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NEW PERTURBATION THEORY WITH CONVERGENT SERIES: CALCULATIONS WITH ARBITRARY VALUES OF COUPLING CONSTANTS

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A new method which allows one to calculate the physical quantities represented by a finite number of terms of their asymptotic PT-expansion is developed. The method is applicable for arbitrary values of coupling constant. In the framework of this method several examples including β-function in scalar $\varphi^4_{(4)}$ field theory are considered and the critical exponent α for the phase transition of He^4 is calculated accurately.

It is widely adopted now that power series in quantum field theory diverge and are nothing but asymptotic expansions in the small range of coupling constant in the vicinity of zero [1].

We propose a quite different approach to approximative calculation of physical quantities represented by the traditional perturbation theory (PT) series. This approach is a direct generalization of the method developed in a number of our recent papers [2–7].

Let us consider the class of alternating series of the form

$$\sum_{n=0}^{\infty} f_n g^n \ . \tag{1}$$

The following three properties describe this class:

(A) *The coefficients f_n satisfy the condition*

$$|f_n| \sim C n! a^n n^b = \tilde{f}_n \ , \tag{2}$$

where a, b, C are some positive constants.

(B) *The series* (1) *is the Taylor expansion at the point of $g = 0$. The function is infinitely differentiable at $g \geq 0$ and has analytical continuation to the right half-plane $\{\mathrm{Re} g > 0\}$. Also it satisfies the following estimate uniformly in $g \in \{\mathrm{Re} g > 0\}$*

$$\left| f(g) - \sum_{n=0}^{N-1} f_n g^n \right| < C_1^N N! N^{\alpha} g^N . \tag{3}$$

Under these conditions the function $f(g)$ is uniquely defined by the series (1).

(C) *Represent* $f(g)$ *in the form*

$$f(g) = \int_0^\infty e^{-gt}\, F(t)\, dt \ , \tag{4}$$

where $F(g)$ *is a distribution. Let us demand the validity of the following inequality*

$$\int\limits_0^\infty x^n\, |F(x)|\, dx \ \leq \ const \left| \int\limits_0^\infty x^n\, F(x)\, dx \right| \ , \quad n \geq 0 \, . \tag{5}$$

It should be noted that the inequality (5) puts rather tough conditions on the series to be considered. In particular, in such a series one cannot change any finite number of terms because (5) does not take place for polynomials.

Rewrite (4) in the following form

$$f(g) = \lim_{R\to\infty} f(g,R) = \lim_{R\to\infty} \int_{-R}^{+R} \tilde{\varphi}_m(\rho) \left(\int_0^\infty \cos\{\rho[gt]^{1/2m}\}\, F(t)\, dt \right) d\rho \ , \tag{6}$$

where

$$\tilde{\varphi}_m(\rho) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \exp\{-r^{2m}\} e^{-i\rho r}\, dr \tag{7}$$

with some integer m.

The regularized function $f(g,R)$ can be expanded into the absolutely convergent series of the form

$$f(g,R) = \sum_{n=0}^\infty \frac{1}{(2n)!} g^{\frac{k}{m}} A_{2n}(m,R) B_{\frac{k}{m}}, \tag{8}$$

where we denote

$$A_{2n}(m,R) = \frac{(-1)^n}{2\pi} \int_{-\infty}^{+\infty} \rho^{2n} \tilde{\varphi}_m(\rho)\, d\rho$$

and

$$B_\alpha = \int_0^\infty t^\alpha\, F(t)\, dt \ .$$

Our method allows one to deal with large g as well. Instead of (6) by the substitution $\rho \to \rho g^{-1/2m}$ we obtain

$$f(g) = \lim_{R\to\infty} f(g,R) =$$

$$= g^{-\frac{1}{2m}} \lim_{R\to\infty} \int_{-R}^{+R} \tilde{\varphi}_m(\rho g^{-\frac{1}{2m}}) \left(\int_0^{\infty} \cos\{\rho t^{1/2m}\} \, F(t)\, dt \right) d\rho \ , \qquad (9)$$

which also can be expanded into the convergent series in inverse powers of g:

$$f(g,R) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} g^{-\frac{2n+1}{2m}} E_{2n}(m,R,\infty) \ , \qquad (10)$$

$$E_{2n}(m,R,N) = \frac{1}{2\pi m} \sum_{k=0}^{N} \frac{(-1)^k}{(2k)!} \Gamma\left(\frac{2n+1}{2m}\right) \frac{2R^{2n+2k+1}}{2n+2k+1} B_{\frac{k}{m}} \ .$$

Now one can approximate the sum $f(g,R)$ of the absolutely convergent series (8) or (10) by some finite number of its terms. In case of large g, the conclusive approximant takes the following form (for the m given):

$$f(g,R,N) = \sum_{n=0}^{mN} \frac{1}{(2n)!} g^{-\frac{2n+1}{2m}} E_{2n}(m,R,mN) \ . \qquad (11)$$

The coefficients B_α with integer indices α are nothing but the coefficients of the traditional perturbation theory. For the coefficients B_α with noninteger α we have worked out new methods of evaluation [7].

We describe here one of these methods for the case $\alpha = k/4$, where k is an integer. Consider the function $\chi(\alpha) = \ln B_\alpha$ and suppose that $\chi^{(n)}(\alpha) < 1$. It is easy to see, that

$$\chi(\alpha_n) \approx \frac{9}{16}\left[\chi(\alpha_n + \frac{1}{8}) + \chi(\alpha_n - \frac{1}{8})\right] - \frac{1}{16}\left[\chi(\alpha_n + \frac{3}{8}) + \chi(\alpha_n - \frac{3}{8})\right] \ ,$$

hence

$$B_{\frac{1}{4}(n-\frac{1}{2})} \approx \left(B_{\frac{n-1}{4}}\right)^{9/16} \left(B_{\frac{n}{4}}\right)^{9/16} \left(B_{\frac{n-2}{4}}\right)^{-1/16} \left(B_{\frac{n+1}{4}}\right)^{-1/16} \ . \qquad (12)$$

Also we have

$$B_{\frac{1}{4}(n-\frac{1}{2})} = C(n) \int_0^{\infty} \left[\int_0^{\infty} \left(\cos(\xi x^{\frac{1}{8}}) - \sum_{k=0}^{n-1} \frac{(-1)^k}{(2k)!} \xi^{2k} x^{\frac{k}{4}} \right) F(x) dx \right] \xi^{-2n} \, d\xi \ , \qquad (13)$$

where

$$C(n) = \frac{(-1)^n \, 2\, (2n-1)!}{\pi} \ .$$

The integral (11) can be approximated by the sum

$$B_{\frac{1}{4}\left(n-\frac{1}{2}\right)} \approx C(n) \sum_{k=0}^{4N-1} \frac{(-1)^k}{(2k)!} \frac{a^{2k-2n+1}}{2k-2n+1} B_{\frac{k}{4}} \ , \tag{14}$$

where a is a free parameter the accuracy of the approximation depends on. From (12) and (14) we obtain the system of equations for the approximate evaluation of B_α with noninteger indices.

With this method the coefficients $B_{k/4}$ for $k = 5, 6, 7, 9, 10, 11, 13$ were calculated in case of the «zero-dimensional» functional integral (see first model example below), where they can be calculated explicitly:

$$B_{\frac{k}{m}} = \Gamma\left(\frac{4k+2}{2m}\right) \ .$$

Comparison of the results of the calculations with our method is given in the Table. The first column here is for the exact values given by the gamma-function and the second one is for those calculated by the method exposed above. We get a very good coinsidence in this case.

Table

k	$B_{k/4}$	$\tilde{B}_{k/4}$	$\frac{\lvert B_{k/4}-\tilde{B}_{k/4}\rvert}{B_{k/4}}$
5	2	1.957	0.0214
6	3.323	3.281	0.01269
7	6	5.89	0.0184
9	24	24.494	0.0206
10	52.343	54.175	0.035
11	120	123.804	0.0317
13	720	693.432	0.0369

Another key point of the method is how to pick the sufficient value of parameter R. The following statement can be proved: For fixed interval of the parameter $g: \ 0 < g < g_0$ and fixed N, there exists a continuous function $R^*(g)$ such that

$$f(g, R^*(g), N) = f(g).$$

If we take a segment $\Delta_R = [R_1, R_2]$ containing the point $R^*(g)$ inside, we can fix any $R \in \Delta_R$ and approximate the function $f(g)$ by the $f(g, R, N)$. In this case the error of the approximation is

$$\delta_f = \left| \max_{\Delta_R} f(g, R, N) - \min_{\Delta_R} f(g, R, N) \right| \ .$$

Now the constructive idea is the following: since R is the cut-off parameter, then there should be a finite segment Δ_R for any given ε and N fixed such that $|f(g,R,N)-f(g)| \le \varepsilon$ uniformly in $R \in \Delta_R$. The boundary values of R on the segment Δ_R are stipulated by regularization features and the number of traditional PT terms available. The calculations performed for several model examples demostrate that such a segment does exist and for diverse cases turns out to be somewhere inside the interval $[5,7]$. As the value of the function $f(g)$ one can take now the mean value:

$$\bar{f} = \frac{1}{2}\left(\max_{\Delta_R} f(g,R,N) + \min_{\Delta_R} f(g,R,N)\right).$$

Thereby the function $f(g)$ is calculated with the accuracy

$$\delta_f = \frac{1}{2}\left[\max_{\Delta_R} f(g,R,N) - \min_{\Delta_R} f(g,R,N)\right].$$

We have verified the method developed above for a number of model examples, where the exact solutions are known, either analytically or numerically, and the coefficients of the PT exhibit factorial growth. Thus, the «zero-dimensional» version of functional integral (see Fig. 1; curve I is the exact solution, curve II is our approximant and curve III is obtained by naïf summation of the first five PT terms)

$$I(g) = \int_{-\infty}^{+\infty} e^{-x^2-gx^4}\,dx = \exp\left\{\frac{1}{8g}\right\} K_{\frac{1}{4}}\left(\frac{1}{8g}\right)\frac{1}{\sqrt{4g}}$$

has the PT coefficients

$$I(g) = \sum_{n=0}^{\infty} a_k(-g)^k, \quad a_k = \frac{1}{(k)!}\frac{\sqrt{\pi}}{2^{4k}}\frac{(4k)!}{(2k)!}.$$

The problem of determining the electron energy levels in the Coulomb field of a nucleus with $Z > 137$ supplies another model example (see Fig. 2). Here a divergent power series appears [10]

$$I(g) = \sum_{k=1}^{\infty} a_k(-g)^k, \quad a_k = \frac{1}{2\pi k}\left[(-1)^k B_{2k} + \frac{\Gamma(k-1/2)}{\Gamma(k)\Gamma(-1/2)} - 1\right],$$

where B_{2k} are Bernulli numbers. On the other hand, for the $I(g)$ one can analytically obtain

$$I(g) = -\frac{1}{\pi}\left\{\psi(g^{-1/2}) + \frac{1}{2}\left[\ln\frac{g}{1+g} + 1 + g^{1/2} - (1+g)^{1/2}\right]\right\},$$

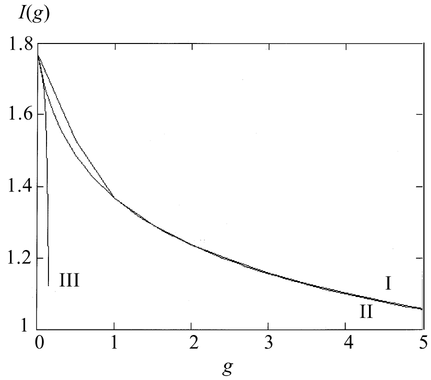

Fig. 1.

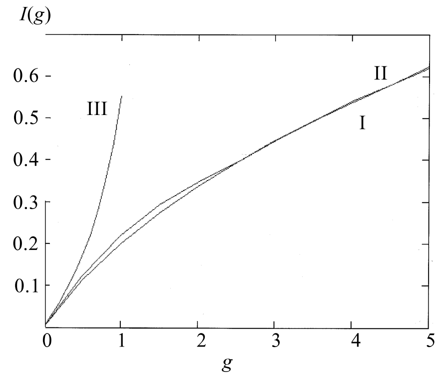

Fig. 2.

where $\psi(x) = \Gamma'(x)/\Gamma(x)$. In this case, it is more appropriate to consider $\exp\{-I(g)\}$ instead of $I(g)$. The coefficients $B_{k/m}$ of our series are

$$B_{1/4} = 0.7436; \quad B_{2/4} = 0.5432; \quad B_{3/4} = 0.3867; \; B_{5/4} = 0.2675;$$

$$B_{6/4} = 0.27; \quad B_{7/4} = 0.2725; \quad B_{9/4} = 0.3346; \quad B_{10/4} = 0.4024;$$

$$B_{11/4} = 0.4795; \quad B_{13/4} = 0.7672; \quad B_{14/4} = 1.022; \quad B_{15/4} = 1.346;$$

$$B_{17/4} = 2.51; \quad B_{18/4} = 3.577; \quad B_{19/4} = 5.123.$$

The result is shown in Fig. 2, where curves are the same as in Fig. 1.

For these examples rather good coincidence with the exact solution in a wide range of the coupling constants has been obtained.

With our method, we have investigated the behavior in the region $0 < g < 20$ of the β-function in the scalar $\varphi^4_{(4)}$ field theory, those first five terms are calculated in the MS-scheme [8]:

$$\beta^{MS} = \frac{5}{3}g^2 - \frac{10}{3}g^3 + 20.043g^4 - 175.257g^5 + 1922.33g^6 \; .$$

It is more convenient to consider the function $f(g) = \exp\{-\beta(g)/g\}$. Calculating the coefficients $B_{k/4}$ in the way outlined above, one obtains

$$B_{1/4} = 1.0; \quad B_{1/2} = 1.1; \quad B_{3/4} = 1.31; \quad B_{5/4} = 2.273;$$

$$B_{3/2} = 3.380; \quad B_{7/4} = 5.445; \quad B_{9/4} = 17.532; \quad B_{5/2} = 34.609;$$

$$B_{11/4} = 72.21; \quad B_{13/4} = 361.8; \quad B_{7/2} = 858.3; \quad B_{15/4} = 2097.9;$$

$$B_{17/4} = 13526; \quad B_{9/2} = 3601 \cdot 10; \quad B_{19/4} = 988 \cdot 10^2$$

(see [7] for more details).

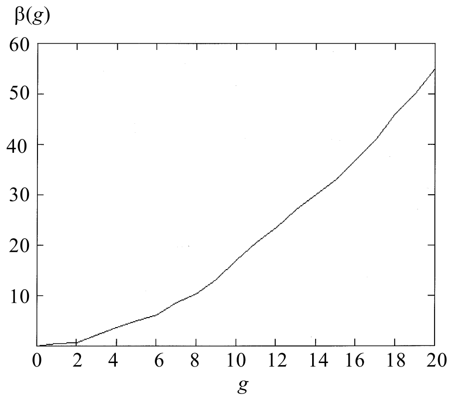

Fig. 3.

We have considered two cases, namely when four ($N = 4$) and five ($N = 5$) terms of traditional PT are taken as known. In our approach, the approximants are very close to each other (see Fig. 3) and both have the error less than 0.1. It should be stressed here that as we consider the β-function to be satisfying the conditions formulated above (1)–(5), the error we obtain is the *absolute* one. So the *true* β-function lies within the closed interval for all $g \in [0, 20]$. This is the crucial point which distinguishes our method principally from others, where one obtains only *relative* error.

The method was also applied for calculation of the critical exponent α for phase transition of He^4. The value of α being calculated by our method shows rather good coincidence with the experimental one, obtained very precisely in experiments [11]

$$\alpha = -0.0129 \pm 0.0008, \qquad \alpha_{exp} = -0.0128 \pm 0.0004 \; .$$

It should be noted that the authors of [12] also got the rather good result using the seven-loop term. Thus, the new approach developed above could be considered as a next step after the traditional PT calculations. It allows one to use more comprehensively the information obtained with the traditional PT theory and to find the numerical values with very high accuracy for arbitrary values of coupling constant.

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