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ABSENCE OF AXIAL ANOMALY IN THE BACKGROUND OF THE BOHM–AHARONOV VECTOR POTENTIAL

Yu.A. Sitenko

Bogolyubov Institute for Theoretical Physics,
National Academy of Sciences, 03143 Kyiv, Ukraine

The problem of the axial anomaly in the presence of the Bohm–Aharonov gauge vector field is exactly solved.

The axial anomaly arises as a violation of the classical conservation law for the axial-current at the quantum level. Since its discovery [1–3] the anomaly has played a more and more significant role in the development of contemporary quantum field theory, and has led to a number of important phenomenological consequences in particle physics. Although the first anomalies were found in studies by means of the diagram technique of perturbations in coupling constant, it was soon recognized that the results do not depend on the use of perturbative methods. The nonperturbative (i.e., not describable in the framework of perturbation theory) nature of the anomalies can be revealed by means of an approach in which the gauge vector field is treated as a classical external one and the problem of quantizing massless fermions in this background is solved. Such a treatment makes it possible to regard the anomaly as a manifestation of nontrivial topology of configurations of the gauge vector field and establish a connection between the anomaly and the topological invariant of the spectrum of the massless Dirac operator in an external-field background.

Singular (or contact, or zero-range) interaction potentials were introduced in quantum mechanics more than sixty years ago [4–6]. A mathematically consistent and rigorous treatment of the subject was developed [7], basing on the notion of self-adjoint extension of a Hermitian operator (for a review see monograph [8]). Singular external-field background can act on the quantized spinor field in a rather unusual manner: a leak of quantum numbers from the singularity point into the vacuum occurs [9–14]. This is due to the fact that a solution to the Dirac equation, unlike that to the Schrödinger one, does not obey a condition of regularity at the singularity point. It is necessary then to specify a boundary condition at this point, and the least restrictive, but still physically acceptable, condition is such that guarantees self-adjointness of the operator of the appropriate dynamical variable.

In the present paper the problem of the axial anomaly in the singular background of the Bohm–Aharonov [15] gauge vector field is comprehensively studied. We show that, contrary to the leak of vacuum quantum numbers, the leak of anomaly from the singularity point does not occur.

Let us consider the effective action functional for quantized massless spinor field $\Psi(x)$ in external classical vector field $V_\mu(x)$ in the Wick-rotated (Euclidean) d-dimensional space-time

$$S^{\text{eff}}[V_\mu(x)] = 3D - \ln \int d\Psi(x)\, d\Psi^\dagger(x)\, \exp[-\int d^d x\, L(x)] =$$

$$= 3D - \ln Det(-i\gamma^\mu \nabla_\mu), \tag{1}$$

where

$$L(x) = 3D - \frac{i}{2}\Psi^\dagger(x)\gamma^\mu[\nabla_\mu\Psi(x)] + \frac{i}{2}[\nabla_\mu\Psi(x)]^\dagger\gamma^\mu\Psi(x) \tag{2}$$

is the Lagrangian density, $\nabla_\mu = 3D\partial_\mu - iV_\mu(x)$ is the covariant differentiation operator, and γ^μ ($\mu = 3D\overline{1,d}$) are the Dirac matrices,

$$[\gamma^\mu, \gamma^\nu]_+ = 3D2g^{\mu\nu}, \qquad \mathrm{tr}\,\gamma^\mu = 3D0, \qquad g_{\mu\nu} = 3Ddiag(1, ..., 1). \tag{3}$$

If there exists matrix Γ anticommuting with the Dirac matrices,

$$[\Gamma, \gamma^\mu]_+ = 3D0, \qquad \mathrm{tr}\,\Gamma = 3D0, \qquad \Gamma^2 = 3DI, \tag{4}$$

then one can define local chiral transformation

$$\Psi(x) \rightarrow e^{i\omega(x)\Gamma}\Psi(x), \qquad \Psi^\dagger(x) \rightarrow \Psi^\dagger(x)e^{i\omega(x)\Gamma},$$
$$V_\mu(x) \rightarrow e^{i\omega(x)\Gamma}V_\mu(x)e^{-i\omega(x)\Gamma} + \partial_\mu\omega(x)\Gamma. \tag{5}$$

The invariance of functional (1) under this transformation corresponds to conservation law

$$\nabla_\mu J^\mu_{d+1}(x) = 3D0, \tag{6}$$

where

$$J^\mu_{d+1}(x) = 3D\mathrm{itr}\,\langle x|\gamma^\mu\Gamma(-i\gamma^\nu\nabla_\nu)^{-1}|x\rangle. \tag{7}$$

However, functional (1), as well as current (7), is ill-defined, suffering from both ultraviolet and infrared divergences. Performing the regularization of divergencies in a way which is consistent with gauge invariance, one may arrive at the violation of conservation law (6) (i.e., at the axial anomaly) [1–3].

An example of a singular background field configuration is provided by that of the Bohm–Aharonov [15] vortex represented by a point for $d = 3D2$, a line for $d = 3D3$, and a $(d-2)$-dimensional hypersurface for $d > 3$:

$$
\begin{aligned}
V^1(x) &= 3D - \Phi^{(0)}\frac{x^2}{(x^1)^2 + (x^2)^2}, \quad V^2(x) = 3D\Phi^{(0)}\frac{x^1}{(x^1)^2 + (x^2)^2}, \\
&V^\nu(x) = 3D0, \qquad \nu = 3D\overline{3, d},
\end{aligned}
\tag{8}
$$

$$
B^{3\cdots d}(x) = 3D2\pi\Phi^{(0)}\delta(x),
\tag{9}
$$

where $\Phi^{(0)}$ is the vortex flux in 2π units, i.e., in the London $(2\pi\hbar ce^{-1})$ units, since we use conventional units $\hbar = 3Dc = 3D1$ and coupling constant e is included into vector potential $V_\mu(x)$.

In the $d = 3D2$ case, the γ-matrices are chosen as $\gamma^1 = 3D\sigma_1, \gamma^2 = 3D\sigma_2$, and, consequently, $\Gamma = 3D\sigma_3$, where σ_1, σ_2 and σ_3 are the Pauli matrices. Then the complete set of solutions to Dirac equation

$$
(-i\gamma^\mu\nabla_\mu - E)\langle x|E\rangle = 3D0
\tag{10}
$$

in background (8) takes the form

$$
\langle x|E\rangle = 3D\sum_{n\in\mathbb{Z}} \left(\begin{array}{c} f_n(r)\exp(in\varphi) \\ g_n(r)\exp[i(n+1)\varphi] \end{array} \right),
\tag{11}
$$

where Z is the set of integer numbers, r and φ are the polar coordinates, and the radial functions, in general, are

$$
\left(\begin{array}{c} f_n(r) \\ g_n(r) \end{array} \right) =
$$

$$
= 3D \left(\begin{array}{c} C_n^{(1)}(E)J_{n-\Phi^{(0)}}(|E|r) + C_n^{(2)}(E)Y_{n-\Phi^{(0)}}(|E|r) \\ i(E/|E|)\left[C_n^{(1)}(E)J_{n+1-\Phi^{(0)}}(|E|r) + C_n^{(2)}(E)Y_{n+1-\Phi^{(0)}}(|E|r)\right] \end{array} \right),
\tag{12}
$$

$J_\rho(u)$ and $Y_\rho(u)$ are the Bessel and the Neumann functions of order ρ. It is clear that the condition of regularity at $r = 3D0$ can be imposed on both f_n and g_n for all n in the case of integer values of quantity $\Phi^{(0)}$ only. Otherwise, the condition of regularity at $r = 3D0$ can be imposed on both f_n and g_n for all but $n = 3Dn_0$, where n_0 is the integer part of the quantity $\Phi^{(0)}$ (i.e., the integer which is less than or equal to $\Phi^{(0)}$); in this case at least one of the functions, f_{n_0} or g_{n_0}, remains irregular, although square integrable, with the asymptotics r^{-p}

$(p < 1)$ at $r \to 0$. The question arises then, what boundary condition, instead of regularity, is to be imposed on f_{n_0} and g_{n_0} at $r = 3D0$ in the latter case?

To answer this question, one has to find the self-adjoint extension for the partial Dirac operator corresponding to the mode with $n = 3Dn_0$. If this operator is defined on the domain of regular at $r = 3D0$ functions, then it is Hermitian, but not self-adjoint, having the deficiency index equal to (1,1). The use of the Weyl–von Neumann theory of self-adjoint operators (see, e.g., Ref. 8) yields that, for the partial Dirac operator to be self-adjoint extended, it has to be defined on the domain of functions satisfying the boundary condition

$$
i \cos\left(\frac{\theta}{2} + \frac{\pi}{4}\right) 2^{1-F} \, \Gamma(1 - F) \lim_{r \to 0} \left(\mu r\right)^{F} f_{n_0}(r)
$$
$$
= 3D \sin\left(\frac{\theta}{2} + \frac{\pi}{4}\right) 2^{F} \, \Gamma(F) \lim_{r \to 0} \left(\mu r\right)^{1-F} g_{n_0}(r), \tag{13}
$$

where $\Gamma(u)$ is the Euler gamma function,

$$
F = 3D\Phi^{(0)} - n_0 \tag{14}
$$

is the fractional part of quantity $\Phi^{(0)}$ ($0 \leq F < 1$), θ is the self-adjoint extension parameter, and $\mu > 0$ is inserted merely for the dimension reasons. Note that Eq.(13) implies that $0 < F < 1$, since in the case of $F = 3D0$ both f_{n_0} and g_{n_0} satisfy the condition of regularity at $r = 3D0$. Note also that, since Eq.(13) is periodic in θ with period 2π, all permissible values of θ can be restricted, without a loss of generality, to range $0 \leq \theta \leq 2\pi$.

The gauge invariant regularization of $\nabla_\mu J^\mu_{d+1}(x)$ can be achieved by means of the zeta function method [16–18], yielding, instead of Eq.(6), the following relation

$$
\nabla_\mu J^\mu_{d+1}(x) = 3D2 \lim_{z \to 0} \lim_{M \to 0} \tilde{\zeta}_x(z|M), \tag{15}
$$

where

$$
\tilde{\zeta}_x(z|M) = 3D\mathrm{tr} \, \langle x| \Gamma \left\{ \nabla^\mu \nabla_\mu + \frac{i}{2}[\gamma^\mu, \gamma^\nu]_- [\nabla_\mu V_\nu(x)] + M^2 \right\}^{-z} |x\rangle \tag{16}
$$

is the modified zeta function density.

In the $d = 3D2$ case, using the explicit form of the solution to the Dirac equation in background (8), it is straightforward to compute the modified zeta function density. As follows already from the preceding discussion, the modified zeta function density vanishes in the case of integer values of $\Phi^{(0)}$ ($F = 3D0$), since this case is indistinguishable from the case of the trivial background

($\Phi^{(0)} = 3D0$). In the case of noninteger values of $\Phi^{(0)}$ ($0 < F < 1$) we get

$$\tilde{\zeta}_x(z|M) = 3D\frac{\sin(F\pi)}{\pi^3}\sin(z\pi)r^{2(z-1)}\int\limits_{|M|r}^{\infty} dw\, w(w^2 - M^2r^2)^{-z}$$

$$\times\left\{K_F^2(w) - K_{1-F}^2(w) + \left[K_F^2(w) + K_{1-F}^2(w)\right]\right.$$

$$\left.\times\tanh\ln\left[\left(\frac{w}{\mu r}\right)^{2F-1}\cotan\left(\frac{\theta}{2} - \frac{\pi}{4}\right)\right]\right\}. \qquad (17)$$

Taking limit $M \to 0$, we get

$$\tilde{\zeta}_x(z|0) = 3D\frac{\sin(F\pi)}{\pi^3}\sin(z\pi)r^{2(z-1)}$$

$$\times\left\{\frac{\sqrt{\pi}}{2}\frac{\Gamma(1-z)}{\Gamma(\frac{3}{2}-z)}(F - \frac{1}{2})\Gamma(F-z)\Gamma(1-F-z)\right.$$

$$+ \int\limits_0^{\infty} dw\, w^{1-2z}\left[K_F^2(w) + K_{1-F}^2(w)\right]\tanh\ln\left[\left(\frac{w}{\mu r}\right)^{2F-1}\cotan\left(\frac{\theta}{2} - \frac{\pi}{4}\right)\right]\right\}; \quad (18)$$

in particular, at half-integer values of the vortex flux:

$$\tilde{\zeta}_x(z|0)\big|_{F=3D\frac{1}{2}} = 3D\frac{\sin\theta}{2\pi^{\frac{3}{2}}}\frac{\Gamma(\frac{1}{2}-z)}{\Gamma(z)}r^{2(z-1)}; \qquad (19)$$

and at $\cos\theta = 3D0$:

$$\tilde{\zeta}_x(z|0) = 3D \pm \frac{\sin(F\pi)}{2\pi^{\frac{3}{2}}}\frac{\Gamma(\frac{3}{2}-z\pm F\mp\frac{1}{2})\Gamma(\frac{1}{2}-z\mp F\pm\frac{1}{2})}{\Gamma(z)\Gamma(\frac{3}{2}-z)}r^{2(z-1)}, \quad (20)$$

$$\theta = 3D\pi(1\mp\frac{1}{2}).$$

Consequently, we obtain

$$\tilde{\zeta}_x(0|0) = 3D0, \qquad x \neq 0. \qquad (21)$$

Thus the anomaly is absent everywhere on the plane with the puncture at $x = 3D0$. This looks rather natural, since two-dimensional anomaly $2\tilde{\zeta}_x(0|0)$ is usually identified with quantity $\frac{1}{\pi}B(x)$, and background field strength $B(x)$ vanishes everywhere on the punctured plane, see Eq. (9) at $d = 3D2$. We see that natural anticipations are confirmed, provided that the boundary conditions

at the puncture are chosen to be physically acceptable, i.e., compatible with the self-adjointness of the Dirac operator.

We might finish here the discussion of the anomaly problem in the background of the Bohm–Aharonov vortex. However, there remains a purely academic question: what is the anomaly in background (8)–(9) on the whole plane (without puncturing $x = 3D0$)? Just due to a confusion persisting in the literature [19, 20], we shall waste now some time to clarify this, otherwise inessential, point.

Background field strength (9), when considered on the whole plane, is interpreted in the sense of a distribution (generalized function), i.e., a functional on a set of suitable test functions $f(x)$:

$$\int d^2x \, f(x) \, \frac{1}{\pi} B(x) = 3Df(0) \, 2\Phi^{(0)};$$ (22)

here $f(x)$ is a continuous function. In particular, choosing $f(x) = 3D1$, one gets

$$\int d^2x \, \frac{1}{\pi} B(x) = 3D2\Phi^{(0)}.$$ (23)

Considering the anomaly on the whole plane, one is led to study different limiting procedures as $r \to 0$ and $z \to 0$ in Eq.(18). So, the notorious question is, whether anomaly $2\tilde{\zeta}_x$ can be interpreted in the sense of a distribution which coincides with distribution $\frac{1}{\pi} B(x)$? The answer is resolutely negative, and this will be immediately demonstrated below.

First, using explicit form (18), we get

$$\int d^2x \, 2\tilde{\zeta}_x(z|0) = 3D \left\{ \begin{array}{ll} \infty, & z \neq 0 \\ 0, & z = 3D0 \end{array} \right. ;$$ (24)

therefore, the anomaly functional cannot be defined on the same set of test functions as that used in Eq.(22) (for example, the test functions have to decrease rapidly enough at large (small) distances in the case of $z > 0$ ($z < 0$)). Moreover, if one neglects the requirement of self-consistency, permitting a different (more specified) set of test functions for the anomaly functional, then even this will not save the situation. Let us take $z > 0$ for definiteness and use the test functions which are adjusted in such a way that the quantity

$$A = 3D \lim_{z \to 0_+} \int d^2x \, f(x) \, 2\tilde{\zeta}_x(z|0)$$ (25)

is finite. Certainly, this quantity can take values in a rather wide range, but it cannot be made equal to the right-hand side of Eq.(23). Really, the only source of the dependence on $\Phi^{(0)}$ in the integral in Eq.(25) is the factor $\tilde{\zeta}_x(z|0)$, and the latter, as is evident from Eq.(18), depends rather on F, than on $\Phi^{(0)}$ itself, thus

forbidding the linear dependence of A on $\Phi^{(0)}$. In particular, let us choose test function $f(x)$ in the form

$$f(x) = 3D \exp(-\tilde{\mu}^2\, r^2), \qquad (26)$$

where $\tilde{\mu}$ is the parameter of the dimension of mass. Then, choosing the case of $\cos\theta = 3D0$ for simplicity and using Eq.(20), one gets that Eq.(25) takes the form

$$A = 3D2\left(F - \frac{1}{2} \pm \frac{1}{2}\right), \qquad \theta = 3D\pi\left(1 \mp \frac{1}{2}\right), \qquad (27)$$

which differs clearly from $2\Phi^{(0)}$.

We have proved that, in a singular background, the conventional relation between the axial anomaly and the background field strength is valid only in the space with punctured singularities; consequently, wherever the field strength is zero the anomaly always is absent. If singularities are not punctured, then the anomaly and the field strength can be interpreted in the sense of distributions, but, contrary to the assertions of the authors of Refs.19,20, the conventional relation is not valid.

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