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WHAT IS THE PARTITION BUNDLE?

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Six-dimensional $(2,0)$ theory can be defined on a large class of six-manifolds endowed with some additional topological and geometrical data. We discuss the nature of the object in such a theory that generalizes the partition function of a more conventional quantum field theory.

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The most elementary definition of the partition function Z of, e.g., quantum mechanics is in a Hamiltonian formulation:

$$Z = \mathrm{Tr}_{\mathcal{H}}\left(\mathrm{e}^{-\beta H + \gamma J + \ldots}\right)$$

with

$$\begin{aligned} \mathcal{H} &= \text{Hilbert space of theory}, \\ H, J, \ldots &= \text{commuting observable operators}, \\ \beta, \gamma, \ldots &= \text{formal parameters}. \end{aligned}$$

But this is of course not generally covariant!

In quantum field theory, we therefore often prefer a Lagrangian formulation, e.g., for Yang–Mills theory with 't Hooft flux v on a four-manifold B:

$$Z_v \quad = \quad \sum_P \delta_{w_1(P),v} \int \mathcal{D}A \ldots \exp\left(-\int_B \mathrm{Tr}(g^{-2} F \wedge *F + \theta F \wedge F) + \ldots\right)$$

with

$$\begin{aligned} P &= \text{gauge bundle}, \\ A &= \text{connection on } P, \\ F &= \text{field strength of } A, \\ g, \theta &= \text{coupling constant, theta angle}. \end{aligned}$$

But what is the counterpart of Z for quantum theories which do not admit a classical description? The best known examples of such theories are the $(2,0)$ *superconformal theories in six dimensions* with the following properties:

• Completely classified by the type

$$\Phi \in \mathrm{ADE} \simeq \{\text{simply laced Lie algebras}\}.$$

• Realized in type IIB string theory at codimension 4 singularity.

• A-series (D-series) realized on coincident M5-branes (with orientifold plane).

• Holographic representation of A-series as M-theory on $AdS_7 \times S^4$.

• $OSp(6,2|4)$ superconformal algebra in flat space with $so(6,2) \oplus sp(4)$ even subalgebra.

But $(2,0)$ theories can also be defined on an arbitrary six-manifold M endowed with some additional data:

• Data related to the geometry of M, namely,

$$\begin{aligned}
\sigma &\in \Sigma \\
&= \{\text{orientations on } M\} \\
&= \text{affine space over } H^0(M,\mathbb{Z}_2) \\
s &\in \mathcal{S} \\
&= \{\text{spin structures on } M\} \\
&= \text{affine space over } H^1(M,\mathbb{Z}_2) \\
[g] &\in \mathcal{G} \\
&= \{\text{conformal structures on } M\} \\
&= \text{infinite dimensional real manifold.}
\end{aligned}$$

• Data related to the $sp(4) \simeq so(5)$ R-symmetry (neglected in this talk).

• Data related to observables defined on two- and four-dimensional submanifolds of M (also neglected here).

These theories are indeed generally covariant in six dimensions. But they cannot be described by any generally covariant Lagrangian. This is intimately related to the unusual properties of a «partition object» Z that generalizes the partition function of a more conventional quantum theory. In fact, these theories do not even have any «fields» that obey classical differential «equations of motion». It *is* possible to write a Lagrangian in certain situations in which general covariance is broken anyway; e.g., compactification may lead to a low-energy theory that is a conventional quantum field theory.

The basic question that will be discussed in this talk is: *What kind of object is Z, and how does it depend on the geometric data* $(\sigma, s, [g])$*?*

A brief summary of the answer is:

• Z is not a number but an element of a finite-dimensional vector space V determined by $(\sigma, s, [g])$.

• A choice of basis of V necessarily breaks six-dimensional general covariance. If this is done, the corresponding components of Z are analogous to the partition functions Z_v for different 't Hooft flux v in Yang–Mills theory.

• As the data are varied, the spaces V fit together to the total space of a vector bundle over $\Sigma \times \mathcal{S} \times \mathcal{G}$.

We will work with the A_{N-1} series of models drawing inspiration from the holographic representation. The holographic dual is a supergravity theory on an open seven-manifold Y of boundary $\partial Y = M$ with action

$$S = N \int_X C \wedge dC + \ldots$$

Here C is an Abelian three-form gauge field. In the gauge where C has no component normal to M, the phase space of the C-field is the intermediate Jacobian torus

$$T = H^3(M, \mathbb{R})/H^3(M, \mathbb{Z}) \simeq (\mathbb{R}/\mathbb{Z})^{2n}$$

with $n = (1/2)b_3(M)$ (half of the third Betti number of M). Geometric quantization of this TFT leads to a holomorphic prequantum line bundle $\mathcal{L}^N$ over T and a finite-dimensional Hilbert space

$$V = H^0(T, \mathcal{L}^N)$$

of holomorphic sections with $\dim_{\mathbb{C}} V = N^n$. The «partition vector» Z is an element of V.

In somewhat more detail, the data $(\sigma, s, [g])$ in the infinite-dimensional space $\Sigma \times \mathcal{S} \times \mathcal{G}$ determines data (ω, u, J) in a finite-dimensional space $\Omega \times \mathcal{U} \times \mathcal{J}$ related to the intermediate Jacobian T:

$$\begin{aligned}
\omega \in \Omega \\
&= \{\text{symplectic structures on } T \\
&\quad \text{induced from the intersection form}\} \\
&= \text{set with 2 elements} \\
u \in \mathcal{U} \\
&= \{\text{nondegenerate quadratic forms on } H^3(M, \mathbb{Z}_2) \text{ polarized by } \omega\} \\
&= \text{set with } 2^{2n} \text{ elements} \\
J \in \mathcal{J} \\
&= \text{translation invariant complex structures on } T\} \\
&= \text{complex space of dimension } \frac{1}{2}n(n+1).
\end{aligned}$$

We define the map

$$\phi\colon \Sigma \times \mathcal{S} \times \mathcal{G} \to \Omega \times \mathcal{U} \times \mathcal{J}.$$

• The symplectic structure ω on $H^3(M, \mathbb{R})$ is given by the wedge product followed by integration over M.

• The nondegenerate quadratic form u on $H^3(M, \mathbb{Z}_2)$ is defined as

$$(-1)^{u(\gamma)} = \exp\left(2\pi i \frac{1}{2} \int\limits_{S^1 \times M} C \wedge dC\right).$$

Here C is an Abelian three-form gauge field on $S^1 \times M$ determined by a straight line from 0 to $\gamma \in H^3(M, \mathbb{Z}) \subset H^3(M, \mathbb{R})$. Because of $1/2$, to make sense of this expression requires a spin structure s on M.

• The complex structure J on $H^3(M, \mathbb{R})$ is given by the Hodge duality operator $*$, which obeys $** = -1$ for an Euclidean signature on M.

The data (ω, u, J) determine a Hermitian line bundle $\mathcal{L}$ over the intermediate Jacobian T:

• The curvature of $\mathcal{L}$ is given by ω.

• The holonomy of $\mathcal{L}$ along the closed curve on T descending from a straight line from 0 to $\gamma \in H^3(M, \mathbb{Z}) \subset H^3(M, \mathbb{Z})$ is given by $(-1)^{u(\gamma)}$.

For the A_{N-1} model, the TFT prequantum line bundle is $\mathcal{L}^N$ and the partition vector Z of the $(2, 0)$ theory is an element of $V = H^0(T, \mathcal{L}^N)$. $\mathcal{L}^N$ is invariant under the translations

$$T_c\colon T \to T$$

parametrized by $c \in (1/N)H^3(M, \mathbb{Z})/H^3(M, \mathbb{Z})$. Clearly,

$$T_c^N = \mathbb{1},$$
$$T_c T_{c'} = T_{c'} T_c.$$

But the induced operators

$$T_c^*\colon V \to V$$

instead fulfill the Heisenberg relations

$$(T_c^*)^N = (-1)^{u(Nc)},$$
$$T_c^* T_{c'}^* = T_{c'}^* T_c^* \exp\left(2\pi i N \int\limits_M c \wedge c'\right).$$

The spin structure s determines the choice of square root signs in the Heisenberg algebra

$$T_c^* T_{c'}^* = \pm \sqrt{\exp\left(2\pi i N \int\limits_M c \wedge c'\right)}\, T_{c+c'}^*.$$

The vector space V carries an irreducible representation of this Heisenberg algebra.

Having understood what happens in a specific geometric situation, our next task is to investigate what happens to the vector space V as the geometric data $(\sigma, s, [g])$ are varied in the space $\Sigma \times S \times \mathcal{G}$. The answer is that the vector space $V = H^0(T, \mathcal{L}^N)$ is the fiber of a rank N^n holomorphic vector bundle over $\Omega \times \mathcal{U} \times \mathcal{J}$. Indeed, we have described a map

$$\phi\colon \Sigma \times S \times \mathcal{G} \to \Omega \times \mathcal{U} \times \mathcal{J}.$$

Pullback by ϕ gives the «partition bundle» over $\Sigma \times S \times \mathcal{G}$.

Eventually, one would like to compute the precise «partition section» Z of this bundle, but this goal is still out of reach. But for the moment, we can at least gain a more explicit understanding of the holomorphic vector bundle over $\Omega \times \mathcal{U} \times \mathcal{J}$: We concentrate on the space of conformal structures on M

$$\mathcal{G} = \overline{\mathcal{G}}/\{\text{mapping class group of } M\}$$

and the space of (translation invariant) complex structures on T

$$\mathcal{J} = \overline{\mathcal{J}}/\{\text{group isomorphic to } Sp_{2n}(\mathbb{Z})\},$$

where the bar denotes the universal covering space. The map $\phi\colon \Sigma \times S \times \mathcal{G} \to \Omega \times \mathcal{U} \times \mathcal{J}$ induces a homomorphism

$$\{\text{mapping class group of } M\} \to Sp_n(\mathbb{Z}).$$

The holomorphic bundle over $\mathcal{J}$ pulls back to a (necessarily trivial) holomorphic bundle over $\overline{\mathcal{J}}$ that we will describe next.

$Sp_{2n}(\mathbb{Z})$ acts on $H^3(M, \mathbb{Z}) \simeq \mathbb{Z}^{2n}$, preserving the symplectic structure ω and permuting the possible quadratic forms u in two orbits:

- The first orbit consists of u which give $H^3(M, \mathbb{Z}_2)$ the structure of a direct sum of n hyperbolic planes. There is then a (non-unique) Lagrangian decomposition

$$H^3(M, \mathbb{Z}) = A \oplus B$$

with

$$u(a + b) = \int_M a \wedge b \quad \text{for} \quad a \in A, \quad b \in B.$$

- The second orbit consists of u which give $H^3(M, \mathbb{Z}_2)$ the structure of a direct sum of $n - 1$ hyperbolic planes and a two-dimensional anisotropic space. (We conjecture that no u on this orbit arise from a spin structure on M as described above.)

The universal covering space $\overline{\mathcal{J}}$ of $\mathcal{J}$ can be identified with the genus n Siegel upper half space (i.e., the set of complex symmetric $n \times n$ matrices with positive definite imaginary part). We describe the holomorphic bundle over $\mathcal{J}$ by explicitly constructing a holomorphic frame, i.e., a frame for the bundle over $\overline{\mathcal{J}}$ subject to certain quasi-periodicity requirements. This can be seen as a kind of vector-valued Siegel modular forms for the subgroup of $Sp_{2n}(\mathbb{Z})$ that stabilizes u that do not seem to have been much considered before in the literature.

With the decomposition $H^3(M, \mathbb{Z}) = A \oplus B$, we can make the following identifications:

• The complex structure J amounts to a map

$$\tau \colon A \to B \otimes \mathbb{C}$$

with a certain self-adjointness property and with positive definite imaginary part.

• The intermediate Jacobian is then

$$T = \frac{B \otimes \mathbb{C}}{B \oplus \tau A}.$$

• An element of $V = H^0(T, \mathcal{L}^N)$ can be identified with a holomorphic function

$$\psi(\tau|.) \colon B \otimes \mathbb{C} \to \mathbb{C}$$

subject to the quasi-periodicity conditions

$$\psi(\tau|z + m + \tau n) = \psi(\tau|z) \exp\left(-i\pi N \int_M n \wedge \tau n + 2n \wedge z\right)$$

for $z \in B \otimes \mathbb{C}$, $n \in A$, and $m \in B$.

We define a holomorphic frame $\{\psi_{[a]}\}$ indexed by $[a] \in (1/N)A/A$. This is uniquely determined (up to a common holomorphic factor) by requiring the following behavior under the Heisenberg translations:

$$\psi_{[a]}(\tau|z + b' + \tau a') = \psi_{[a+a']}(\tau|z) \times$$
$$\times \exp\left(-i\pi N \int_M a' \wedge \tau a' + 2a' \wedge z - 2a \wedge b'\right)$$

for $a' \in (1/N)A$ and $b' \in (1/N)B$. The solution is

$$\psi_{[a]}(\tau|z) = \frac{1}{\theta(\tau|0)} \sum_{n \in A} \exp\left(i\pi N \int_M (n + a) \wedge \tau(n + a) + 2(n + a) \wedge z\right).$$

(Here $\theta(\tau|z) = \sum_{n \in A} \exp(n \wedge \tau n + n \wedge z)$ is the Riemann theta function.)

Finally, we must investigate how the frame $\{\psi_{[a]}\}$ for $[a] \in (1/N)A/A$ behaves under a symplectic transformation S: With $H^3(M, \mathbb{Z}) = A \oplus B$, we write

$$S = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} : \begin{pmatrix} B \to B & A \to B \\ B \to A & A \to A \end{pmatrix}.$$

The action on a section ψ of $H^0(T, \mathcal{L}^N)$ is

$$S\psi(\tau|z) = \psi(S\tau|Sz) \exp\left(-\frac{N}{2}\gamma z \wedge Sz\right),$$

where

$$\begin{aligned} \tau &\mapsto S\tau = (\alpha\tau + \beta)(\gamma\tau + \delta)^{-1}, \\ z &\mapsto Sz = (\gamma\tau + \delta)^{*-1} z. \end{aligned}$$

One thus finds the automorphic transformation law

$$\psi_{[a]}(\tau|z) = \frac{\sqrt[8]{1}}{N^n} \sum_{[b] \in \frac{1}{N} B/B} S\psi_{[-\gamma b + \delta a]}(\tau|z) \times \\ \times \exp\left(-i\pi N \int_M \delta a \wedge \beta a + 2\beta a \wedge \gamma b + \gamma b \wedge \alpha b\right).$$

These results have been reported in my paper [1], where more details and a complete set of references can be found.

REFERENCES

1. *Henningson M.* The Partition Bundle of Type A_{N-1} $(2,0)$ Theory // JHEP. 2011. V. 1104. P. 090; arXiv:1012.4299 [hep-th].