad (25)$$

This particular geometry is called the hyper-Kähler (HK) geometry.

- $\mathcal{N} = (4, 0)$, $d = 2$: Torsion is a closed 3-form, complex structures are annihilated by covariant derivatives (with a connexion including torsion) and form the quaternionic algebra. This geometry is called the hyper-Kähler with torsion (HKT) geometry.

In both cases, the prepotentials of the geometry are known. They have been studied by A. Galperin, E. Ivanov, S. Kalitzin, V. Ogievetsky and E. Sokatchev [8] for the HK geometry and by F. Delduc, S. Kalitzin, E. Sokatchev [9] for the HKT geometry. Both cases require making use of harmonic superspace.

HARMONIC SUPERSPACE [10, 11]

The coordinates of $\mathcal{N} = 4$ superspace in one dimension may be written as $(t, \theta^i, \bar{\theta}_i)$, where θ^i , $i = 1, 2$, is a pair of complex Grassmann variables. In the harmonic approach, one adds to these variables another set of bosonic variables (u_i^\pm) , $i = 1, 2$, called harmonic variables. They should be such that the 2 by 2 matrix

$$\begin{pmatrix} u_1^+ & u_1^- \\ u_2^+ & u_2^- \end{pmatrix} \quad (26)$$

belongs to the group $SU(2)$. All fields depend on harmonic variables and have definite charges under the right action of the diagonal $U(1)$ subgroup of $SU(2)$. In harmonic superspace, one can find a subspace invariant under all four supersymmetries

$$(t_A, \theta^+ = \theta^i u_i^+, \bar{\theta}^+ = \bar{\theta}^i u_i^+, u_i^\pm). \tag{27}$$

This subspace is called analytic superspace. It involves only half of the original Grassmann variables. To describe HK or HKT geometry, one needs a set of $2n$ charge 1 analytic superfields (which will be called hypermultiplets)

$$q^{+a}(t_A, \theta^+, \bar{\theta}^+, u^\pm), \quad a = 1, \dots, 2n. \tag{28}$$

In the HK case, the prepotential is a charge 4 scalar function $\mathcal{L}^{+4}(q^{+a}, u^\pm)$. In the HKT case, the analytic prepotential carries an $Sp(2n)$ index and has charge 3, $\mathcal{L}^{+3a}(q^{+b}, u^\pm)$. Notice that HK is a special case of HKT, with

$$\mathcal{L}^{+3a} = \Omega^{ab} \frac{\partial}{\partial q^{+b}} \mathcal{L}^{+4}, \tag{29}$$

where Ω is a $2n$ by $2n$ constant regular antisymmetric matrix (also called a symplectic metric). Since in harmonic superspace there are new coordinates, the harmonic variables u_i^\pm , there also appear new derivatives, which are consistent with the constraints on the harmonic variables. They are called harmonic derivatives and read

$$D^{++} = u^{+i} \frac{\partial}{\partial u^{-i}}, \quad D^{--} = u^{-i} \frac{\partial}{\partial u^{+i}}, \quad D^0 = u^{+i} \frac{\partial}{\partial u^{+i}} - u^{-i} \frac{\partial}{\partial u^{-i}}. \tag{30}$$

Their commutation relations give back the Lie algebra of $SU(2)$. An important point about the derivative operator D^{++} is that it acts inside the analytic subspace.

SUPERFIELD CONSTRAINTS AND ACTION

In two dimensions, the field equations of an HKT nonlinear sigma model read

$$D^{++} q^{+a} = \mathcal{L}^{+3a}(q^+, u^\pm). \tag{31}$$

When restricted to one dimension, this equation is no longer dynamical. It puts to zero some component fields inside q^{+a} , but it does not restrict the time dependence of those component fields which survive. It is a harmonic constraint that restricts the $SU(2)$ content of the superfields. It may be shown that the content of the superfields q^{+a} , subject to the constraints (31) (as well as to some self-consistent reality condition), is just that of n $(\mathbf{4}, \mathbf{4}, \mathbf{0})$ multiplets of $d = 1$,

$\mathcal{N} = 4$ supersymmetry in one dimension. Each $(4, 4, 0)$ multiplet contains 4 real bosons and 4 real fermions.

Since the constraints (31) are not dynamical, we need some extra input from which the equations of motion of the fields may be obtained. The most general action leading to equations for the physical bosons which are of second order in time derivatives reads

$$S = \int dt d^4\theta du \mathcal{L}(q^{+a}, q^{-a}, u^\pm), \quad q^{-a} = D^{--}q^{+a}. \quad (32)$$

Since this action is integrated over the whole superspace, it is not required that the integrand lives on the analytic subspace. Indeed, the Lagrangian density \mathcal{L} depends on the non-analytic superfields q^{-a} . One may add to this action a term which is an integral on the analytic subspace only

$$S_{WZ} = \int du dt_A d^2\theta^{(-2)} \mathcal{L}^{+2}(q^{+a}, u^\pm). \quad (33)$$

It is called a Wess–Zumino term, and physically it describes the coupling of the particle to an external magnetic field. The major part of the article [1] is devoted to extracting the geometry experienced by the physical fields from the non-linear constraints (31) and the action (32).

COMPONENTS, BRIDGES AND METRIC

One first expands the superfields q^{+a} in powers of the Grassmann variables θ^+ , $\bar{\theta}^+$

$$q^{+a} = f^{+a}(t, u) + \theta^+ \chi^a(t, u) + \bar{\theta}^+ \bar{\chi}^a(t, u) + \theta^+ \bar{\theta}^+ A^{-a}(t, u). \quad (34)$$

It may be shown that the component A^{-a} is fully determined by the other components as a consequence of the constraint (31). The remaining components are not yet ordinary fields. They depend not only on time, but also on harmonic variables u_i^\pm . This dependence is restricted as a consequence of the constraint (31)

$$D^{++}f^{+a} = \mathcal{L}^{+3a}(f^+, u^\pm), \quad D^{++}\chi^a - \frac{\partial \mathcal{L}^{+3a}(f^+, u^\pm)}{\partial f^{+b}} \chi^b = 0. \quad (35)$$

The first of the equations (35) means that f^{+a} is determined if one knows its lowest order term in harmonic variables. One has to separate

$$f^{+a}(t, u) = f^{ia}(t)u_i^+ + v^{+a}(f^{jb}(t), u^\pm). \quad (36)$$

The $4n$ fields $f^{ia}(t)$ form coordinates of the manifold under study. The functions v^{+a} may be interpreted as a bridge between two different sets of coordinates.

The fermionic components $\chi^a(t, u)$ and its complex conjugate also satisfy a complicated harmonic equation, which is the second equation of (35). It may be simplified by introducing a frame bridge, which is a $2n \times 2n$ matrix M satisfying

$$D^{++} M_a^b + \frac{\partial \mathcal{L}^{+3c}}{\partial f^{+a}} M_c^b = 0. \quad (37)$$

Then the fermionic field $\chi^a = M_b^a \chi^b$ is independent of harmonic variables, $D^{++} \chi^a = 0$, and thus depends only on time. The frame bridge is used to define the harmonic independent vielbeins e_{ib}^{ka} as

$$\frac{\partial f^{+a}}{\partial f^{ib}} M_a^a = -e_{ib}^{ka} u_k^+, \quad (38)$$

as well as the symplectic metric

$$G_{ab} = \int du (M^{-1})_{\underline{a}}^a (M^{-1})_{\underline{b}}^b (\partial_{+[a} \partial_{-b]} \mathcal{L} + \dots). \quad (39)$$

One finally gets the local expression for the Riemannian metric on the manifold

$$g_{ia kb} = G_{\underline{cd}} \epsilon_{\underline{lt}} e_{ia}^{\underline{lc}} e_{kb}^{\underline{td}}. \quad (40)$$

Notice that the tangent space metric $G_{\underline{cd}}$ is not constant, so the vielbeins e_{ib}^{ka} do not define an orthonormal frame. On the contrary, complex structures are constant in the tangent space and read in the coordinate space as

$$(J_{(\underline{lk})})_{jc}^{ia} = i e_{(\underline{lb}}^{ia} e_{jc}^{\underline{tb}} \epsilon_{\underline{k}\underline{t}}. \quad (41)$$

Finally, the component action reads

$$S = \int dt \left[\frac{1}{2} g_{ia kb} \dot{f}^{ia} \dot{f}^{kb} - \frac{i}{4} G_{[\underline{a}\underline{b}]} (\nabla \bar{\chi}^{\underline{a}} \chi^{\underline{b}} - \bar{\chi}^{\underline{a}} \nabla \chi^{\underline{b}}) - \frac{1}{16} (\epsilon^{\underline{ik}} \nabla_{i[\underline{a}} \nabla_{\underline{k}\underline{b}]} G_{[\underline{c}\underline{d}]}) \bar{\chi}^{\underline{a}} \bar{\chi}^{\underline{b}} \chi^{\underline{c}} \chi^{\underline{d}} \right].$$

The salient points of the results that were obtained in [1] are that complex structures form a quaternionic algebra, that they are covariantly constant and that torsion is in general not closed. A geometry with such properties was called weak HKT in a paper by P. Howe and G. Papadopoulos in 1996 [13]. A novel feature which is brought in by the harmonic superspace approach is that this weak HKT geometry is solved in terms of two unconstrained prepotentials, the general one $\mathcal{L}(f^+, f^-, u^\pm)$ and the analytic one $\mathcal{L}^{+3a}(f^+, u^\pm)$.

Some particular cases may arise. If the Lagrangian in (32) is quadratic, $\mathcal{L} = \Omega_{ab}q^{+a}q^{-b}$, then the torsion is closed and the geometry is HKT. If, moreover, the analytic prepotential \mathcal{L}^{+3a} is a derivative,

$$\mathcal{L}^{+3a} = \Omega^{ab} \frac{\partial}{\partial q^{+b}} \mathcal{L}^{+4}, \quad (42)$$

then the geometry is HK. If, however, one restricts the analytic prepotential as in (42) but keeps a general Lagrangian \mathcal{L} , one gets a geometry intimately connected to the hyper-Kähler geometry encoded in \mathcal{L}^{+4} , but which includes torsion. In particular, if the manifold has dimension 4, the HKT metric is conformal to the HK metric, with a conformal factor which is a harmonic function on the HK manifold (i.e., satisfies the covariant Laplace–Beltrami equation on this manifold, which just amounts to the torsion closedness condition in this case) [14]. If the conformal factor is arbitrary, one faces a weak HKT geometry. In the simplest case $\mathcal{L}^{+4} = 0$, the metric is conformal to the flat \mathbb{R}^4 metric, while the torsion closedness condition is just the \mathbb{R}^4 Laplace equation for the conformal factor [4, 15, 16].

BEYOND WEAK HKT

Thus, a set of hypermultiplets of the same kind does not allow one to describe in superspace the most general geometry allowed by $\mathcal{N} = 4$ supersymmetry in one dimension. We conjecture that the description of this general case requires the simultaneous use of two different types of hypermultiplets. The automorphism group of the $\mathcal{N} = 4$ supersymmetry algebra is $SO(4) \simeq SU(2) \times SU(2)$. One of these two $SU(2)$ groups acts on the harmonic variables. One may define two types of hypermultiplets, depending on which $SU(2)$ group is associated with harmonic variables. Very probably, when using the two types together, one may describe the general $\mathcal{N} = 4$ geometry.

A computation in support of this conjecture was done in $\mathcal{N} = 2$ superspace [1]. Starting from chiral superfields z^α, y^a ($\alpha = 1, \dots, 2n, a = 1, \dots, 2m$), one can write 2 extra supersymmetry transformations as

$$\delta z^\alpha = \epsilon J_\beta^\alpha D z^\beta, \quad \delta y^a = \bar{\epsilon} \tilde{J}_b^a D u^b.$$

Then the z coordinates and the y coordinates belong to different $(4, 4, 0)$ representations of $\mathcal{N} = 4$ supersymmetry. We have checked that, generically, the complex structures (in the full target space of complex dimension $2(n+m)$) do not form the quaternionic algebra, and only symmetrized covariant derivatives of complex structures vanish. It remains to show that one indeed can get the most general geometry in this way. For the particular case of two linear $(4, 4, 0)$

multiplets of different sorts (thus corresponding to 8-dimensional target space), the most general component action was constructed in [17], proceeding from $\mathcal{N} = 4$ superfield formalism. The set of relevant target metrics encompasses some examples which were explicitly given earlier in [18] and were argued in [4] to correspond to the general geometry.

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