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ON AFFINE EXTENSION OF SPLINT ROOT SYSTEMS

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Splint of root system of simple Lie algebra appears naturally in the study of (regular) embeddings of reductive subalgebras. It can be used to derive branching rules. Application of splint properties drastically simplifies calculations of branching coefficients. We study affine extension of splint root system of simple Lie algebra and obtain relations on theta and branching functions.

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INTRODUCTION

The term *splint* was introduced by D. Richter in [1], where the classification of splints for simple Lie algebras was obtained. The fan $\Gamma \subset \Delta$ was introduced in [2] as a subset of root system describing recurrent properties of branching coefficients for maximal embeddings. Injection fan is an efficient tool to study branching rules. Later this construction was generalized to nonmaximal embeddings and affine Lie algebras in [3, 4]. In paper [5], we have shown that the existence of a splint for a root system of a simple Lie algebra leads to simplifications in reduction procedures of a Lie algebra module to modules of a subalgebra. This effect is based on the injection fan and singular element properties of Lie algebra modules.

In the present note, we discuss possible applications of splint in a root system of simple Lie algebra related to representation theory of affine Lie algebras. We discuss the structure of injection fan for affine Lie algebras and show that it admits a decomposition similar to that used in [5] for simple Lie algebras. Such a decomposition leads to equations for theta functions. We study graded branching of affine Lie algebra modules reduced to a finite-dimensional subalgebra and discuss consequences of splint in this case.

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SPLINTS AND AFFINE LIE ALGEBRAS

Consider simple Lie algebra $\mathfrak{g}$ with a root system Δ. Let $\mathfrak{a}_1 \subset \mathfrak{g}$ be its reductive subalgebra of the same rank, such that $\Delta_{\mathfrak{a}} \equiv \Delta_1 \subset \Delta$ and $Q_{\mathfrak{a}} \subset Q$, where Q is the root lattice. Irreducible highest-weight modules of $\mathfrak{g}$ and $\mathfrak{a}$ are denoted by L^{μ} and $L^{\nu}_{\mathfrak{a}}$, correspondingly. The Weyl character formula for irreducible modules is $\operatorname{ch} L^{\mu} = \dfrac{\Psi^{(\mu)}}{\prod\limits_{\alpha\in\Delta} (1 - \mathrm{e}^{-\alpha})}$, where $\Psi^{(\mu)} = \sum\limits_{w\in W} \epsilon(w)\, \mathrm{e}^{w(\mu+\rho)-\rho}$ is a singular element of the module and W — the Weyl group of $\mathfrak{g}$. Formal character of irreducible module admits the decomposition

$$\operatorname{ch} L^{\mu} = \sum_{\nu\in P_{\mathfrak{a}}} b^{\mu}_{\nu} \operatorname{ch} L^{\nu}_{\mathfrak{a}}, \tag{1}$$

where P, $P_{\mathfrak{a}}$ are weight lattices of $\mathfrak{g}$ and $\mathfrak{a}$. We want to study the affine extension of this situation: $\mathfrak{g} \subset \hat{\mathfrak{g}}$, $\mathfrak{a} \subset \hat{\mathfrak{a}}$, $\mathfrak{a} \subset \hat{\mathfrak{g}}$, $\hat{\Delta} \subset \hat{\Delta}$ and $\operatorname{ch} L^{\hat{\mu}}_{\hat{\mathfrak{g}}} = \sum\limits_{\hat{\nu}} b^{\hat{\mu}}_{\hat{\nu}} \operatorname{ch} L^{\hat{\nu}}_{\hat{\mathfrak{a}}}$. For weights of an affine Lie algebra $\hat{\mathfrak{g}}$ we have $\hat{\mu} = (\mu, k, n)$, where μ is a weight of $\mathfrak{g}$, k — the level of the module and n — the grade of the weight $\hat{\mu}$

Definition 2.1. *Embedding ϕ of a root system Δ_1 into a root system Δ is a bijective map of roots of Δ_1 to a (proper) subset of Δ that commutes with vector composition law in Δ_1 and Δ.*

$$\phi : \Delta_1 \longrightarrow \Delta, \quad \phi \circ (\alpha + \beta) = \phi \circ \alpha + \phi \circ \beta, \quad \alpha, \beta \in \Delta_1.$$

Note that the image $\operatorname{Im}(\phi)$ must not inherit the root system properties except the addition rules equivalent to the addition rules in Δ_1 (for pre-images). Two embeddings ϕ_1 and ϕ_2 can splinter Δ when the latter can be presented as a disjoint union of images $\operatorname{Im}(\phi_1)$ and $\operatorname{Im}(\phi_2)$.

ϕ induces an injection of formal algebras: $\mathcal{E}_0 \hookrightarrow \mathcal{E}$ and for the image $\mathcal{E}_i = \operatorname{Im}_{\phi}(\mathcal{E}_0)$ one can consider its inverse ϕ^{-1}: $\mathcal{E}_i \longrightarrow \mathcal{E}_0$.

Definition 2.2. *A root system Δ «splinters» as (Δ_1, Δ_2) if there are two embeddings ϕ_1: $\Delta_1 \hookrightarrow \Delta$ and ϕ_2: $\Delta_2 \hookrightarrow \Delta$, where (a) Δ is the disjoint union of the images of ϕ_1 and ϕ_2 and (b) neither the rank of Δ_1 nor the rank of Δ_2 exceeds the rank of Δ.*

It is equivalent to say that (Δ_1, Δ_2) is a «splint» of Δ, and we shall denote this by $\Delta \approx (\Delta_1, \Delta_2)$. Each component Δ_1 and Δ_2 is a «stem» of the splint.

We consider the case when one of the stems $\Delta_1 = \Delta_{\mathfrak{a}}$ is a root subsystem. As is shown in [5], the second stem $\Delta_{\mathfrak{s}} := \Delta_2 = \Delta \setminus \Delta_{\mathfrak{a}}$ can be translated into a product $\prod\limits_{\beta\in\Delta^{+}_{\mathfrak{s}}} (1 - \mathrm{e}^{-\beta}) = -\sum\limits_{\gamma\in P} s(\gamma)\, \mathrm{e}^{-\gamma}$, and it defines an injection fan $\Gamma_{\mathfrak{a}} \hookrightarrow \mathfrak{g}$ [2–4].

Since the singular element of L^{μ} can be written as $\Psi_{\mathfrak{g}}^{(\mu)} = \mathrm{e}^{-\rho} \sum_{w \in W_{\mathfrak{a}}} \epsilon(w) w \circ (\mathrm{e}^{\rho_{\mathfrak{a}}} \Psi^{\widetilde{\mu}+\rho_{\mathfrak{s}}})$ for branching coefficients, we get the identity [5]:

$$b_{(\mu-\phi(\widetilde{\mu}-\widetilde{\nu}))}^{(\mu)} = M_{(\mathfrak{s})\widetilde{\nu}}^{\widetilde{\mu}}. \tag{2}$$

Here the highest weight $\widetilde{\mu}$ is totally defined by the weight μ, they have the same Dynkin numbers: $\mu = \sum m_k \omega_k \Longrightarrow \widetilde{\mu} = \sum m_k \omega_{(\mathfrak{s})k}$. So branching coefficients coincide with weight multiplicities of $\mathfrak{s}$-modules.

Now we consider affine extension of this setup, $\hat{\mathfrak{a}} \subset \hat{\mathfrak{g}}$. Since $\operatorname{rank}\mathfrak{g} \leqslant \operatorname{rank}\mathfrak{a} + \operatorname{rank}\mathfrak{s}$ for the Weyl denominators, we get

$$\prod_{\alpha \in \hat{\Delta}_1^+} (1-\mathrm{e}^{-\alpha})^{\operatorname{mult}(\alpha)} \prod_{\beta \in \hat{\Delta}_2^+} (1-\mathrm{e}^{\phi\circ\beta})^{\operatorname{mult}(\beta)} = \\ = \prod_{\gamma \in \hat{\Delta}^+} (1-\mathrm{e}^{-\gamma})^{\operatorname{mult}(\gamma)} \prod_{n=0}^{\infty} (1-\mathrm{e}^{-n\delta})^{\operatorname{rank}\mathfrak{a}+\operatorname{rank}\mathfrak{s}-\operatorname{rank}\mathfrak{g}}.$$

Using a specialization [6–8] and the definition of the Dedekind eta function $\eta(\tau) = q^{1/24} \prod_{n=1}^{\infty} (1-q^n)$, where $q = \mathrm{e}^{2\pi i \tau}$, we can rewrite this identity as the relation imposed on theta functions $\Theta_{\widehat{\lambda}=(\lambda,k,0)}^{(\hat{\mathfrak{g}})}(\tau, z) = \sum_{\xi \in Q_{\mathfrak{g}}+\frac{\lambda}{k}} \exp\left[2\pi i k\left(\frac{1}{2}(\xi,\xi)\tau + +(\xi,z)\right)\right]$:

$$\eta(\tau)^{\dim(\mathfrak{a})} \prod_{\alpha \in \Delta_1^+} \frac{\Theta_{\alpha}^{(\hat{A}_1)}(\tau,z)}{\eta(\tau)} \eta(\tau)^{\dim(\mathfrak{s})} \prod_{\beta \in \Delta_2^+} \frac{\Theta_{\phi\circ\beta}^{(\hat{A}_1)}(\tau,z))}{\eta(\tau)} = \\ = \eta(\tau)^{\operatorname{rank}(\mathfrak{a})+\operatorname{rank}(\mathfrak{s})-\operatorname{rank}(\mathfrak{g})} \eta(\tau)^{\dim(\mathfrak{g})} \prod_{\alpha \in \Delta^+} \frac{\Theta_{\alpha}^{(\hat{A}_1)}(\tau,z))}{\eta(\tau)}. \tag{3}$$

Here $z \in P_{\geqslant 0} \otimes \mathbb{C}$. Using the Weyl denominator identity this relation can be rewritten as a nontrivial relation connecting theta functions of algebras $\hat{\mathfrak{g}}, \hat{\mathfrak{s}}, \hat{\mathfrak{a}}$:

$$\left(\sum_{v \in W_{\mathfrak{a}}} \epsilon(v) \Theta_{v\rho_{\mathfrak{a}}}^{(\hat{\mathfrak{a}})}(\tau,z)\right) \left(\sum_{u \in W_{\mathfrak{s}}} \epsilon(u) \Theta_{\phi\circ(u\rho_{\mathfrak{s}})}^{(\hat{\mathfrak{s}})}(\tau,z)\right) = \left(\sum_{w \in W} \epsilon(w) \Theta_{w\rho_{\mathfrak{g}}}^{(\hat{\mathfrak{g}})}(\tau,z)\right). \tag{4}$$

Now consider the branching of $\hat{\mathfrak{g}}$-module to $\mathfrak{g}$-modules. For formal characters we can write the following expression:

$$\operatorname{ch} L_{\hat{\mathfrak{g}}}^{\hat{\mu}} = \sum_{n=0}^{\infty} \mathrm{e}^{-n\delta} \sum_{\nu \in P} b_{\nu}^{(\hat{\mu})}(n) \operatorname{ch} L_{\mathfrak{g}}^{\nu}. \tag{5}$$

Rewriting this equation for weight multiplicities we get $m^{(\hat{\mu})}_{\hat{\nu}=(\nu,k,n)} = \sum_{\xi\in P} b^{(\hat{\mu})}_{\xi} \times \times (n) m^{(\xi)}_{\nu}$. We can introduce branching functions similarly to the case of branching for affine subalgebra [6, 8]: $b^{(\hat{\mu})}_{\nu}(q) = \sum_{n=0}^{\infty} b^{(\hat{\mu})}_{\nu}(n) q^n$. These branching functions are connected to q-dimension of module $\dim_q L^{\hat{\mu}}_{\hat{\mathfrak{g}}} = \sum_{n=0}^{\infty} q^n \sum_{\nu\in P} b^{(\hat{\mu})}_{\nu}(n) \times \times \dim L^{\nu}_{\mathfrak{g}} = \sum_{\nu\in P} b^{(\hat{\mu})}_{\nu}(q) \dim L^{\nu}_{\mathfrak{g}}$. It is well known that q-dimension is a modular function for some $\Gamma \subset SL_2(\mathbb{Z})$ [9], so branching functions $b^{(\hat{\mu})}_{\nu}(q)$ have modular properties.

For string functions of a module $L^{\hat{\mu}}$ we have

$$\sigma^{(\hat{\mu})}_{\nu}(q) = \sum_{\xi\in P} m^{(\xi)}_{\nu} b^{(\hat{\mu})}_{\xi}(q). \tag{6}$$

Introduce an ordering of the set of weights ξ as follows: attribute to a weight (ρ,ξ) its product (ρ,ξ) with the Weyl vector ρ. Then relation (6) can be written in the matrix form $\sigma(q) = Mb(q)$ or as an inverse relation $b(q) = M^{-1}\sigma(q)$. Here $\sigma(q)$ and $b(q)$ are infinite columns of string and branching functions. Matrix M contains multiplicities of weights in $\mathfrak{g}$-modules similar to that of Table 1 in [10]. The inverse matrix M^{-1} encodes recurrent relations imposed on weight multiplicities [11].

Now consider the branching of $\hat{\mathfrak{g}}$-modules in $\mathfrak{a}$-modules and assume the existence of a splint $\Delta^+_{\mathfrak{g}} = \Delta^+_{\mathfrak{a}} \cup \phi(\Delta^+_{\mathfrak{s}})$. We decompose $\mathfrak{g}$-modules in equation (5) into $\mathfrak{a}$-modules using property (2):

$$\begin{aligned}\operatorname{ch} L^{\hat{\mu}}_{\hat{\mathfrak{g}}} &= \sum_{n=0}^{\infty} \mathrm{e}^{-n\delta} \sum_{\nu\in P_{\mathfrak{a}}} b^{(\hat{\mu})}_{(\hat{\mathfrak{g}}\downarrow\mathfrak{a})\nu}(n) \operatorname{ch} L^{\nu}_{\mathfrak{a}} = \\ &= \sum_{n=0}^{\infty} \mathrm{e}^{-n\delta} \sum_{\nu\in P} b^{(\hat{\mu})}_{(\hat{\mathfrak{g}}\downarrow\mathfrak{g})\nu}(n) \sum_{\xi\in P_{\mathfrak{a}}} b^{(\nu)}_{(\mathfrak{g}\downarrow\mathfrak{a})\xi} \operatorname{ch} L^{\xi}_{\mathfrak{a}} = \\ &= \sum_{n=0}^{\infty} \mathrm{e}^{-n\delta} \sum_{\nu\in P} b^{(\hat{\mu})}_{(\hat{\mathfrak{g}}\downarrow\mathfrak{g})\nu}(n) \sum_{\xi\in P_{\mathfrak{a}}} M^{\widetilde{\nu}}_{\widetilde{\nu}-\phi^{-1}(\nu-\xi)} \operatorname{ch} L^{\xi}_{\mathfrak{a}}.\end{aligned} \tag{7}$$

We see that the similar matrix relation holds for branching functions $b_{(\hat{\mathfrak{g}}\downarrow\mathfrak{a})}(q) = M_{\mathfrak{s}}\ b_{(\hat{\mathfrak{g}}\downarrow\mathfrak{g})}(q)$, and we can write $\sigma(q) = M_{\mathfrak{a}}\ b_{(\hat{\mathfrak{g}}\downarrow\mathfrak{a})}(q)$. So if we know branching coefficients for the embedding $\mathfrak{g} \subset \hat{\mathfrak{g}}$ (for example, see [12]) we can easily obtain branching functions for the embedding $\mathfrak{a} \subset \hat{\mathfrak{g}}$.

CONCLUSION

We have demonstrated that splint in affine Lie algebras leads to new relations between theta functions and branching functions for branching to finite-dimensional subalgebras, which can be useful for computations. Further question is to generalize this analysis to affine subalgebras and to apply the results to branching in the study of CFT coset models.

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