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ON p-ADIC PATH INTEGRAL

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Dedicated to the memory of N.N.Bogoliubov

Feynman's path integral is generalized to quantum mechanics on p-adic space and time. Such p-adic path integral is analytically evaluated for quadratic Lagrangians. Obtained result has the same form as that one in ordinary quantum mechanics.

1. It is well known that dynamical evolution of any one-dimensional quantum-mechanical system, described by a wave function $\Psi(x,t)$, is given by

$$\Psi(x'',t'') = \int \mathcal{K}(x'',t'';x',t')\Psi(x',t')dx', \tag{1}$$

where $\mathcal{K}(x'',t'';x',t')$ is the kernel of the corresponding unitary operator acting as follows:

$$\Psi(t'') = \mathcal{U}(t'',t')\Psi(t'). \tag{2}$$

$\mathcal{K}(x'',t'';x',t')$ is also called Green's function, or the quantum-mechanical propagator, and the probability amplitude to go a particle from a point (x',t') to a point (x'',t''). One can easily deduce the following three general properties:

$$\int \mathcal{K}(x'',t'';x,t)\mathcal{K}(x,t;x',t')dx = \mathcal{K}(x'',t'';x't'), \tag{3}$$

$$\int \mathcal{K}^*(x'',t'';x',t')\mathcal{K}(x'',t'';x,t')dx'' = \delta(x'-x), \tag{4}$$

$$\mathcal{K}(x'',t'';x',t'') = lim_{t'\to t''}\mathcal{K}(x'',t'';x',t') = \delta(x''-x'). \tag{5}$$

Since all information on quantum dynamics can be deduced from the propagator $\mathcal{K}(x'',t'';x',t')$ it can be regarded as the basic ingredient of quantum theory. In Feynman's formulation [1] of quantum mechanics, $\mathcal{K}(x'',t'';x',t')$ was postulated to be the path integral

$$\mathcal{K}(x'',t'';x',t') = \int \exp\left(\frac{2\pi i}{h}\int_{t'}^{t''} L(\dot q,q,t)dt\right)\mathcal{D}q, \tag{6}$$

where $x'' = q(t'')$ and $x' = q(t')$, and h is the Planck constant.

In its original form, the path integral (6) is the limit of the corresponding multiple integral of $n-1$ variables $q_i = q(t_i)$, $(i = 1, 2, ..., n-1)$, when $n \to \infty$. For the half of century of its history, the path integral has been a subject of permanent interest in theoretical and mathematical physics. At present days (see, e.g., [2]) it is one of the most profound and promising approaches to foundations of quantum theory (in particular, quantum field theory and superstring theory). Feynman's path integral is inevitable in formulation of p-adic [3] and adelic [4] quantum mechanics. It is worth noting that just Feynman's path integral approach enables natural foundation of quantum theory on p-adic and adelic spaces.

2. Recall that the set of rational numbers Q plays an important role in mathematics as well as in physics. From algebraic point of view, Q is the simplest number field of characteristic 0. The usual absolute value and p-adic valuation (p is any of prime numbers) exhaust all possible nontrivial norms on Q [5]. Completion of Q with respect to metrics induced by these norms leads to the field of real numbers R and the fields of p-adic numbers Q_p, $(p = 2, 3, 5, ...)$. Thus Q is dense in R and all Q_p. From physical point of view, all numerical results of measurements are rational numbers. However, theoretical models of physical systems are traditionally constructed using real and complex numbers. One can ask the following question: Why real (and complex) numbers are so good in description of usual physical phenomena, and, is there any aspect of physical reality which has to be described by p-adic numbers. Construction of p-adic models and their appropriate interpretation can gradually give answer to this question. Since 1987, there have been many publications (for a review, see, e.g., [6–8]) on possible applications of p-adic numbers in modern theoretical and mathematical physics. For a systematic approach to this subject, p-adic [3] and adelic [4] quantum mechanics have been formulated.

Recall also that any p-adic number $x \in Q_p$ can be presented as the following infinite expansion

$$x = p^\nu (x_0 + x_1 p + x_2 p^2 + \cdots), \quad \nu \in Z,$$

where $x_i = 0, 1, ..., p-1$ are digits. We will use the Gauss integral [7]

$$\int_{Q_p} \chi_p(\alpha x^2 + \beta x) dx = \lambda_p(\alpha) \mid 2\alpha \mid_p^{-\frac{1}{2}} \chi_p\left(-\frac{\beta^2}{4\alpha}\right), \quad \alpha \neq 0,$$

where $\chi_p(a) = \exp(2\pi i \{a\}_p)$ is the additive character, and $\{a\}_p$ is the fractional part of $a \in Q_p$. $\lambda_p(x)$ is a complex-valued arithmetic function (for a definition, see [7]) with the following properties:

$$\lambda_p(0) = 1, \ \lambda_p(a^2 x) = \lambda_p(x),$$

$$\lambda_p(x)\lambda_p(y) = \lambda_p(x+y)\lambda_p(x^{-1}+y^{-1}), \ \lambda_p^*(x)\lambda_p(x) = 1.$$

3. p-Adic quantum mechanics, we are interested in, contains complex-valued functions of p-adic arguments. There is not the corresponding Schrödinger equation, but Feynman's path integral approach seems to be quite natural. Feynman's path integral for p-adic propagator $\mathcal{K}_p(x'', t''; x', t')$, where $\mathcal{K}_p$ is complex-valued and x'', x', t'', t' are p-adic variables, is a direct p-adic generalization of (6), i.e.,

$$\mathcal{K}_p(x'', t''; x', t') = \int \chi_p \left(-\frac{1}{h} \int_{t'}^{t''} L(\dot{q}, q, t) dt \right) \mathcal{D}q, \tag{7}$$

where $\chi_p(a)$ is p-adic additive character. The Planck constant h in (6) and (7) is the same rational number. Integral $\int_{t'}^{t''} L(\dot{q}, q, t)dt$ we consider as the difference of antiderivative (without pseudoconstants) of $L(\dot{q}, q, t)$ in final (t'') and initial (t') times. $\mathcal{D}q = \prod_{i=1}^{n-1} dq(t_i)$, where $dq(t_i)$ is the p-adic Haar measure. Thus, p-adic path integral is the limit of the multiple Haar integral when $n \to \infty$. To calculate (7) in this way one has to introduce some order on $t \in Q_p$, and it is successfully done in Ref. 9. On previous investigations of p-adic path integral one can see [10]. Our main task here is derivation of the exact result for p-adic Feynman's path integral (7) for the general case of Lagrangians $L(\dot{q}, q, t)$, which are quadratic polynomials in $\dot{q}$ and q, without making time discretization.

A general quadratic Lagrangian can be written as follows:

$$L(\dot{q}, q, t) = \frac{1}{2} \frac{\partial^2 L_0}{\partial \dot{q}^2} \dot{q}^2 + \frac{\partial L_0}{\partial \dot{q}} \dot{q} + \frac{\partial^2 L_0}{\partial \dot{q} \partial q} \dot{q} q + L_0 + \frac{\partial L_0}{\partial q} q + \frac{1}{2} \frac{\partial^2 L_0}{\partial q^2} q^2, \tag{8}$$

where index 0 denotes that the Taylor expansion of $L(\dot{q}, q, t)$ is around $\dot{q} = q = 0$. The Euler–Lagrange equation of motion is

$$\frac{\partial^2 L_0}{\partial \dot{q}^2} \ddot{q} + \frac{d}{dt} \left(\frac{\partial^2 L_0}{\partial \dot{q}^2} \right) \dot{q} + \left[\frac{d}{dt} \left(\frac{\partial^2 L_0}{\partial \dot{q} \partial q} \right) - \frac{\partial^2 L_0}{\partial q^2} \right] q = \frac{\partial L_0}{\partial q} - \frac{d}{dt} \left(\frac{\partial L_0}{\partial \dot{q}} \right). \tag{9}$$

General solution of (9) is

$$q \equiv x(t) = C_1 x_1(t) + C_2 x_2(t) + w(t), \tag{10}$$

where $x_1(t)$ and $x_2(t)$ are two linearly independent solutions of the corresponding homogeneous equation, and $w(t)$ is a particular solution of the complete equation (9). Note that $x(t)$ denotes the classical trajectory. Imposing the boundary conditions $x' = x(t')$ and $x'' = x(t'')$, constants of integration C_1 and C_2 become:

$$C_1 \equiv C_1(t'', t') = \frac{(x' - w')x_2'' - (x'' - w'')x_2'}{x_2'' x_1' - x_1'' x_2'}, \tag{11a}$$

$$C_2 \equiv C_2(t'', t') = \frac{(x'' - w'')x_1' - (x' - w')x_1''}{x_2'' x_1' - x_1'' x_2'}. \tag{11b}$$

Since $C_1(t'',t')$ and $C_2(t'',t')$ are linear in x'' and x', the corresponding classical action $\bar{S}(x'',t'';x',t') = \int_{t'}^{t''} L(\dot{x},x,t)dt$ is quadratic in x'' and x'. Note that the above expressions have the same form in R and Q_p.

Quantum fluctuations lead to deviations of classical trajectory and any quantum path may be presented as $q(t) = x(t) + y(t)$, where $y' = y(t') = 0$ and $y'' = y(t'') = 0$. The corresponding Taylor expansion of $S[q]$ around classical path $x(t)$ is

$$S[q] = S[x+y] = S[x] + \frac{1}{2!}\delta^2 S[x] = S[x] + \frac{1}{2}\int_{t'}^{t''}\left(\dot{y}\frac{\partial}{\partial\dot{q}} + y\frac{\partial}{\partial q}\right)^{(2)} L(\dot{q},q,t)dt, \tag{12}$$

where we used $\delta S[x] = 0$. We have now

$$\mathcal{K}_p(x'',t'';x',t') =$$

$$= \chi_p\left(-\frac{1}{h}S[x]\right)\int \chi_p\left(-\frac{1}{2h}\int_{t'}^{t''}\left(\dot{y}\frac{\partial}{\partial\dot{q}} + y\frac{\partial}{\partial q}\right)^{(2)} L(\dot{q},q,t)dt\right)\mathcal{D}y \tag{13}$$

with $y'' = y' = 0$ and $S[x] = \bar{S}(x'',t'';x',t')$.

Note that $\mathcal{K}_p(x'',t'';x',t')$ has the form

$$\mathcal{K}_p(x'',t'';x',t') = N_p(t'',t')\chi_p\left(-\frac{1}{h}\bar{S}(x'',t'';x',t')\right), \tag{14}$$

where $N_p(t'',t')$ does not depend on end points x'' and x'. To calculate $N_p(t'',t')$ we use conditions (3) and (4). Substituting $\mathcal{K}_p(x'',t'';x',t')$ into (4) we obtain (for details, see [10]):

$$N_p(t'',t') = \left|\frac{1}{h}\frac{\partial^2\bar{S}}{\partial x''\partial x'}(x'',t'';x',t')\right|_p^{\frac{1}{2}} A_p(t'',t'), \tag{15}$$

where $|A_p(t'',t')|_\infty = 1$, ($|\cdot|_p$ and $|\cdot|_\infty$ denote p-adic and absolute value, respectively). Replacing (15) in equation (3) we get conditions:

$$A_p(t'',t)A_p(t,t')\lambda_p(\alpha) = A_p(t'',t'), \tag{16}$$

$$\left|\frac{1}{h}\frac{\partial^2\bar{S}}{\partial x''\partial x}(x'',t'';x,t)\right|_p^{\frac{1}{2}}\left|\frac{1}{h}\frac{\partial^2\bar{S}}{\partial x\partial x'}(x,t;x',t')\right|_p^{\frac{1}{2}}|2\alpha|_p^{-\frac{1}{2}} =$$

$$= \left|\frac{1}{h}\frac{\partial^2\bar{S}}{\partial x''\partial x'}(x'',t'';x',t')\right|_p^{\frac{1}{2}}, \tag{17}$$

where

$$\alpha = -\frac{1}{2h}\left[\frac{\partial^2 \bar{S}}{\partial x^2}(x'', t''; x, t) + \frac{\partial^2 \bar{S}}{\partial x^2}(x, t; x', t')\right]. \tag{18}$$

Analyzing the above formulae we obtain [10]

$$A_p(t'', t') = \lambda_p\left(-\frac{1}{2h}\frac{\partial^2 \bar{S}}{\partial x'' \partial x'}(x'', t''; x', t')\right). \tag{19}$$

For details of a quite rigorous derivation of (19), see [11].

As the final result we have

$$\mathcal{K}_p(x'', t''; x', t') = \lambda_p\left(-\frac{1}{2h}\frac{\partial^2 \bar{S}}{\partial x'' \partial x'}\right)\left|\frac{1}{h}\frac{\partial^2 \bar{S}}{\partial x'' \partial x'}\right|_p^{\frac{1}{2}} \chi_p\left(-\frac{1}{h}\bar{S}(x'', t''; x', t')\right) \tag{20}$$

which is the p-adic Feynman path integral for quadratic Lagrangians. The corresponding path integral of ordinary quantum mechanics [10] can be transformed into the same form as (20), i.e., in such case index p is replaced by index ∞. This supports Volovich's conjecture [12] that fundamental physical laws should be invariant under interchange of number fields Q_p and R.

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