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THE APPLICATION OF THE BOGOLIUBOV–MITROPOLSKY METHOD FOR INVESTIGATION OF THE MASSIVE PARTICLE MOTION IN THE GRAVITATIONAL FIELDS OF RADIATING STARS

I.P. Denisova, A.A. Zubrilo

Department of Applied Mathematics and Informatics,
K.E.Tsiolkovsky Moscow State Aviation Technological University, Moscow, Russia

The work demonstrates how fundamental concept of the Bogoliubov–Mitropolslky method can be used to study the laws that govern the particle motion in gravitational field of a radiating star.

Academician N.N.Bogoliubov, an eminent Russian scientist, has developed many of the latest research lines in mathematics, physics, and mechanics. Among his developments, the asymptotic methods [1–4] for integration of nonlinear equations have been used extensively in the nonlinear oscillation theory. We shall demonstrate the techniques for using the fundamental concept of the Bogoliubov–Mitropolsky method in examining the laws that govern the particle motion in the gravitational field of radiating star.

The Schwarzschild solution [5] is normally used at large to study the processes in gravitational fields of stars. But, no matter how high merited the solution is, it describes an extremely idealized situation disregarding the possible star rotation, electric charge, nonspherical distribution and motion of matter, and radiation from the star. Therefore, in line with studying the Schwarzschild solution proper and the characteristic processes therein (particle motion in the field, collapse), attempts were made to solve the Einstein equations for more general astrophysical situations. Since the Einstein equations are the system of the nonlinear partial differential equations, whose right-hand sides comprise the matter energy-momentum tensor (which, in turn, is much affected by gravity), it has become quite clear that the above problems cannot be solved analytically.

At the same time, some of the exact solutions [5] for these equations have been found. Among them, we are most interested in the metrics of a radiating star obtained first by Vaidya [6] in 1949. This solution describes a gravitational field beyond a spherical star with a spherically-symmetric radiation of massless particles. The present-day approaches treat such a radiation to be the electromagnetic radiation and neutrino radiation from the conventional «steady-state» stars and from Supernova outburst.

According to Einstein's equations, the interval for this case has the form:

$$ds^2 = \left(1 - \frac{2M(w)}{r}\right) dw^2 + 2dwdr - r^2[d\theta^2 + \sin^2\theta d\varphi^2], \tag{1}$$

where the system of units, in which $G = c = 1$, is used.

Contrary to the Schwarzschild metrics, the star's mass $M(w)$, entering this expression depends on the Eddington–Finkelstein [5] retarded time w, thus forming the function predestined by the star radiation law: $dM(w)/dw = -f(w)$, where $f(w)$ is the quantity of energy emitted from a star in unit time.

Since the stellar evolution stages are ranging from Supernova explosions (when the radiation from the stars is extremely high) to slow-dying white dwarfs, neutron stars, and black holes, the function $M = M(w)$ for a star is qualitatively of different forms for different stages.

The equations of geodesics for the metric (1) are complicated, in comparison with the equations of geodesics in Schwartzschild's space-time:

$$\frac{du^0}{ds} + r(u^\varphi)^2 - \frac{M}{r^2}(u^0)^2 = 0, \tag{2}$$

$$\frac{du^r}{ds} + \left[\frac{M}{r^2}\left(1 - \frac{2\ M}{r}\right) - \frac{1}{r}\frac{dM}{dw}\right](u^0)^2 + \frac{2M}{r^2}u^0u^r + (2M - r)(u^\varphi)^2 = 0,$$

$$\frac{du^\varphi}{ds} + \frac{2}{r}u^r u^\varphi = 0.$$

All the persistent attempts [7] made throughout the last five decades to integrate the above equations have failed. Therefore, our first step in studying the massive particle motion in the gravitational field of a radiating star is to examine the particular case of steady and moderate radiation from a star, where the function $M = M(w)$ is «slowly» varying function of the time w.

To emphasize a slowness of a modification M we shall enter «slow» time $\tau = \varepsilon w$, where $\varepsilon \ll 1$, and we shall consider that $M = M(\tau)$. In terms of the Bogoliubov–Mitropolsky classification [4], then, the system of differential equations (2) for finite motion of a massive particle coincides with the equations of nonlinear oscillations with slowly-varying parameter $M(\tau)$, (the term «slowly» is used in the sense as indicated by Bogoliubov and Mitropolsky, i.e., the star mass $M(\tau)$ can change but little within the period of orbital particle motion).

The system of equations (2) has two first integrals. Really, as the metric (1) does not depend on an angle φ, then a component u_φ of 4-velocity vector of massive particle should be a constant: $u_\varphi = \alpha = \mathrm{const}$. From here follows, that

$$u^\varphi = \frac{d\varphi}{ds} = g^{\varphi\varphi} u_\varphi = -\frac{\alpha}{r^2}\,. \tag{3}$$

Another integral of a system (2) can be received if to take into account that the quadrate of 4-vector of any massive particle should be equal to unity: $u^i u^k g_{ik} = 1$.

In result we shall have:

$$(u^0)^2 \left(1 - \frac{2M}{r}\right) + 2\ u^0\ u^r = 1 + \frac{\alpha^2}{r^2}\,. \tag{4}$$

Thus, from two remaining equations (2) one by virtue of a relation (4) will be a consequence of another and can be dropped. Remaining independent equation of the system (2) we shall reduce in the most simple form. For it, the radial component of 4-velocity vector shall present as:

$$u^r = \mathcal{E} - \left(1 - \frac{2M}{r}\right) u^0\,, \tag{5}$$

where $\mathcal{E}$ is some new slowly varying unknown function having sense of an energy of particle per unit of its mass.

Then from relations (5) and (4) we shall have:

$$u^0 = \frac{r\left[\mathcal{E} \pm \sqrt{\mathcal{E}^2 - \left(1 - \frac{2M}{r}\right)\left(1 + \frac{\alpha^2}{r^2}\right)}\right]}{(r - 2M)}, \tag{6}$$

$$u^r = \mp\sqrt{\mathcal{E}^2 - \left(1 - \frac{2M}{r}\right)\left(1 + \frac{\alpha^2}{r^2}\right)}.$$

As at the movement to gravitating centre $u^r < 0$, the upper sign in these expressions corresponds to the coordinate's origin, and lower sign — from it.

Substituting a relation (5) in the second equation of the system (2) and using the first equation, it is simple to receive:

$$d\mathcal{E} = -\frac{u^0}{r} dM = -\frac{\left[\mathcal{E} \pm \sqrt{\mathcal{E}^2 - \left(1 - \frac{2M}{r}\right)\left(1 + \frac{\alpha^2}{r^2}\right)}\right]}{(r - 2M)} dM. \tag{7}$$

From here directly follows, that the energy of particle is enlarged ($d\mathcal{E} > 0$) if the star loses a mass on radiation ($dM < 0$), and decreases ($d\mathcal{E} < 0$) if there is an absorption by a star of the radiation which has come from spatial infinity ($dM > 0$).

Using relations (3) and (6), we shall receive a system of the differential equations:

$$\frac{dw}{dr} = \frac{u^0}{u^r} = \mp \frac{r}{(r-2M)} \left\{ \frac{\mathcal{E}(\tau)}{\sqrt{\mathcal{E}^2(\tau) - \left(1 - \frac{2M}{r}\right)\left(1 + \frac{\alpha^2}{r^2}\right)}} \pm 1 \right\}, \tag{8}$$

$$\frac{d\varphi}{dr} = \frac{u^\varphi}{u^r} = \pm \frac{\alpha}{r^2 \sqrt{\mathcal{E}^2(\tau) - \left(1 - \frac{2M}{r}\right)\left(1 + \frac{\alpha^2}{r^2}\right)}},$$

$$\frac{ds}{dr} = \frac{1}{u^r} = \mp \frac{1}{\sqrt{\mathcal{E}^2(\tau) - \left(1 - \frac{2M}{r}\right)\left(1 + \frac{\alpha^2}{r^2}\right)}}.$$

Thus, to get the laws that govern the massive particle motion in the gravitational field of a radiating star, we must integrate the equations (8) making allowance for expression (7).

Without restricting the generality, the initial conditions can be prescribed at the moment $w = 0$, assuming that the massive particle was at the point $r = r_0,\ \varphi = \varphi_0$ just at that moment and had an energy $\mathcal{E} = \mathcal{E}_0$ per its mass unit.

Examine equation (7) and the first equation of system (8). The asymptotic solutions for these two equations will be constructed in terms of the ideology of the Bogoliubov–Mitropolsky method [4, p.339-340] concerning the existence of «fast» and «slow» times. However, the straightforward application of this technique to equations (7) and (8) has not been a success in virtue of the peculiar character of the celestial mechanics problems. Namely, the dominant asymptotic part and the correction thereto cannot be singled out in the radicand because, in the case of finite motion, this expression is strictly vanishing at the trajectory turning points and differs from zero between these points.

Therefore, the asymptotic solutions for the system of equations (7) and (8) will be constructed by iterations. The iteration procedure can readily be realized if the equation

$$\mathcal{E}_0^2 - \left(1 - \frac{2M_0}{r}\right)\left(1 + \frac{\alpha^2}{r^2}\right) = 0 \tag{9}$$

does not have any multiple roots. This condition is met when the initial energy of a particle per the particle mass unit obeys the relation

$$\mathcal{E}_0^2 \neq \frac{36M_0^2\alpha + \alpha^3 \pm \sqrt{(\alpha^2 - 12M_0^2)^3}}{54M_0^2\alpha}.$$

Since the function $M = M(w)$ is slowly varying, the zero-order approximation in small parameter ε of equation (7) yields $\mathcal{E} = \mathcal{E}_0 = \mathrm{const}$.

Substituting this value to the right-hand side of the first of the equations (8), we can find function $w(r)$ in the same approximation:

$$w_{(0)}(r) = \mp \int\limits_{r_0}^{r} \frac{xdx}{[x-2M_0]} \Big\{ \frac{\mathcal{E}_0}{\sqrt{\mathcal{E}_0^2 - \left(1-\frac{2M_0}{x}\right)\left(1+\frac{\alpha^2}{x^2}\right)}} \pm 1 \Big\}.$$

Let's reduce now equation (7) to the form:

$$\frac{d\mathcal{E}}{dr} = \pm \varepsilon r \frac{\left[\mathcal{E}(r) \pm \sqrt{\mathcal{E}^2(r) - \left(1 - \frac{2M\left(\varepsilon w(r)\right)}{r}\right)\left(1+\frac{\alpha^2}{r^2}\right)}\right]^2 \frac{dM\left(\varepsilon w(r)\right)}{d\tau}}{[r-2M\left(\varepsilon w(r)\right)]^2 \sqrt{\mathcal{E}^2(r) - \left(1 - \frac{2M\left(\varepsilon w(r)\right)}{r}\right)\left(1+\frac{\alpha^2}{r^2}\right)}}. \quad (10)$$

For deriving its solution in the following approximation on small parameter ε it is enough to make replacement: $w(r) = w_0(r),\ \mathcal{E}(r) = \mathcal{E}_0,\ M = M_0$ in the right-hand side of this expression. In result we shall have:

$$\mathcal{E}(r)_{(1)} = \mathcal{E}_0 \pm \varepsilon \int\limits_{r_0}^{r} \frac{xdx \left[\mathcal{E}_0 \pm \sqrt{\mathcal{E}_0^2 - \left(1-\frac{2M_0}{x}\right)\left(1+\frac{\alpha^2}{x^2}\right)}\right]^2 \frac{dM\left(\varepsilon w_{(0)}(x)\right)}{d\tau}}{[x-2M_0]^2 \sqrt{\mathcal{E}_0^2 - \left(1-\frac{2M_0}{x}\right)\left(1+\frac{\alpha^2}{x^2}\right)}}.$$

Substituting this expression in the right-hand side of the first of the equations (8), we shall find expression for $w(r) = w_{(1)}(r)$ to within ε, inclusive:

$$w_{(1)}(r) = \mp \int\limits_{r_0}^{r} \frac{xdx \left[\mathcal{E}_{(1)}(x) \pm \sqrt{\mathcal{E}_{(1)}^2(x) - \left(1 - \frac{2M\left(\varepsilon w_0(x)\right)}{x}\right)\left(1+\frac{\alpha^2}{x^2}\right)}\right]}{[x-2M\left(\varepsilon w_0(x)\right)] \sqrt{\mathcal{E}_{(1)}^2(x) - \left(1 - \frac{2M\left(\varepsilon w_0(x)\right)}{x}\right)\left(1+\frac{\alpha^2}{x^2}\right)}}.$$

Sticking to the above iteration procedure, we can construct the expressions for $w(r) = w_{(N)}(r)$ and $\mathcal{E}(r) = \mathcal{E}_{(N)}(r)$, that satisfy the first equation of system (8) and equation (10) up to N-th order in ε. After that, by substituting the resultant expressions $w(r) = w_{(N)}(r)$ and $\mathcal{E}(r) = \mathcal{E}_{(N)}(r)$ in the right-hand sides of the remaining equations of system (8) and by integrating the expressions, we find the trajectory equation $\varphi = \varphi(r)$ and the dependence of the massive particle proper time on r up to ε^N, inclusive:

$$\varphi(r) = \varphi_{(N)}(r) = \varphi_0 \pm \alpha \int\limits_{r_0}^{r} \frac{dx}{x^2 \sqrt{\mathcal{E}_{(N)}^2(x) - \left(1 - \frac{2M\left(\varepsilon w_{(N)}(x)\right)}{x}\right)\left(1+\frac{\alpha^2}{x^2}\right)}},$$

$$s(r) = s_{(N)}(r) = \mp \int\limits_{r_0}^{r} \frac{dx}{\sqrt{\mathcal{E}^2_{(N)}(x) - \left(1 - \frac{2M\left(\varepsilon w_{(N)}(x)\right)}{x}\right)\left(1 + \frac{\alpha^2}{x^2}\right)}}.$$

The above expressions get applicable up to the trajectory turning points, i.e., up to the point $r = r_{\mathrm{rev}}$, that solves equation (9). To extend the trajectory to beyond this point, the resultant values $\mathcal{E}(r_{\mathrm{rev}}),\ w(r_{\mathrm{rev}}),\ \varphi(r_{\mathrm{rev}})$, and $s(r_{\mathrm{rev}})$ must be taken to be the initial conditions, whereupon the above iteration procedure is to be repeated to construct the asymptotic solution for a next trajectory branch.

Thus, the Bogoliubov–Mitropolsky method makes it possible to construct a massive particle trajectory in the gravitational field of a radiating star and to study the law that governs the massive particle motion along the found trajectory.

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