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INTEGRABILITY OF THE BFKL DYNAMICS AND POMERON TRAJECTORIES IN A THERMOSTAT

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We consider scattering amplitudes in QCD at high energies $\sqrt{s}$ and fixed momentum transfers $q = \sqrt{-t}$ with a nonzero temperature T in the t channel. In the s channel the temperature leads to a compactification of the impact parameter plane. We find that the thermal BFKL Hamiltonian in the leading logarithmic approximation proceeds to have the property of the holomorphic separability. Moreover, there exists an integral of motion allowing one to construct the Pomeron wave function for arbitrary T in the coordinate and momentum representations. The holomorphic Hamiltonian for n-reggeized gluons at $T \neq 0$ in the multicolour limit $N_c \to \infty$ turns out to be equal to the local Hamiltonian for an integrable Heisenberg spin model. Further, the two-gluon Baxter function coincides with the corresponding wave function in the momentum representation. We calculate the spectrum of the Pomeron Regge trajectories at a finite temperature with taking into account the QCD running coupling. The important effect of the t-channel temperature is the appearance of a confining potential between gluons.

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1. INTRODUCTION

The scattering amplitudes $A(s,t)$ at high energies $2E = \sqrt{s}$ and fixed momentum transfers $q = \sqrt{-t}$ were calculated in the leading logarithmic approximation (LLA) $\alpha_s \ln s \sim 1$, $\alpha_s = g^2/4\pi \to 0$ (g is the QCD coupling constant) by summing the largest contributions $\sim (\alpha_s \ln s)^n$ to all orders of perturbation theory by Balitsky, Fadin, Kuraev, and Lipatov (BFKL) [1]. The BFKL Pomeron turns out to be a composite state of two reggeized gluons in LLA (it takes place also in the next-to-leading approximation [2]). The Pomeron wave function $\Psi(\boldsymbol{\rho}_1, \boldsymbol{\rho}_2)$ in the two-dimensional impact-parameter space $\boldsymbol{\rho}$ satisfies the stationary Schrödinger equation

$$E\,\Psi(\boldsymbol{\rho}_1, \boldsymbol{\rho}_2) = H_{12}\Psi(\boldsymbol{\rho}_1, \boldsymbol{\rho}_2), \quad H_{12} = H_{\rm kin} + H_{\rm pot}. \tag{1.1}$$

The intercept Δ of the BFKL Pomeron, related to the high energy asymptotics $\sigma_t \sim s^{\Delta}$ of the total cross section, is proportional to the ground state energy E of the Hamiltonian H_{12}

$$\Delta = -\frac{\alpha_s N_c}{2\pi} E. \tag{1.2}$$

The kinetic energy

$$H_{\text{kin}} = \ln |p_1|^2 + \ln |p_2|^2$$

is the sum of two gluon Regge trajectories, and the potential energy

$$H_{\text{pot}} = \frac{1}{p_1 p_2^*} \ln |\rho_{12}|^2 p_1 p_2^* + \frac{1}{p_2 p_1^*} \ln |\rho_{12}|^2 p_2 p_1^* - 4\psi(1), \quad \rho_{12} = \rho_1 - \rho_2$$

is related by similarity transformations to the two-dimensional Green function $\ln |\rho_{12}|^2$. We introduced here the complex gluon coordinates $\rho_r = x_r + iy_r$ and the corresponding momenta $p_r = i\partial/\partial(\rho_r)$. Further, $\psi(z) = \Gamma'(z)/\Gamma(z)$. The BFKL equation is used for the description of the deep-inelastic lepton–hadron scattering together with the DGLAP equation [3] (see, for example, [4]). It is invariant under the Möbius transformations [5]

$$\rho_r \to \frac{a\rho_r + b\rho_r}{c\rho_r + d\rho_r} \tag{1.3}$$

with arbitrary complex parameters a, b, c, d, and H_{12} has the property of holomorphic separability (see [4] and [6])

$$H_{12} = h_{12} + h_{12}^*, \quad h_{12} = \sum_{r=1}^{2} \left[\ln p_r + \frac{1}{p_r} \ln(\rho_{12})\, p_r - \psi(1) \right]. \tag{1.4}$$

Note, that the above properties are valid only for the Möbius class of functions $\Psi(\boldsymbol{\rho}_1, \boldsymbol{\rho}_2)$ vanishing at $|\rho_{12}| \to 0$ [7]. For more singular functions, the transformations (1.3) schould be generalized [8].

The wave functions Ψ belong to the principal series of unitary representations of the Möbius group with conformal weights $m = 1/2 + i\nu + n/2$ and $\widetilde{m} = 1/2 + i\nu - n/2$ expressed in terms of the anomalous dimension $\gamma = 1 + 2i\nu$ and the integer conformal spin n being quantum numbers of the local gauge-invariant operators [5]. The conformal weights are related to the eigenvalues $m(m-1)$ and $\widetilde{m}(\widetilde{m}-1)$ of the Casimir operators M^2 and M^{2*}, where

$$M^2 = \left(\sum_{r=1}^{2} M_3^{(r)} \right)^2 + \\ + \frac{1}{2} \left(\sum_{r=1}^{2} M_+^{(r)} \sum_{s=1}^{2} M_-^{(s)} + \sum_{r=1}^{2} M_-^{(r)} \sum_{s=1}^{2} M_+^{(s)} \right) = \rho_{12}^2 p_1 p_2. \tag{1.5}$$

Here $\mathbf{M}^{(r)}$ are the Möbius group generators

$$M_3^{(r)} = \rho_r\,\partial_r, \quad M_+^{(r)} = \partial_r, \quad M_-^{(r)} = -\rho_r^2\,\partial_r, \tag{1.6}$$

and $\partial_r = \partial/\partial\rho_r$.

The eigenfunctions of H_{12} can be considered as three-point functions of a two-dimensional conformal field theory and have the property of holomorphic factorization [5],

$$f_{m,\widetilde{m}}(\boldsymbol{\rho}_1, \boldsymbol{\rho}_2; \boldsymbol{\rho}_0) = \langle 0|\ \varphi(\boldsymbol{\rho}_1)\,\varphi(\boldsymbol{\rho}_1)\,O_{m,\widetilde{m}}(\boldsymbol{\rho}_0)\ |0\rangle = \\ = \left(\frac{\rho_{12}}{\rho_{10}\,\rho_{20}}\right)^m \left(\frac{\rho_{12}^*}{\rho_{10}^*\,\rho_{20}^*}\right)^{\widetilde{m}}. \tag{1.7}$$

One can calculate the energy inserting this Ansatz in the BFKL equation [1]

$$E_{m,\widetilde{m}} = \varepsilon_m + \varepsilon_{\widetilde{m}}, \quad \varepsilon_m = \psi(m) + \psi(1-m) - 2\psi(1). \tag{1.8}$$

The minimum of $E_{m,\widetilde{m}}$ is obtained at $m = \widetilde{m} = 1/2$ leading to a large intercept $\Delta = 4(\alpha_s/\pi)N_c \ln 2$ for the BFKL Pomeron. In the next-to-leading approximation the effective intercept is comparatively small ($\Delta \sim 0.2$ for the QCD case [9]).

The increase of total cross sections at large energies is related to the growth of the gluon number at small values of the Bjorken variable x. This phenomenon is especially important for the Brookhaven experiment devoted to the search for the quark–gluon plasma in the heavy ion collisions [10,11]. In these collisions the temperature of the hadron matter grows and the linear potential responsible for the quark confinement disappears, which leads to the suppression of the production of the ψ meson and other hadrons [11]. A similar deconfinement effect should also exist for Pomeron and other reggeons constructed from gluons. To study these phenomena in our previous short paper [12], we constructed the BFKL equation for the Pomeron at an arbitrary temperature T in the center-of-mass system of the t channel (where $\sqrt{t} = 2\epsilon$) and investigated the integrability properties of the reggeon dynamics in a thermostat for composite states of n-reggeized gluons in the multi-colour QCD [12]. Here we consider these problems in more detail with taking into account also effects of the running coupling constant in QCD.

2. GLUON REGGE TRAJECTORY AND BFKL KERNEL AT $T \neq 0$

The gluon Regge trajectory $j(t)$ in the leading logarithmic approximation (LLA) is given below:

$$\omega(t) = j(t) - 1 = \\ = -\frac{g^2}{16\pi^3}\,N_c \int d^2k \frac{\mathbf{q}^2 + \lambda^2}{(\mathbf{k}^2 + \lambda^2)((\mathbf{q}-\mathbf{k})^2 + \lambda^2)}, \quad t = -\mathbf{q}^2, \tag{2.1}$$

where g is the coupling constant for the Yang–Mills theory with the gauge group $SU(N_c)$, and λ is a fictitious gluon mass introduced to regularize infrared divergencies. Note, however, that in the gauge theory with the scalar field described by the $N_c \times N_c$-Hermitian matrix, the Higgs mechanism leads to an appearance of the nonzero mass λ for the vector bosons W, and the above expression for $\omega(t)$ describes the boson Regge trajectory [1]. In this case the trajectory goes through the physical point $j = 1$ at the particle mass $\sqrt{t} = \lambda$. For $N_c = 2$, the considered model coincides with the electroweak theory at a vanishing Weinberg angle $\theta_W = 0$ [13].

Let us consider the Regge kinematics in which the total particle energy $\sqrt{s}$ is asymptotically large in comparison with the temperature T. In this case, one can neglect the temperature effects in the propagators of the initial and intermediate particles in the direct channels s and u. But the momentum transfer $|q|$ is considered to be of the order of T (note, that q is the vector orthogonal to the momenta of initial particles $q \approx q_\perp$). As is well known [10], the particle wave functions $\psi(x)$ at temperature T are periodic in the Euclidean time $\tau = it$ with period $1/T$.

We introduce the temperature T in the center-of-mass frame of the t channel. Technically it corresponds to a compactification of the Euclidean time τ. Thus, the Euclidean energies of the intermediate gluons in the t channel become quantized as

$$k_4^{(l)} = 2\pi l T. \tag{2.2}$$

As an example, one can consider in QCD the gauge-invariant correlation function for four photon fields having the negative virtualities $p_i^2 < 0$ $(i = 1, 2, 3, 4)$. Other invariants s, t, u of the amplitude are chosen in such a way, that the corresponding kinematics can be realized by the Euclidean momenta p_i. In this case in the Feynman matrix elements one can integrate over the Euclidean momenta of virtual quarks and gluons quantized as in Eq. (2.2) in the center-of-mass system of the t channel. This amplitude can be continued analytically to the Regge kinematics corresponding to large s and fixed negative t and photon virtualities. In fact, the continuation is performed in the t-channel scattering angle θ which is not modified by the temperature effects.

In the s channel the invariant t remains to be negative and the momentum transfer q is an Euclidean vector. Moreover, the analytically continued 4-momenta of the t-channel gluons can be considered as Euclidean vectors. After the analytic continuation of the scattering amplitude to the s channel, the t-channel Euclidean time τ in fact coincides with one of two components y of the impact-parameter coordinate $\boldsymbol{\rho}$, which leads to the compactification of the impact parameter space

$$\boldsymbol{\rho} = (x, y), \quad 0 < y < T^{-1} \tag{2.3}$$

and to the simple quantization condition for the corresponding components of the transverse gluon momenta

$$\mathbf{k} = (k_x, k_y), \quad \mathbf{q} = (q_x,\, q_y), \quad k_y = 2\pi T l, \quad q_y = 2\pi T r, \tag{2.4}$$

where $r, l = 0, \pm 1, \pm 2, \ldots$

Therefore we have for the Regge trajectory of the t-channel gluon in a thermostat the following expression (cf. (2.1)):

$$\omega(-\mathbf{q}^2) = -\frac{g^2}{8\pi^2}\, N_c\, \Omega(\mathbf{q}), \tag{2.5}$$

where

$$\Omega(\mathbf{q}) = 2\pi T \sum_{r'=-\infty}^{\infty} \times$$
$$\times \int_{-\infty}^{\infty} \frac{d\,k_x}{2\pi} \frac{q_x^2 + 4\pi^2\, T^2 r^2 + \lambda^2}{(k_x^2 + 4\pi^2 T^2 (r')^2 + \lambda^2)((q-k)_x^2 + 4\pi^2 T^2 (r-r')^2 + \lambda^2)}. \tag{2.6}$$

By calculating the integral over k with residues and assuming that $\lambda \ll q \sim 2\pi T$, one can derive the final result

$$\Omega(\mathbf{q}) = \frac{2\pi\, T}{\lambda} - 2\,\psi(1) + \frac{1}{2}\left(\psi\left(1 + \frac{r}{2} + i\frac{q_x}{4\pi T}\right) + \psi\left(1 + \frac{r}{2} - i\frac{q_x}{4\pi T}\right) + \right.$$
$$\left. + \psi\left(1 - \frac{r}{2} + i\frac{q_x}{4\pi T}\right) + \psi\left(1 - \frac{r}{2} - i\frac{q_x}{4\pi T}\right)\right), \tag{2.7}$$

where $\psi(x) = \Gamma'(x)/\Gamma(x)$. The singular contribution $2\pi T/\lambda$ was obtained from the zero mode $r' = 0$, and to sum over nonvanishing r', we used the well-known relation

$$\psi(1-x) - \psi(1) = \sum_{r'=1}^{\infty} \left(\frac{1}{r'} - \frac{1}{r'-x}\right).$$

Note, that in another limit $\lambda \sim q \ll 2\pi T$, the main contribution can be obtained from the zero modes $r = r' = 0$:

$$\Omega_{2+1}(\mathbf{q}) = 2\pi \frac{T}{\lambda} \frac{\mathbf{q}^2 + \lambda^2}{\mathbf{q}^2 + 4\lambda^2}. \tag{2.8}$$

If one will redefine the factor g^2 by including in it the temperature T, this expression for $\Omega_z(\mathbf{q})$ describes the vector boson Regge trajectory for $(2+1)$-dimensional non-Abelian gauge theory [14]. Thus, for large T, there is a compactification of one space dimension.

In the general case $\lambda \sim q \sim 2\pi T$, the result is more complicated

$$\Omega_g(\mathbf{q}) = 2\pi T \times$$
$$\times \sum_{r'=-\infty}^{\infty} \frac{(q_x^2 + 4\pi^2 T^2 r^2 + \lambda^2)(q_x^2 + 4\pi^2 T^2 r^2 - 8\pi^2 T^2 r r')}{\sqrt{\lambda^2 + 4\pi^2 T^2 r'^2}\,\left((q_x^2 + 4\pi^2 T^2 r^2 - 8\pi^2 T^2 r r')^2 + 4 q_x^2 (\lambda^2 + 4\pi^2 T^2 r'^2)\right)}. \tag{2.9}$$

It is important, that the gluon Regge trajectory (2.6) has the property of holomorphic separability

$$\Omega(\mathbf{q}) = \beta\left(\frac{q_x + i2\pi Tr}{2}\right) + \beta\left(\frac{q_x - i2\pi Tr}{2}\right), \tag{2.10}$$

where

$$\beta(z) = \frac{\pi T}{\lambda} + \frac{1}{2}\left(\psi\left(1 + i\frac{z}{2\pi T}\right) + \psi\left(1 - i\frac{z}{2\pi T}\right) - 2\psi(1)\right). \tag{2.11}$$

Now we consider the contribution to the kernel of the BFKL equation from the emission of a gluon in the intermediate state of the s channel. For the gluon with a definite helicity and momentum k, the reggeon–reggeon–gluon vertex is equal to the following expression (or to complex conjugated one) [6]:

$$\Gamma \sim \frac{q_1\, q_1'^*}{k}, \quad \mathbf{k} = \mathbf{q}_1 - \mathbf{q}_1',$$

where q_1, q_1' are complex components of transverse momenta $\mathbf{q}_1$, $\mathbf{q}_1'$ of neighbouring reggeized gluons, and k is a complex component of the transverse momentum of the produced gluon.

The corresponding term in the BFKL equation is proportional to the product of two vertices Γ and can be presented as an operator with the integral kernel (see [6])

$$K = -\frac{q_1\, q_2^*}{|q_1|^2\, |q_2|^2}\,\frac{1}{\mathbf{k}^2 + \lambda^2}\, q_1'^*\, q_2' + \mathrm{h.c.},$$

where the factors $|q_1|^{-2}$ and $|q_2|^{-2}$ correspond to the propagators of two reggeized gluons being constituents of the BFKL Pomeron. The effective gluon propagator $D(k) = 1/(|k|^2 + \lambda^2)$ (regularized by the gluon mass λ) results from the product of the nonlocal effective vertices $\Gamma \sim 1/k$. In a more traditional form of taking into account also the masses λ in other vector boson propagators one can write the integral kernel for various t-channel states i of the colour group as follows (see [1]):

$$K_i(\mathbf{q}_1, \mathbf{q}_2; \mathbf{q}_1', \mathbf{q}_2') = -\frac{c_i}{\mathbf{k}^2 + \lambda^2}\left(\frac{\mathbf{q}_1'^2 + \lambda^2}{\mathbf{q}_1^2 + \lambda^2} + \frac{\boldsymbol{q}_2'^2 + \lambda^2}{\mathbf{q}_2^2 + \lambda^2}\right) + \\ + \frac{c_i \mathbf{q}^2 + \left(\frac{3}{2}c_i - \frac{1}{2}\right)\lambda^2}{(\mathbf{q}_1^2 + \lambda^2)(\mathbf{q}_2^2 + \lambda^2)}. \tag{2.12}$$

In the case of the $SU(2)$ gauge model with the Higgs mechanism, the coefficients c_i for $i = 1$ (singlet), $i = 3$ (triplet) and $i = 5$ (quintet) states are, respectively,

$$c_1 = 2, \quad c_3 = 1, \quad c_5 = -1. \tag{2.13}$$

In the general case the nonstationary BFKL equation in LLA can be written in the form:

$$\frac{\partial f^{(i)}(\mathbf{q}_1,\mathbf{q}_2)}{\partial t} = \left(\Omega_g(\mathbf{q}_1)+\Omega_g(\mathbf{q}_2)\right) f^{(i)}(\mathbf{q}_1,\mathbf{q}_2) + \\ + \int \frac{d^2 q_1'}{2\pi}\, K_i(\mathbf{q}_1,\mathbf{q}_2;\mathbf{q}_1',\mathbf{q}_2')\, f^{(i)}(\mathbf{q}_1',\mathbf{q}_2'), \quad (2.14)$$

where

$$t = -\frac{g^2 N_c}{8\pi^2}\ln s, \quad \mathbf{q}_2 = \mathbf{q}-\mathbf{q}_1, \quad \mathbf{q}_2' = \mathbf{q}-\mathbf{q}_1',$$

and it is implied that

$$\int \frac{d^2 q_1'}{2\pi} \equiv 2\pi T \sum_{r'=-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{d(q_1')_x}{2\pi}, \quad (q_1')_y = 2\pi T r', \qquad (2.15)$$

where $r' = 0, \pm 1, \pm 2, \ldots$ In particular, for the W-boson channel with the isospin 1 we obtain the bootstrap solution corresponding to the boson reggeization

$$f^{(3)}(y;\mathbf{q}_1,\mathbf{q}_2) = f^{(3)}(0;\mathbf{q}_1,\mathbf{q}_2)\,\exp\left(y\omega_g(\mathbf{q})\right), \quad \omega_g(\mathbf{q}) = -\frac{g^2}{8\pi^2} N_c \Omega_g(\mathbf{q}),$$

providing that the initial conditions are

$$f^{(3)}(0;\mathbf{q}_1,\mathbf{q}_2) = \frac{\phi(\mathbf{q})}{(\mathbf{q}_1^2+\lambda^2)\,(\mathbf{q}_2^2+\lambda^2)},$$

and the function $\phi(\mathbf{q})$ does not depend on the gluon virtualities $\mathbf{q}_1^2$ and $\mathbf{q}_2^2$.

For asymptotically high temperatures $T \to \infty$, the BFKL equation becomes one-dimensional

$$\frac{1}{2\pi T}\frac{\partial f^{(i)}(q_1,q_2)}{\partial t} = \frac{1}{\lambda}\left(\frac{q_1^2+\lambda^2}{q_1^2+4\lambda^2} + \frac{q_2^2+\lambda^2}{q_2^2+4\lambda^2}\right) f^{(i)}(q_1,q_2) + \\ + \frac{c_i q^2 + \left(\frac{3}{2}c_i - \frac{1}{2}\right)\lambda^2}{(q_1^2+\lambda^2)(q_2^2+\lambda^2)} \int_{-\infty}^{\infty} \frac{dq_1'}{2\pi} f^{(r)}(q_1',q_2') - \\ - \int_{-\infty}^{\infty} \frac{dq_1'}{2\pi} \frac{c_i}{(q_1-q_1')^2+\lambda^2}\left(\frac{(q_1')^2+\lambda^2}{(q_1)_1^2+\lambda^2} + \frac{(q_2')^2+\lambda^2}{(q_2)^2+\lambda^2}\right) f^{(r)}(q_1',q_2') \quad (2.16)$$

and can be solved exactly with the use of the Green function constructed in terms of the eigenfunctions and eigenvalues of the integral kernel [14, 15].

3. BFKL EQUATION IN QCD AT FINITE T

In the BFKL Hamiltonian for QCD, the gluon Regge trajectories lead to the kinetic energy and the contribution from the real gluon emission is responsible for the potential energy $V(\boldsymbol{\rho}_{12})$ with $\boldsymbol{\rho}_{12} = \boldsymbol{\rho}_1 - \boldsymbol{\rho}_2$, where $\boldsymbol{\rho}_1$ and $\boldsymbol{\rho}_2$ are impact parameters of two gluons in a Pomeron. Going to the coordinate representation at a finite temperature T, one can perform the Fourier transformation of the effective propagator for the produced gluon in the limit $\lambda \to 0$:

$$V(\boldsymbol{\rho}) = -2\pi T \sum_{r=-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{dk}{\pi} \frac{\mathrm{e}^{i(kx+2\pi Tr\,\tau)}}{k^2+4\pi^2T^2r^2+\lambda^2} = -\frac{2\pi T}{\lambda}\,\mathrm{e}^{-\lambda|x|} +$$

$$+ \ln|1-\mathrm{e}^{-2\pi T\,(|x|+i\tau)}|^2 = -\frac{2\pi T}{\lambda} + \ln|2\,\sinh(\pi T\,\rho)|^2 . \quad (3.1)$$

Note, that the gluon Green function can be written in another form

$$V(\boldsymbol{\rho}) = -\frac{2\pi T}{\lambda} + \ln\,|2\pi T\rho|^2 + \sum_{r=1}^{\infty} \ln\left|1+\frac{T^2\rho^2}{r^2}\right|^2 \quad \rho = x + i\tau.$$

At large, T we obtain the confining potential $V(\boldsymbol{\rho}) \sim 2\pi T\,|x|$. The confinement is related to the conservation of the flow of the chromoelectric field through the surface surrounding the colour charge in the compactified impact parameter plane.

It is important, that the potential energy has the property of holomorphic separability

$$V(\boldsymbol{\rho}) = v(\rho) + v(\rho^*), \quad v(\rho) = -\frac{\pi T}{\lambda} + \ln\left(2\,\sinh(\pi T\rho)\right).$$

By adding the virtual corrections from the Regge trajectories of two reggeized gluons and the contributions from the real gluon emission, we cancel infrared divergencies at $\lambda \to 0$ and write the homogeneous BFKL equation as the Schrödinger equation (cf. (1.4))

$$E\Psi(\boldsymbol{\rho}_1,\boldsymbol{\rho}_2) = H\Psi(\boldsymbol{\rho}_1,\boldsymbol{\rho}_2), \quad \omega = -\frac{g^2\,N_c}{8\pi^2}\,E \quad (3.2)$$

with the Hamiltonian

$$H = \sum_{s=1,2} \Omega(\mathbf{p}_s) + \frac{1}{p_1^* p_2}\,V(\boldsymbol{\rho}_{12})\,p_1^* p_2 + \frac{1}{p_1 p_2^*}\,V(\boldsymbol{\rho}_{12})\,p_1 p_2^*.$$

One can easily verify that the BFKL Hamiltonian H in a thermostat respects the property of holomorphic separability [6]

$$H = h + h^* \quad (3.3)$$

with the holomorphic Hamiltonian

$$h = \frac{1}{2} \sum_{s=1,2} \left(\psi \left(1 + i \frac{p_s}{2\pi T} \right) + \psi \left(1 - i \frac{p_s}{2\pi T} \right) - 2\psi(1) \right) + \\ + \frac{1}{p_1} \ln \left(2 \sinh \left(\pi T \rho_{12} \right) \right) p_1 + \frac{1}{p_2} \ln \left(2 \sinh \left(\pi T \rho_{12} \right) \right) p_2, \quad (3.4)$$

where we introduced the holomorphic coordinates $\rho_s = x_s + i\tau_s$ with $\rho_{12} = \rho_1 - \rho_2$ and their conjugated momenta according to the definition

$$p_s = \frac{1}{2} \left(i \frac{\partial}{\partial x_s} + \frac{\partial}{\partial \tau_s} \right), \quad [p_s, \rho_r] = i\delta_{r,s}.$$

Here it is implied that imaginary parts of momenta p_s are quantized as $\mathrm{Im}\, p = \pi T r$. Note, that the above definition of p corresponds to the relation

$$p = \frac{p_x - ip_y}{2}. \quad (3.5)$$

For a large temperature the holomorphic Hamiltonian is simplified as follows:

$$\lim_{T\to\infty} h = \sum_{s=1,2} \left(-\frac{\psi''(1)}{2} \left(\frac{p_s}{2\pi T} \right)^2 + \pi T \frac{1}{p_r} |\rho_{12}|\, p_r \right).$$

Using the holomorphic separability (3.3) one can derive the holomorphic factorization of the Pomeron wave function [6]

$$\Psi(\boldsymbol{\rho}_1, \boldsymbol{\rho}_2) = \sum_{k,l} a_{kl}\, \Psi_k(\rho_1, \rho_2)\, \Psi_l(\rho_1^*, \rho_2^*), \quad (3.6)$$

where k, l enumerate different solutions of the corresponding holomorphic and antiholomorphic Schrödinger equations for the same energies ϵ and $\widetilde{\epsilon}$, respectively. The coefficients a_{kl} are chosen from the single-valuedness constraint for the wave function $\Psi(\boldsymbol{\rho}_1, \boldsymbol{\rho}_2)$.

At small T we return to the known result for the holomorphic BFKL Hamiltonian [6]

$$\lim_{T\to 0} h = \ln(p_1\, p_2) + \frac{1}{p_1} \ln(\rho_{12})\, p_1 + \frac{1}{p_2} \ln(\rho_{12})\, p_2 - 2\psi(1). \quad (3.7)$$

Its eigenfunctions in the coordinate space are infinite-dimensional representations of the Möbius group with the conformal weight $m = 1/2 + i\nu + n/2$ [5]

$$\Psi(\rho_{10}, \rho_{20}) = \left(\frac{\rho_{12}}{\rho_{10}\rho_{20}} \right)^m. \quad (3.8)$$

The corresponding eigenvalues are

$$\epsilon_m = \psi(m) + \psi(1-m) - 2\psi(1). \tag{3.9}$$

The eigenfunctions can be written also in the momentum representation

$$\Psi(p_1, p_2) \sim \int dk\, k^{-1-m} \left((p_1 - k)(p_2 + k)\right)^{-1+m}, \tag{3.10}$$

where the closed contour of integration over k is situated between the singularities of the integrand. There are two independent solutions of such a type and both of them can be expressed in terms of the hypergeometric functions.

Some properties of the BFKL dynamics in the thermostat are the same as at a zero temperature. As an example, we point out the bootstrap relation. Namely, the BFKL equation with gluon quantum numbers in the crossed channel has the pole solution corresponding to a reggeized gluon. Note, that to obtain the bootstrap relation, one should restore in the gluon propagators the λ dependence.

On the other hand, some important features of the BFKL equation at $T = 0$ are modified. For example, the Möbius invariance in the usual form is lost due to the fact, that the temperature T has a nonzero dimension $\sim m$, but this symmetry is realized in another form, as it will be demonstrated below. Moreover, the model with n-reggeized gluons in the crossed channel at $N_c \to \infty$ turns out to be integrable at a finite temperature [12].

4. MEROMORPHIC PROPERTIES OF EIGENFUNCTIONS

It is convenient to measure coordinates in units of $1/(2\pi T)$ and momenta in units of $2\pi T$ by rescaling

$$2\pi T \rho_s \to \rho_s, \ \frac{p_s}{2\pi T} \to p_s. \tag{4.1}$$

The potential $V(\boldsymbol{\rho})$ in these variables has the form

$$V(\boldsymbol{\rho}) = -\frac{1}{\lambda} + \ln\left|2\sinh\frac{\rho}{2}\right|^2, \quad \rho = x + i\tau, \tag{4.2}$$

where λ is measured in units of $2\pi T$. Note, that the kinetic energy for each of gluons $i = 1, 2$ can be written as an integral operator acting on the wave function as follows:

$$\left(\Omega(\mathbf{p}) - \frac{1}{\lambda}\right)\Psi(\boldsymbol{\rho}) = -\int_{-\infty}^{\infty} dx' \times$$
$$\times \int_{-\pi}^{\pi} \frac{d\tau'}{4\pi} \left|\coth\frac{x - x' + i(\tau - \tau')}{2}\right|^2 \left(\Psi(\boldsymbol{\rho}') - \Psi(\boldsymbol{\rho})\right).$$

In the new variables, the holomorphic Hamiltonian simplifies:

$$h = \frac{1}{2} \sum_{s=1,2} \left(\psi(1 + i\,p_s) + \psi(1 - i\,p_s) - 2\psi(1)\right) + \\ + \sum_{s=1,2} \frac{1}{p_s} \ln\left(2\,\sinh\left(\frac{\rho_{12}}{2}\right)\right) p_s. \quad (4.3)$$

The Pomeron wave function $\Psi(\boldsymbol{\rho}_1,\, \boldsymbol{\rho}_2)$ should be a periodic function of the second components y_s of the vectors $\boldsymbol{\rho}_s$ with the period equal to 2π. Respectively, second components $i\partial/(\partial y_s)$ of the momenta $\mathbf{p}_s$ are integer numbers $r = 0, \pm 1, \pm 2, \ldots$ But due to the holomorphic separability (3.3), the BFKL equation has an additional symmetry. Indeed, H is a periodic function of ρ_{12} and ρ_{12}^* separately. If we analytically continue $\Psi(\rho_1, \rho_2)$ and $\Psi(\rho_1^*, \rho_2^*)$ as functions of the real parts of ρ_{12} and ρ_{12}^* in the complex plane, they will be quasi-periodic functions along the imaginary axis and in expression (3.6) the corresponding phases compensate.

It is convenient to introduce in the t channel the center-of-mass coordinate R, the total momentum Q and the corresponding relative variables z and p:

$$R = \frac{\rho_1 + \rho_2}{2} = X + iY, \quad z = \frac{\rho_{12}}{2} = x + iy, \\ Q = p_1 + p_2 = \frac{K + iN}{2}, \quad p = p_1 - p_2 = \frac{k + in}{2}, \quad (4.4)$$

with the reverse relations

$$p_1 = \frac{Q + p}{2}, \quad p_2 = \frac{Q - p}{2}. \quad (4.5)$$

They satisfy the commutation relations

$$[Q, R] = i, \quad [p, z] = i.$$

Note, that in terms of the usual component (p_x, p_y) of gluon momenta we have (cf. (3.5))

$$Q = \frac{1}{2}\left((p_1 + p_2)_x - i(p_1 + p_2)_y\right), \\ p = \frac{1}{2}\left((p_1 - p_2)_x - i(p_1 - p_2)_y\right), \quad t = -4|Q|^2. \quad (4.6)$$

Because H does not depend on R, the total momenta Q, Q^* are conserved and below they will be considered as C numbers. To simplify notation, sometimes the dependence from them will not be shown explicitly.

The eigenvalues of N and n are expressed through the integer numbers r_1, r_2 for the second components of the momenta $\mathbf{p}_1$, $\mathbf{p}_2$ as follows:

$$N = r_1 + r_2, \quad n = r_1 - r_2.$$

We can write the total BFKL Hamiltonian in the new variables using the relation

$$\psi(1-x) = \psi(x) + \pi \cot(\pi x)$$

as follows:

$$H = \psi(1+i\,p_1) + \psi(i\,p_2) + \psi(-i\,p_1^*) + \psi(1-i\,p_2^*) - \\ - 4\psi(1) + 2\ln|2\,\sinh\,z|^2 + \frac{i}{2}\left(\frac{1}{p_2} - \frac{1}{p_1}\right)(\coth\,z - 1) + \\ + \frac{i}{2}\left(\frac{1}{p_2^*} - \frac{1}{p_1^*}\right)(\coth\,z^* - 1). \quad (4.7)$$

To find an asymptotic behaviour of the solution of the Schrödinger equation at large $x = \mathrm{Re}\,z \to +\infty$, the integration contour in the Fourier transformation over k,

$$\Psi(x,n) = \int_{-\infty}^{\infty} \frac{dk}{\pi}\,\mathrm{e}^{-ikx}\,f(k,n),$$

should be shifted into the low half of the complex plane k up to the singularities of the function $f(k,n)$. This function has the simple poles at

$$k = k_{\pm}^{(l)}(s,n) = \pm K + i(-4s - n \pm N), \quad s = 0,1,2,\ldots$$

Indeed, at large positive x we have

$$2\,\ln|2\sinh\,z|^2 \to 4x = -4i\,\frac{\partial}{\partial k},$$

and therefore one can obtain an approximate solution of the Schrödinger equation near these singularities in the form

$$f(k) \sim \frac{\Gamma(-ip_1^*)\,\Gamma(ip_2)}{\Gamma(1+ip_1)\Gamma(1-ip_2^*)} = \frac{\Gamma(ip_2^*)\,\Gamma(-ip_1)}{\Gamma(1-ip_2)\Gamma(1+ip_1^*)}. \quad (4.8)$$

One can write the Hamiltonian in another form

$$H = \psi(1+i\,p_2) + \psi(i\,p_1) + \psi(-i\,p_2^*) + \psi(1-i\,p_1^*) - \\ - 4\psi(1) + 2\ln\,|2\,\sinh\,z|^2 + \frac{i}{2}\left(\frac{1}{p_2} - \frac{1}{p_1}\right)(\coth\,z + 1) + \\ + \frac{i}{2}\left(\frac{1}{p_2^*} - \frac{1}{p_1^*}\right)(\coth\,z^* + 1). \quad (4.9)$$

Therefore, by pushing $x \to -\infty$ we can find another approximate solution for $f(k)$:

$$f(k) \sim \frac{\Gamma(-ip_2^*)\,\Gamma(ip_1)}{\Gamma(1+ip_2)\Gamma(1-ip_1^*)} = \frac{\Gamma(ip_1^*)\,\Gamma(-ip_2)}{\Gamma(1-ip_1)\Gamma(1+ip_2^*)} \tag{4.10}$$

near the poles in the upper semiplane situated at the points

$$k = k_{\pm}^{(u)}(s,n) = \pm K + i(4s - n \pm N), \quad s = 0, 1, 2, \ldots,$$

complex conjugated with respect to $k_{\pm}^{(l)}$.

Thus, the solution of the Schrödinger equation in the momentum representation is a meromorphic function of the variable k

$$\Psi(K,N,k,n) = \sum_{s=0}^{\infty} \sum_{\sigma=\pm} \left(\frac{c_{\sigma}^{(u)}(s,n)}{k - k_{\sigma}^{(u)}(s,n)} + \frac{c_{\sigma}^{(d)}(s,n)}{k - k_{\sigma}^{(d)}(s,n)} \right). \tag{4.11}$$

There is a simple interpretation of the pole singularities of $\Psi(K,N,k,n)$ in terms of analytic properties of the vertex $\Gamma(\mathbf{p}_1^2, \mathbf{p}_2^2, \mathbf{Q}^2)$ considered as a function of the gluon virtualities $M_1^2 = -\mathbf{p}_1^2, M_2^2 = -p_2^2$. This function is obtained from the above solution Ψ by removing the gluon propagators

$$\Gamma(\mathbf{p}_1^2, \mathbf{p}_2^2, \mathbf{Q}^2) = |p|_1^2\,|p|_2^2\,\Psi(K,N,k,n).$$

If we consider inhomogeneous BFKL equation, the Born contribution $|p|_1^{-2}\,|p|_2^{-2}$ for Ψ has the poles at $|p|_1^2 = 0$ corresponding to the singularities at

$$k = -K \pm i(n+N)$$

and at $|p|_2^2 = 0$ corresponding to the singularities at

$$k = K \pm i(n-N).$$

The first iteration of this inhomogeneous term obtained by applying to it the BFKL kernel gives (after the cancelation of the second order poles) the single poles in all points $k = k_{\sigma}^{u,l}(s)$ ($\sigma = \pm 1$) (corresponding to the cuts in $|p|_1^2$ at $\sigma = -1$ and in $|p|_2^2$ at $\sigma = 1$ for $T = 0$). Note, that for the gluon quantum numbers in the t channel these poles are cancelled leading to the result depending only on Q in accordance with the bootstrap property. The subsequent iteration of the inhomogeneous BFKL equation does not give new singularitiues.

The recurrent relations for the residues of the poles are discussed in the next section.

5. RELATIONS FOR RESIDUES

Assuming, that the Pomeron wave functions have the property of the holomorphic factorization (3.6), we can restrict ourselves to the problem of constructing the holomorphic and antiholomorphic wave functions. For example, let us write the Hamiltonian (4.9) in the separable form:

$$H = h' + \widetilde{h'}^*, \tag{5.1}$$

where

$$h' = \psi(1 + i\,p_2) + \psi(i\,p_1) - 2\psi(1) + 2\ln(2\,\sinh\,z) + \\ + \frac{i}{2}\left(\frac{1}{p_2} - \frac{1}{p_1}\right)(\coth\,z + 1), \tag{5.2}$$

$$\widetilde{h'}^* = \psi(-i\,p_2^*) + \psi(1 - i\,p_1^*) - 2\psi(1) + 2\ln(2\,\sinh\,z^*) + \\ + \frac{i}{2}\left(\frac{1}{p_2^*} - \frac{1}{p_1^*}\right)(\coth\,z^* + 1). \tag{5.3}$$

Note, that the Hamiltonian h' can be written as an integral operator in the coordinate space

$$h' = \rho_{12} - \sum_{k=1}^{\infty}\left(\mathrm{e}^{k\rho_{12}} - 1\right)\left(-\frac{1}{\partial_1} + \frac{1}{\partial_2} - \frac{2}{k}\right)\mathrm{e}^{-k\rho_{12}}.$$

Therefore the eigenvalue equation for it has the form

$$\epsilon\Psi(z) = 2z\Psi(z) + \\ + 4\int^z dz'\left(\frac{\mathrm{e}^{2(z-z')}}{1 - \mathrm{e}^{2(z-z')}}(\Psi(z') - \Psi(z)) - \frac{\mathrm{e}^{-2z'}}{1 - \mathrm{e}^{-2z'}}(\Psi(z') - \Psi(0))\right).$$

Here the boundary conditions are not specified, because there is an uncertainty in a separation of the holomorphic and antiholomorphic contributions in the total Hamiltonian H.

The dispersion representation for the holomorphic solution of the equation

$$h'\varphi^{(1)}(p_1, p_2) = \epsilon\varphi^{(1)}(p_1, p_2),$$

with the poles at $p_1 = il$ ($l = 0, 1, 2, \ldots$) (corresponding to the asymptotics of $\Psi(x, n)$ at $x \to \infty$), can be written as follows:

$$\varphi^{(1)}(p_1, p_2) = \sum_{l=0}^{\infty}\frac{r(Q, l)}{p_1 - il}, \quad Q = p_1 + p_2. \tag{5.4}$$

It is convenient to introduce the singular and regular terms in their expansion near the poles

$$\lim_{p_1 \to il} \varphi^{(1)}(p_1, p_2) = \frac{r(Q,l)}{p_1 - il} + a(Q,l), \quad a(Q,l) = {\sum_{l'=0}^{\infty}}' \frac{r(Q,l')}{i(l-l')}, \tag{5.5}$$

where $\sum'$ means that the singular term at $l' = l$ is omitted.

On the other hand, putting our Ansatz in the eigenvalue equation for the Hamiltonian h' (5.2) from the condition of cancelling the poles of the first and second orders, we obtain the following relations between $r(Q,l)$ and $a(Q,l)$:

$$\begin{aligned} -i\,a(Q,l) = (\psi(1+l) + \psi(1+iQ+l) - 2\psi(1) - \epsilon)\, r(Q,l) + \\ + \sum_{s=1}^{l} \left(-\frac{2}{s} - i\frac{1}{Q-il} + \frac{1}{l} \right) r(Q, l-s). \end{aligned} \tag{5.6}$$

For $l = 0$ an additional term appears in the right-hand side of the above relations:

$$\begin{aligned} -ia(Q,0) = (\psi(1+iQ) - \psi(1) - \epsilon)\, r(Q,0) + \\ + \sum_{s=1}^{\infty} (\psi(1+s) - \psi(1)) r(Q,s), \end{aligned} \tag{5.7}$$

where we took into account that

$$\sum_{s=0}^{\infty} r(Q,s) = 0.$$

Note, that the above equations can be written only in terms of the residues $r(Q,l)$:

$$\begin{aligned} 0 = (\psi(1+l) + \psi(1+iQ+l) - 2\psi(1) - \epsilon)\, r(Q,l) - \\ - {\sum_{s=0}^{\infty}}' \frac{r(Q,s)}{|l-s|} + \left(\frac{1}{l} + \frac{1}{l+iQ} \right) \sum_{s=0}^{l-1} r(Q,s) \end{aligned} \tag{5.8}$$

for $l = 1, 2, \ldots$ and

$$\begin{aligned} 0 = (\psi(1+iQ) - \psi(1) - \epsilon)\, r(Q,0) - \sum_{s=1}^{\infty} \frac{r(Q,s)}{s} + \\ + \sum_{s=1}^{\infty} (\psi(1+s) - \psi(1)) r(Q,s) \end{aligned} \tag{5.9}$$

for $l = 0$.

We can present the holomorphic Hamiltonian in other form:

$$h' = \psi(i\,p_2) + \psi(1 + i\,p_1) - 2\psi(1) + 2\ln(2\,\sinh\,z) + \\ + \frac{i}{2}\left(\frac{1}{p_2} - \frac{1}{p_1}\right)(\coth\,z - 1), \quad (5.10)$$

and therefore one can construct the new eigenfunction of h' with the poles at $p_2 = il$ considering the limit $x \to -\infty$:

$$\varphi^{(2)}(p_1, p_2) = \sum_{l=0}^{\infty} \frac{r(Q,l)}{p_2 - il}\,, \quad (5.11)$$

where the residues $r(Q,l)$ coincide with those for $\varphi^{(1)}(p_1, p_2)$.

There is another representation of the total Hamiltonian H as a sum of holomorphic and antiholomorphic functions

$$H = h'' + \widetilde{h''}^{*}, \quad (5.12)$$

where

$$h'' = \psi(1 - i\,p_2) + \psi(-i\,p_1) - 2\psi(1) + 2\ln(2\sinh\,z) + \\ + \frac{i}{2}\left(\frac{1}{p_2} - \frac{1}{p_1}\right)(\coth\,z - 1), \quad (5.13)$$

$$\widetilde{h''}^{*} = \psi(i\,p_2^*) + \psi(1 + i\,p_1^*) - 2\psi(1) + 2\ln(2\sinh\,z^*) + \\ + \frac{i}{2}\left(\frac{1}{p_2^*} - \frac{1}{p_1^*}\right)(\coth\,z^* - 1). \quad (5.14)$$

It means that we can find the eigenfunctions of the holomorphic Hamiltonian h'' with the poles in the points $p_1 = -il$ and $p_2 = -il$:

$$\varphi^{(1)}(-p_1, -p_2) = \sum_{l=0}^{\infty} \frac{r(-Q,l)}{-p_1 - il}, \quad \varphi^{(2)}(-p_1, -p_2) = \sum_{l=0}^{\infty} \frac{r(-Q,l)}{-p_2 - il}. \quad (5.15)$$

Note, that for $Q = 0$, the above equations for $r(Q,l)$ and $a(Q,l)$ are simplified as follows:

$$-i\,a(0,l) = (2\psi(1+l) - 2\psi(1) - \epsilon)\,r(0,l) + \sum_{s=1}^{l}\left(-\frac{2}{s} + \frac{2}{l}\right) r(0, l-s) \quad (5.16)$$

for $l = 1, 2, \dots$ and

$$-ia(0,0) = -\epsilon\, r(0,0) + \sum_{s=1}^{\infty} (\psi(1+s) - \psi(1))\, r(0,s) \tag{5.17}$$

for $l = 0$.

The problem how to construct the Pomeron wave function having the property of the holomorphic factorization will be considered below. In the next section we shall derive the holomorphic integral of motion. It will give us a possibility of constructing more simple recurrent relations for residues of the functions $\phi^{(1),(2)}$.

6. SMALL-T EXPANSION AND THE INTEGRAL OF MOTION

As was shown above, the BFKL Hamiltonian at finite T can be written as a sum of the holomorphic and antiholomorpic contributions

$$H = h + h^*, \quad \rho = \rho_{12},$$

where according to Eq. (4.3)

$$h = \frac{1}{2} \sum_{s=1,2} (\psi(1+i\,p_s) + \psi(1-i\,p_s) - 2\psi(1)) + \\ + \sum_{s=1,2} \frac{1}{p_s} \ln\left(2\sinh\left(\frac{\rho}{2}\right)\right) p_s. \tag{6.1}$$

One should solve correspondingly the holomorphic and antiholomorphic Schrödinger equations

$$\epsilon\Psi(\rho) = h\,\Psi(\rho), \quad \widetilde{\epsilon}\widetilde{\Psi}(\rho^*) = h^*\,\widetilde{\Psi}(\rho^*), \quad E = \epsilon + \widetilde{\epsilon}.$$

Let us consider the small-T behaviour of h. For this purpose, it is helpful to use the well-known expansions of the functions appearing in expression (6.1):

$$\psi(1+z) = \ln z + \frac{1}{2z} - \sum_{k=1}^{\infty} \frac{B_{2k}}{2k\, z^{2k}},$$

$$2\ln\left(2\sinh\frac{\rho}{2}\right) = 2\ln\rho + \sum_{k=1}^{\infty} \frac{B_{2k}\,\rho^{2k}}{k\,(2k)!},$$

$$\frac{1}{e^{\rho}-1} = \frac{1}{\rho} - \frac{1}{2} + \sum_{k=1}^{\infty} \frac{B_{2k}\,\rho^{2k-1}}{k\,(2k)!},$$

where B_{2k} are the Bernoulli numbers. Thus, for small ρ and large p_1, p_2 (corresponding to large T in the initial variables), we obtain

$$h = h_0 + \Delta h, \quad h_0 = \ln(p_1 p_2) + 2\ln\,\rho + i\left(\frac{1}{p_2} - \frac{1}{p_1}\right)\frac{1}{\rho} - 2\psi(1), \tag{6.2}$$

where

$$\Delta h = \sum_{k=1}^{\infty} \frac{B_{2k}}{2k}\left((-1)^{k+1}\left(\frac{1}{p_1^{2k}} + \frac{1}{p_2^{2k}}\right) + \frac{2}{(2k)!}\rho^{2k} + \right.$$
$$\left. + i\left(\frac{1}{p_2} - \frac{1}{p_1}\right)\frac{\rho^{2k-1}}{(2k-1)!}\right). \tag{6.3}$$

Several terms of this expansion are given below:

$$\Delta h = \frac{1}{12}\left(\frac{1}{p_1^2} + \frac{1}{p_2^2}\right) + \frac{i}{12}\left(\frac{1}{p_2} - \frac{1}{p_1}\right)\rho + \frac{\rho^2}{12} +$$
$$+ \frac{1}{120}\left(\frac{1}{p_1^4} + \frac{1}{p_2^4}\right) - \frac{i}{720}\left(\frac{1}{p_2} - \frac{1}{p_1}\right)\rho^3 - \frac{\rho^4}{1440} +$$
$$+ \frac{1}{252}\left(\frac{1}{p_1^6} + \frac{1}{p_2^6}\right) + \frac{i}{30240}\left(\frac{1}{p_2} - \frac{1}{p_1}\right)\rho^5 + \frac{\rho^6}{90720} +$$
$$+ \frac{1}{240}\left(\frac{1}{p_1^8} + \frac{1}{p_2^8}\right) - \frac{i}{1209600}\left(\frac{1}{p_2} - \frac{1}{p_1}\right)\rho^7 - \frac{\rho^8}{4838400} + \ldots \tag{6.4}$$

Note, that the odd terms in the expansion in T are absent. Using the above expressions we can attempt to find small-T corrections to the Casimir operator of the Möbius group $A_0 = M^2$ being the integral of motion for $T = 0$:

$$A = A_0 + \Delta A, \quad A_0 = \rho^2\, p_1 p_2, \quad [h_0,\, A_0] = 0.$$

In the next-to-leading order for h we have

$$\left[\frac{1}{12}\left(\frac{1}{p_1^2} + \frac{1}{p_2^2}\right) + \frac{i}{12}\left(\frac{1}{p_2} - \frac{1}{p_1}\right)\rho + \frac{\rho^2}{12},\, \rho^2\, p_1 p_2\right] =$$
$$= \frac{1}{6}\left(\frac{p_2}{p_1^3} + \frac{p_1}{p_2^3}\right) + \frac{1}{3}\left(\frac{1}{p_1^2} + \frac{1}{p_2^2}\right) - \frac{i}{3}\left(\frac{p_2}{p_1^2} - \frac{p_1}{p_2^2}\right)\rho +$$
$$+ \frac{i}{2}\left(\frac{1}{p_2} - \frac{1}{p_1}\right)\rho + \left(1 - \frac{1}{4}\left(\frac{p_2}{p_1} + \frac{p_1}{p_2}\right)\right)\rho^2 - \frac{i}{6}(p_2 - p_1)\rho^3.$$

On the other hand, with the use of the relations

$$\left[\ln\left(p_1p_2\right) + 2\ln\rho + i\left(\frac{1}{p_2} - \frac{1}{p_1}\right)\frac{1}{\rho} - 2\psi(1),\, p_1p_2\rho^4\right] =$$
$$= 24i\left(\frac{p_2}{p_1^4} - \frac{p_1}{p_2^4}\right)\frac{1}{\rho} + 30\left(\frac{p_2}{p_1^3} + \frac{p_1}{p_2^3}\right) - 20i\left(\frac{p_2}{p_1^2} - \frac{p_1}{p_2^2}\right)\rho -$$
$$- 2i\left(\frac{1}{p_1} - \frac{1}{p_2}\right)\rho - \left(9\left(\frac{p_2}{p_1} + \frac{p_1}{p_2}\right) + 4\right)\rho^2 + 2i(p_2 - p_1)\rho^3,$$

$$\left[\ln\left(p_1p_2\right) + 2\ln\rho + i\left(\frac{1}{p_2} - \frac{1}{p_1}\right)\frac{1}{\rho} - 2\psi(1),\, (p_1 - p_2)\rho^3\right] =$$
$$= -6\left(\frac{p_2}{p_1^4} - \frac{p_1}{p_2^4}\right)\frac{1}{\rho} + 6\left(\frac{1}{p_1^3} - \frac{1}{p_2^3}\right)\frac{1}{\rho} + 8i\left(\frac{p_2}{p_1^3} + \frac{p_1}{p_2^3}\right)\left(\frac{1}{p_1^2} + \frac{1}{p_2^2}\right) +$$
$$+ 6\left(\frac{p_2}{p_1^2} - \frac{p_1}{p_2^2}\right)\rho - 4\left(\frac{1}{p_1} - \frac{1}{p_2}\right)\rho - 3i\left(\frac{p_2}{p_1} + \frac{p_1}{p_2}\right)\rho^2 + 2i\rho^2,$$

$$\left[\ln\, p_1p_2) + 2\ln\rho + i\left(\frac{1}{p_2} - \frac{1}{p_1}\right)\frac{1}{\rho} - 2\psi(1),\, \rho^2\right] =$$
$$= 2i\left(\frac{1}{p_1} - \frac{1}{p_2}\right)\rho - 3\left(\frac{1}{p_1^2} + \frac{1}{p_2^2}\right) - 2i\left(\frac{1}{p_1^3} - \frac{1}{p_2^3}\right)\frac{1}{\rho},$$

we obtain that

$$\left[h_0,\, -\frac{1}{12}\left(p_1p_2\rho^4 + 4i(p_1 - p_2)\rho^3 + 12\rho^2\right)\right] =$$
$$+ \frac{1}{6}\left(\frac{p_2}{p_1^3} + \frac{p_1}{p_2^3}\right) + \frac{1}{3}\left(\frac{1}{p_1^2} + \frac{1}{p_2^2}\right) - \frac{i}{3}\left(\frac{p_2}{p_1^2} - \frac{p_1}{p_2^2}\right)\rho - \frac{i}{2}\left(\frac{1}{p_1} - \frac{1}{p_2}\right)\rho -$$
$$- \frac{1}{4}\left(\frac{p_2}{p_1} + \frac{p_1}{p_2}\right)\rho^2 - \frac{i}{6}\left(p_2 - p_1\right)\rho^3 + \rho^2.$$

It means, that indeed up to the next-to-next-to-leading order of the perturbation theory in T^2 we have the integral of motion

$$[h,\, A] = 0, \quad A = \rho^2 p_1p_2 + \frac{1}{12}\left(p_1p_2\rho^4 + 4i(p_1 - p_2)\rho^3 + 12\rho^2\right) + \ldots =$$
$$= g(\rho)\, p_1\, p_2,$$

where

$$g(\rho) = \rho^2 + \frac{\rho^4}{12} + \ldots$$

One can assume* that $g(\rho)$ is the periodic function equal to

$$g(\rho) = 4\left(\sinh\,\frac{\rho}{2}\right)^2.$$

Thus, the integral of motion is

$$A = 4\left(\sinh\,\frac{\rho}{2}\right)^2\, p_1\, p_2. \tag{6.5}$$

In the next section we shall prove that A indeed commutes with the holomorphic Hamiltonian.

7. CONFORMAL RELATION BETWEEN MODELS WITH $T \neq 0$ AND $T = 0$

The Pomeron wave function can be constructed explicitly in the coordinate space. For this purpose, we use the conformal transformation of gluon coordinates and momenta

$$\rho_r = \ln\,\rho'_r, \quad p_r = \rho'_r p'_r \tag{7.1}$$

of holomorphic Hamiltonian (6.1). In new variables h is reduced to the usual BFKL Hamiltonian (1.4) for the vanishing temperature:

$$h = \ln\,(p'_1\, p'_2) + \frac{1}{p'_1}\,\ln\,(\rho'_{12})\,\, p'_1 + \frac{1}{p'_2}\,\ln\,(\rho'_{12})\,\, p'_2 - 2\psi(1), \tag{7.2}$$

To verify this fact it is enough to use the following operator identity (see [4]):

$$\frac{1}{2}\left[\psi\left(1 + z\frac{\partial}{\partial z}\right) + \psi\left(-z\frac{\partial}{\partial z}\right)\right] = \ln\,z + \ln\,\frac{\partial}{\partial z}, \tag{7.3}$$

satisfying the hermiticity symmetry, as well as the known properties of the ψ function. Note, however, that this identity does not allow one to fix in h the contributions proportional to the periodic functions of the type $\coth\,(\pi p_r)$, which is related to the ambiguity in the presention of H as a sum of holomorphic and antiholomorphic operators (cf., for example, (5.1) and (5.12)).

Analogously in the new variables the integral of motion, Eq. (6.5), is reduced to the Casimir operator of the Möbius group

$$A = -(\rho'_{12})^2\frac{\partial}{\partial\rho'_1}\,\frac{\partial}{\partial\rho'_2}. \tag{7.4}$$

*We thank V. N. Velizhanin, who verified this assumption in higher orders of the expansion in T.

Its eigenfunctions are well known (see (3.8)). Thus, the Pomeron wave function at a nonzero temperature having the property of single-valuedness and periodicity takes the form

$$\Psi_{(m,\widetilde{m})}(\boldsymbol{\rho}_1,\, \boldsymbol{\rho}_2,\, \boldsymbol{\rho}_0) = \left(\frac{\sinh \dfrac{\rho_{12}}{2}}{2\, \sinh \dfrac{\rho_{10}}{2}\, \sinh \dfrac{\rho_{20}}{2}} \right)^m \left(\frac{\sinh \dfrac{\rho_{12}^*}{2}}{2\, \sinh \dfrac{\rho_{10}^*}{2}\, \sinh \dfrac{\rho_{20}^*}{2}} \right)^{\widetilde{m}} . \tag{7.5}$$

The orthogonality and completeness relations for these functions can be easily obtained from the analogous results for $T = 0$ (see [5]) using the above conformal transformation.

In summary, the exponential transformation Eq. (7.1), which in dimensional variables takes the form

$$\rho' = \frac{1}{2\pi T}\, \mathrm{e}^{2\pi T \rho}, \tag{7.6}$$

maps the reggeon dynamics from zero temperature to temperature T. This mapping explicitly exhibits the periodicity $\rho \to \rho + i/T$ for a thermal state. It must be noticed that such a class of mappings is known to describe thermal situations for quantum fields in accelerated frames and in black-hole backgrounds [33].

As is well known [17], the BFKL equation at $T = 0$ can be generalized to composite states of n-reggeized gluons. In the multicolour limit $N_c \to \infty$, the BKP equations are significantly simplified thanks to their conformal invariance [5], holomorphic separability [6], and an existence of integrals of motion [16]. The generating function for the holomorphic integrals of motion coincides with the transfer matrix for an integrable lattice spin model [18, 19]. The transfer matrix is the trace of the monodromy matrix

$$t(u) = L_1(u)\, L_2(u) \ldots L_n(u), \tag{7.7}$$

satisfying the Yang–Baxter equations [19]. The integrability of the n-reggeon dynamics in multicolour QCD is valid also at nonzero temperature T, where, according to the above arguments we should take the L-operator in the form of the following matrix:

$$L_k = \begin{pmatrix} u + p_k & \mathrm{e}^{-\rho_k}\, p_k \\ -\mathrm{e}^{\rho_k}\, p_k & u - p_k \end{pmatrix}. \tag{7.8}$$

The integrals of motion for the n-reggeon composite state now have the form (cf. [19])

$$q_r = \sum_{\{i_1 i_2 \cdots i_r\}} (\mathrm{e}^{\rho_{i_1}} - \mathrm{e}^{\rho_{i_2}}) \cdots (\mathrm{e}^{\rho_{i_r}} - \mathrm{e}^{\rho_{i_1}})\, (\mathrm{e}^{-\rho_{i_1}} p_{i_1}) \cdots (\mathrm{e}^{-\rho_{i_r}} p_{i_r}), \quad [q_r, h = 0]. \tag{7.9}$$

The holomorphic Hamiltonian is the local Hamiltonian of the integrable Heisenberg model with the spins being the unitary transformed generators of the Möbius group (cf. [20, 21])

$$M_3^k = \partial_k \quad , \quad M_+^k = \mathrm{e}^{-\rho_k}\,\partial_k, \quad M_-^k = -\mathrm{e}^{\rho_k}\,\partial_k. \tag{7.10}$$

These operators are generators of the conformal transformations

$$\rho_k \to \ln \frac{a\,\mathrm{e}^{\rho_k} + b}{c\,\mathrm{e}^{\rho_k} + d}, \tag{7.11}$$

leaving unchanged the cylinder with the identification of the points $\mathrm{Im}\,\rho \equiv \mathrm{Im}\,\rho + 2\pi$ on the complex ρ plane.

Because the Hamiltonian at a nonzero temperature can be obtained by a unitary transformation from the zero temperature Hamiltonian, the spectrum of intercepts for multigluon states is the same as for zero temperature [22–26], and the wave functions of the composite states can be calculated by the substitution $\rho_k \to \mathrm{e}^{\rho_k}$. Nevertheless, formally the spin model with this Hamiltonian is not symmetric under all possible rotations and therefore it is not the XXX magnet considered in [20, 21].

8. HOLOMORPHIC SOLUTION IN THE MIXED REPRESENTATION

Instead of solving the Schrödinger equation, we can search for a solution of the eigenvalue equation for the integral of motion (6.5)

$$A\,\Psi(\rho_1,\,\rho_2) = m(m-1)\,\Psi(\rho_1,\,\rho_2). \tag{8.1}$$

Extracting from $\psi(\rho_1,\,\rho_2)$ the plane wave corresponding to the motion of the gluon center of mass with the total momentum $Q^* = (p_1^* + p_2^*)/2$ (see (4.6))

$$\psi(\rho_1,\,\rho_2) = \mathrm{e}^{iQ^*R}\,\Psi(\rho), \quad R = \frac{\rho_1 + \rho_2}{2}, \quad \mathbf{QR} = Q^*R + QR^*, \tag{8.2}$$

one can write the equation for the holomorphic wave function depending on the relative coordinate $\rho = \rho_1 - \rho_2$ as follows:

$$\left(\frac{Q^{*2}}{4} + \frac{\partial^2}{(\partial\rho)^2}\right)\Psi(\rho) = \frac{m(m-1)}{4(\sinh\,\rho/2)^2}\,\Psi(\rho). \tag{8.3}$$

Note, that the crossed channel invariant is (4.6)

$$t = -4|Q|^2. \tag{8.4}$$

By using the new variable

$$x = \frac{1 - \coth \rho/2}{2}, \tag{8.5}$$

we obtain

$$\left(\frac{Q^{*2}}{4} + x(1-x)\partial\, x(1-x)\partial \right) \Psi = -x(1-x)\, m(m-1)\, \Psi.$$

Extracting from $\psi(\rho)$ an additional factor

$$\Psi(\rho) = \left(x(x-1)\right)^{iQ^*/2}\; g(x) \tag{8.6}$$

one can verify that the function $g(x)$ satisfies the hypergeometric equation

$$\left(x(1-x)\frac{d^2}{dx^2} + (\gamma - (\alpha+\beta+1)x)\,\frac{d}{dx} - \alpha\beta \right) g(x) = 0$$

for

$$\alpha = m + iQ^*, \quad \beta = 1 - m + iQ^*, \quad \gamma = 1 + iQ^*.$$

Independent solutions of these equations are $F(\alpha,\beta;\gamma;x)$ and $x^{1-\gamma}F(\alpha-\gamma+1,\beta-\gamma+1;2-\gamma;x)$. But because for $\rho \to 0$ we have

$$x \approx -\rho^{-1} \to -\infty,$$

near $\rho = 0$ it is natural to use their linear combinations

$$\begin{aligned} g_1(x) &= (-x)^{-iQ^*-m}\, F\left(iQ^* + m, m; 2m; \frac{1}{x}\right), \\ g_2(x) &= (-x)^{-iQ^*+m-1}\, F\left(iQ^* - m + 1, 1 - m; 2(1-m); \frac{1}{x}\right) \end{aligned} \tag{8.7}$$

related to each other by the substitution $m \leftrightarrow 1 - m$.

Thus, the functions

$$\begin{aligned} \Psi_1(\rho) &= \left(1 - \frac{1}{x}\right)^{iQ^*/2} (-x)^{-m}\, F\left(iQ^* + m, m; 2m; \frac{1}{x}\right), \\ \Psi_2(\rho) &= \left(1 - \frac{1}{x}\right)^{iQ^*/2} (-x)^{m-1}\, F\left(iQ^* - m + 1, 1 - m; 2(1-m); \frac{1}{x}\right) \end{aligned} \tag{8.8}$$

are eigenfunctions of the integral of motion and of the holomorphic Hamiltonian.

We can easily verify, that indeed $\Psi_{1,2}$ are the eigenfunctions of h in the particular case $Q^* = 0$, where we can expand them at small ρ

$$\lim_{Q\to 0} \Psi_1(\rho) = \rho^m \left(1 - \frac{1}{24}\,\frac{m(m-1)}{2m+1}\,\rho^2 + \right.$$
$$\left. + \frac{1}{5760}\,\frac{m(m-1)(5m^2+7m+6)}{(2m+1)(2m+3)}\rho^4 + \ldots\right).$$

The Schrödinger equation here can be written as follows:

$$h\Psi_{(m)} = \epsilon_m\,\Psi_{(m)}.$$

It is easy to find the action of various terms in h on the powers ρ^{m+2k}, for example,

$$\frac{1}{p_1^s}\rho^{m+2k} = (-1)^s\frac{1}{p_2^s}\,\rho^{m+2k} = (-i)^s\,\prod_{r=1}^{s}\frac{1}{m+2k+r}\,\rho^{m+2k+s}.$$

We use also the following relation for the action of the free Green function at $T = 0$ on these terms:

$$\frac{1}{\epsilon_m - h_0}\,\rho^{m+2k} = -\frac{1}{2}\left(\sum_{r=0}^{2k-1}\frac{1}{m+r}\right)^{-1}\rho^{m+2k}.$$

Thus, introducing the small-ρ expansion of the solution at $Q = 0$

$$f_{(m)}(\rho) = \rho^{-m}\Psi_{(m)} = \sum_{t=0}^{\infty} c_{(m)}^t\,\rho^{2t}, \quad c_m^0 = 1,$$

one can obtain the following recurrence relation for the coefficients $c_{(m)}^t$ of the Taylor series:

$$-2\left(\sum_{r=0}^{2t-1}\frac{1}{m+r}\right)c_{(m)}^t =$$
$$= \sum_{k=1}^{t}\frac{B_{2k}}{k}\left(-\prod_{r=1}^{2k}\frac{1}{m+2t-2k+r} + \frac{1}{(2k)!} - \frac{1}{(m+2t)(2k-1)!}\right)c_{(m)}^{t-k}. \tag{8.9}$$

It gives us a possibility to obtain the small-temperature expansion of the holomorphic wave function

$$f_{(m)}(\rho) = 1 - \frac{1}{24}\,\frac{m(m-1)}{2m+1}\,\rho^2 + \frac{1}{5760}\,\frac{m(m-1)(5m^2+7m+6)}{(2m+1)(2m+3)}\rho^4 + \ldots$$

in an agreement with the small-ρ expansion of the explicit solution $\Psi_1(\rho)$ at $Q = 0$.

9. SINGLE-VALUEDNESS OF THE POMERON WAVE FUNCTION

The eigenfunctions $\Psi_{1,2}(\rho)$ of the integral of motion A for a general case $Q^* \neq 0$ are given below (see (8.8)):

$$\begin{aligned} &\Psi_1(\rho) = \Psi_{(m)}(\rho, Q^*) = (1-y)^{iQ^*/2}(-y)^m\, F(iQ^* + m, m; 2m; y),\\ &\Psi_2(\rho) = \Psi^{(1-m)}(\rho, Q^*), \end{aligned} \tag{9.1}$$

where

$$y = \frac{2}{1 - \coth\, \rho/2} = 1 - \mathrm{e}^{\rho}. \tag{9.2}$$

Note, that the strip $|\mathrm{Im}\,\rho| < \pi$ corresponding to the impact-parameter space at a finite temperature is mapped to all complex plane ρ in such a way, that its boundaries $\mathrm{Im}\,\rho = \pm\pi$ are mapped respectively into the lower and upper sides of the cut $y > 1$. The eigenfunctions

$$\Psi_{\pm}^{\mathrm{neg}}(\rho, Q^*) = y^m (1-y)^{\pm\, iQ^*/2} F(\pm iQ^* + m, m; \pm iQ^* + 1; 1-y) \tag{9.3}$$

for $\mathrm{Re}\,\rho < 0$ describe the quasi-periodic (Bloch) solutions of the Schrödinger equation having the property

$$\Psi_{\pm}^{\mathrm{neg}}(\rho + \pi i, Q^*) = \mathrm{e}^{\mp Q^* \rho}\, \Psi_{\pm}^{\mathrm{neg}}(\rho - \pi i, Q^*). \tag{9.4}$$

In the region $\mathrm{Re}\,\rho > 0$ such Bloch functions are

$$\begin{aligned} &\Psi_{\pm}^{\mathrm{pos}}(\rho, Q^*) = \\ &\qquad = y^m (1-y)^{\mp\, iQ^*/2 - m} F\left(\pm iQ^* + m, m; \pm iQ^* + 1; \frac{1}{1-y}\right). \end{aligned} \tag{9.5}$$

Using the property of the single-valuedness of the Pomeron wave function $\Psi_{(m,\widetilde{m})}(\boldsymbol{\rho}, \mathbf{Q})$ near $\boldsymbol{\rho} = 0$, we can present it as follows:

$$\begin{aligned} \Psi_{(m,\widetilde{m})}(\boldsymbol{\rho}, \mathbf{Q}) = \Psi_{(m)}(\rho, Q^*)\, \Psi^{(\widetilde{m})}(\rho^*, Q) + \\ + d_{m,\widetilde{m}}(Q, Q^*)\, \Psi^{(1-m)}(\rho, Q^*)\, \Psi^{(1-\widetilde{m})}(\rho^*, Q), \end{aligned} \tag{9.6}$$

where

$$m = \frac{1}{2} + i\nu + \frac{n}{2}, \quad \widetilde{m} = \frac{1}{2} + i\nu - \frac{n}{2}.$$

The coefficient $d_{m,\widetilde{m}}(Q, Q^*)$ can be found from the single-valuedness of Ψ near $\rho \to \pm\infty$ and its periodicity in respect to the shift $\mathrm{Im}\,\rho \to \mathrm{Im}\,\rho + 2\pi i$.

With the use of the known relation for the hypergeometric functions

$$\Psi_{(m)}(\rho, Q^*) = \\ = (-y)^m \left(\frac{\Gamma(2m)\Gamma(-iQ^*)}{\Gamma(m-iQ^*)\Gamma(m)} (1-y)^{iQ/2} F(iQ^*+m, m; iQ^*+1; 1-y) + \right. \\ \left. + \frac{\Gamma(2m)\Gamma(iQ^*)}{\Gamma(m+iQ^*)\Gamma(m)} (1-y)^{-iQ^*/2} F(-iQ^*+m, m; -iQ^*+1; 1-y) \right) \quad (9.7)$$

and similar relations for $\Psi^{(1-m)}(\rho, Q^*)$, we obtain that near the point $\boldsymbol{\rho} = -\infty$ (corresponding to $y = 1$) the interference terms disappear and $\Psi_{(m,\widetilde{m})}(\boldsymbol{\rho}, \mathbf{Q})$ has the single-valuedness property provided that

$$d_{m,\widetilde{m}}(Q, Q^*) = \\ = -\frac{\Gamma(2m)\,\Gamma(2\widetilde{m})\,\Gamma(1-m)\,\Gamma(1-\widetilde{m})}{\Gamma(2-2m)\,\Gamma(2-2\widetilde{m})\,\Gamma(m)\,\Gamma(\widetilde{m})} \frac{\Gamma(1-m-iQ^*)\,\Gamma(1-\widetilde{m}+iQ)}{\Gamma(m-iQ^*)\,\Gamma(\widetilde{m}+iQ)}. \quad (9.8)$$

In an explicit way the wave function near $y = 1$ (or $\rho = -\infty$) can be written as follows:

$$\Psi_{(m,\widetilde{m})}(\boldsymbol{\rho}, \mathbf{Q}) = 2\cos(\pi\widetilde{m}) \frac{\Gamma(2m)\,\Gamma(2\widetilde{m})}{\Gamma(m)\,\Gamma(\widetilde{m})} \frac{\Gamma(-iQ^*)\,\Gamma(1-\widetilde{m}+iQ)}{\Gamma(m-iQ^*)\,\Gamma(1+iQ)} y^m y^{*\,\widetilde{m}} \times \\ \times (1-y)^{iQ^*/2}(1-y^*)^{iQ/2} F(iQ^*+m, m; iQ^*+1; 1-y) \times \\ \times F(iQ+\widetilde{m}, \widetilde{m}; iQ+1; 1-y^*) + \\ + 2\cos(\pi m) \frac{\Gamma(2m)\,\Gamma(2\widetilde{m})}{\Gamma(m)\,\Gamma(\widetilde{m})} \frac{\Gamma(1-m-iQ^*)\,\Gamma(iQ)}{\Gamma(1-iQ^*)\,\Gamma(\widetilde{m}+iQ)} y^m y^{*\,\widetilde{m}} \times \\ \times (1-y)^{-iQ^*/2}(1-y^*)^{-iQ/2} F(-iQ^*+m, m; -iQ^*+1; 1-y) \times \\ \times F(-iQ+\widetilde{m}, \widetilde{m}; -iQ+1; 1-y^*). \quad (9.9)$$

With the use of the following relation for the hypergeometric functions:

$$\Psi_{(m)}(\rho, Q^*) = \left(-\frac{y}{1-y}\right)^m \left(\frac{\Gamma(2m)\Gamma(-iQ^*)}{\Gamma(m-iQ^*)\Gamma(m)} (1-y)^{-iQ^*/2} \times \right. \\ \times F\left(iQ^*+m, m; iQ^*+1; \frac{1}{1-y}\right) + \frac{\Gamma(2m)\Gamma(iQ^*)}{\Gamma(m+iQ^*)\Gamma(m)} (1-y)^{iQ^*/2} \times \\ \left. \times F\left(-iQ^*+m, m; -iQ^*+1; \frac{1}{1-y}\right) \right) \quad (9.10)$$

and the above expression for $d_{m,\widetilde{m}}(Q,Q^*)$, we obtain also the single-valued expression for the wave function near $y=\infty$ (or $\rho=\infty$)

$$\Psi_{(m,\widetilde{m})}(\boldsymbol{\rho},\mathbf{Q}) = 2\cos(\pi\widetilde{m})\,\frac{\Gamma(2m)\,\Gamma(2\widetilde{m})}{\Gamma(m)\,\Gamma(\widetilde{m})}\,\frac{\Gamma(-iQ^*)\,\Gamma(1-\widetilde{m}+iQ)}{\Gamma(m-iQ^*)\,\Gamma(1+iQ)}\times$$
$$\times\left(\frac{y}{y-1}\right)^m\left(\frac{y^*}{y^*-1}\right)^{\widetilde{m}}(1-y)^{-iQ^*/2}(1-y^*)^{-iQ/2}\times$$
$$\times F\left(iQ^*+m,m;iQ^*+1;\frac{1}{1-y}\right)F\left(iQ+\widetilde{m},\widetilde{m};iQ+1;\frac{1}{1-y^*}\right)+$$
$$+\,2\cos(\pi m)\,\frac{\Gamma(2m)\,\Gamma(2\widetilde{m})}{\Gamma(m)\,\Gamma(\widetilde{m})}\,\frac{\Gamma(1-m-iQ^*)\,\Gamma(iQ)}{\Gamma(1-iQ^*)\,\Gamma(\widetilde{m}+iQ)}\,y^m y^{*\,\widetilde{m}}\times$$
$$\times(1-y)^{iQ^*/2}(1-y^*)^{iQ/2}\,F\left(-iQ^*+m,m;-iQ^*+1;\frac{1}{1-y}\right)\times$$
$$\times F\left(-iQ+\widetilde{m},\widetilde{m};-iQ+1;\frac{1}{1-y^*}\right). \quad (9.11)$$

By comparing the above expressions for the wave function $\Psi_{(m,\widetilde{m})}(\boldsymbol{\rho},\mathbf{Q})$ one can verify its symmetry to the transformations

$$\rho\to-\rho,\quad \rho\to\rho^*,\ m\to\widetilde{m}$$

and

$$\rho\to\rho+2\pi i,$$

corresponding to the single-valuedness of the solution on the cylinder.

We shall use below explicit expressions for $\Psi_{(m,\widetilde{m})}(\boldsymbol{\rho},\mathbf{Q})$ to calculate the Pomeron Regge trajectories. Note, the Pomeron wave function in the mixed representation can be obtained also by the Fourier transformation of expression (7.5) in the coordinate representation

$$\Psi_{(m,\widetilde{m})}(\boldsymbol{\rho},\mathbf{Q})\sim\int d^2R\,e^{i\mathbf{qR}}\left(\frac{\sinh\rho/2}{2\sinh(R+\rho/2)\,\sinh(R-\rho/2)}\right)^m\times$$
$$\times\left(\frac{\sinh\rho^*/2}{2\sinh(R^*+\rho^*/2)\,\sinh(R^*-\rho^*/2)}\right)^{\widetilde{m}}, \quad (9.12)$$

where $Q=(q_x+iq_y)/2$, $t=-|q|^2$ and the integration is performed over the strip $|R_2|<\pi$.

10. BAXTER–SKLYANIN REPRESENTATION FOR POMERON at $T \neq 0$

Let us consider the eigenvalue equation for the integral of motion in the momentum space

$$4\left(\sinh\frac{\rho}{2}\right)^2 p_1\, p_2\, \psi(p_1, p_2) = m(m-1)\, \psi(p_1, p_2). \tag{10.1}$$

With the use of the relations

$$[\rho, p_1] = -i, \quad [\rho, p_2] = i \tag{10.2}$$

and the Taylor expansion

$$\mathrm{e}^{\pm\rho}\psi(p_1, p_2) = \psi(p_1 \mp i, p_2 \pm i), \tag{10.3}$$

we can rewrite Eq. (10.1) as the finite difference equation

$$\begin{aligned}(p_1 - i)\,(p_2 + i)\,\psi(p_1 - i, p_2 + i) - 2\,p_1\,p_2\,\psi(p_1, p_2) + \\ + (p_1 + i)\,(p_2 - i)\,\psi(p_1 + i, p_2 - i) = m(m-1)\,\psi(p_1, p_2).\end{aligned} \tag{10.4}$$

Let us see that this equation is just the Baxter equation for a two lattice sites inhomogeneous model, that is, a lattice model where there are inhomogeneities associated to each lattice site [27, 28]. In the Pomeron case the lattice consists of just two sites. We obtain in terms of the variables

$$Q = \frac{p_1 + p_2}{2}, \quad P = \frac{p_1 - p_2}{2}$$

the equation

$$\begin{aligned}\phi_Q(P - i)\ \psi_Q(P - i) + \phi_Q(P + i)\,\psi_Q(P + i) = \\ = [m(m-1) + 2\phi_Q(P)]\,\psi_Q(P),\end{aligned} \tag{10.5}$$

where

$$\phi_Q(P) \equiv Q^2 - P^2.$$

We recognize in Eq. (10.5) the Baxter equation for a two lattice sites inhomogeneous model with inhomogeneities $\pm Q$. Here, P plays the role of spectral parameter, $\psi_Q(P)$ stands for Baxter's Q function and $m(m-1) + 2\phi_Q(P)$ is the eigenvalue of the transfer matrix [27, 28].

Notice, that Eq. (10.5) is invariant under the change:

$$\psi_Q(P) \to Z(P)\,\psi_Q(P),$$

where $Z(P)$ is a periodic function of P with period i.

Initially we investigate its particular case $q = (p_1 + p_2)/2 = 0$, where the wave function depends only on the relative momentum $P = (p_1 - p_2)/2$ and satisfies the equation

$$(P-i)^2\,\psi(P-i) - 2\,P^2\,\psi(P) + (P+i)^2\,\psi(P+i) = -m(m-1)\,\psi(P). \quad (10.6)$$

Obviously, that after changing notations

$$P \to \lambda,\ \psi(P) \to Q(\lambda)$$

it coincides with the Baxter equation [27, 28] in the Pomeron case $n = 2$

$$(\lambda+i)^n\,Q(\lambda+i) + (\lambda-i)^n\,Q(\lambda-i) = \Lambda(\lambda)\,Q(\lambda), \quad (10.7)$$

where $\Lambda(u)$ is the eigenvalue of the transfer matrix [19]

$$\Lambda(u) = 2\,u^n - m(m-1)\,u^{n-2} + q_3\,u^{n-3} + \ldots + q_n \quad (10.8)$$

for the Heisenberg spin model [20, 21] corresponding to the BFKL dynamics in the multicolour QCD.

The Baxter function $Q(\lambda)$ according to Sklyanin [29] enters in the factorized expression for the holomorphic wave function of the composite state of n-reggeized gluons in the so-called Baxter–Sklyanin representation [25] (see also [26])

$$\psi(\lambda_1, \lambda_2, \ldots, \lambda_{n-1}) = \prod_{r=1}^{n-1} Q(\lambda_r)\ \Omega_0(\lambda_1, \lambda_2, \ldots, \lambda_{n-1}), \quad (10.9)$$

where $\Omega_0(\lambda_1, \lambda_2, \ldots, \lambda_{n-1})$ is the wave function of a pseudovacuum state. We should find a complete set of such holomorphic solutions to be able to construct the wave function in the total space $(\mathbf{q}, \boldsymbol{\lambda}_1, \ldots, \boldsymbol{\lambda}_{n-1})$ with the use of the holomorphic factorization [25, 26]. The unitary transformation for the transition from the momentum to the Baxter–Sklyanin representation can be constructed [25, 26].

For the Pomeron case $n = 2$, the Baxter function and the pseudovacuum state are well known [25]

$$Q(\lambda, m) = -\frac{\pi^2 m(1-m)}{\sin \pi m}\,{}_3F_2(1-i\lambda, 2-m, 1+m; 2, 2; 1), \quad \Omega_0(\lambda) \sim \lambda, \quad (10.10)$$

where ${}_3F_2(-i\lambda, 2-m, 1+m; 2, 2; 1)$ is the generalized hypergeometric function defined by its Taylor expansion

$${}_3F_2(\alpha_1, \alpha_2, \alpha_3; \beta_1, \beta_2; z) = \\ = \sum_{k=0}^{\infty} \frac{(\alpha_1)_k(\alpha_2)_k(\alpha_3)_k}{(\beta_1)_k(\beta_2)_k}\,\frac{z^k}{k!}, \quad (\alpha)_k = \frac{\Gamma(\alpha+k)}{\Gamma(\alpha)}. \quad (10.11)$$

The Baxter function is a meromorphic function

$$Q(\lambda, m) = \sum_{l=0}^{\infty} \frac{r_l(m)}{\lambda - i\,l}, \tag{10.12}$$

where

$$r_l(m) = -\frac{\sin\,\pi m}{i\pi} Q(-il, m), \quad r_0(m) = i\pi. \tag{10.13}$$

The residues satisfy the recurrent relation obtained directly from the Yang–Baxter equation (10.7) [25]

$$(l+1)^2\, r_{l+1}(m) = (2l^2 + m(m-1))\, r_l(m) - (l-1)^2\, r_{l-1}(m). \tag{10.14}$$

Note, that the Hamiltonian in the Baxter–Sklyanin representation coincides with the expression for h in the momentum representation at a finite temperature and $q = 0$ (see (4.3))

$$h = \psi(1+i\,p)+\psi(1-i\,p)-2\psi(1)+\frac{1}{p}\,\ln\left(2\sinh\left(\frac{\lambda}{2}\right)\right)p, \quad p = i\frac{\partial}{\partial\lambda}. \tag{10.15}$$

Using this result one can obtain easily the holomorphic Pomeron energy in terms of derivatives of $Q(\lambda, m)$ at $\lambda = \pm i$ (cf. [25]).

Taking into account that the second independent solution of the Baxter equation is $Q(-\lambda, m)$, we can construct the wave function in the Baxter representation from the condition of its normalizability [25]

$$\Phi_{m,\widetilde{m}}(\boldsymbol{\lambda}) \sim |\lambda|^2 \big(Q(\lambda^*, \widetilde{m})\, Q(\lambda, m) + \\ + (-1)^{m-\widetilde{m}} Q(-\lambda^*, \widetilde{m})\, Q(-\lambda, m)\big). \tag{10.16}$$

It is remarkable that the wave function $\Psi_{m,\widetilde{m}}(\mathbf{p})$ for the Pomeron in the momentum space for $\mathbf{Q} = 0$ coincides with the Baxter function for the Heisenberg spin model. As a result, we obtain for it the following expression:

$$\Psi_{m,\widetilde{m}}(\mathbf{p}) \sim Q_{m,\widetilde{m}}(\mathbf{p}) \sim Q(p^*, \widetilde{m})\, Q(p, m) + (-1)^{m-\widetilde{m}}\, Q(-p^*, \widetilde{m})\, Q(-p, m). \tag{10.17}$$

Note, that the above two expressions Φ and Ψ differ by the factor coinciding with the pseudovacuum state $\Omega_0(\boldsymbol{\rho}) \sim |\rho|^2$, which is a consequence of the fact that Φ and Ψ are normalized with a different weight. Indeed, the integral of motion is Hermitian on the functions normalized as follows:

$$||\Psi||^2 = \int d^2p\, |\Psi(\mathbf{p})|^2\, |p|^4, \tag{10.18}$$

where in $\int d^2p$ the integration over $\Re p$ and the summation over $\mathrm{Im}\, p = 0, \pm 1, \pm 2 \ldots$ are implied.

On the other hand, we can obtain for the Baxter function $Q_{m,\widetilde{m}}(\boldsymbol{\lambda})$ the integral representation

$$Q_{m,\widetilde{m}}(\boldsymbol{\lambda}) \sim \int d^2\rho\, \mathrm{e}^{i\mathbf{k}\boldsymbol{\rho}}\, \Psi_{(m,\widetilde{m})}(\boldsymbol{\rho}, \mathbf{Q}), \tag{10.19}$$

where the integration is performed over the strip $|\rho_2| < \pi$. The wave function $\Psi_{(m,\widetilde{m})}(\boldsymbol{\rho}, \mathbf{Q})$ in the mixed representation was calculated in the previous section (see, for example, (9.12)).

11. POMERON WAVE FUNCTION IN THE MOMENTUM REPRESENTATION

11.1. Hypergeometric Representations. We can obtain the Pomeron wave function in the momentum representation by Fourier transformation of the wave function in the mixed representation Eqs. (9.5). That is,

$$\Psi(P, Q) = \int d\rho^*\, \mathrm{e}^{-iP\rho^*}\, \Psi(\rho^*, Q). \tag{11.1}$$

The wave function $\Psi(\rho^*, Q) \equiv \Psi(\rho, Q^*)^*$ can be expressed in terms of associated Legendre functions:

$$\Psi_+(\rho^*, Q) = \mathrm{e}^{-iQ\rho^*}\, (1 - \mathrm{e}^{-\rho^*})^m\, {}_2F_1(m + 2iQ, m; 1 + 2iQ; \mathrm{e}^{-\rho^*}) = \\ = \Gamma(1 + 2iQ)\, P_{m-1}^{-2iQ}\left(\coth\frac{\rho^*}{2}\right). \tag{11.2}$$

It is convenient to use the variable $t \equiv 1/(1 - \mathrm{e}^{\rho^*})$ to perform the integral in Eq. (11.1),

$$\Psi_+(P, Q) = \Gamma(1 + iQ)\, \mathrm{e}^{-\pi(Q+P)} \times \\ \times \int_0^1 dt\, t^{iP-1}\, (1 - t)^{-iP-1} P_{m-1}^{-2iQ}(1 - 2t) = \frac{i\pi m(1 - m)\, \mathrm{e}^{-\pi(Q+P)}}{(1 + 2iQ)\sinh[\pi(P + Q)]} \times \\ \times {}_3F_2(2 - m, m + 1, 1 + i[P + Q]; 2 + 2iQ, 2; 1), \tag{11.3}$$

where we used Eq. (A.2) in Appendix, and ${}_3F_2$ stands for the generalized hypergeometric function defined in Eq. (10.11). Notice that Eq. (11.3) is invariant under $m \leftrightarrow 1 - m$, as it must be. In the $Q = 0$ limit, we recover the result Eq. (10.10) found in [25] upon the identification $P = -\lambda$.

By analogy with Eq. (10.10), we consider the solution of the Baxter equation

$$\psi_+(P,Q) \equiv i\pi \frac{\mathrm{e}^{\pi(P+Q)}\,\sinh[\pi(P+Q)]}{\sin[\pi m]}\,\Psi_+(P,Q) = -\frac{\pi^2 m(1-m)}{(1+2iQ)\sin(\pi m)}\times$$
$$\times\,{}_3F_2(2-m, m+1, 1+i[P+Q]; 2+2iQ, 2; 1). \quad (11.4)$$

Notice, that we can always multiply a solution of the Baxter equation (10.5) by a periodic function of P with period i. Now we have

$$\psi_+(P,0) = Q(\lambda,m)|_{\lambda=-P}\,.$$

Using the transformation formulas for the functions ${}_3F_2(a,b,c;d,e;1)$, Eq. (11.4) can be written as [37]

$$\psi_+(P,Q) = \frac{\pi^2\,m(1-m)}{\Gamma(2+2iQ)\,\sin(\pi m)}\frac{\Gamma(i[Q-P])}{\Gamma(1-i[Q+P])}\times$$
$$\times\,{}_3F_2(1-m+2iQ, m+2iQ, 1+i[P+Q]; 2+2iQ, 1+2iQ; 1). \quad (11.5)$$

We have from Eqs. (10.11) and (11.3) the following series representation for the momentum wave function:

$$\psi_+(P,Q) = -\frac{\pi}{\Gamma(1+i\,[P+Q])}\times$$
$$\times\sum_{n=1}^{\infty}\frac{\Gamma(n+m)\Gamma(n+1-m)\Gamma(n+i\,[P+Q])}{(n-1)!\,n!\,\Gamma(n+1+2iQ)}. \quad (11.6)$$

The late terms in this series behave as $n^{-1-i[Q-P]}$. Hence, this is a convergent series for $\mathrm{Im}\,(P-Q)>0$.

In order to analytically continue the wave function to the lower P–Q plane, we integrate term by term in Eq. (11.3) the expansion of $P^{-iQ}_{m-1}(1-2t)$ around $t=1$ with the result (see Appendix):

$$\psi_+(P,Q) = -\frac{i\pi^2}{\sinh\,2\pi Q}\frac{\Gamma(1+2iQ)}{\Gamma(1-i\,[P+Q])\Gamma(m+2iQ)\Gamma(1-m+2iQ)}\times$$
$$\times\sum_{n=0}^{\infty}\left[\frac{\Gamma(n+m)\Gamma(n+1-m)\Gamma(n-i\,[P+Q])}{(n-1)!\,n!\,\Gamma(n+1-2iQ)}-\right.$$
$$\left.-\frac{\Gamma(n+m+2iQ)\Gamma(n+1-m+2iQ)\Gamma(n+i[Q-P])}{\Gamma(n+2iQ)\,n!\,\Gamma(n+1+2iQ)}\right]. \quad (11.7)$$

The late terms in this series behave as $n^{-1-i(P-Q)}$. Hence, this is a convergent series for $\mathrm{Im}(P-Q) < 0$. Equation (11.7) can be explicitly expressed in terms of ${}_3F_2$ functions as follows:

$$\psi_+(P,Q) = \frac{\pi}{\sin[\pi m]} \frac{\Gamma(2iQ-1)\Gamma(2iQ+1)}{\Gamma(m+2iQ)\Gamma(1-m+2iQ)} \times$$
$$\times\, {}_3F_2(2-m, m+1, 1+i[P+Q]; 2+2iQ, 2; 1) + \frac{\Gamma(1-2iQ)\Gamma(i\,[Q-P])}{\Gamma(1-i\,[P+Q])} \times$$
$$\times\, {}_3F_2(1-m+2iQ, m+2iQ, i[Q-P]; 2iQ, 1+2iQ; 1). \quad (11.8)$$

Equation (11.7) explicitly displays simple poles at

$$p_2 = Q - P = 0, i, 2i, 3i, \ldots, li, \ldots$$

We can obtain the linearly independent solution by changing $P \to -P$ in Eqs. (11.2)–(11.7), and $\psi_+(-P,Q)$ exhibits a semi-infinite sequence of poles at

$$p_1 = P + Q = 0, i, 2i, 3i, \ldots, li, \ldots$$

Alternatively, Eq. (11.5) explicitly displays simple poles in the upper half planes p_1 and p_2 in the Gamma functions factors $\Gamma(ip_2)\Gamma(ip_1)$.

Furthermore, one also gets an independent solution by changing i by $-i$ in Eqs. (11.2)–(11.7).

We can explicitly compute the residues of $\psi_+(P,Q)$ at $p_2 = il$, $l = 0, 1, 2, \ldots$ from Eq. (11.7) with the result,

$$r_l(m,Q) \equiv \lim_{p_2 \to il} [p_2 - il]\, \psi_+(Q-p_2, Q) =$$
$$= -\pi(2iQ)_l \sum_{n=0}^{l} \frac{(-1)^n}{n!\,(l-n)!} \frac{(m+2iQ)_n\,(1-m+2iQ)_n}{(2\,iQ)_n\,(1+2iQ)_n}, \quad (11.9)$$

where $(\alpha)_k$ is defined in Eq. (10.11). These residues can be explicitly expressed in terms of the functions ${}_3F_2$ as follows:

$$r_l(m,Q) = -\pi\, \frac{(2iQ)_l}{l!}\, {}_3F_2(1-m+2iQ, m+2iQ, -l; 2iQ, 1+2iQ; 1). \quad (11.10)$$

Alternatively, we can write these residues using [37] as

$$r_l(m,Q) = \pi\, \frac{m(1-m)}{1+2iQ}\, {}_3F_2(2-m, m+1, 1-l; 2+2iQ, 2; 1). \quad (11.11)$$

Equation (11.9) yields for the first residues:

$$r_0(m,Q) = -\pi, \quad \frac{r_1(m,Q)}{r_0(m,Q)} = -\frac{m(1-m)}{1+2iQ}, \qquad\qquad \frac{r_2(m,Q)}{r_1(m,Q)} = -\frac{m(1-m)+2}{4(1+iQ)} + 1. \tag{11.12}$$

11.2. Pole Expansion. For $q \neq 0$, we can search for the solutions $\psi^{(1)}$ and $\psi^{(2)}$ of the eigenvalue equation (10.4) with poles in the variables p_1 and p_2, respectively (cf. (5.4) and (5.11)):

$$\psi^{(s)}(p_1,p_2) = \sum_{l=0}^{\infty} \frac{r_l(m,Q)}{p_s - il}, \quad r_0(m,Q) \neq 0, \tag{11.13}$$

where the residues $r_l(m,Q)$ do not depend on s and satisfy the recurrence relations (cf. (5.6))

$$(l-1)\,(2iQ+l-1)\,r_{l-1}(m,Q) + (l+1)\,(2iQ+l+1)\,r_{l+1}(m,Q) = \\ = [2l\,(2iQ+l) + m(m-1)]r_l(m,Q). \tag{11.14}$$

The quantities $r_l(m,Q)$ given by Eq. (11.9) indeed obey these recurrence relations.

Two other independent solutions can be obtained by the substitution

$$p_{1,2} \to -p_{1,2} \tag{11.15}$$

and have the following Mittag-Löffler representation:

$$\psi_m^{(s)}(-p_1,-p_2) = \sum_{l=0}^{\infty} \frac{r_l(m,-Q)}{-p_s - il}, \quad r_l(m,-Q) \neq 0. \tag{11.16}$$

Note, that for $Q=0$ the functions $\psi_m^{(1)}(p_1,p_2)$ and $\psi_m^{(2)}(-p_1,-p_2)$ coincide.

We can find the Pomeron wave function in the momentum space with the use of the holomorphic factorization analogously to the case $Q=0$ (see (10.17)). For this purpose, one should know the values of the functions $\psi_m^{(1)}(p_1,p_2)$ and $\psi_m^{(2)}(p_1,p_2)$ in the regular points $p_2 \to il$ and $p_1 \to il$, respectively. These quantities coincide with each other

$$R_l(m,Q) \equiv \psi_m^{(1)}(2Q-il, il) = \psi_m^{(2)}(il, 2Q-il) = \sum_{k=0}^{\infty} \frac{r_k(m,Q)}{2Q - i(k+l)}$$

and according to Eq. (10.4) satisfy the recurrence relations

$$(l+1)\,(2iQ+l+1)\,R_{l+1}(m,Q)+(l-1)\,(2iQ+l-1)\,R_{l-1}(m,Q)= \\ =\left[2l\,(2iQ+l)+m(m-1)\right]\,R_l(m,q). \quad (11.17)$$

It is obvious, that $R_2(m,Q)$ is expressed only in terms of $R_1(m,Q)$

$$2\,(2iQ+2)\,R_2(m,Q)=\left[2\,(2iQ+1)+m(m-1)\right]\,R_1(m,Q). \quad (11.18)$$

Therefore by comparing (11.17) with (11.14) we obtain that $R_l(m,Q)$ and $r_l(m,Q)$ are proportional

$$R_l(m,Q)=\frac{R_1(m,Q)}{r_1(m,Q)}\,r_l(m,Q). \quad (11.19)$$

These equations are fulfilled by the hypergeometric representations obtained in Subsec. 11.1. Identifying Eqs. (11.4) and (11.13) yields,

$$\psi^{(2)}(p_1,p_2)=-\frac{\pi^2 m(1-m)}{(1+2iQ)\sin(\pi m)}\,{}_3F_2(2-m,m+1,1+i[P+Q];2+2iQ,2;1).$$

Therefore,

$$R_l(m,Q)=-\frac{\pi^2 m(1-m)}{(1+2iQ)\sin(\pi m)}\,{}_3F_2(2-m,m+1,1-l;2+2iQ,2;1).$$

Using now Eq. (11.11) shows that Eq. (11.19) is fulfilled. In addition, we get

$$\frac{R_l(m,Q)}{r_l(m,Q)}=\frac{R_1(m,Q)}{r_1(m,Q)}=-\frac{\pi}{\sin(\pi m)}. \quad (11.20)$$

The Pomeron wave function in the momentum representation should be constructed with the use of the holomorphic factorization in such a way that it can be normalized (cf. (10.18))

$$\int d^2p_1\,d^2p_2\,|p_1|^2\,|p_2|^2\,|\Psi_{m,\widetilde{m}}(\mathbf{p}_1,\mathbf{p}_2)|^2<\infty, \quad (11.21)$$

where $\int d^2p$ is the implied integration over p_x and the summation over $p_y=r$. The normalizability requirement means, that $\Psi_{m,\widetilde{m}}(\mathbf{p}_1,\mathbf{p}_2)$ should not contain the poles at $p_{1,2}=\pm i\,l$ for $l\neq 0$. Taking into account also its symmetry to the transformation $p_r\to -p_r$, we obtain

$$\Psi_{m,\widetilde{m}}\sim\psi^{(1)}_m(p_1,p_2)\,\psi^{(2)}_{\widetilde{m}}(-p_1^*,-p_2^*)+ \\ +\frac{r_1(m,Q)\,R_1(\widetilde{m},-Q^*)}{r_1(\widetilde{m},-Q^*)\,R_1(m,Q)}\,\psi^{(2)}_m(p_1,p_2)\,\psi^{(1)}_{\widetilde{m}}(-p_1^*,-p_2^*)+(p_r\to -p_r). \quad (11.22)$$

The residues of the poles at $p_1 = -p_1^* = il$ for $l = 1, 2, \ldots$ are zero by construction. For the residues of the poles at $p_2 = -p_2^* = il$ ($l = 1, 2, \ldots$), we obtain

$$\lim_{p_2 \to il} (p_2 - il)\, \Psi_{m,\widetilde{m}}(\mathbf{p}_1, \mathbf{p}_2) \sim -r_l(\widetilde{m}, -Q^*)\, R_l(m, Q) + \\ + \frac{r_1(m,Q)\, R_1(\widetilde{m}, -Q^*)}{r_1(\widetilde{m}, -Q^*)\, R_1(m,Q)}\, r_l(m,Q)\, R_l(\widetilde{m}, -Q^*). \quad (11.23)$$

Due to the recurrence relation it is enough to verify the vanishing of this expression at $l = 1$, which gives the condition

$$\left(\frac{r_1(m,Q)\, R_1(\widetilde{m}, -Q^*)}{r_1(\widetilde{m}, -Q^*)\, R_1(m,Q)} \right)^2 = 1. \quad (11.24)$$

Using Eq. (11.20) and the relation $\sin(\pi m) = (-1)^n \ \sin(\pi \widetilde{m})$ yields

$$\frac{r_1(m,Q)\, R_1(\widetilde{m}, -Q^*)}{r_1(\widetilde{m}, -Q^*)\, R_1(m,Q)} = (-1)^n,$$

which demonstrates Eq. (11.24).

12. DIPOLE PICTURE AND BALITSKY–KOVCHEGOV EQUATION

There is a fruitful physical interpretation of the nonstationary BFKL equation in LLA as an equation for the amplitude N being a function of the rapidity $Y = \ln\, s$

$$N(\boldsymbol{\rho}_1, \boldsymbol{\rho}_2, Y) = N_{\boldsymbol{\rho}_1, \boldsymbol{\rho}_2}(Y),$$

and describing the scattering of the dipole constructed from two gluons with the transverse coordinates $\boldsymbol{\rho}_1$ and $\boldsymbol{\rho}_2$ off a target. The Balitsky–Kovchegov equation taking into account also a nonlinear effect of the dipole pair production takes the form

$$\frac{\partial N_{\boldsymbol{\rho}_1, \boldsymbol{\rho}_2}}{\partial Y} = \\ = \bar{\alpha}_s \int \frac{d^2 \rho_0}{2\pi}\, \frac{\boldsymbol{\rho}_{12}^2}{\boldsymbol{\rho}_{10}^2 \boldsymbol{\rho}_{20}^2} \left(N_{\boldsymbol{\rho}_1, \boldsymbol{\rho}_0} + N_{\boldsymbol{\rho}_2, \boldsymbol{\rho}_0} - N_{\boldsymbol{\rho}_1, \boldsymbol{\rho}_2} - N_{\boldsymbol{\rho}_1, \boldsymbol{\rho}_0}\, N_{\boldsymbol{\rho}_2, \boldsymbol{\rho}_0} \right), \quad (12.1)$$

where $\bar{\alpha}_s = \alpha_s\, N_c / \pi$.

In the linear approximation the above equation can be written as a Schrödinger equation

$$\frac{\partial N_{\boldsymbol{\rho}_1, \boldsymbol{\rho}_2}}{\partial t} = H\, N_{\boldsymbol{\rho}_1, \boldsymbol{\rho}_2} \quad (12.2)$$

with an Hamiltonian H. Here we introduce the time by the definition

$$t = -\frac{2\pi}{\alpha_s\, N_c}\, Y.$$

The action of the Hamiltonian on the dipole scattering amplitude is the following integral transformation:

$$H\, N_{\boldsymbol{\rho}_1,\boldsymbol{\rho}_2} = \int \frac{d^2\rho_0}{\pi}\, \frac{\boldsymbol{\rho}_{12}^2}{\boldsymbol{\rho}_{10}^2 \boldsymbol{\rho}_{20}^2} \left(N_{\boldsymbol{\rho}_1,\boldsymbol{\rho}_2} - N_{\boldsymbol{\rho}_1,\boldsymbol{\rho}_0} - N_{\boldsymbol{\rho}_2,\boldsymbol{\rho}_0}\right).$$

On the other hand, it can be written in the operator form

$$H = 2\ln\left(|\rho_{12}|^2\right) + |\rho_{12}|^2\, \ln\left(|p_1|^2\, |p_2|^2\right) |\rho_{12}|^{-2} - 4\psi(1),$$

where the following relations were used:

$$\int \frac{d^2\rho_0}{\pi}\, \frac{\boldsymbol{\rho}_{12}^2}{(\boldsymbol{\rho}_{10}^2 + \lambda^{-2})\,(\boldsymbol{\rho}_{20}^2 + \lambda^{-2})} = 2\ln\left(\boldsymbol{\rho}_{12}^2 \lambda^2\right),$$

$$\int \frac{d^2\rho_0}{\pi}\, \frac{\boldsymbol{\rho}_{12}^2}{\boldsymbol{\rho}_{10}^2\,(\boldsymbol{\rho}_{20}^2 + \lambda^{-2})}\, N_{\boldsymbol{\rho}_1,\boldsymbol{\rho}_0} = \boldsymbol{\rho}_{12}^2\, \ln\, \frac{|p_2|^2}{\lambda^2}\, \frac{1}{\boldsymbol{\rho}_{12}^2}\, N_{\boldsymbol{\rho}_1,\boldsymbol{\rho}_2}.$$

The Hamiltonian in the operator form has the property of the holomorphic separability

$$H = h + h^*, \quad , \; h = \rho_{12}\, \ln\left(p_1\, p_2\right) \rho_{12}^{-1} + 2\ln\left(\rho_{12}\right) - 2\psi(1).$$

It can be easily verified (see [7] and [16]), that h coincides in the Möbius representation with the usual holomorphic Hamiltonian

$$h = \ln\left(p_1 p_2\right) + p_1^{-1}\, \ln\left(\rho_{12}\right) p_1 + p_2^{-1}\, \ln\left(\rho_{12}\right) p_2 - 2\psi(1).$$

The important physical property of the dipole picture is the vanishing of interactions for the dipole with small sizes $\rho_{12} \to 0$, which is realized in the Möbius representation [7] (see, however, [8]). It turns out, that this feature is valid also at finite temperature.

Indeed, in this case the holomorphic Hamiltonian

$$\widetilde{h}\psi(1+ip_1) + \psi(1-ip_2) - 2\psi(1) + 2\ln\left(2\sinh\, \frac{\rho_{12}}{2}\right) + i\left(\frac{1}{p_2} - \frac{1}{p_1}\right)\, \frac{\mathrm{e}^{-\rho_{12}}}{1-\mathrm{e}^{-\rho_{12}}}$$

can be presented as follows:

$$h' = \rho_{12} - \sum_{k=1}^{\infty} \left(1 - \mathrm{e}^{-k\rho_{12}}\right) \left(\frac{1}{k+ip_1} + \frac{1}{k-ip_2} - \frac{2}{k}\right)$$

with the use of the relations

$$i\left(\frac{1}{p_2}-\frac{1}{p_1}\right)\frac{\mathrm{e}^{-\rho_{12}}}{1-\mathrm{e}^{-\rho_{12}}}=\sum_{k=1}^{\infty}\mathrm{e}^{-k\rho_{12}}\left(\frac{1}{k+ip_1}+\frac{1}{k-ip_2}\right),$$

$$2\ln\left(2\sinh\frac{\rho_{12}}{2}\right)=\rho_{12}-2\sum_{k=1}^{\infty}\frac{\mathrm{e}^{-k\rho_{12}}}{k},$$

$$\psi(1+z)-\psi(1)=\sum_{k=1}^{\infty}\left(\frac{1}{k}-\frac{1}{k+z}\right).$$

The fact, that the Hamiltonian h' vanishes at $\rho_{12}\to 0$, means that the eigenfunction $\Psi(\rho_{12})$ does not have the regular contribution $\sim \mathrm{const}$ for small ρ_{12} (it is not valid, however, for the Odderon wave function, as demonstrated in [7,8]).

With the use of the conformal transformation (7.1) we can obtain from (12.1) the generalized Balitsky–Kovchegov equation in the case of a nonzero temperature

$$\frac{\partial N_{\boldsymbol{\rho}_1,\boldsymbol{\rho}_2}}{\partial Y}=\bar{\alpha}_s\int\frac{d^2\rho_0}{2\pi}\,\frac{|\sinh\,\rho_{12}/2|^2}{4\,|\sinh\,\rho_{10}/2|^2\,|\sinh\,\rho_{20}/2|^2}\times$$
$$\times\left(N_{\boldsymbol{\rho}_1,\boldsymbol{\rho}_0}+N_{\boldsymbol{\rho}_2,\boldsymbol{\rho}_0}-N_{\boldsymbol{\rho}_1,\boldsymbol{\rho}_2}-N_{\boldsymbol{\rho}_1,\boldsymbol{\rho}_0}\,N_{\boldsymbol{\rho}_2,\boldsymbol{\rho}_0}\right),\quad (12.3)$$

where $N_{\boldsymbol{\rho}_1,\boldsymbol{\rho}_2}$ is an averaged number of the dipoles in a hadon with the color charges situated at the points $\boldsymbol{\rho}_1$ and $\boldsymbol{\rho}_2$, and the integration over ρ_0 is performed over the strip $0<\mathrm{Im}\,\rho_0=y<2\pi$.

For a very large temperature T, the coordinate y is completely compactified. As a result, for $T\to\infty$ in the linear approximation we obtain the BFKL equation in the $(2+1)$-dimensions [14, 15]. Moreover, the nonlinear Balitsky–Kovchegov equation in this limit turns out to be exactly solvable [31]. It will be interesting to develop the perturbation theory in the parameter $1/T$ to relate these nonlinear equations for the $2+1$ and $3+1$ cases.

Note, however, that in the Balitsky–Kovchegov approach one takes into account only the so-called fan diagrams for the Pomeron interactions among all possible reggeon diagrams appearing from the high energy effective action [32]. This action gives a possibility to construct the field theory describing the reggeized gluon interactions. In the reggeon field theory the rapidity $y=\ln s$ is considered as a time and the gluon fields depend only on the two-dimensional transverse coordinates $\boldsymbol{\rho}$.

13. POMERON TRAJECTORIES WITH THE RUNNING COUPLING CONSTANT

It was shown above, that independent wave functions describing the Pomeron at temperature T as a composite state of two gluons with the total momentum Q^*

and the relative coordinate ρ can be expressed in terms of hypergeometric functions [12] (see (9.1))

$$\Psi^{(1)}_{(m)}(\rho, Q^*) = \mathrm{e}^{i/2Q^*\rho}(\mathrm{e}^{\rho}-1)^m F(iQ^*+m, m; 2m; 1-\mathrm{e}^{\rho}),$$
$$\Psi^{(2)}_{(m)}(\rho, Q^*) \equiv \Psi^{(1)}_{(1-m)}(\rho, Q^*). \tag{13.1}$$

For $\rho \to 0$, the singularities of $\Psi^{(r)}(\rho, Q^*)$ at $1-\mathrm{e}^{\rho}=1$ and $1-\mathrm{e}^{\rho}=\infty$ correspond to the points $\rho=-\infty$ and $\rho=\infty$, respectively.

The analytic continuation of $\Psi^{(r)}$ along the imaginary axes from $\rho=0$ to $\rho=2\pi i$ is equivalent to the continuation of these eigenfunctions in a circle passed in a clock-wise direction around the singularity at $\rho=-\infty$. The monodromy matrix expressing the analytically continued solutions in terms of the initial ones was calculated above through the quasi-periodic (Bloch) solutions (see (9.3) and (9.5)) $\Psi^{(\pm)}_m(\rho, Q^*)$ as

$$\Psi^{(\pm)}_m(\rho, Q^*) \equiv (1-\mathrm{e}^{-\rho})^m F(m \mp iQ^*, m; 1 \mp iQ^*\,\mathrm{e}^{-\rho})\,\mathrm{e}^{\pm i/2Q^*\rho}. \tag{13.2}$$

They obey the quasi-periodicity property

$$\Psi^{(\pm)}_m(\rho+2\pi i, Q^*) = \mathrm{e}^{\mp\pi Q^*}\,\Psi^{(\pm)}_m(\rho, Q^*). \tag{13.3}$$

Notice the symmetry property $\Psi^{(\pm)}_m(\rho, Q^*) = \Psi^{(\pm)}_{1-m}(\rho, Q^*)$. The wave function $\Psi^{(1)}_{(m)}(\rho, Q^*)$ can be expressed in terms of the Bloch solutions $\Psi^{(\pm)}_m(\rho, Q^*)$ as

$$\Psi^{(1)}_{(m)}(\rho, Q^*) = \frac{2^{2\,m-1}}{\sqrt{\pi}}\Gamma\left(m+\frac{1}{2}\right) \times$$
$$\times\left[\frac{\Gamma(iQ^*)}{\Gamma(m+iQ^*)}\,\Psi^{+}_m(\rho, Q^*) + \frac{\Gamma(-iQ^*)}{\Gamma(m-iQ^*)}\,\Psi^{-}_m(\rho, Q^*)\right], \tag{13.4}$$

and there is a similar formula for $\Psi^{(2)}_{(m)}(\rho, Q^*)$.

The Pomeron wave functions can be constructed as a bilinear combination of holomorphic and antiholomorphic eigenfunctions $\Psi^{(r)}(\rho, Q^*)$ and $\Psi^{(r)}(\rho^*, Q)$ (cf. (9.6))

$$\Psi_{(m,\widetilde{m})}(\boldsymbol{\rho}, \mathbf{Q}) = \chi^{(1)}_{(m)}(\rho, Q^*)\chi^{(1)}_{(\widetilde{m})}(\rho^*, Q) +$$
$$+ c(m, \widetilde{m}, Q, Q^*)\chi^{(2)}_{(m)}(\rho, Q^*)\chi^{(2)}_{(\widetilde{m})}(\rho^*, Q), \tag{13.5}$$

where

$$\chi^{(1)}_{(m)}(\rho, Q^*) = 2^{1-2\,m}\frac{\Gamma(m+iQ^*)}{\Gamma(m+1/2)}\Psi^{(1)}_{(m)}(\rho, Q^*), \quad \chi^{(2)}_{(m)}(\rho, Q^*) = \chi^{(1)}_{(1-m)}(\rho, Q^*),$$

and $N = 2\mathrm{Im}\,Q$ is an integer.

The property of the wave function single-valuedness in the cylinder topology corresponding to the periodicity on the boundaries of the strip $0 < \operatorname{Im}\rho_{12} < 2\pi$ was imposed by requiring the vanishing of cross-terms in the asymptotic behaviour $\rho, \rho^* \to \infty$. This condition determines $c(m, \widetilde{m}, Q, Q^*)$ to be (cf. (9.8))

$$c(m, \widetilde{m}, Q, Q^*) = -\frac{\sin\pi(m - iQ^*)}{\sin\pi(m + iQ^*)},$$

where we used Eq. (13.4) and

$$\lim_{\rho\to\infty} \Psi_m^{(\pm)}(\rho, Q^*) = \mathrm{e}^{\pm i/2 Q^* \rho}.$$

In addition, notice that

$$\frac{\sin\pi(m - iQ^*)}{\sin\pi(m + iQ^*)} = \frac{\sin\pi(\widetilde{m} - i\,Q)}{\sin\pi(\widetilde{m} + iQ)}.$$

In summary, the Pomeron wave function for a fixed coupling constant can be written as

$$\Psi_{(m,\widetilde{m})}(\boldsymbol{\rho}, \mathbf{Q}) = \chi^{(1)}_{(m)}(\rho, Q^*)\chi^{(1)}_{(\widetilde{m})}(\rho^*, Q) - \\ - \frac{\sin\,\pi(m - iQ^*)}{\sin\,\pi(m + iQ^*)}\chi^{(2)}_{(m)}(\rho, Q^*)\chi^{(2)}_{(\widetilde{m})}(\rho^*, Q), \quad (13.6)$$

up to a normalization constant.

The Pomeron wave function at temperature T takes the following form for small distances ρ:

$$f(\boldsymbol{\rho}, \mathbf{Q}) \overset{\vec{\rho}\to 0}{=} \rho^m(\rho^*)^{\tilde{m}} + \mathrm{e}^{i\delta_{m,\tilde{m}}(\mathbf{Q})}\,\rho^{1-m}(\rho^*)^{1-\tilde{m}} = \\ = |\rho|^{1+2i\nu}\left(\frac{\rho}{\rho^*}\right)^{n/2} + \mathrm{e}^{i\delta_{m,\tilde{m}}(\mathbf{Q})}\,|\rho|^{1-2i\nu}\left(\frac{\rho^*}{\rho}\right)^{n/2}, \quad (13.7)$$

where

$$m = \frac{1}{2} + i\nu + \frac{n}{2}, \quad \tilde{m} = \frac{1}{2} + i\nu - \frac{n}{2}, \\ \mathrm{e}^{i\delta_{m,\tilde{m}}(\mathbf{Q})} = \mathrm{e}^{i\delta^0_{m,\tilde{m}}(\mathbf{Q})}\,\mathrm{e}^{i\delta^T_{m,\tilde{m}}(\mathbf{Q})} \quad (13.8)$$

and

$$\mathrm{e}^{i\delta^0_{m,\tilde{m}}(\mathbf{Q})} = (-1)^n\left(\frac{|Q|}{4}\right)^{-4i\nu}\left(\frac{Q}{Q^*}\right)^n\frac{\Gamma(m+1/2)}{\Gamma(3/2-m)}\frac{\Gamma(\tilde{m}+1/2)}{\Gamma(3/2-\tilde{m})}, \\ \mathrm{e}^{i\delta^T_{m,\tilde{m}}(\mathbf{Q})} = |Q|^{4i\nu}(-1)^n\left(\frac{Q^*}{Q}\right)^n\frac{\Gamma(1-m+iQ)}{\Gamma(m+iQ)}\frac{\Gamma(1-\tilde{m}-iQ^*)}{\Gamma(\tilde{m}-iQ^*)}. \quad (13.9)$$

Since $Q = Q_{\rm phys}/2\pi T$ for $T \to 0$ and fixed $Q_{\rm phys}$ we have $Q \to \infty$. We find for the zero temperature limit from Eq. (13.9)

$$\lim_{T\to 0} \delta^T_{m,\tilde m}(\mathbf{Q}) = 0.$$

Using that

$$Q = Q_R + \frac{i}{2}N, \quad Q^* = Q_R - \frac{i}{2}N \tag{13.10}$$

and $\operatorname{Im} Q_R = 0$, we find for the states with a vanishing conformal spin $n = 0$

$$\mathrm{e}^{i\delta^T_{m,\tilde m}(\mathbf{Q})} = \left[Q_R^2 + \left(\frac{N}{2}\right)^2\right]^{2i\nu} \times \\ \times \frac{\Gamma\left(\frac{1-N}{2} - i\nu + iQ_R\right)\Gamma\left(\frac{1-N}{2} - i\nu - iQ_R\right)}{\Gamma\left(\frac{1-N}{2} + i\nu + iQ_R\right)\Gamma\left(\frac{1-N}{2} + i\nu - iQ_R\right)}. \tag{13.11}$$

Now we consider the case of the running coupling constant using the method of [5]. The Pomeron wave function must be an eigenfunction of the operator

$$\mathcal{E}(Q) \equiv \alpha_s(Q)\chi\left(-\frac{i}{2}\frac{d}{d\ln|\rho|}\right), \tag{13.12}$$

where

$$\chi(\nu) = -\frac{6}{\pi}\operatorname{Re}\left[\gamma + \psi\left(\frac{1}{2} + i\nu\right)\right]$$

for the vanishing conformal spin $n = 0$. In the above expression, γ is the Euler constant and $\psi(x) = \partial \ln \Gamma(x)/\partial x$ stands for the digamma function. The QCD running coupling constant in one-loop approximation for three flavours u, d, s takes the form

$$\alpha_s(Q) = \frac{4\pi}{9\,\log(4Q^2/\Lambda^2)}, \tag{13.13}$$

where $\Lambda \approx 200$ MeV is the QCD parameter.

The scale dependence of the running coupling constant $\alpha_s(Q)$ makes the calculation of eigenfunctions and eigenvalues of Eq. (13.12) a nontrivial task. In [5], such an eigenvalue problem is solved in a semiclassical approximation in the zero temperature case. We use the same method here to treat the nonzero temperature case.

The semiclassical quantization condition takes now the form

$$\phi(\nu_k) = k + \frac{1}{4}, \quad k = 0, 1, 2, \ldots, \tag{13.14}$$

where

$$\phi(\nu) \equiv \frac{4}{9\alpha_s(Q)} \left[\frac{1}{\chi(\nu)} \int\limits_0^\nu dx\, \chi(x) - \nu \right] - \frac{1}{2\pi}\, \delta_\nu^T(\mathbf{Q}), \tag{13.15}$$

and $\delta_\nu^T(\mathbf{Q}) = \delta_{m,\tilde{m}}^T(\mathbf{Q})$ is presented in Eq. (13.9).

The corresponding eigenvalue of the operator $\mathcal{E}(Q)$ is given approximately by

$$\omega_k(Q) = \alpha_s(Q)\chi(\nu_k), \tag{13.16}$$

where ν_k is found from Eqs. (13.14), (13.15). Thus, the eigenvalues and eigenfunctions depend parametrically on $Q = Q_{\rm phys}/2\,\pi\,T$. In particular, we recover the zero temperature limit for $Q \to \infty$ [5]. The characteristic scale in temperature follows from the running QCD coupling Eq. (13.13) and is given by

$$T_{\rm char} \equiv \frac{1}{4\,\pi} \exp\left(\frac{2\pi}{9\alpha_s(Q)} \right).$$

We investigated initially the case $N = 0$ in Eq. (13.10). In Fig. 1, the Pomeron Regge trajectories as functions of $\alpha(Q)$, $t = |Q|^2/4$ are constructed for $k = 0$ in (13.14) for various temperatures. The similar results for the cases $k = 1$ and $k = 2$ are presented in Figs. 2 and 3, respectively. In Fig. 4, the values of the Regge trajectories for t, corresponding to $\alpha_s = 0.5$, are drawn as functions of $T/T_{\rm char}$ for $k = 0$ and for $N = 0$ and $N = 2$. We see from this figure that for $T \to 0$ the rotational invariance is restored since the results for $N = 0$ and $N = 2$ coincide. It is interesting, that the eigenvalue ω_0 firstly grows with the temperature and then goes down. The vanishing of the intercept at large temperatures $T \to \infty$

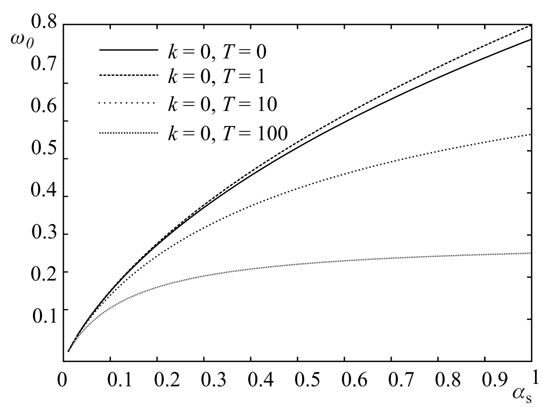

Fig. 1. The first eigenvalue ω_0 vs. α_s for several values of the temperature

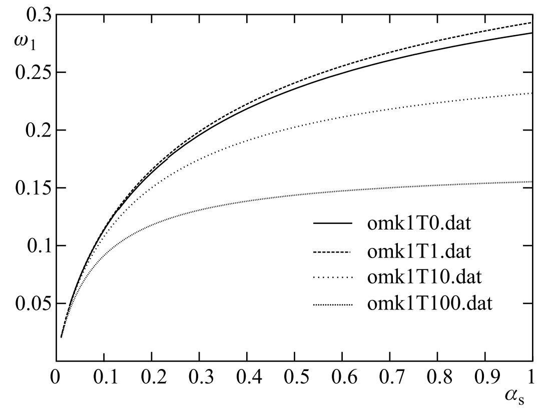

Fig. 2. The second eigenvalue ω_1 vs. α_s for several values of the temperature

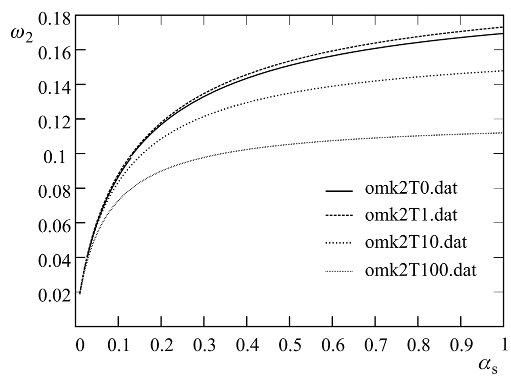

Fig. 3. The third eigenvalue ω_2 vs. α_s for several values of the temperature

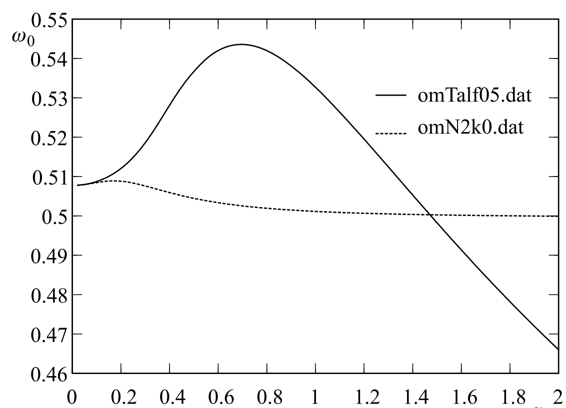

Fig. 4. The first eigenvalue ω_0 vs. $T/T_{\rm char}$ for $\alpha_s = 0.5$ for $N = 0$ and $N = 2$

is related to the effect of the asymptotic freedom because the QCD coupling constant tends to zero providing that the typical gluon virtualities grow with temperature, but the functional form of this dependence is not known (cf. [38]). Its initial growth is presumably related to the fact, that in the case $N = 0$ the t-channel temperature leads to a compactification of the transverse coordinates y_i orthogonal to the momentum transfer $\mathbf{q}$, which leads to a confinement phenomenon similar to that in the dual Meissner effect resulting in a compression of the chromo-electric field in a string stretched between colour objects.

APPENDIX

We compute here the integral in Eq. (11.3).

We have from [36] that

$$\int_0^1 dx\,(1-x)^{\mu-\lambda-1}x^{\sigma+\lambda-1}P_\nu^{2\lambda}(1-2x) = \\ = \frac{e^{i\pi\lambda}}{\Gamma(1-2\lambda)}B(\sigma,\mu)\,{}_3F_2(-\nu,\nu+1,\sigma;1-2\lambda,\sigma+\mu;1). \quad (A.1)$$

We can compute here the $\sigma+\mu \to 0$ limit using the series expansion Eq. (10.11) for the generalized hypergeometric function with the result

$$\lim_{\sigma+\mu\to 0}\frac{1}{\Gamma(\sigma+\mu)}\,{}_3F_2(-\nu,\nu+1,\sigma;1-2\lambda,\sigma+\mu;1) = \\ = \frac{\sigma\nu(\nu+1)}{2\lambda-1}\,{}_3F_2(1-\nu,\nu+2,\sigma+1;2-2\lambda,2;1). \quad (A.2)$$

We derive here the analytic continuation of the Pomeron wave function $\psi_+(P,Q)$ to the upper P–Q plane.

The associated Legendre functions appearing in Eq. (11.3) can be expressed in terms of hypergeometric functions as [36]

$$P_{m-1}^{-2iQ}(1-2t) = \frac{1}{\Gamma(1+2iQ)}\left(1-\frac{1}{t}\right)^{-iQ}\,{}_2F_1(m,1-m;1+2iQ;t). \quad (A.3)$$

In order to expand this function in powers of $t-1$, we relate the hypergeometric function ${}_2F_1(m,1-m;1+i\,Q;t)$ with hypergeometric functions with argument $1-t$ [36],

$$
\begin{aligned}
&{}_2F_1(m,1-m;1+2iQ;t)= \\
&=\frac{\Gamma(1+2iQ)\Gamma(2iQ)}{\Gamma(m+2iQ)\Gamma(1-m+2iQ)}\,{}_2F_1(m,1-m;1-2iQ;1-t)+(1-t)^{2iQ}\times \\
&\quad\times\frac{\Gamma(1+2iQ)\Gamma(-2iQ)}{\Gamma(m)\Gamma(1-m)}\,{}_2F_1(m+2iQ,1-m+2iQ;1+2iQ;1-t). \quad \text{(A.4)}
\end{aligned}
$$

These two hypergeometric functions are expressed with the standard hypergeometric series [36],

$$
{}_2F_1(a,b;c;z)=\sum_{n=0}^{\infty}\frac{(a)_n(b)_n z^n}{(c)_n n!}. \quad \text{(A.5)}
$$

Inserting Eqs.(A.3)–(A.5) into Eq. (11.3) and integrating term by term yields Eq. (11.7).

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