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THE ANGULAR MOMENTUM CONTROVERSY: WHAT IS IT ALL ABOUT AND DOES IT MATTER?

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A controversy has arisen as to how to define quark and gluon angular momentum, important in understanding the internal structure of the nucleon. For a review of the controversy, see [1]. I survey some of the ideas put forward and try to assess their physical implications.

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INTRODUCTION

Since all the controversial issues in QCD already arise in QED, I shall mainly discuss QED for simplicity. For a theory invariant under space–time and Lorentz transformations, from the Lagrangian, via Noether's theorem, QED textbooks derive a conserved energy-momentum density

$$t^{\mu\nu} \quad \partial_\mu t^{\mu\nu} = 0 \tag{1}$$

and a conserved angular-momentum density

$$\mathcal{M}^{\mu\nu\lambda} \quad \partial_\mu \mathcal{M}^{\mu\nu\lambda} = 0. \tag{2}$$

I shall call these *canonical*: $t^{\mu\nu}_{\rm can}$ and $\mathcal{M}^{\mu\nu\lambda}_{\rm can}$. The total 4-momentum is

$$P^\mu_{\rm can} = \int d^3x\, t^{0\mu}_{\rm can}(x), \tag{3}$$

and the total angular momentum is

$$M^{ij}_{\rm can} \equiv \int d^3x \mathcal{M}^{0ij}_{\rm can}(x) \quad \text{with} \quad J^k_{\rm can} = \frac{1}{2}\epsilon_{kij}\, M^{ij}_{\rm can}. \tag{4}$$

These are the generators of translations and rotations. For fields $\phi_r(x)$:

$$i\,[P^\mu_{\rm can},\, \phi_r(x)] = \partial^\mu\, \phi_r(x) \tag{5}$$

and

$$i[M^{ij}_{\rm can}, \phi_r(x)] = (x^i\partial^j - x^j\partial^i)\phi_r(x) + (\Sigma^{ij})_r{}^s \phi_s(x), \tag{6}$$

where $(\Sigma^{ij})_r{}^s$ is the appropriate spin matrix. One finds

$$\mathbf{J}_{\rm can} = \int d^3x\, \psi^\dagger \boldsymbol{\gamma}\gamma_5 \psi + \int d^3x\, \psi^\dagger [\mathbf{x} \times (-i\boldsymbol{\nabla})]\psi + \int d^3x\, (\mathbf{E} \times \mathbf{A}) + \\ + \int d^3x\, E^i [\mathbf{x} \times \boldsymbol{\nabla} A^i] = \mathbf{S}_{\rm can}(\text{el}) + \mathbf{L}_{\rm can}(\text{el}) + \mathbf{S}_{\rm can}(\gamma) + \mathbf{L}_{\rm can}(\gamma). \quad (7)$$

This has the nice features that a) it looks like sum of free electron plus free photon terms, b) the photon angular momentum is split into *spin* and *orbital* parts (recall that we talk about gluon *spin* in QCD!), and c) one finds that the total energy looks like electron energy plus photon energy plus $H_{\rm int}$. On the negative side, *only* the electron spin term is gauge invariant (GI) and all textbooks on QED say: *The angular momentum of the photon cannot be split in a gauge invariant way into a spin part and an orbital part*. Does it matter if the individual terms are *not* GI? Ji argues yes: if experimentally measurable, the operators should be GI. I say no: what you measure are matrix elements. The physical matrix elements must be GI. This issue is unresolved and I will not discuss it further. But clearly one has to explain how one can measure gluon spin!

1. THE BELINFANTE ENERGY-MOMENTUM AND ANGULAR-MOMENTUM DENSITIES

One can define the Belinfante energy-momentum density which is symmetric: $t^{\nu\mu}_{\rm bel}(x) = t^{\mu\nu}_{\rm bel}(x)$, is gauge invariant, and differs from the canonical one by a spatial divergence:

$$t^{0\mu}_{\rm bel}(x) = t^{0\mu}_{\rm can}(x) + \text{spatial divergence}. \quad (8)$$

It follows that

$$P^\mu_{\rm bel} \equiv \int d^3x\, t^{0\mu}_{\rm bel}(x) = P^\mu_{\rm can} \quad (9)$$

if the fields vanish at infinity. Similarly for the Belinfante angular-momentum density

$$\mathcal{M}^{0ij}_{\rm bel}(x) = \mathcal{M}^{0ij}_{\rm can}(x) + \text{spatial divergence} \quad (10)$$

so that

$$\mathbf{J}_{\rm bel} = \mathbf{J}_{\rm can} \quad (11)$$

if the fields vanish at infinity. Does it make sense to talk about fields vanishing at infinity? Classical: yes. The field strength has a numerical value, but be careful.

Quantum: no. What do you mean by an *operator* vanishing? I will return to this presently. What does $\mathbf{J}_{\text{bel}}$ look like?

$$\mathbf{J}_{\text{bel}} = \int d^3x\,\psi^\dagger \boldsymbol{\gamma}\gamma_5\psi + \int d^3x\,\psi^\dagger[\mathbf{x}\times(-i\mathbf{D})]\psi + \int d^3x\,\mathbf{x}\times(\mathbf{E}\times\mathbf{B}) = \\ = \mathbf{S}_{\text{bel}}(\text{el}) + \mathbf{L}_{\text{bel}}(\text{el}) + \mathbf{J}_{\text{bel}}(\gamma), \quad (12)$$

where the covariant derivative is

$$\mathbf{D} = \boldsymbol{\nabla} - ie\mathbf{A}. \quad (13)$$

Note that each term is gauge invariant, but that $\mathbf{J}_{\text{bel}}(\gamma)$ is *not* split into spin and orbital parts. There are several delicate questions involved in the above, even at classical level. Applying the above to a free classical electromagnetic field, one gets

$$\mathbf{J}_{\text{can}} = \underbrace{\int d^3x\,(\mathbf{E}\times\mathbf{A})}_{\text{spin term}} + \underbrace{\int d^3x\,E^i(\mathbf{x}\times\boldsymbol{\nabla}A^i)}_{\text{orbital term}} \quad (14)$$

and

$$\mathbf{J}_{\text{bel}} = \int d^3x\,[\mathbf{x}\times(\mathbf{E}\times\mathbf{B})]. \quad (15)$$

Consider a left-circularly polarized (= positive helicity) beam propagating along OZ,

$$A^\mu = \left(0,\, \frac{E_0}{\omega}\,\cos(kz-\omega t),\, \frac{E_0}{\omega}\,\sin(kz-\omega t),\, 0\right).$$

Then $\mathbf{E}$, $\mathbf{B}$, and $\mathbf{A}$ all rotate in the XY plane. Consider the component of $\mathbf{J}$ along OZ. The Poynting vector $\mathbf{E}\times\mathbf{B}$ is along OZ so that $[\mathbf{x}\times(\mathbf{E}\times\mathbf{B})_z = 0$ and thus one gets the wrong result $J_{\text{bel},\,z} = 0$. For the canonical version, $\boldsymbol{\nabla}\,A_{x,y} \propto \mathbf{e}_{(z)}$ so the orbital term gives 0. One finds

$$J_{\text{can},\,z} \text{ per unit volume} = \frac{E_0^2}{\omega}.$$

For one photon per unit volume $E_0^2 = \hbar\omega$ so that

$$J_{\text{can},\,z} \text{ per photon} = \hbar,$$

i.e., the canonical spin term gives the expected result.

For the quantum case, what does it mean to say an operator vanishes at infinity? The equivalence of canonical and Belinfante momentum and angular momentum depended on being able to neglect integrals of spatial divergences. Usually we are interested in expectation values of these operators, i.e., their forward matrix elements. For these it may be possible to justify neglecting the contribution at infinity.

1.1. Spatial Divergence of a Local Operator. A local operator $O(x)$ is defined at one space–time point and must satisfy the law of translation, i.e.,

$$\mathrm{e}^{ia\cdot P} O(x)\, \mathrm{e}^{-ia\cdot P} = O(x+a). \tag{16}$$

For the spatial divergence of a local operator we have

$$\langle \mathbf{p}'|\, \partial_j O(x) |\mathbf{p}\rangle = \frac{\partial}{\partial x^j} \langle \mathbf{p}'|\, O(x)\, |\mathbf{p}\rangle = \frac{\partial}{\partial x^j} \langle \mathbf{p}'|\, \mathrm{e}^{-ix\cdot P} O(0)\, \mathrm{e}^{ix\cdot P} |\mathbf{p}\rangle =$$
$$= \left[\frac{\partial}{\partial x^j}\, \mathrm{e}^{-i\mathbf{x}\cdot(\mathbf{p}-\mathbf{p}')}\right] \langle \mathbf{p}'|\, O(0)\, |\mathbf{p}\rangle = i(p'^j - p^j)\langle \mathbf{p}'|\, O(0)\, |\mathbf{p}\rangle\, \mathrm{e}^{-i\mathbf{x}\cdot(\mathbf{p}-\mathbf{p}')}. \tag{17}$$

Therefore as $\mathbf{p}'y \to \mathbf{p}$

$$\langle \mathbf{p}\, |\partial_j O(x)\, |\mathbf{p}\rangle = 0 \quad \text{if} \quad \langle \mathbf{p}\, |\, O(0)\, |\mathbf{p}\rangle \quad \text{is nonsingular.} \tag{18}$$

1.2. Spatial Divergence of a Compound Operator. In the angular momentum case the spatial divergence involves an operator of the form $xO(x)$. While this *is* defined at one space–time point, it is *not* a local operator. To see this, suppose that $Q(x) = xO(x)$ is a local operator. Then

$$Q(x) = \mathrm{e}^{-ix\cdot P} Q(0)\, \mathrm{e}^{ix\cdot P} = 0 \quad \text{for all } x, \text{ since} \quad Q(0) = 0. \tag{19}$$

It is then much more difficult to show that one can neglect the expectation value of the spatial divergence of a compound operator. It can be done, but requires use of localized wave packets, as demonstrated by Shore and White. Because of possible dangers, I will use the notation

$$P^\mu_{\mathrm{bel}} \sim P^\mu_{\mathrm{can}} \qquad \text{and} \qquad \mathbf{J}_{\mathrm{bel}} \sim \mathbf{J}_{\mathrm{can}}, \tag{20}$$

when operators differ by a spatial divergence. It is crucially important to note that if

$$\mathbf{J}_{\mathrm{can}} = \mathbf{J}_{\mathrm{can}}(\mathrm{el}) + \mathbf{J}_{\mathrm{can}}(\gamma) \quad \text{and} \quad \mathbf{J}_{\mathrm{bel}} = \mathbf{J}_{\mathrm{bel}}(\mathrm{el}) + \mathbf{J}_{\mathrm{bel}}(\gamma), \tag{21}$$

then

$$\mathbf{J}_{\mathrm{bel}}(\mathrm{el}) \not\sim \mathbf{J}_{\mathrm{can}}(\mathrm{el}) \quad \text{and} \quad \mathbf{J}_{\mathrm{bel}}(\gamma) \not\sim \mathbf{J}_{\mathrm{can}}(\gamma), \tag{22}$$

because these individually do *not* differ by a spatial divergence. Analogous statements hold for QCD. Hence *we can no longer talk about* $\mathbf{J}(\mathrm{quark})$ and $\mathbf{J}(\mathrm{gluon})$. *We must specify which scheme* we are using for $\mathbf{J}$ and for momentum $\mathbf{P}$. Note that this is no worse than the realization that for PDFs we must specify the factorization scheme: $q(x)_{\mathrm{MS}}$, $q(x)_{\overline{\mathrm{MS}}}$, $q(x)_{\mathrm{DIS}}$.

2. THE CONTROVERSY

Given that, in QCD, $\Delta G(x)$ is measurable, Chen, Lu, Sun, Wang, and Goldman (Chen et al.) [2] insist that it *must* be possible to split the photon angular momentum into a spin part and an orbital part in a GI way! They put $\mathbf{A} = \mathbf{A}_{\rm phys} + \mathbf{A}_{\rm pure}$ with

$$\boldsymbol{\nabla} \cdot \mathbf{A}_{\rm phys} = 0, \quad \boldsymbol{\nabla} \cdot \mathbf{A}_{\rm pure} = 0 \tag{23}$$

(often called the transverse $\mathbf{A}_\perp$ and longitudinal $\mathbf{A}_\parallel$ parts, respectively). Adding a spatial divergence to $\mathbf{J}_{\rm can}$ they get

$$\begin{aligned}\mathbf{J}_{\rm chen} = &\int d^3x\, \psi^\dagger \boldsymbol{\gamma}\gamma_5 \psi + \int d^3x\, \psi^\dagger [\mathbf{x} \times (-i\mathbf{D}_{\rm pure})]\psi + \\ &+ \int d^3x\, (\mathbf{E} \times \mathbf{A}_{\rm phys}) + \int d^3x\, E^i [\mathbf{x} \times \boldsymbol{\nabla} A^i_{\rm phys}] = \\ &= \mathbf{S}_{\rm ch}({\rm el}) + \mathbf{L}_{\rm ch}({\rm el}) + \mathbf{S}_{\rm ch}(\gamma) + \mathbf{L}_{\rm ch}(\gamma),\end{aligned} \tag{24}$$

where $D^\mu = \partial^\mu - ieA^\mu_{\rm pure}$. Under a gauge transformation:

$$\mathbf{A}_{\rm pure} \to \mathbf{A}_{\rm pure} + \boldsymbol{\nabla}\Lambda, \quad \mathbf{A}_{\rm phys} \to \mathbf{A}_{\rm phys} \tag{25}$$

so each term in $\mathbf{J}_{\rm chen}$ is indeed GI. Are all textbooks of past 50 years therefore wrong? No! $\mathbf{A}_{\rm phys}$ is not a normal local field. It is *nonlocal*

$$\mathbf{A}_{\rm phys} = \mathbf{A} - \frac{1}{\nabla^2} \boldsymbol{\nabla}(\boldsymbol{\nabla} \cdot \mathbf{A}), \tag{26}$$

recall that $\frac{1}{\nabla^2} f(x) \equiv \frac{1}{4\pi} \int d^3x' \, \frac{f(\mathbf{x}')}{|\mathbf{x} - \mathbf{x}'|}$. What does this actually mean physically? Since Chen et al. is GI, we can choose a gauge $\mathbf{A}_{\rm pure} = 0$, i.e., $\mathbf{A}_{\rm phys} = \mathbf{A}$ which implies that $\boldsymbol{\nabla} \cdot \mathbf{A} = 0$, which is the Coulomb gauge! Thus

$$\mathbf{J}_{\rm chen} \equiv \mathbf{J}_{\rm can}\big|_{\rm Coulomb\ gauge} \tag{27}$$

so Chen et al. is a GI Extension of the Canonical case in the Coulomb gauge. It involves nonlocal fields and A^μ does not transform as a 4-vector under Lorentz transformations. The physical content is exactly the same as in the canonical case in the Coulomb gauge.

3. FURTHER DEVELOPMENTS

Wakamatsu [3] proposed an elegant covariant generalization of Chen et al. He actually has 2 versions. I only have time for Wakamatsu II, but Wakamatsu I is an interesting variant in which a «potential angular momentum» term is moved from the photon to the electron angular momentum. Split

$$A^\mu = A^\mu_{\rm phys} + A^\mu_{\rm pure} \quad \text{with} \quad F^{\mu\nu}_{\rm pure} = 0. \tag{28}$$

Now Wakamatsu does *not* give a specific formula for $A^\mu_{\rm pure}$, and Lorcé [4] has stressed that there exist *Stueckelberg* transformations

$$A^\mu_{\rm pure} \to A^\mu_{\rm pure} + \partial^\mu C(x), \quad A^\mu_{\rm phys} \to A^\mu_{\rm phys} - \partial^\mu C(x) \quad \text{any} \quad C(x) \tag{29}$$

which are *not* gauge transformations since $A^\mu \to A^\mu$, implying that there are an *infinite* number of possible $A^\mu_{\rm pure}$. This can be seen easily: in QED, $A^\mu_{\rm pure} = \partial^\mu \Lambda(x)$ for *any* Λ; in QCD $A^\mu_{\rm pure} = U^{-1}\partial^\mu U$ *for any* $SU(3)$ matrix U. Thus Wakamatsu II is actually an infinite family of schemes and the physical content will depend upon the choice of $A^\mu_{\rm pure}$. Suppose we uniquely specify the scheme $\mathbf{J}^F_{\rm wak\ II}$ by fixing $A^\mu_{\rm pure}$ via

$$A^\mu_{\rm pure}\big|_F = F(A^\mu), \tag{30}$$

where F is some given function. Since the scheme is gauge invariant, choose the gauge which makes $A^\mu_{\rm pure}|_F = 0$. Call it «Gauge F». Then

$$A^\mu_{\rm phys} = A^\mu\big|_{\rm Gauge\ F}, \tag{31}$$

and from the expression for $\mathbf{J}_{\rm wak\ II}$ one sees that

$$\mathbf{J}^F_{\rm wak\ II} = \mathbf{J}_{\rm can}\big|_{\rm Gauge\ F}. \tag{32}$$

Thus the family of schemes Wakamatsu II is identical to the canonical scheme in various choices of gauge.

Hatta gave a precise concrete example of $A^\mu_{\rm pure}$, i.e., a specific choice of the function $F(A^\mu)$ and

$$A^\mu_{\rm pure}\big|_{\rm Hatta} = 0 \tag{33}$$

corresponds to the lightcone gauge $A^+ = 0$. Thus

$$\mathbf{J}_{\rm Hatta} = \mathbf{J}_{\rm can}\big|_{{\rm Gauge}\ A^+=0}.$$

There have been several other papers which I have no time to discuss: Bashinsky and Jaffe; Stoilov; Cho, Ge, and Zhang; Zhang and Pak; Zhou and Huang; Xiang-Song Chen; Lorcé. For access to this literature, see [4].

4. WHICH SCHEME SHOULD YOU LOVE AND TRUST?

I think there are only two fundamental schemes: Canonical and Belinfante. Belinfante is favoured by Ji and collaborators. Pros: each term is gauge invariant; nucleon expectation values can be related to GPDs (for the longitudinal polarized case, see Ji [5]; for the transverse case, see Leader [6]).

Cons: photon (gluon) angular momentum is *not* split into spin and orbital parts; operators do *not* generate rotations.

My preference: a seductively desirable criterion would be that momentum should be the generator of translations; angular momentum should be the generator of rotations. But there is no way to guarantee this. Beware! If someone produces an expression for, say, $\mathbf{J}_{\rm el}$ and claims

$$[J^i_{\rm el},\, J^j_{\rm el}] = i\epsilon_{ijk} J^k_{\rm el}, \tag{34}$$

then ask them kindly to prove this. It is impossible! In an interacting theory $\mathbf{J}_{\rm el} = \mathbf{J}_{\rm el}(t)$ because the quanta exchange momentum and angular momentum. To evaluate the commutator, you would have to completely solve the field theory! So:

4.1. My Minimal Criterion for Favouring a Scheme. At *equal times* the operators should satisfy

$$\begin{aligned} &i\,[P^j_{\rm el}(t),\, \psi_r(t,\mathbf{x})] = \partial^j\, \psi_r(t,\mathbf{x}), \\ &i[M^{ij}_{\rm el}(t), \psi_r(t,\mathbf{x})] = (x^i\partial^j - x^j\partial^i)\psi_r(t,\mathbf{x}) + (\Sigma^{ij})^s_r \psi_s(t,\mathbf{x}). \end{aligned} \tag{35}$$

This leads to the canonical scheme (favoured by me and by Jaffe–Manohar).
Pros: at equal times the operators are the generators of rotations; the photon (gluon) angular momentum is split into spin and orbital parts; the operators have the same form as for free field case; the operators in the gauge $A^+ = 0$ can be related to PDFs and GPDs; it gets the right answer for a circularly polarized classical plane light wave. Cons: the terms are not gauge invariant.

4.2. A Key Issue: the Polarized Gluon Density $\Delta G(x)$. $\Delta G(x)$ is measurable, so $\Delta G(x)$ must be gauge invariant. In what sense does it correspond to the spin of the gluon? In my view, the parton model is a *picture* of QCD in the gauge $A^+ = 0$, so all is well since one can show that

$$\Delta G(x) = \langle \hat{\mathbf{P}} \cdot \mathbf{S}_{\rm can}(\text{gluon})\,\rangle|_{\text{Gauge } A^+=0}, \tag{36}$$

where $\langle \ldots \rangle$ means expectation value in a longitudinally polarized nucleon.

5. SUMMARY

There are, I believe, only two fundamental schemes: Canonical and Belinfante.

Though not gauge invariant, I prefer the Canonical because the operators generate rotations, at least at equal times.

All the new gauge invariant schemes involve nonlocal fields and correspond either to the Canonical version viewed in a particular choice of gauge, or to a mixed scheme in which, however, the nucleon matrix elements agree with the Canonical ones in the $A^+ = 0$ gauge. Thus the new schemes, in my opinion, do not contain any new physics.

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