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COULOMB SYMMETRY AND QUANTUM SPACE

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From the point of view of Bogoliubov axiomatic approach change of the geometry of the momentum space leads to the modification of extension of the S-matrix off the mass shell [1,2] as compared with the standard procedure when the geometry of p-space is Euclidean, i.e., to the different dynamical description [7]. Snyder quantum space (QS) which will be considered here is based on a modification of the geometry of momentum space. We can think that some background interaction exists, which modifies the geometry of the momentum space. We refer the reader to review articles [5,7,8,9] for further references including ones to original papers by H.Snyder, I.E.Tamm, W.Pauli, Y.C.Ningh, Yu.A.Golfand, V.G.Kadyshevsky and others. In turn changing the geometry of momentum space naturally suggests quantum configurational space concept, because boosts generalizing translations of p-space don't commute. The explicit character of Snyder's approach to space-time quantization has a remarkable consequence: we can define the spectrum of a commutative set of operators constructed from $\hat{x}_\mu$ and other generators of isometry group of the momentum space. As has been shown in [7], the formulation of the generalized causality condition and QFT in terms of the points of this quantum space-time is as comprehensive as it is in the usual QFT with the Minkovskian space-time. In this approach the structure of the singular field-theoretic functions is entirely reconstructed as compared to the standard QFT, and the corresponding perturbation theory is free of ultraviolet divergences.

It was noticed long ago [5] that Fock theory of Hydrogen atom in momentum space of constant curvature can be considered as the nonrelativistic version of Snyder QS. Following V.A.Fock [3] we consider the non-Euclidean geometry of momentum space. The Coulomb field performs the role of the background interaction mentioned above, which provides the non-Euclidean geometry of momentum space. The modified shifts of the last (which are up to some similarity transformation the Runge–Lenz vector's components) can be considered as non-relativistic analogues of Snyder's coordinates. The present work is a further development of [5] based on modern ideas of noncommutative differential geometry and noncommutative differential calculus [11–16].

For the continuous part of the energy spectrum it is pseudo-Euclidean 3-dimensional space of negative curvature (Lobachevsky space). Introducing 4-dimensional projective momenta P_μ $(\mu = 0, 1, 2, 3)$ we have

$$\vec{P} = \frac{2q\,\vec{p}}{\vec{p}^{\,2} - q^2}, \quad P_0 = \frac{\vec{p}^{\,2} + q^2}{\vec{p}^{\,2} - q^2}, \quad q = \sqrt{2\mu E}, \quad P_0^2 - \vec{P}^{\,2} = 1. \tag{1}$$

Equation (1) describes the two-sheet hyperboloid (the upper sheet corresponds to $1 \le P_0 < +\infty$, the lower one to $-\infty < P_0 \le -1$). It is convenient to use the hyperspherical coordinates $\vec{P} = \sinh\alpha\ \vec{n}$ $\;P_0 = \pm\cosh\alpha$, $\vec{n} = (\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta)$. The Schrödinger equation in momentum space is manifestly invariant under the isometry group of the Lobachevsky momentum space (1) which is Lorentz group. Generators of Lorentz group boosts

$$\hat{x}_i = -i\left[P_0\frac{\partial}{\partial P^i} - P_i\frac{\partial}{\partial P^0}\right] \tag{2}$$

up to some similarity transformation coincide with the additional integrals of motion of the Coulomb problem, i.e., Runge–Lenz invariants. From the other side their similarity to Snyder coordinates [8] is evident.

As it was shown by V.A.Fock [3] the solutions of the Schrödinger equation in momentum space are the eigen-functions of the Laplace–Beltrami operator on the Lobachevsky space (1), or the Casimir operator of the Lorentz group:

$$\left(\vec{x}^{\,2} - \frac{1}{\hbar^2}\vec{L}^{\,2}\right)\Phi_r(P) = \left(1 + \frac{r^2}{a^2}\right)\Phi_r(P), \tag{3}$$

where $\vec{L}$ is the vector of angular momentum operators. Atomic units of length (Bohr radius), and energy, momentum are correspondingly $a = \hbar^2/\mu e^2$, $e_a = \mu e^4/\hbar^2$, $\pi_a = \mu e^2/\hbar$. The solutions of (3) are the matrix elements of unitary irreducible (infinite dimensional) representations of the Lorentz group. For the principal series of unitary representations of Lorentz group the parameter r runs over the interval $0 \le r < \infty$ which coincides with the physically admitted region of variation. Let us consider quantities

$$\Phi_r(P) = <\vec{r}|\vec{P}> = \left|P_0 - \vec{P}\vec{n}\right|^{-1-i\frac{r}{a}}, \quad \vec{r} = r\,\vec{n}, \quad \vec{n}^{\,2} = 1. \tag{4}$$

The expression (4) from the one side is the solution of the equation (3), from the other side it is the generating function for the radial solutions of the Schrödinger equation in the momentum space. The expression (4) plays the role of the plane wave in quantum r-space. For plane waves (4) the completeness and orthogonality conditions are valid:

$$\frac{1}{(2\pi)^3}\int <\vec{r}|\vec{P}><\vec{P}|\vec{r'}>\, d\Omega_P = \delta\left(\vec{r}-\vec{r'}\right), \tag{5}$$

$$\frac{1}{(2\pi)^3}\int <\vec{P}|\vec{r}><\vec{r}|\vec{P'}>\, d^3r = \delta\left(\vec{P}-\vec{P'}\right) P_0.$$

The plane wave in quantum space obeys the following equations off the energy shell, i.e., for $E_p \neq E_q$ or $p \neq q$:

$$\hat{H}_0<\vec{r}|\vec{P}>= E_P <\vec{r}|\vec{P}>, \tag{6}$$

where

$$\hat{H}_0=\hat{p}_0= e_a\left[\cosh\left(ia\frac{\partial}{\partial r}\right)+\frac{ia}{r}\sinh\left(ia\frac{\partial}{\partial r}\right)-\frac{a^2\Delta_{\theta,\phi}}{r^2}e^{ia\frac{\partial}{\partial r}}-1\right] \tag{7}$$

and $E_P = e_a\left(|P_0|-1\right) = 2e_a\sinh^2(\alpha/2)$. This is differential-difference Schrödinger equation describing a free motion of a particle in QS. Another strong argument for the idea that plane wave (4) and equation (7) describe the free motion in the quantum r-space is the existence of three more differential-difference operators $\hat{p}_i$ for which (4) is the eigenfunction with eigenvalues equal to the momentum components

$$\hat{p}_i<\vec{r}|\vec{P}>= P_i <\vec{r}|\vec{P}>, \tag{8}$$

where

$$\hat{p}_1= \pi_a\left\{\sin\theta\cos\phi\left(e^{ia\frac{\partial}{\partial r}}-\hat{H}_0\right)-i\frac{a}{r}\left(\cos\theta\cos\phi\frac{\partial}{\partial\theta}-\frac{\sin\phi}{\sin\theta}\frac{\partial}{\partial\phi}\right)e^{ia\frac{\partial}{\partial r}}\right\},$$

$$\hat{p}_2= \pi_a\left\{\sin\theta\sin\phi\left(e^{ia\frac{\partial}{\partial r}}-\hat{H}_0\right)-i\frac{a}{r}\left(\cos\theta\sin\phi\frac{\partial}{\partial\theta}+\frac{\cos\phi}{\sin\theta}\frac{\partial}{\partial\phi}\right)e^{ia\frac{\partial}{\partial r}}\right\},$$

$$\hat{p}_3= \pi_a\left\{-\cos\theta\left(e^{ia\frac{\partial}{\partial r}}-\hat{H}_0\right)+i\frac{a}{r}\sin\theta\frac{\partial}{\partial\theta}e^{ia\frac{\partial}{\partial r}}\right\}. \tag{9}$$

Generators of Lorentz group are also deformed by typical finite-difference operators. For example

$$\hat{M}_{12}=\hat{r}_1\hat{p}_2-\hat{r}_2\hat{p}_1= -i\hbar\frac{\partial}{\partial\phi}e^{ia\frac{\partial}{\partial r}}. \tag{10}$$

Let us show that the finite-difference Schrödinger equation (6) is naturally described in terms of noncommutative differential calculus [11–16]. This calculus can be most easily introduced on a ground of the theory of differential forms as its deformation. Our case corresponds to the differential calculus over an associative algebra A over $\mathbb{R}$ or $\mathbb{C}$. The necessity to consider an algebra over $\mathbb{C}$ follows from the form of finite-difference Schrödinger equation, containing shifts by the imaginary quantity ia. This is general property of finite-difference Schrödinger equation (6) corresponding to the continuous part of the spectrum of hydrogen atom, requiring to consider the wave functions in the complex r-plane. Finite linear combinations of elements of A and finite products are again elements of A. The multiplication is associative. A differential calculus on A is a $\mathbb{Z}$-graded associative algebra over $\mathbb{C}$

$$\Omega\left(A\right)=\sum_{r=0}\oplus\,\Omega^{r}\left(A\right), \tag{11}$$

$$\Omega^{0}\left(A\right)=A, \quad \Omega^{r}\left(A\right)=\{0\}\,\forall r<0. \tag{12}$$

The elements of $\Omega^{r}\left(A\right)$ are called r-forms. There exist an exterior derivative operator d which satisfies conditions

$$d^{2}=0 \tag{13}$$

and

$$d\left(\omega\omega'\right)=\left(d\omega\right)\omega'+\left(-1\right)^{r}\omega d\omega', \tag{14}$$

where ω and ω' are r- and r'-forms, respectively. A is the commutative algebra generated by the coordinate functions. Correspondingly we introduce right and left (see ([8])) δ-operations

$$\overrightarrow{\delta}=\overrightarrow{*}\ \ d\ \ \overrightarrow{*}, \qquad \overleftarrow{\delta}=\overleftarrow{*}\ \ d\ \ \overleftarrow{*}\,. \tag{15}$$

Let us concentrate on the one-dimensional case. It is generated by canonical coordinate function of one variable $\psi(r)=r$. One of the simplest deformations of the ordinary differential calculus on A is

$$[dr,\, r]=\frac{ia}{2}\ dr. \tag{16}$$

Equation (16) can be generalized to the total algebra A as

$$dr\ \psi(r)=\psi\left(r+\frac{ia}{2}\right)dr. \tag{17}$$

Then we can introduce the generalized derivatives (left and right) corresponding to our deformed differential calculus. For the left derivative we write

$$d\psi(r) = \left(\overrightarrow{\partial}\ \psi(r)\right) dr = dr \left(\overleftarrow{\partial}\ \psi(r)\right). \tag{18}$$

From Leibniz rule we have

$$\begin{gathered} d\left(\psi(r)\ \varphi(r)\right) = dr \left(\overrightarrow{\partial}\ \left(\psi(r)\ \varphi(r)\right)\right) = \left(d\psi(r)\right)\ \varphi(r) + \psi(r)\ \left(d\varphi(r)\right) = \\ = dr\ \left(\overrightarrow{\partial}\ \psi(r)\right)\ \varphi(r) + \psi(r)\ dr\ \left(\overrightarrow{\partial}\ \varphi(r)\right) \end{gathered} \tag{19}$$

after using (17)

$$d\left(\psi(r)\ \varphi(r)\right) = dr\ \left(\overrightarrow{\partial}\ \psi(r)\right) \varphi(r) + dr\ \psi(r + \frac{ia}{2}) \left(\overrightarrow{\partial}\ \varphi(r)\right). \tag{20}$$

Now from the commutativity of functions $\psi(r)\,\varphi(r) = \varphi(r)\,\psi(r)$ it follows also that equivalent of the Leibniz rule for the left derivative is valid:

$$\begin{gathered} d\left(f(r)\ g(r)\right) = dr\ \left(\overrightarrow{\partial}\ \left(f(r)\ g(r)\right)\right) = \left(dg(r)\right)\ f(r) + g(r)\ \left(df(r)\right) = \\ = dr\ \left(\overrightarrow{\partial}\ g(r)\right)\ f(r) + dr\ g(r + \tfrac{ia}{2})\ \left(\overrightarrow{\partial}\ f(r)\right). \end{gathered} \tag{21}$$

Then the following expressions for the left and right partial derivatives can be established (cf. [8])

$$\overrightarrow{\partial}\ \psi(r) = \frac{\psi(r + \frac{ia}{2}) - \psi(r)}{\frac{ia}{2}},\ \ \overleftarrow{\partial}\ \psi(r) = \frac{\psi(r) - \psi(r - \frac{ia}{2})}{\frac{ia}{2}}. \tag{22}$$

Now let us write down the Hamiltonian operator (7) and the angular momentum operator (10) in terms of noncommutative differential calculus. First we introduce left and right deformation operators in the form

$$\overrightarrow{q} = e^{\overrightarrow{s}} = \left(1 + \frac{ia}{2} \overrightarrow{*}\ d\right) = e^{\frac{ia}{2}\frac{\partial}{\partial r}},\quad \overleftarrow{q} = e^{\overleftarrow{s}} = \left(1 - \frac{ia}{2} \overleftarrow{*}\ d\right) = e^{-\frac{ia}{2}\frac{\partial}{\partial r}}. \tag{23}$$

Hamiltonian operator now can be written in the form

$$\hat{H}_0 = \frac{1}{2}\left(\overrightarrow{H}_0 + \overleftarrow{H}_0\right), \tag{24}$$

where

$$\overrightarrow{H}_0 = \left(h + \frac{a^2 \Delta_{\theta,\varphi}}{r^2} \right) e^{2\overrightarrow{s}}, \quad \overleftarrow{H}_0 = h^* e^{2\overleftarrow{s}}, \quad h = 1 + \frac{ia}{r}. \tag{25}$$

The rotation generator (10) can be written in the form

$$\hat{M}_{12} = \left(-i\hbar \frac{\partial}{\partial \phi} \right) \overrightarrow{q}^{\,2}. \tag{26}$$

Operator $\hat{M}_{12}$ transfers simply into the standard generator $-i\hbar\partial/\partial\phi$ in the limit $r/a \to \infty$, which corresponds to great values of impact parameter. The Poincare Lie algebra is also deformed by typical factors $e^{2\overrightarrow{s}}$, $e^{2\overleftarrow{s}}$. For example

$$\left[\hat{M}_{12}, \hat{p}_1 \right]_{\overrightarrow{q}^{\,2}} = i\hbar \hat{p}_2, \tag{27}$$

where

$$\left[\hat{A}, \hat{B} \right]_{\overrightarrow{q}} = \hat{A} \; e^{\overrightarrow{s}} \; \hat{B} - \hat{B} \; e^{\overrightarrow{s}} \; \hat{A}. \tag{28}$$

From the usual point of view the interaction term $V(r)$ when introduced into the differential–difference Schrödinger equation (6) corresponds to the perturbed Coulomb potential. Let us consider an example of integrable case for the Schrödinger equation with interaction. We write the ladder operators

$$a^{\pm} = \mp \frac{i}{\sqrt{2}\pi_a \cos \frac{r}{2a}} e^{\pm \frac{1}{2} \left(\frac{r}{\lambda_0} \right)^2} \hat{p} \; e^{\mp \frac{1}{2} \left(\frac{r}{\lambda_0} \right)^2}, \tag{29}$$

where $\hat{p} = \frac{1}{2} \left(\overrightarrow{p} + \overleftarrow{p} \right)$ is the non-commutative differential operator of radial momentum introduced in [8], ω is the frequency, λ_0 is a parameter of dimension of length: $\lambda_0 = \sqrt{\hbar/\mu\omega}$. The ladder operators (29) obey the deformed commutation relation

$$\left[a^-, a^+ \right]_q = q a^- a^+ - q^{-1} a^+ a^- = 2 \left(q^{-1} - q \right), \tag{30}$$

q is a dimensionless quantity, parameter of deformation, which is expressed in terms of physical parameters:

$$q = \exp \left\{ -\frac{a^2}{4\lambda_0^2} \right\} = \exp \left\{ -\frac{\hbar\omega}{4e_a} \right\} = \exp \left\{ -\frac{\omega\hbar^3}{4\mu e^4} \right\}. \tag{31}$$

The energy spectrum is

$$E_n = 2e_a \left\{ \exp \left(\frac{\hbar\omega}{2e_a} (n + 1/2) \right) - \cosh \frac{\hbar\omega}{4e_a} \right\}. \tag{32}$$

This integrable case can be easily identified with the well known q-oscillator.

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