ФИЗИКА ЭЛЕМЕНТАРНЫХ ЧАСТИЦ И АТОМНОГО ЯДРА
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ALTARELLI–PARISI EQUATION IN NONEQUILIBRIUM QCD

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The Q^2 evolution of fragmentation function in nonequilibrium QCD by using DGLAP evolution equation may be necessary to study hadron formation from quark–gluon plasma at RHIC and LHC. In this paper we study splitting functions in nonequilibrium QCD by using Schwinger–Keldysh closed-time path integral formalism. For quarks and gluons with arbitrary nonequilibrium distribution functions $f_q(\mathbf{p})$ and $f_g(\mathbf{p})$, we derive expressions for quark and gluon splitting functions in nonequilibrium QCD at leading order in α_s. We make a comparison of these splitting functions with those obtained by Altarelli and Parisi in vacuum.

Обсуждается необходимость учета Q^2-эволюции функции фрагментации в неравновесной КХД, задаваемой уравнением эволюции ДГЛАП, при изучении механизма образования адронов из кварк-глюонной плазмы на установках RHIC и LHC.

В представленной статье изучаются функции расщепления в неравновесной КХД в рамках приближения замкнутого времени Швингера–Келдыша в формализме функциональных интегралов. Для кварков и глюонов с произвольными неравновесными функциями распределения $f_q(\mathbf{p})$ и $f_g(\mathbf{p})$ получены выражения для кварковой и глюонной функций расщепления в неравновесной КХД в ведущем порядке по α_s. Полученные функции сравниваются с результатом вычислений Альтарелли и Паризи в вакууме.

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INTRODUCTION

RHIC and LHC heavy-ion colliders are the best facilities to study quark–gluon plasma in the laboratory. Since two nuclei travel almost at a speed of light, the QCD matter formed at RHIC and LHC may be in nonequilibrium. In order to make meaningful comparison of the theory with the experimental data on hadron production, it may be necessary to study nonequilibrium-nonperturbative QCD at RHIC and LHC. This, however, is a difficult problem.

Nonequilibrium quantum field theory can be studied by using Schwinger–Keldysh closed-time path (CTP) formalism [1,2]. However, implementing CTP in nonequilibrium at RHIC and LHC is a very difficult problem, especially due to the presence of gluons in nonequilibrium and hadronization, etc. Recently, one-loop resumed gluon propagator in nonequilibrium in covariant gauge is derived in [3,4].

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High-p_T hadron production at high-energy e^+e^-, ep, and pp colliders is studied by using the Collins–Soper fragmentation function [5, 6]. For a high-p_T parton fragmenting to hadron, Collins and Soper derived an expression for the fragmentation function based on the field theory and factorization properties in QCD at high energy [7]. This fragmentation function is universal in the sense that, once its value is determined from one experiment, it explains the data from other experiments.

Recently we have derived parton-to-hadron fragmentation function in nonequilibrium QCD by using Schwinger–Keldysh closed-time path integral formalism [8]. This can be relevant at RHIC and LHC heavy-ion colliders to study hadron production from quark–gluon plasma. We have considered a high-p_T parton in medium at initial time τ_0 with arbitrary nonequilibrium (non-isotropic) distribution function $f(\mathbf{p})$ fragmenting to hadron. The special case $f(\mathbf{p}) = \dfrac{1}{\exp\,(p_0/T) \pm 1}$ corresponds to the finite temperature QCD in equilibrium.

We have found the following definition of the parton-to-hadron fragmentation function in nonequilibrium QCD by using closed-time path integral formalism [8]. For a quark (q) with arbitrary nonequilibrium distribution function $f_q(\mathbf{k})$ at initial time, the quark-to-hadron fragmentation function is given by

$$D_{H/q}(z, p_T) = \\ = \frac{1}{2z\;[1+f_q(\mathbf{k})]} \int dx^- \frac{d^{d-2}x_T}{(2\pi)^{d-1}}\, \mathrm{e}^{ik^+x^- + i\,p_T\,x_T/z} \frac{1}{2}\mathrm{tr}_{\mathrm{Dirac}}\;\frac{1}{3}\mathrm{tr}_{\mathrm{color}} \times \\ \times\,[\gamma^+\langle \mathrm{in}|\psi(x^-,x_T)\;\Phi[x^-,x_T]\;a_H^\dagger(P^+,0_T)\;a_H(P^+,0_T)\;\Phi[0]\;\bar{\psi}(0)|\mathrm{in}\rangle], \quad (1)$$

where $z(=P^+/k^+)$ is the longitudinal momentum fraction of the hadron with respect to the parton, and p_T is the transverse momentum of the hadron. For a gluon (g) with arbitrary nonequilibrium distribution function $f_g(\mathbf{k})$ at initial time, the gluon-to-hadron fragmentation function is given by

$$D_{H/g}(z, p_T) = -\frac{1}{2zk^+\;[1+f_g(\mathbf{k})]} \int dx^- \frac{d^{d-2}x_T}{(2\pi)^{d-1}}\, \mathrm{e}^{ik^+x^- + ip_T\,x_T/z} \frac{1}{8}\sum_{a=1}^{8} \times \\ \times\,[\langle \mathrm{in}|F_a^{+\mu}(x^-,x_T)\;\Phi[x^-,x_T]\;a_H^\dagger(P^+,0_T)\;a_H(P^+,0_T)\;\Phi[0]\;F_{\mu a}^{+}(0)|\mathrm{in}\rangle]. \quad (2)$$

In the above equations $|\mathrm{in}\rangle$ is the initial state of the nonequilibrium quark (gluon) medium. The path ordered exponential

$$\Phi[x^\mu] = \mathcal{P}\;\exp\left[ig\int_{-\infty}^{0} d\lambda\; n\cdot A^a(x^\mu + n^\mu\lambda)\;T^a\right] \quad (3)$$

is the Wilson line [5, 7, 9].

Equations (1) and (2) can be compared with the following definition of the Collins–Soper fragmentation function in vacuum [5]:

$$D_{H/q}(z,p_T) = \frac{1}{2z}\int dx^- \frac{d^{d-2}x_T}{(2\pi)^{d-1}}\, \mathrm{e}^{ik^+x^- + ip_T\, x_T/z} \frac{1}{2}\mathrm{tr}_{\mathrm{Dirac}}\ \frac{1}{3}\mathrm{tr}_{\mathrm{color}} \times$$

$$\times\, [\gamma^+ \langle 0|\psi(x^-,x_T)\ \Phi[x^-,x_T]\ a_H^\dagger(P^+,0_T) a_H(P^+,0_T)\ \Phi[0]\ \bar{\psi}(0)|0\rangle] \quad (4)$$

and

$$D_{H/g}(z,p_T) = -\frac{1}{2zk^+}\int dx^- \frac{d^{d-2}x_T}{(2\pi)^{d-1}}\, \mathrm{e}^{ik^+x^- + ip_T\, x_T/z} \frac{1}{8}\sum_{a=1}^{8} \times$$

$$\times\, [\langle 0|F_a^{+\mu}(x^-,x_T)\ \Phi[x^-,x_T]\ a_H^\dagger(P^+,0_T) a_H(P^+,0_T)\ \Phi[0]\ F_{\mu a}^{+}(0)|0\rangle]. \quad (5)$$

Since the fragmentation function is a nonperturbative quantity, we do not have theoretical tools in QCD to calculate it yet. The normal procedure at high-energy pp, ep, and e^+e^- colliders is to extract it at some initial momentum scale μ_0 and then evolve it to another scale μ by using the DGLAP evolution equation [10–12]

$$\mu\frac{\partial}{\partial\mu} D_{i\to j}(z) = \sum_j \int_z^1 \frac{dy}{y} P_{ij}\left(\frac{z}{y},\mu\right) D_{j\to j}(y). \quad (6)$$

In the above equation, $P_{ij}(z)$ is the splitting function of a parton j into a parton i which is related to the probability of a parton j emitting a parton i with longitudinal momentum fraction z. The quark and gluon splitting functions $P_{ij}(z)$ in vacuum are evaluated by Altarelli and Parisi in [11] at the leading order in coupling constant α_s.

In order to apply this procedure at high-energy heavy-ion colliders at RHIC and LHC one needs to prove factorization of fragmentation function in nonequilibrium QCD. Recently we have proved factorization theorem in nonequilibrium QED in [13] and in nonequilibrium QCD in [14].

In this paper we will evaluate the quark and gluon splitting functions in nonequilibrium QCD at the leading order in coupling constant α_s by using closed-time path integral formalism. We find that these splitting functions depend on nonequilibrium distribution functions of quarks and gluons in the QCD medium. The Q^2 evolution of the nonequilibrium fragmentation functions (Eqs. (1) and (2)) can be studied from Eq. (6) by using nonequilibrium splitting functions.

We find the following expressions for the quark and gluon splitting functions in nonequilibrium QCD at leading order in coupling constant α_s:

$$\begin{aligned}
P_{gq}(z) &= C_2(R)\,[1+f_q(k)]^2 \times \\
&\quad\times [1+f_g(k_T,zk)]^2\,[1+f_q(-k_T,(1-z)k)]^2 \left[\frac{1+(1-z)^2}{z}\right], \\
P_{qq}(z) &= C_2(R)\,[1+f_q(k)]^2 \times \\
&\quad\times [1+f_g(-k_T,(1-z)k)]^2\,[1+f_q(k_T,zk)]^2 \left[\frac{1+z^2}{1-z}\right], \\
P_{gg}(z) &= 2C_A\,[1+f_g(k)]^2\,[1+f_g(k_T,zk)]^2 \times \\
&\quad\times [1+f_g(-k_T,(1-z)k)]^2 \left[\frac{1-z}{z}+\frac{z}{1-z}+z(1-z)\right],
\end{aligned} \tag{7}$$

where k is the momentum of initial parton (which is assumed to be along longitudinal direction); k_T is the transverse momentum of the emitted parton, and z is the longitudinal momentum fraction of the initial parton carried by the emitted parton.

Equation (7) can be compared with the following expressions for the splitting functions in vacuum obtained by Altarelli and Parisi [11] at the leading order in coupling constant α_s:

$$\begin{aligned}
P_{gq}(z) &= C_2(R)\ \frac{1+(1-z)^2}{z}, \\
P_{qq}(z) &= C_2(R)\ \frac{1+z^2}{1-z}, \\
P_{gg}(z) &= 2C_A\ \left[\frac{1-z}{z}+\frac{z}{1-z}+z(1-z)\right].
\end{aligned} \tag{8}$$

We will present derivation of Eq. (7) in this paper.

The paper is organized as follows. In Sec. 1 we briefly review the derivation of quark and gluon splitting functions in vacuum. In Sec. 2 we describe Schwinger–Keldysh closed-time path integral formalism in nonequilibrium QCD relevant to our calculation. In Sec. 3 we derive quark and gluon splitting functions P_{ij} in nonequilibrium QCD by using closed-time path integral formalism. The last section contains conclusions.

1. QUARK AND GLUON SPLITTING FUNCTIONS IN VACUUM

In this section we briefly review the derivation of quark and gluon splitting functions P_{ij} in vacuum. We will present our calculation in the S-matrix approach. Hence, our derivation is slightly different from [11].

Consider a quark with momentum p_A emitting a gluon with momentum p_B in the process $q(p_A) \to g(p_B) + q(p_C)$. The S-matrix element for this process is given by

$$S^{(1)} = ig \int d^4x N \left[\bar{\psi}(x) \not{A}^a(x)\, T^a \psi(x) \right], \tag{9}$$

where (the normalization is from [15])

$$\psi(x) = \psi^+(x) + \psi^-(x) = \\ = \sum_{\text{spin}} \sum_p \sqrt{\frac{m}{VE_p}} \left[a_q(p) u(p)\, \mathrm{e}^{-ip\cdot x} + a^\dagger_{\bar{q}}(p) v(p)\, \mathrm{e}^{ip\cdot x} \right],$$

$$\bar{\psi}(x) = \bar{\psi}^+(x) + \bar{\psi}^-(x) = \\ = \sum_{\text{spin}} \sum_p \sqrt{\frac{m}{VE_p}} \left[a_{\bar{q}}(p) \bar{v}(p)\, \mathrm{e}^{-ip\cdot x} + a^\dagger_q(p) \bar{u}(p)\, \mathrm{e}^{ip\cdot x} \right], \tag{10}$$

$$A^\mu(x) = A^{\mu+}(x) + A^{\mu-}(x) = \\ = \sum_{\text{spin}} \sum_p \sqrt{\frac{1}{2VE_p}} \left[a_g(p) \epsilon^\mu(p)\, \mathrm{e}^{-ip\cdot x} + a^\dagger_g(p) \epsilon^\mu(p)\, \mathrm{e}^{ip\cdot x} \right].$$

In the above equations, $a_q(p)$, $a_{\bar{q}}(p)$, and $a_g(p)$ are annihilation operators for quark, antiquark, and gluons, respectively. In Eq. (10) the suppression of color indices is understood. The initial and final states are

$$\begin{aligned} |i\rangle &= |q(p_A)\rangle = a^\dagger_q(p_A)|0\rangle, \\ |f\rangle &= |q(p_C), \quad g(p_B)\rangle = a^\dagger_q(p_C) a^\dagger_g(p_B)|0\rangle, \end{aligned} \tag{11}$$

where $p_C = p_A - p_B$. Hence we find

$$|\langle f|S^{(1)}|i\rangle|^2 = \left[\frac{V}{(E_C + E_B - E_A)} \right]^2 \frac{m}{VE_C} \frac{m}{VE_A} \frac{1}{2VE_B} \sum_{\text{spin}} |M|^2, \tag{12}$$

where

$$M = ig\bar{u}(p_C)\, \gamma_\mu u(p_A)\, \epsilon^{a\mu}(p_B)\, T^a. \tag{13}$$

For massless quarks, we find

$$W_{gq} = |\langle f|S^{(1)}|i\rangle|^2 \frac{V d^3p_C}{(2\pi)^3} = \\ = C_2(R) g^2 \frac{d^3p_C}{(2\pi)^3} \frac{E_B}{2E_A E_C (2E_B)^2} \frac{1}{(E_C + E_B - E_A)^2} \times \\ \times \mathrm{Tr}[\not{p}_C \gamma^i \not{p}_A \gamma^j] \left(\delta^{ij} - \frac{p_B^i p_B^j}{\mathbf{p}_B^2}\right), \quad (14)$$

which gives the quark-to-gluon splitting function

$$P_{gq}(z) = C_2(R) \frac{1 + (1-z)^2}{z}. \quad (15)$$

Similarly we find the quark-to-quark splitting function

$$P_{qq}(z) = C_2(R)\ \frac{1 + z^2}{1 - z} \quad (16)$$

and the gluon-to-gluon splitting function

$$P_{gg}(z) = 2C_A\ \left[\frac{1-z}{z} + \frac{z}{1-z} + z(1-z)\right]. \quad (17)$$

2. NONEQUILIBRIUM QCD USING CLOSED-TIME PATH FORMALISM

Unlike pp collisions, the ground state at RHIC and LHC heavy-ion collisions (due to the presence of a QCD medium at initial time $t = t_{\rm in}$ (say $t_{\rm in} = 0$)) is not a vacuum state $|0\rangle$ any more. We denote $|{\rm in}\rangle$ as the initial state of the nonequilibrium QCD medium at $t_{\rm in}$. The nonequilibrium distribution function $f(\mathbf{k})$ of a parton (quark or gluon), corresponding to such an initial state is given by

$$\langle a^\dagger(\mathbf{k}) a(\mathbf{k}')\rangle = \langle {\rm in}|a^\dagger(\mathbf{k}) a(\mathbf{k}')|{\rm in}\rangle = f(\mathbf{k}) \delta^{(3)}_{\mathbf{k}\mathbf{k}'}, \quad (18)$$

where we have assumed space translational invariance at initial time.

Finite temperature field theory formulation is a special case of this when $f(\mathbf{k}) = \dfrac{1}{\exp\,(k_0/T) \pm 1}$.

2.1. Quarks in Nonequilibrium. The nonequilibrium (massless) quark propagator at initial time $t = t_{\rm in}$ is given by (suppression of color indices is understood)

$$G(k)_{ij} = \\ = \not{k} \begin{pmatrix} \frac{1}{k^2+i\epsilon} + 2\pi\delta(k^2)f_q(\mathbf{k}) & -2\pi\delta(k^2)\theta(-k_0) + 2\pi\delta(k^2)f_q(\mathbf{k}) \\ -2\pi\delta(k^2)\theta(k_0) + 2\pi\delta(k^2)f_q(\mathbf{k}) & -\frac{1}{k^2-i\epsilon} + 2\pi\delta(k^2)f_q(\mathbf{k}) \end{pmatrix}, \quad (19)$$

where $i, j = +, -$, and $f_q(\mathbf{k})$ is the arbitrary nonequilibrium distribution function of quark.

2.2. Gluons in Nonequilibrium. We work in the frozen ghost formalism [3, 4], where the nonequilibrium gluon propagator at initial time $t = t_{\rm in}$ is given by (the suppression of color indices is understood)

$$G^{\mu\nu}(k)_{ij} = -i\left[g^{\mu\nu} + (\alpha-1)\frac{k^\mu k^\nu}{k^2}\right] G^{\rm vac}_{ij}(k) - iT^{\mu\nu}G^{\rm med}_{ij}(k), \quad (20)$$

where $i, j = +, -$. The transverse tensor is given by

$$T^{\mu\nu}(k) = g^{\mu\nu} - \frac{(k\cdot u)(u^\mu k^\nu + u^\nu k^\mu) - k^\mu k^\nu - k^2 u^\mu u^\nu}{(k\cdot u)^2 - k^2}, \quad (21)$$

with the flow velocity of the medium u^μ. $G^{\rm vac}_{ij}(k)$ are the usual vacuum propagators of the gluon

$$G^{\rm vac}_{ij}(k) = \begin{pmatrix} \frac{1}{k^2+i\epsilon} & -2\pi\delta(k^2)\theta(-k_0) \\ -2\pi\delta(k^2)\theta(k_0) & -\frac{1}{k^2-i\epsilon} \end{pmatrix}, \quad (22)$$

and the medium part of the propagators is given by

$$G^{\rm med}_{ij}(k) = 2\pi\delta(k^2)f_g(\mathbf{k})\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}. \quad (23)$$

2.3. Ratio of Characteristic Relaxation Time of the Nonequilibrium State to the QCD Evolution Time. The typical relaxation time in the nonequilibrium QCD plasma can be written as [16–18]

$$\tau_c = \frac{1}{n\hat{\sigma}_{\rm tr}}, \quad (24)$$

where

$$n = \int d^3k f(\mathbf{k}) \quad (25)$$

is the parton number density in terms of the nonequilibrium parton distribution function $f(\mathbf{k})$, and $\hat{\sigma}_{\rm tr}$ is the typical transport cross section of the partonic collisions in the nonequilibrium QCD plasma which depends on the nonequilibrium parton distribution function $f(\mathbf{k})$.

Consider, for example, the $gg \to gg$ scattering. The leading-order partonic differential cross section in vacuum is given by

$$\frac{d\hat{\sigma}}{d\hat{t}} = \frac{9\pi\alpha_s^2}{2\hat{s}^2}\left[3 - \frac{\hat{u}\hat{t}}{\hat{s}^2} - \frac{\hat{s}\hat{t}}{\hat{u}^2} - \frac{\hat{s}\hat{u}}{\hat{t}^2}\right], \tag{26}$$

which in the infrared limit $\hat{t} \to 0$ diverges

$$\frac{d\hat{\sigma}}{d\hat{t}} = \frac{9\pi\alpha_s^2}{2\hat{t}^2}. \tag{27}$$

However, in the medium, the medium modified resumed gluon propagator removes this infrared divergence and the typical finite-differential cross section becomes [19]

$$\frac{d\hat{\sigma}}{d\hat{t}} = \frac{9\pi\alpha_s^2}{8}\left[\frac{1}{(\Pi_L - \hat{t})(\bar{\Pi}_L - \hat{t})} + \frac{1}{(\Pi_T - \hat{t})(\bar{\Pi}_L - \hat{t})} + \right.$$
$$\left. + \frac{1}{(\Pi_L - \hat{t})(\bar{\Pi}_T - \hat{t})} + \frac{1}{(\Pi_T - \hat{t})(\bar{\Pi}_T - \hat{t})}\right], \tag{28}$$

where Π_L and Π_T are the longitudinal and transverse components of the gluon self-energy which depend on the nonequilibrium distribution function $f(\mathbf{k})$. One can see that even at the one-loop level of the self-energy the magnetic screening mass is nonzero [3, 4, 20, 21] as long as the nonequilibrium distribution function $f(\mathbf{k})$ is nonisotropic, i.e., it depends on the direction of $\mathbf{k}$ of the parton, which is the case at the early stage of the heavy-ion collisions at RHIC and LHC. The expressions of the medium modified resumed gluon propagator at the one-loop level of self-energy in nonequilibrium in covariant gauge is recently derived in [3, 4].

The transport cross section [16, 19]

$$\hat{\sigma}_{\rm tr} = \int_{-\hat{s}/2}^{0} d\hat{t}\, \frac{d\hat{\sigma}}{d\hat{t}} \sin^2\theta_{\rm cm} = \int_{-\hat{s}/2}^{0} d\hat{t}\, \frac{d\hat{\sigma}}{d\hat{t}}\frac{4\hat{u}\hat{t}}{\hat{s}^2} \tag{29}$$

for this process can be obtained by using Eq. (28). Since gluons are dominate part of the total parton production at the early stage of the heavy-ion collisions at RHIC and LHC, one can get an estimate of the relaxation time of the nonequilibrium state from Eqs. (24), (25), (28), and (29). For example, the typical value of the maximum relaxation time in nonequilibrium state found in [16] is ~ 1.5 fm at RHIC and LHC heavy-ion colliders.

The typical QCD evolution time associated with the DGLAP evolution equation of the fragmentation function is given by

$$t = \ln Q, \tag{30}$$

where

$$Q = \mu \tag{31}$$

is the energy scale determined by the hard process probing fragmentation function [22].

3. QUARK AND GLUON SPLITTING FUNCTIONS IN NONEQUILIBRIUM QCD

In this section we evaluate quark and gluon splitting functions P_{ij} in nonequilibrium QCD. Similar to the vacuum case in [11] (see Eq. (11)) we define the state $|i\rangle$ and $|f\rangle$ in nonequilibrium QCD as follows:

$$\begin{aligned} |i\rangle &= |q(p_A)\rangle = a_q^\dagger(p_A)|\text{in}\rangle, \\ |f\rangle &= |q(p_C), \quad g(p_B)\rangle = a_q^\dagger(p_C)a_g^\dagger(p_B)|\text{in}\rangle, \end{aligned} \tag{32}$$

where $|\text{in}\rangle$ is the initial state of the nonequilibrium QCD medium. It has to be remembered that for evaluating the Feynman diagrams and S matrix we work in the interaction picture, where the fields $\psi(x)$ and $A_\mu(x)$ obey the free field equations in terms of creation and annihilation operators as given by Eq. (10). From Eqs. (32) and (9) we find

$$\langle f|S^{(1)}|i\rangle = ig\int d^4x \langle \text{in}|a_q(p_C)\, a_g(p_B) N[\bar{\psi}(x) \not{A}^a(x)\, T^a \psi(x)]\, a_q^\dagger(p_A)|\text{in}\rangle, \tag{33}$$

which gives by using Eq. (10)

$$\begin{aligned} \langle f|S^{(1)}|i\rangle = ig\int d^4x \sum_{\text{spin}} \sum_{p,p',p''} \sqrt{\frac{m}{VE_p}}\sqrt{\frac{m}{VE_{p''}}}\sqrt{\frac{1}{2VE_{p'}}}\times \\ \times \langle \text{in}|a_q(p_C)a_g(p_B)[a_q^\dagger(p)\bar{u}(p)\, e^{ip\cdot x}][\gamma_\mu a_g^\dagger(p')\epsilon^{a\mu}(p')\, e^{ip'\cdot x}T^a]\times \\ \times [a_q(p'')u(p'')\, e^{-ip''\cdot x}]a_q^\dagger(p_A)|\text{in}\rangle. \end{aligned} \tag{34}$$

Performing x integration we find

$$\langle f|S^{(1)}|i\rangle = ig\sum_{\text{spin}}\sum_{p,p',p''} \frac{V}{(E+E'-E'')}\sqrt{\frac{m}{VE_p}}\sqrt{\frac{m}{VE_{p''}}}\sqrt{\frac{1}{2VE_{p'}}}\times$$
$$\times \langle \text{in}|a_q(p_C)a_g(p_B)[a_q^\dagger(p)\bar{u}(p)][\gamma_\mu a_g^\dagger(p')\epsilon^{a\mu}(p')T^a]\times$$
$$\times [a_q(p'')u(p'')]a_q^\dagger(p_A)|\text{in}\rangle. \quad (35)$$

In the interaction picture, the commutation relations are the same as that for free field operators

$$[a(p), a^\dagger(p')] = \delta_{pp'},$$
$$[a(p), a(p')] = [a^\dagger(p), a^\dagger(p')] = 0, \quad (36)$$

which gives

$$\langle f|S^{(1)}|i\rangle = ig\sum_{\text{spin}}\sum_{p,p',p''} \frac{V}{(E+E'-E'')}\sqrt{\frac{m}{VE_p}}\sqrt{\frac{m}{VE_{p''}}}\sqrt{\frac{1}{2VE_{p'}}}\times$$
$$\times \langle \text{in}|[a_q^\dagger(p)a_q(p_C)+\delta_{pp_C}]\bar{u}(p)\gamma_\mu[a_g^\dagger(p')a_g(p_B)+\delta_{p'p_B}]\times$$
$$\times \epsilon^{a\mu}(p')T^a[a_q^\dagger(p_A)a_q(p'')+\delta_{p''p_A}]u(p'')|\text{in}\rangle. \quad (37)$$

For our purpose of evaluating Feynman diagrams in momentum space, we use Eq. (18)

$$\langle \text{in}|a_p^\dagger a_{p'}|\text{in}\rangle = f(\mathbf{p})\delta^{(3)}_{\mathbf{pp'}}, \quad (38)$$

where we have assumed the space-translational invariance at initial time $t = t_{\text{in}} = 0$. Using Eq. (38) and summing over p, p', and p'', we find from Eq. (37)

$$|\langle f|S^{(1)}|i\rangle|^2 = \left[\frac{V}{(E_C+E_B-E_A)}\right]^2 \frac{m}{VE_C}\frac{m}{VE_A}\frac{1}{2VE_B}\times$$
$$\times [1+f_q(\mathbf{p}_C)]^2\,[1+f_g(\mathbf{p}_B)]^2\,[1+f_q(\mathbf{p}_A)]^2 \sum_{\text{spin}}|M|^2, \quad (39)$$

where

$$M = ig\bar{u}(p_C)\,\gamma_\mu u(p_A)\,\epsilon^{a\mu}(p_B)T^a. \quad (40)$$

From now onwards, we can follow exactly the same steps as in the vacuum case (see Sec. 1, the derivations after Eq. (12)) to find the probability

$$W = C_2(R)\frac{\alpha_s}{2\pi}\,[1+f_q(\mathbf{p}_C)]^2\times$$
$$\times [1+f_g(\mathbf{p}_B)]^2\,[1+f_q(\mathbf{p}_A)]^2\,\frac{1+(1-z)^2}{z}\,dz\,d(\ln p_T^2). \quad (41)$$

3.1. Quark-to-Gluon Splitting Function in Nonequilibrium QCD. From Eq. (41) we find the quark-to-gluon splitting function in nonequilibrium QCD at leading order in α_s:

$$P_{gq}(z) = C_2(R)\ [1 + f_q(p)]^2\ [1 + f_g(p_T, zp)]^2 \times \\ \times [1 + f_q(-p_T, (1-z)p)]^2\ \frac{1 + (1-z)^2}{z} \quad (42)$$

which reproduces Eq. (7).

In the above equation, p is the momentum of initial quark (which is assumed to be along longitudinal direction); p_T is the transverse momentum of the gluon, and z is the longitudinal momentum fraction of the initial quark carried by the gluon.

3.2. Quark-to-Quark Splitting Function in Nonequilibrium QCD. The quark-to-quark splitting function (in the process $q(p_A) \to q(p_B) + g(p_C)$) can be obtained from the quark-to-gluon splitting function (in the process $q(p_A) \to g(p_B) + q(p_C)$) with the replacement $z \to (1 - z)$

$$P_{qq}(z) = P_{Gq}(1 - z), \quad z < 1. \quad (43)$$

Hence we find from Eq. (42) the quark-to-quark splitting function in nonequilibrium QCD

$$P_{qq}(z) = C_2(R)\ [1 + f_q(p)]^2 \times \\ \times [1 + f_g(-p_T, (1-z)p)]^2 [1 + f_q(p_T, zp)]^2\ \frac{1 + z^2}{1 - z} \quad (44)$$

which reproduces Eq. (7).

3.3. Gluon-to-Gluon Splitting Function in Nonequilibrium QCD. Similarly, using three-gluon vertex and carrying out the similar algebra, we find gluon-to-gluon splitting function in nonequilibrium QCD

$$P_{gg}(z) = 2C_A\ [1 + f_g(p)]^2\ [1 + f_g(p_T, zp)]^2 \times \\ \times [1 + f_g(-p_T, (1-z)p)]^2\ \left[\frac{1-z}{z} + \frac{z}{1-z} + z(1-z)\right] \quad (45)$$

which reproduces Eq. (7).

The splitting functions in nonequilibrium QCD as given by Eqs. (42), (44), and (45) can be used to study DGLAP evolution equation of fragmentation function in nonequilibrium QCD [8, 14] to study high-p_T hadron production from quark–gluon plasma at RHIC and LHC.

CONCLUSIONS

RHIC and LHC heavy-ion colliders are the best facilities to study quark–gluon plasma in the laboratory. Since two nuclei travel almost at a speed of light, the QCD matter formed at RHIC and LHC may be in nonequilibrium. Since the fragmentation function is a non-perturbative quantity, we do not have theoretical tools in QCD to calculate it yet. The normal procedure at high-energy pp, ep, and e^+e^- colliders is to extract it at some initial momentum scale μ_0 and then evolve it to another scale μ by using the DGLAP evolution equation which involves splitting function P_{ji} of a parton j into a parton i. The quark and gluon splitting functions in vacuum are evaluated by Altarelli and Parisi in [11] at the leading order in coupling constant α_s. In order to apply this procedure at high-energy heavy-ion colliders at RHIC and LHC one needs to prove factorization of fragmentation function in nonequilibrium QCD. Recently, we have proved factorization theorem in nonequilibrium QED in [13] and in nonequilibrium QCD in [14].

In this paper we have evaluated the quark and gluon splitting functions in nonequilibrium QCD at the leading order in coupling constant α_s by using closed-time path integral formalism. For quarks and gluons with arbitrary nonequilibrium distribution functions $f_q(\mathbf{p})$ and $f_g(\mathbf{p})$, we have derived expressions for quark and gluon splitting functions in nonequilibrium QCD. We have found that the quark and gluon splitting functions depend on nonequilibrium distribution functions $f_q(\mathbf{p})$ and $f_g(\mathbf{p})$. We have made a comparison of these splitting functions with those obtained by Altarelli and Parisi in vacuum.

The splitting functions in nonequilibrium QCD can be used to study DGLAP evolution equation of fragmentation function in nonequilibrium QCD [8, 14] to study high-p_T hadron production from quark–gluon plasma [16, 23] at RHIC and LHC.

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