REMARKS TOWARDS THE SPECTRUM OF THE HEISENBERG SPIN CHAIN TYPE MODELS

Č. Burdík [1,*], *J. Fuksa* [1,2,**], *A. P. Isaev* [2,***],
S. O. Krivonos [2,****], *O. Navrátil* [3,*****]

[1] Department of Mathematics, Faculty of Nuclear Sciences and Physical Engineering,
Czech Technical University in Prague, Prague
[2] Joint Institute for Nuclear Research, Dubna
[3] Department of Mathematics, Faculty of Transportation Sciences,
Czech Technical University in Prague, Prague

*E-mail: burdices@kmlinux.fjfi.cvut.cz
**E-mail: fuksa@theor.jinr.ru
***E-mail: isaevap@theor.jinr.ru
****E-mail: krivonos@theor.jinr.ru
*****E-mail: navratil@fd.cvut.cz

ФИЗИКА ЭЛЕМЕНТАРНЫХ ЧАСТИЦ И АТОМНОГО ЯДРА
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REMARKS TOWARDS THE SPECTRUM OF THE HEISENBERG SPIN CHAIN TYPE MODELS

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[1] Department of Mathematics, Faculty of Nuclear Sciences and Physical Engineering, Czech Technical University in Prague, Prague
[2] Joint Institute for Nuclear Research, Dubna
[3] Department of Mathematics, Faculty of Transportation Sciences, Czech Technical University in Prague, Prague

The integrable closed and open chain models can be formulated in terms of generators of the Hecke algebras. In this review paper, we describe in detail the Bethe ansatz for the integrable XXX and XXZ closed chain models. We find the Bethe vectors for two-component and inhomogeneous models. We also find the Bethe vectors for the fermionic realization of the integrable XXX and XXZ closed chain models by means of the algebraic and coordinate Bethe ansatzes. Special modification of the XXZ closed spin chain model ("small polaron model") is considered. Finally, we discuss some questions relating to the general open Hecke chain models.

Интегрируемые модели открытых и замкнутых цепочек могут быть сформулированы в терминах генераторов алгебр Гекке. В данной обзорной статье мы описываем в деталях анзац Бете для интегрируемых, замкнутых XXX- и XXZ-моделей. Построены векторы Бете для двухкомпонентных и неоднородных моделей. С использованием алгебраического и координатного анзацев Бете найдены векторы Бете для фермионных реализаций интегрируемых XXX- и XXZ-моделей. Обсуждаются также свойства общих открытых цепочек Гекке.

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1. INTRODUCTION

A braid group $\mathcal{B}_L$ in the Artin presentation is generated by invertible elements T_i $(i = 1, \ldots, L-1)$ subject to the relations:

$$T_i\, T_{i+1}\, T_i = T_{i+1} T_i T_{i+1}, \quad T_i T_j = T_j T_i \quad \text{for} \quad i \neq j \pm 1. \tag{1.1}$$

*E-mail: burdices@kmlinux.fjfi.cvut.cz
**E-mail: fuksa@theor.jinr.ru
***E-mail: isaevap@theor.jinr.ru
****E-mail: krivonos@theor.jinr.ru
*****E-mail: navratil@fd.cvut.cz

The A-type Hecke algebra $H_L(q)$ (see, e.g., [1]) is a quotient of the group algebra of $\mathcal{B}_L$ by additional Hecke relations

$$T_i^2 = (q - q^{-1})T_i + 1 \quad (i = 1, \ldots, L-1), \tag{1.2}$$

where q is the parameter (deformation parameter). Let x be the parameter (spectral parameter) and we define elements

$$T_k(x) := x^{-1/2}T_k - x^{1/2}T_k^{-1} \in H_n(q), \tag{1.3}$$

which are called baxterized elements. By using (1.1) and (1.2), one can check that the baxterized elements (1.3) satisfy the Yang–Baxter equation in the braid group form

$$T_k(x)T_{k+1}(xy)T_k(y) = T_{k+1}(y)T_k(xy)T_{k+1}(x) \tag{1.4}$$

and

$$T_k(x)T_k(y) = (q - q^{-1})T_k(xy) + (x^{-1/2} - x^{1/2})(y^{-1/2} - y^{1/2}). \tag{1.5}$$

Equations (1.4) and (1.5) are baxterized analogs of the first relation in (1.1) and the Hecke condition (1.2).

The Hamiltonian of the open Hecke chain model of the length L is

$$\mathcal{H}_L = \sum_{k=1}^{L-1} T_k \in H_L(q) \tag{1.6}$$

(see, e.g., [2] and references therein). Any representation ρ of the Hecke algebra gives an integrable open spin chain with the Hamiltonian $\rho(\mathcal{H}_L) = \sum\limits_{k=1}^{L-1} \rho(T_k)$. Define the closed Hecke algebra $\hat{H}_L(q)$ by adding additional generator T_L to the set $\{T_1, \ldots, T_{L-1}\}$ such that T_L satisfies the same relations (1.1) and (1.2) for any i and $j \neq i \pm 1$, where we have to use the periodic condition $T_{L+k} = T_k$. Then the closed Hecke chain of the length L is described by the Hamiltonian $\mathcal{H}_L = \sum\limits_{k=1}^{L} T_k \in \hat{H}_L(q)$ and any representation ρ of $\hat{H}_L(q)$ leads to the integrable closed spin chain with the Hamiltonian

$$\rho(\mathcal{H}_L) = \sum_{k=1}^{L} \rho(T_k). \tag{1.7}$$

In Secs. 2–4, special representations $\rho = \rho_R$ of the algebra $H_L(q)$, called the R-matrix representations, are considered. In the case of $GL_q(2)$-type R-matrix representation ρ_R, the Hamiltonian (1.7) coincides with the XXZ spin

chain Hamiltonian. It is clear that in the case of $q = 1$ we recover the XXX spin chain. The integrable structures for XXX spin chain are introduced in Subsec. 2.1. We discuss some results of the algebraic Bethe ansatz for these models. In Sec. 3, we formulate the so-called two-component model (see [3, 4] and references therein). The two-component model was introduced to avoid problems with computation of correlation functions for local operators attached to some site x of the chain. Using this approach, we obtain in Sec. 4 the explicit formulas for the Bethe vectors, which show the equivalence of the algebraic and coordinate Bethe ansatzes.

In Sec. 5, we generalize the results of Secs. 2–4 to the case of inhomogeneous XXX spin chain.

The realization of the XXX spin chains in terms of free fermions is considered in Secs. 6–8. Here we explicitly construct the Bethe vectors for XXX spin chains in the sectors of one, two, and three magnons. In Sec. 9, we discuss another special representation ρ of the Hecke algebra $H_L(q)$ which we call the fermionic representation. In this representation, the Hamiltonian (1.7) describes the so-called "small polaron model" (see [5] and references therein). In Secs. 10 and 11, we construct the Bethe vectors and obtain the Bethe ansatz equations for the "small polaron model" and for the XXZ closed spin chains by means of the coordinate Bethe ansatz and compare the results with those obtained by means of the algebraic Bethe ansatz in Sec. 2. We show that the Hamiltonian of the "small polaron model" has the different spectrum compared with the XXZ model in the sector of an even number of magnons.

Finally, in Sec. 12, we discuss the general open Hecke chain models which are formulated in terms of the elements of the Hecke algebra $H_n(q)$. We present the characteristic polynomials (in the case of the finite length of the chain) which define the spectrum of the Hamiltonian of this model in some special irreducible representations of $H_n(q)$. The method of construction of irreducible representations of the algebra $H_n(q)$ is formulated at the end of Sec. 12.

In Appendix, we give some details of our calculations.

2. ALGEBRAIC BETHE ANSATZ

At the beginning, we describe some basic features of the agebraic Bethe ansatz. The method was formulated as a part of the quantum inverse scattering method proposed by Faddeev, Sklyanin, and Takhtadjan [6, 7]. The main object of this method is the Yang–Baxter algebra generated by matrix elements of the monodromy matrix. The main rules for the Yang–Baxter algebra were elaborated in the very first papers [9–11]. Many quantum integrable systems were described in terms of this method, cf. [13–15]. We strongly recommend the review paper [8] for introductory reading and [12] for a more detailed review.

2.1. L-Operator and Transfer Matrix for XXX Spin Chain. Suppose we have a chain of L sites. The local Hilbert space h_j corresponds to the jth site. For our purposes, it is sufficient to suppose $h_j = \mathbb{C}^2$. The total Hilbert space of the chain is

$$\mathscr{H} = \prod_{j=1}^{L} \otimes h_j. \tag{2.1}$$

The basic tool of the algebraic Bethe ansatz is the Lax operator. For its definition, we need an auxiliary vector space $V_a = \mathbb{C}^2$. The Lax operator is a parameter-dependent object acting on the tensor product $V_a \otimes h_i$,

$$L_{a,i}: \quad V_a \otimes h_i \to V_a \otimes h_i, \tag{2.2}$$

explicitly defined as

$$L_{a,i}(\lambda) = \left(\lambda + \frac{1}{2}\right) \mathbb{I}_{a,i} + \sum_{\alpha=1}^{3} \sigma_a^\alpha S_i^\alpha, \tag{2.3}$$

where $S_i^\alpha = (1/2)\sigma_i^\alpha$ is the spin operator on the ith site, $\sigma_a^\alpha = (\sigma_a^x, \sigma_a^y, \sigma_a^z)$ are Pauli sigma-matrices which act in the space V_a (σ_i^α are Pauli sigma-matrices which act in the space h_i), and $\mathbb{I}_{a,i}$ is the identity matrix in $V_a \otimes h_i$. Operator $L_{a,i}(\lambda)$ can be expressed as a matrix in the auxiliary space

$$L_{a,i}(\lambda) = \begin{pmatrix} \lambda + \dfrac{1}{2} + S_i^z & S_i^- \\ S_i^+ & \lambda + \dfrac{1}{2} - S_i^z \end{pmatrix}. \tag{2.4}$$

Its matrix elements form an associative algebra of local operators in the quantum space h_i.

Introducing the permutation operator P,

$$P = \frac{1}{2}\left(\mathbb{I} \otimes \mathbb{I} + \sum_{\alpha=1}^{3} \sigma^\alpha \otimes \sigma^\alpha\right), \tag{2.5}$$

(here $\mathbb{I}$ denotes a 2×2 unit matrix), we can rewrite the Lax operator as

$$L_{a,i}(\lambda) = \lambda \mathbb{I}_{a,i} + P_{a,i}. \tag{2.6}$$

Assume two Lax operators $L_{a,i}(\lambda)$ resp. $L_{b,i}(\mu)$ in the same quantum space h_i but in different auxiliary spaces V_a resp. V_b. The product of $L_{a,i}(\lambda)$ and $L_{b,i}(\mu)$ makes sense in the tensor product $V_a \otimes V_b \otimes h_i$. It turns out that there

is an operator $R_{ab}(\lambda-\mu)$ acting nontrivially in $V_a \otimes V_b$ such that the following equality holds:

$$R_{ab}(\lambda-\mu)L_{a,i}(\lambda)L_{b,i}(\mu) = L_{b,i}(\mu)L_{a,i}(\lambda)R_{ab}(\lambda-\mu). \tag{2.7}$$

Relation (2.7) is called the fundamental commutation relation. The explicit expression for $R_{ab}(\lambda-\mu)$ is

$$R_{ab}(\lambda-\mu) = (\lambda-\mu)\,\mathbb{I}_{a,b} + P_{a,b}, \tag{2.8}$$

where $\mathbb{I}_{a,b}$ resp. $P_{a,b}$ is identity resp. permutation operator in $V_a \otimes V_b$. In the matrix form we get for $R_{ab}(\lambda-\mu)$

$$R_{ab}(\lambda-\mu) = \begin{pmatrix} \lambda-\mu+1 & 0 & 0 & 0 \\ 0 & \lambda-\mu & 1 & 0 \\ 0 & 1 & \lambda-\mu & 0 \\ 0 & 0 & 0 & \lambda-\mu+1 \end{pmatrix}. \tag{2.9}$$

The operator $R_{ab}(\lambda-\mu)$ is called the R-matrix. It satisfies the Yang–Baxter equation

$$R_{ab}(\lambda-\mu)R_{ac}(\lambda)R_{bc}(\mu) = R_{bc}(\mu)R_{ac}(\lambda)R_{ab}(\lambda-\mu) \tag{2.10}$$

in $V_a \otimes V_b \otimes V_c$.

Comparing (2.6) and (2.8), we see that the Lax operator and the R-matrix are the same.

We define a monodromy matrix

$$T_a(\lambda) = L_{a,1}(\lambda)L_{a,2}(\lambda)\cdots L_{a,L}(\lambda) \tag{2.11}$$

as a product of the Lax operators along the chain, i.e., over all quantum spaces h_i. As a matrix in the auxiliary space V_a, the monodromy matrix

$$T_a(\lambda) = \begin{pmatrix} A(\lambda) & B(\lambda) \\ C(\lambda) & D(\lambda) \end{pmatrix} \tag{2.12}$$

defines an algebra of global operators $A(\lambda), B(\lambda), C(\lambda), D(\lambda)$ on the Hilbert space $\mathscr{H}$. It is called the Yang–Baxter algebra. The monodromy matrix $T_a(\lambda)$ is a step from local observables S_i^α on h_i to global observables on $\mathscr{H}$.

The trace

$$\tau(\lambda) \equiv \mathrm{Tr}_a T_a(\lambda) = A(\lambda) + D(\lambda) \tag{2.13}$$

of $T_a(\lambda)$ in the auxiliary space V_a is called the transfer matrix. It constitutes a generating function for commutative conserved charges. Assume the Lax

operators $L_a = L_{a,i}(\lambda), L_b = L_{b,i}(\mu), L'_a = L_{a,i+1}(\lambda), L'_b = L_{b,i+1}(\mu)$, and $R_{ab} = R_{ab}(\lambda - \mu)$ in the tensor product $V_a \otimes V_b \otimes \mathscr{H}$, then

$$R_{ab}L_aL'_aL_bL'_b = R_{ab}L_aL_bL'_aL'_b = L_bL_aR_{ab}L'_aL'_b = \\ = L_bL_aL'_bL'_aR_{ab} = L_bL'_bL_aL'_aR_{ab}. \quad (2.14)$$

Here we used (2.10) and the fact that operators acting nontrivially in different vector spaces commute. Hence, we can deduce commutation relations between the elements of the monodromy matrix

$$R_{ab}(\lambda - \mu)T_a(\lambda)T_b(\mu) = T_b(\mu)T_a(\lambda)R_{ab}(\lambda - \mu). \quad (2.15)$$

Equation (2.15) is a consequence of (2.7) for global observables $A(\lambda), B(\lambda), C(\lambda)$, and $D(\lambda)$. We call it the global fundamental commutation relation. The commutativity of transfer matrices obviously follows from (2.15). After multiplying (2.15) by $R^{-1}_{ab}(\lambda - \mu)$ we get

$$T_a(\lambda)T_b(\mu) = R^{-1}_{ab}(\lambda - \mu)T_b(\mu)T_a(\lambda)R_{ab}(\lambda - \mu). \quad (2.16)$$

Taking the trace over auxiliary spaces V_a and V_b, we obtain

$$\tau(\lambda)\tau(\mu) = \tau(\mu)\tau(\lambda). \quad (2.17)$$

Obviously, the monodromy matrix (2.11) is a polynomial of degree L with respect to the parameter λ

$$T_a(\lambda) = \left(\lambda + \frac{1}{2}\right)^L \mathbb{I} + \left(\lambda + \frac{1}{2}\right)^{L-1} \sum_{i=1}^{L} \sum_{\alpha=1}^{3} \sigma_a^\alpha \otimes S_i^\alpha + \dots \quad (2.18)$$

Therefore, the transfer matrix $\tau(\lambda)$ is also a polynomial of degree L

$$\tau(\lambda) = 2\left(\lambda + \frac{1}{2}\right)^L + \sum_{k=0}^{L-2} \lambda^k Q_k. \quad (2.19)$$

The term of order λ^{L-1} vanishes because Pauli matrices are traceless. Due to commutativity (2.17) of transfer matrices also

$$[Q_j, Q_k] = 0. \quad (2.20)$$

We see that the transfer matrix is a generating function for a set of commuting observables.

The Hamiltonian of the system appears naturally amongst the observables Q_k. From the definition of the Lax operator (2.3) we see that

$$L_{a,i}(-1/2) = P_{a,i}. \quad (2.21)$$

Hence,

$$T_a(-1/2) = P_{a,1}P_{a,2}\cdots P_{a,L} = P_{L-1,L}\cdots P_{2,3}P_{1,2}P_{a,1}. \tag{2.22}$$

If we differentiate $T_a(\lambda)$ with respect to λ, we get

$$\frac{dT_a(\lambda)}{d\lambda}\Big|_{\lambda=-1/2} = \sum_{k=1}^{L} P_{a,1}\cdots \underbrace{P_{a,k}}_{\text{missing}} \cdots P_{a,L} = \\ = \sum_{k=1}^{L} P_{L-1,L}\cdots P_{k-1,k+1}\cdots P_{1,2}P_{a,1}. \tag{2.23}$$

Remind that $\operatorname{Tr}_a P_{a,j} = \mathbb{I}_j$. Differentiating the logarithm of the transfer matrix we get:

$$\frac{d}{d\lambda}\ln\,\tau(\lambda)\Big|_{\lambda=-1/2} = \frac{d\tau(\lambda)}{d\lambda}\tau^{-1}(\lambda)\Big|_{\lambda=-1/2} = \tag{2.24}$$

$$= \sum_{k=1}^{L}(P_{L-1,L}\cdots P_{k-1,k+1}\cdots P_{1,2})(P_{1,2}P_{2,3}\cdots P_{L-1,L}) = \sum_{k=1}^{L} P_{k,k+1}. \tag{2.25}$$

The Hamiltonian of the system is

$$H = \sum_{k=1}^{L}\sum_{\alpha=1}^{3} S_k^\alpha S_{k+1}^\alpha = \frac{1}{2}\sum_{k=1}^{L} P_{k,k+1} - \frac{L}{4}, \tag{2.26}$$

where we set $S_{n+L} = S_n$ resp. $P_{L,L+1} = P_{L,1}$. We can see that

$$H = \frac{1}{2}\frac{d}{d\lambda}\ln\tau(\lambda)\Big|_{\lambda=-1/2} - \frac{L}{4}. \tag{2.27}$$

This is the reason why we can say that the transfer matrix $\tau(\lambda)$ is a generating function for commuting conserved charges.

Remark. Let S_i^α be the generators of the Lie algebra $su(2)$ in ith site

$$[S_i^\alpha, S_j^\beta] = i\varepsilon^{\alpha\beta\gamma}S_i^\gamma\,\delta_{ij}, \tag{2.28}$$

and we take generators S_i^α in any representation of $su(2)$ which acts in the space h_i. Then equations (2.3) and (2.4) define L-operator for the integrable chain model with arbitrary spin in each site. Relations (2.7) with R-matrix (2.8) are equivalent to the defining relations (2.28). Formulas (2.11), (2.12), (2.13), (2.17), (2.18), (2.19), and (2.20) are valid for this generalized spin chain models as well.

2.2. Some Remarks on the XXZ Chain. The fundamental R-matrix for the quantum group $GL_q(N)$ is [16, 17]

$$\hat{R} = q \sum_{i=1}^{N} e_{ii} \otimes e_{ii} + \sum_{i \neq j} e_{ij} \otimes e_{ji} + (q - q^{-1}) \sum_{i<j} e_{ii} \otimes e_{jj}, \tag{2.29}$$

where e_{ij} is the $N \times N$ matrix unity. In a particular case of $GL_q(2)$, the R-matrix (2.29) can be written in terms of Pauli matrices

$$\hat{R} = \frac{1}{2}\left(\sigma^x \otimes \sigma^x + \sigma^y \otimes \sigma^y + \frac{q + q^{-1}}{2}\sigma^z \otimes \sigma^z\right) + \\ + \frac{q - q^{-1}}{4}\left(\sigma^z \otimes \mathbb{I}_2 - \mathbb{I}_2 \otimes \sigma^z\right) + \frac{3q - q^{-1}}{4}\mathbb{I}_2 \otimes \mathbb{I}_2. \tag{2.30}$$

Here and below we use notation $\mathbb{I}_N$ for the $N \times N$ unit matrix. The fundamental R-matrix (2.29) satisfies the Hecke condition (1.2)

$$\hat{R}^2 = (q - q^{-1})\hat{R} + \mathbb{I}_N \otimes \mathbb{I}_N. \tag{2.31}$$

If we define

$$\hat{R}^{(q)}_{kk+1} = \mathbb{I}_N^{\otimes(k-1)} \otimes \hat{R} \otimes \mathbb{I}_N^{\otimes(L-k-1)}, \tag{2.32}$$

we obtain the R-matrix representation ρ_R of the Hecke algebra (1.1), (1.2)

$$\rho_R \,:\, T_k \to \hat{R}^{(q)}_{kk+1}. \tag{2.33}$$

Then, the baxterized R-matrix is (see Eq. (1.3))

$$\hat{R}_{kk+1}(\mu) = \rho_R\left(\mu^{-1/2}T_k - \mu^{1/2}T_k^{-1}\right) = \mu^{-1/2}\hat{R}^{(q)}_{kk+1} - \mu^{1/2}(\hat{R}^{(q)}_{kk+1})^{-1} = \\ = (\mu^{-1/2} - \mu^{1/2})\hat{R}^{(q)}_{kk+1} + \mu^{1/2}(q - q^{-1}). \tag{2.34}$$

This R-matrix is a solution of the Yang–Baxter equation in the braid group form

$$\hat{R}_{kk+1}(\lambda)\hat{R}_{k+1k+2}(\lambda \cdot \mu)\hat{R}_{kk+1}(\mu) = \hat{R}_{k+1k+2}(\mu)\hat{R}_{kk+1}(\lambda \cdot \mu)\hat{R}_{k+1k+2}(\lambda). \tag{2.35}$$

Note that if there is a solution of equation (2.35), the solution of the equation

$$R_{k,k+1}(\lambda)R_{k,k+2}(\lambda\mu)R_{k+1,k+2}(\mu) = R_{k+1,k+2}(\mu)R_{k,k+2}(\lambda\mu)R_{k,k+1}(\lambda) \tag{2.36}$$

can be easily found as

$$R_{k,k+1}(\lambda) = \hat{R}_{k,k+1}(\lambda)P_{k,k+1}. \tag{2.37}$$

The R-matrix $R_{k,k+2}(\lambda)$ has to be defined as $P_{k+1,k+2}R_{k,k+1}(\lambda)P_{k+1,k+2}$. The validity of (2.36) is very important for correct definition of the transfer matrix. We are able to define the Lax operator as the R-matrix

$$L_{a,i}(\lambda) = R_{a,i}(\lambda) \tag{2.38}$$

and the monodromy matrix in the form (2.11). Commutativity of the transfer matrix is just a matter of proving

$$R_{ab}(\mu)T_a(\lambda\mu)T_b(\lambda) = T_b(\lambda)T_a(\lambda\mu)R_{ab}(\mu). \tag{2.39}$$

The R-matrices (2.29), (2.34) for $N = 2$ are the basic building blocks for the XXZ spin chain. Let us write (2.37) for $N = 2$ as follows:

$$R_{kk+1}(\lambda) = \mathbb{I}_2^{\otimes(k-1)} \otimes R(\lambda) \otimes \mathbb{I}_2^{\otimes(L-k-1)}, \tag{2.40}$$

where $R(\lambda) = (\lambda^{-1/2}\hat{R} - \lambda^{1/2}\hat{R}^{-1})P$, the matrix $\hat{R}$ is given in (2.30), and $R(\lambda)$ has the matrix form

$$R(\lambda) = \begin{pmatrix} \lambda^{-1/2}q - \lambda^{1/2}q^{-1} & 0 & 0 & 0 \\ 0 & \lambda^{-1/2} - \lambda^{1/2} & \lambda^{-1/2}(q - q^{-1}) & 0 \\ 0 & \lambda^{1/2}(q - q^{-1}) & \lambda^{-1/2} - \lambda^{1/2} & 0 \\ 0 & 0 & 0 & \lambda^{-1/2}q - \lambda^{1/2}q^{-1} \end{pmatrix} \tag{2.41}$$

which is important to write the commutation relations (2.39) in components. We see that the form (2.41) of the R-matrix is not symmetric to transposition, as usually appears in literature, cf. [8, 4], etc. We use the Drinfel'd–Reshetikhin twist to symmetrize it, cf. [18].

The R-matrix $R_{ab}(\lambda)$ acts in the tensor product of the auxiliary spaces $V_a \otimes V_b$. The monodromy matrix $T_a(\lambda)$ acts in $V_a \otimes \mathscr{H}$. Let U be a diagonal matrix. It can be easily seen that $[U \otimes \mathbb{I}_b + \mathbb{I}_a \otimes U, R_{ab}(\lambda)] = 0$. We introduce the twisted R-matrix resp. the monodromy matrix

$$\tilde{R}_{ab}(\lambda) = (\lambda^U \otimes \mathbb{I}_b)R_{ab}(\lambda)(\lambda^{-U} \otimes \mathbb{I}_b), \tag{2.42}$$

$$\tilde{T}_a(\lambda) = (\lambda^U \otimes \mathbb{I}_{\mathscr{H}})T_a(\lambda)(\lambda^{-U} \otimes \mathbb{I}_{\mathscr{H}}). \tag{2.43}$$

If

$$R_{ab}(\lambda)T_a(\lambda\mu)T_b(\mu) = T_b(\mu)T_a(\lambda\mu)R_{ab}(\lambda) \tag{2.44}$$

is satisfied, then also

$$\tilde{R}_{ab}(\lambda)\tilde{T}_a(\lambda\mu)\tilde{T}_b(\mu) = \tilde{T}_b(\mu)\tilde{T}_a(\lambda\mu)\tilde{R}_{ab}(\lambda). \tag{2.45}$$

In other words, the global fundamental commutation relations remain unchanged.

This twist differs slightly from the twist proposed in [4]. The author supposes a matrix ω, whose tensor square commutes with R-matrix $[\omega \otimes \omega, R_{ab}(\lambda)] = 0$, and concludes that the matrix $\hat{T}_a(\lambda) = \omega T_a(\lambda)$ satisfies (2.44) as the original untwisted matrix $T_a(\lambda)$. In both cases, the crucial premise for usability of the twist is the commutativity of $\omega \otimes \omega$, or its infinitesimal form $U \otimes \mathbb{I} + \mathbb{I} \otimes U$, with the R-matrix.

Below we will consider only the case of $N = 2$. Taking $U = \begin{pmatrix} 1/4 & 0 \\ 0 & -1/4 \end{pmatrix}$ in (2.42), where $R_{ab}(\lambda)$ is given by (2.41), we get

$$\tilde{R}(\lambda) = \begin{pmatrix} \lambda^{-1/2}q - \lambda^{1/2}q^{-1} & 0 & 0 & 0 \\ 0 & \lambda^{-1/2} - \lambda^{1/2} & q - q^{-1} & 0 \\ 0 & q - q^{-1} & \lambda^{-1/2} - \lambda^{1/2} & 0 \\ 0 & 0 & 0 & \lambda^{-1/2}q - \lambda^{1/2}q^{-1} \end{pmatrix} \tag{2.46}$$

which corresponds to the R-matrix appearing in [8, 4]. Moreover, it is easy to see that

$$\tilde{T}_a(\lambda) = \tilde{R}_{a,1}(\lambda)\tilde{R}_{a,2}(\lambda) \cdots \tilde{R}_{a,L}(\lambda). \tag{2.47}$$

It can also be seen that

$$\tilde{T}_a(\lambda) = \begin{pmatrix} \tilde{A}(\lambda) & \tilde{B}(\lambda) \\ \tilde{C}(\lambda) & \tilde{D}(\lambda) \end{pmatrix} = \begin{pmatrix} A(\lambda) & \lambda^{1/2}B(\lambda) \\ \lambda^{-1/2}C(\lambda) & D(\lambda) \end{pmatrix}, \tag{2.48}$$

where $A,\ B,\ C,\ D$ correspond to the original monodromy matrix $T_a(\lambda)$. Moreover, one can easily realize that

$$\tilde{A}(\lambda) = \frac{1}{2}(\lambda^{-1/2} - \lambda^{1/2} + q\lambda^{-1/2} - q^{-1}\lambda^{1/2})\,\mathbb{I}_2 + \\ + \frac{1}{2}(\lambda^{1/2} - \lambda^{-1/2} + q\lambda^{-1/2} - q^{-1}\lambda^{1/2})\sigma^z, \tag{2.49}$$

$$\tilde{B}(\lambda) = (q - q^{-1})\sigma^-, \tag{2.50}$$

$$\tilde{C}(\lambda) = (q - q^{-1})\sigma^+, \tag{2.51}$$

$$\tilde{D}(\lambda) = \frac{1}{2}(\lambda^{-1/2} - \lambda^{1/2} + q\lambda^{-1/2} - q^{-1}\lambda^{1/2})\,\mathbb{I}_2 + \\ + \frac{1}{2}(\lambda^{-1/2} - \lambda^{1/2} - q\lambda^{-1/2} + q^{-1}\lambda^{1/2})\sigma^z, \tag{2.52}$$

where $\sigma^{\pm} = (1/2)(\sigma^x \pm i\sigma^y)$. The twisted R-matrix $\tilde{R}_{k,k+1}(\lambda)$ resp. the twisted monodromy matrix $\tilde{T}_a(\lambda)$ will be used throughout the text.

2.3. Global Fundamental Commutation Relations. Global commutation relations are determined by Eq. (2.15) resp. (2.44) for XXX resp. XXZ in the tensor product $V_a \otimes V_b \otimes \mathscr{H}$. They are explicitly expressed by multiplication

of matrices in the tensor product of the auxiliary spaces $V_a \otimes V_b$. After simple factorization, the R-matrices (2.9) resp. (2.46) can be written uniformly in the following way:

$$R_{ab}(\lambda) = \begin{pmatrix} f(\lambda) & 0 & 0 & 0 \\ 0 & 1 & g(\lambda) & 0 \\ 0 & g(\lambda) & 1 & 0 \\ 0 & 0 & 0 & f(\lambda) \end{pmatrix}, \tag{2.53}$$

where for the XXX chain we have

$$f(\lambda) = \frac{\lambda + 1}{\lambda}, \quad g(\lambda) = \frac{1}{\lambda}, \tag{2.54}$$

and for XXZ chain we have

$$f(\lambda) = \frac{\lambda^{-1/2} q - \lambda^{1/2} q^{-1}}{\lambda^{-1/2} - \lambda^{1/2}}, \quad g(\lambda) = \frac{q - q^{-1}}{\lambda^{-1/2} - \lambda^{1/2}}. \tag{2.55}$$

We take the monodromy matrix (2.43) resp. (2.47) for the XXZ chain. For more comfort, we omit the tilde over the corresponding operators. The matrices $T_a(\lambda)$ resp. $T_b(\mu)$ take the form

$$T_a(\lambda) = \begin{pmatrix} A(\lambda) & & B(\lambda) & \\ & A(\lambda) & & B(\lambda) \\ C(\lambda) & & D(\lambda) & \\ & C(\lambda) & & D(\lambda) \end{pmatrix} \tag{2.56}$$

resp.

$$T_b(\mu) = \begin{pmatrix} A(\mu) & B(\mu) & & \\ C(\mu) & D(\mu) & & \\ & & A(\mu) & B(\mu) \\ & & C(\mu) & D(\mu) \end{pmatrix}. \tag{2.57}$$

Multiplying and comparing the left- and right-hand sides of (2.15) resp. (2.45), we obtain the set of commutation relations. Comparing the matrix elements on the positions $(1,1)$, $(1,4)$, $(4,1)$, $(4,4)$, we obtain

$$[A(\lambda), A(\mu)] = [B(\lambda), B(\mu)] = [C(\lambda), C(\mu)] = [D(\lambda), D(\mu)] = 0. \tag{2.58}$$

Comparing $(2,3)$ resp. $(2,2)$, we get

$$[B(\lambda), C(\mu)] = g(\lambda, \mu) \left(D(\mu) A(\lambda) - D(\lambda) A(\mu) \right), \tag{2.59}$$

$$[A(\lambda), D(\mu)] = g(\lambda, \mu) \left(C(\mu) B(\lambda) - C(\lambda) B(\mu) \right). \tag{2.60}$$

From a comparison of the matrix elements $(1,3)$, $(3,4)$, $(2,1)$ resp. $(4,3)$ we obtain

$$A(\mu)B(\lambda) = f(\lambda,\mu)B(\lambda)A(\mu) + g(\mu,\lambda)B(\mu)A(\lambda), \tag{2.61}$$
$$D(\mu)B(\lambda) = f(\mu,\lambda)B(\lambda)D(\mu) + g(\lambda,\mu)B(\mu)D(\lambda), \tag{2.62}$$
$$A(\mu)C(\lambda) = f(\mu,\lambda)C(\lambda)A(\mu) + g(\lambda,\mu)C(\mu)A(\lambda), \tag{2.63}$$
$$D(\mu)C(\lambda) = f(\lambda,\mu)C(\lambda)D(\mu) + g(\mu,\lambda)C(\mu)D(\lambda), \tag{2.64}$$

where for the XXX chain we have

$$f(\lambda,\mu) = f(\lambda-\mu) = \frac{\lambda-\mu+1}{\lambda-\mu}, \quad g(\lambda,\mu) = g(\lambda-\mu) = \frac{1}{\lambda-\mu}, \tag{2.65}$$

and for XXZ chain we have

$$f(\lambda,\mu) = f(\lambda/\mu) = \frac{\mu q - \lambda q^{-1}}{\mu-\lambda}, \quad g(\lambda,\mu) = g(\lambda/\mu) = \sqrt{\lambda\mu}\,\frac{q-q^{-1}}{\mu-\lambda}. \tag{2.66}$$

We see that $g(\mu,\lambda) = -g(\lambda,\mu)$.

2.4. Eigenstates of the Transfer Matrix. The uncovering of the spectrum of the transfer matrix $\tau(\lambda) = A(\lambda) + D(\lambda)$ is now the natural next step. In Subsec. 2.3, we get four operators $A(\lambda)$, $B(\lambda)$, $C(\lambda)$, and $D(\lambda)$ under commutation relations (2.58)–(2.65). They generate an associative algebra. Relations (2.58)–(2.65) together with an assumption that the Hilbert space $\mathscr{H}$ has the structure of the Fock space, are sufficient to find the spectrum $\tau(\lambda)$. From the beginning, we work on the Hilbert space $\mathscr{H} = (\mathbb{C}^2)^{\otimes L}$, i.e., we choose a specific representation. But the content of this chapter is valid in general, i.e., also for other representations.

To uncover the Fock space structure in $\mathscr{H}$, let us find a pseudovacuum vector $|0\rangle \in \mathscr{H}$ such that $C(\lambda)|0\rangle = 0$ which is an eigenvector of the operators $A(\lambda)$ and $D(\lambda)$

$$A(\lambda)|0\rangle = \alpha(\lambda)|0\rangle, \quad D(\lambda)|0\rangle = \delta(\lambda)|0\rangle. \tag{2.67}$$

Let us remind that there is a state $|0\rangle_k$ in each h_k such that the corresponding Lax operator is of the upper triangular form

$$L_{a,k}(\lambda)|0\rangle_k = \begin{pmatrix} a(\lambda) & \text{something} \\ 0 & d(\lambda) \end{pmatrix} |0\rangle_k, \tag{2.68}$$

where $a(\lambda)$, $d(\lambda)$ are the functions of the parameter λ. We see that for XXX

$$a(\lambda) = \lambda + 1, \quad d(\lambda) = \lambda \tag{2.69}$$

and for XXZ

$$a(\lambda) = \lambda^{-1/2}q - \lambda^{1/2}q^{-1}, \quad d(\lambda) = \lambda^{-1/2} - \lambda^{1/2}. \tag{2.70}$$

The vector $|0\rangle \in \mathscr{H}$ is of the form

$$|0\rangle = |0\rangle_1 \otimes |0\rangle_2 \otimes \ldots \otimes |0\rangle_L. \tag{2.71}$$

In our particular representation, $h_k = \mathbb{C}^2$, we have $|0\rangle_k = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$. It can be easily seen that

$$T_a(\lambda)|0\rangle = \begin{pmatrix} a(\lambda)^L & \text{something} \\ 0 & d(\lambda)^L \end{pmatrix} |0\rangle. \tag{2.72}$$

We have found that the state $|0\rangle \in \mathscr{H}$ satisfies

$$C(\lambda)|0\rangle = 0, \qquad A(\lambda)|0\rangle = \alpha(\lambda)|0\rangle, \qquad D(\lambda)|0\rangle = \delta(\lambda)|0\rangle, \tag{2.73}$$

where

$$\alpha(\lambda) = a(\lambda)^L, \quad \delta(\lambda) = d(\lambda)^L, \tag{2.74}$$

i.e., $|0\rangle \in \mathscr{H}$ is the eigenstate of the transfer matrix $\tau(\lambda) = A(\lambda) + D(\lambda)$.

Other eigenstates of the transfer matrix (2.13) are of the form

$$|\{\lambda\}\rangle = |\lambda_1, \ldots, \lambda_M\rangle \equiv B(\lambda_1)B(\lambda_2)\cdots B(\lambda_M)|0\rangle \equiv |M\rangle. \tag{2.75}$$

They are called the Bethe vectors. For $M \in \mathbb{N}$, we will call the Bethe vector $|\lambda_1, \ldots, \lambda_M\rangle$ the M-magnon state. It turns out that there have to be some restrictions on the parameters $\{\lambda\} = \{\lambda_1, \ldots, \lambda_M\}$ to get the eigenstates of the transfer matrix. First, we note that in view of commutativity of the operators B (2.58) we have

$$|\lambda_1, \ldots, \lambda_M\rangle = |\sigma(\lambda_1, \ldots, \lambda_M)\rangle \tag{2.76}$$

for any permutation $\sigma \in S_M$ of $\{\lambda_1, \ldots, \lambda_M\}$. Then, using (2.61), (2.73), and (2.76), we deduce

$$\begin{aligned}
A(\lambda)\,|\lambda_1, \ldots, \lambda_M\rangle &= A(\lambda)\,B(\lambda_1)\cdots B(\lambda_M)|0\rangle = \\
&= \big(f(\lambda_1,\lambda)B(\lambda_1)A(\lambda) + g(\lambda,\lambda_1)B(\lambda)A(\lambda_1)\big)B(\lambda_2)\cdots B(\lambda_M)|0\rangle = \\
&= \big(f(\lambda_1,\lambda) + g(\lambda,\lambda_1)P_{\lambda\lambda_1}\big)B(\lambda_1)A(\lambda)B(\lambda_2)\cdots B(\lambda_M)|0\rangle = \\
&= \ldots = \prod_{k=1}^{M}\big(f(\lambda_k,\lambda) + g(\lambda,\lambda_k)P_{\lambda\lambda_k}\big)\alpha(\lambda)|\lambda_1, \ldots, \lambda_M\rangle = \\
&= \alpha(\lambda)\prod_{k=1}^{M} f(\lambda_k,\lambda)|\lambda_1, \ldots, \lambda_M\rangle + \\
&\quad + \sum_{i=1}^{M}\Phi_i(\lambda,\lambda_1,\ldots,\lambda_M)|\lambda_1,\ldots,\lambda_{i-1},\lambda,\lambda_{i+1},\ldots\rangle,
\end{aligned} \tag{2.77}$$

where $P_{\lambda\lambda_k}$ is a permutation operator of the parameters λ and λ_k, and it is clear that

$$\Phi_1(\lambda,\lambda_1,\dots,\lambda_M)=\alpha(\lambda_1)g(\lambda,\lambda_1)\prod_{k=2}^{M}f(\lambda_k,\lambda_1). \tag{2.78}$$

Since the left-hand side of (2.77) is symmetric under all permutations of $\{\lambda_1,\dots,\lambda_M\}$, we obtain

$$\Phi_i(\lambda,\lambda_1,\dots,\lambda_M)=\alpha(\lambda_i)g(\lambda,\lambda_i)\prod_{\substack{k=1\\k\neq i}}^{M}f(\lambda_k,\lambda_i),\quad \forall i=1,\dots,M. \tag{2.79}$$

In the same way by using (2.62), (2.73), and (2.76), we deduce

$$\begin{aligned}D(\lambda)\,|\lambda_1,\dots,\lambda_M\rangle &= D(\lambda)\,B(\lambda_1)\cdots B(\lambda_M)|0\rangle = \\ &= \prod_{k=1}^{M}\big(f(\lambda,\lambda_k)+g(\lambda_k,\lambda)P_{\lambda\lambda_k}\big)\delta(\lambda)|\lambda_1,\dots,\lambda_M\rangle = \\ &= \delta(\lambda)\prod_{k=1}^{M}f(\lambda,\lambda_k)|\lambda_1,\dots,\lambda_M\rangle + \\ &+\sum_{i=1}^{M}\Psi_i(\lambda,\lambda_1,\dots,\lambda_M)|\lambda_1,\dots,\lambda_{i-1},\lambda,\lambda_i,\dots,\lambda_M\rangle,\end{aligned} \tag{2.80}$$

where

$$\Psi_i(\lambda,\lambda_1,\dots,\lambda_M)=\delta(\lambda_i)g(\lambda_i,\lambda)\prod_{\substack{k=1\\k\neq i}}^{M}f(\lambda_i,\lambda_k),\quad \forall i=1,\dots,M. \tag{2.81}$$

The combination of (2.77) and (2.80) gives that $|\lambda_1,\dots,\lambda_M\rangle$ is the eigenvector of the transfer matrix (2.13) $\tau(\lambda)=A(\lambda)+D(\lambda)$

$$\begin{gathered}(A(\lambda)+D(\lambda))|\{\lambda\}\rangle=\Lambda(\lambda,\{\lambda\})|\{\lambda\}\rangle,\quad \{\lambda\}\equiv\{\lambda_1,\dots,\lambda_M\},\\ \Lambda(\lambda,\{\lambda\})=\alpha(\lambda)\prod_{i=1}^{M}f(\lambda_i,\lambda)+\delta(\lambda)\prod_{i=1}^{M}f(\lambda,\lambda_i)\end{gathered} \tag{2.82}$$

if the set of parameters $\{\lambda_1,\dots,\lambda_M\}$ satisfies the so-called Bethe equations:

$$\begin{aligned}&\Phi_i(\lambda,\lambda_1,\dots,\lambda_M)+\Psi_i(\lambda,\lambda_1,\dots,\lambda_M)=0\ \Rightarrow\\ &\Rightarrow\alpha(\lambda_i)g(\lambda,\lambda_i)\prod_{\substack{k=1\\k\neq i}}^{M}f(\lambda_k,\lambda_i)+\delta(\lambda_i)g(\lambda_i,\lambda)\prod_{\substack{k=1\\k\neq i}}^{M}f(\lambda_i,\lambda_k)=0.\end{aligned} \tag{2.83}$$

If the Bethe equations are satisfied, we call the Bethe vectors $|\lambda_1,\dots,\lambda_M\rangle$ on-shell, otherwise off-shell.

For the XXX chain, using explicit formulas (2.65) and (2.73), we write (2.82) and (2.83) in the form

$$\begin{aligned}(A(\lambda)+D(\lambda))\,|\lambda_1,\dots,\lambda_M\rangle &= \Lambda(\lambda,\lambda_i)\,|\lambda_1,\dots,\lambda_M\rangle,\\ \Lambda(\lambda,\{\lambda\}) &= (\lambda+1)^L\prod_{i=1}^{M}\frac{\lambda_i-\lambda+1}{\lambda_i-\lambda}+\lambda^L\prod_{i=1}^{M}\frac{\lambda-\lambda_i+1}{\lambda-\lambda_i}\end{aligned} \tag{2.84}$$

if the set of parameters $\{\lambda_1,\dots,\lambda_M\}$ satisfies the Bethe equations in the following form:

$$\left(\frac{\lambda_k+1}{\lambda_k}\right)^L=\prod_{\substack{j=1\\ j\neq k}}^{M}\frac{\lambda_k-\lambda_j+1}{\lambda_k-\lambda_j-1}=-\prod_{j=1}^{M}\frac{\lambda_k-\lambda_j+1}{\lambda_k-\lambda_j-1}. \tag{2.85}$$

The Bethe equations for the XXZ chain possess the following form:

$$\left(\frac{q-\lambda_k q^{-1}}{1-\lambda_k}\right)^L=(-1)^{M-1}\prod_{\substack{j=1\\ j\neq k}}^{M}\frac{\lambda_j q-\lambda_k q^{-1}}{\lambda_k q-\lambda_j q^{-1}}=\prod_{\substack{j=1\\ j\neq k}}^{M}\frac{\lambda_k q^{-1}-\lambda_j q}{\lambda_k q-\lambda_j q^{-1}}. \tag{2.86}$$

Setting $\lambda_j=q^{-2\alpha_j}$ and $q=\mathrm{e}^{h/2}$ in (2.86), we get (2.85) in the limit $h\to 0$. The corresponding eigenvalue is

$$\begin{aligned}\Lambda(\lambda,\{\lambda\})=(\lambda^{-1/2}q-\lambda^{1/2}q^{-1})^L\prod_{i=1}^{M}\frac{\lambda_i q-\lambda q^{-1}}{\lambda_i-\lambda}+\\ +(\lambda^{-1/2}-\lambda^{1/2})^L\prod_{i=1}^{M}\frac{\lambda q-\lambda_i q^{-1}}{\lambda-\lambda_i}.\end{aligned} \tag{2.87}$$

3. GENERALIZATION OF THE TWO-COMPONENT MODEL

In the literature, cf. [3, 4], etc., there appears the so-called two-component model. The two-component model was introduced to avoid problems with computation of correlation functions for local operators attached to some site x of the chain in the algebra of global operators (2.12) $A(\lambda)$, $B(\lambda)$, $C(\lambda)$, and $D(\lambda)$ defined on the chain as a whole.

We divide the chain $[1,\dots,L]$ into two components $[1,\dots,x]$ and $[x+1,\dots,L]$. Then we have the Hilbert space splitted into two parts $\mathscr{H}=\mathscr{H}_1\otimes\mathscr{H}_2$, where $\mathscr{H}_1=h_1\otimes\ldots\otimes h_x$ and $\mathscr{H}_2=h_{x+1}\otimes\ldots\otimes h_L$. We see that pseudovacuum

$|0\rangle \in \mathscr{H}$ is of the form $|0\rangle = |0\rangle_1 \otimes |0\rangle_2$, where $|0\rangle_1 \in \mathscr{H}_1$ and $|0\rangle_2 \in \mathscr{H}_2$. We define on $V_a \otimes \mathscr{H}_1 \otimes \mathscr{H}_2$ the monodromy matrix for each component

$$T_1(\lambda) = L_{a,1}(\lambda) \cdots L_{a,x}(\lambda) = \begin{pmatrix} A_1(\lambda) & B_1(\lambda) \\ C_1(\lambda) & D_1(\lambda) \end{pmatrix} \tag{3.1}$$

resp.

$$T_2(\lambda) = L_{a,x+1}(\lambda) \cdots L_{a,L}(\lambda) = \begin{pmatrix} A_2(\lambda) & B_2(\lambda) \\ C_2(\lambda) & D_2(\lambda) \end{pmatrix}. \tag{3.2}$$

Each of these monodromy matrices satisfies exactly the same commutation relations (2.15) as the original undivided monodromy matrix (2.11). Moreover, we have

$$A_j(\lambda)|0\rangle_j = \alpha_j(\lambda)|0\rangle_j, \quad D_j(\lambda)|0\rangle_j = \delta_j(\lambda)|0\rangle_j, \quad C_j(\lambda)|0\rangle_j = 0. \tag{3.3}$$

Operators corresponding to different components mutually commute. From construction, it is easy to see that

$$\alpha(\lambda) = \alpha_1(\lambda)\alpha_2(\lambda), \quad \delta(\lambda) = \delta_1(\lambda)\delta_2(\lambda). \tag{3.4}$$

For the whole chain $[1, \ldots, L]$, the full monodromy matrix T is

$$\begin{aligned} T(\lambda) = \begin{pmatrix} A(\lambda) & B(\lambda) \\ C(\lambda) & D(\lambda) \end{pmatrix} &= T_1(\lambda)\, T_2(\lambda) = \\ &= \begin{pmatrix} A_1(\lambda)A_2(\lambda) + B_1(\lambda)C_2(\lambda) & A_1(\lambda)B_2(\lambda) + B_1(\lambda)D_2(\lambda) \\ C_1(\lambda)A_2(\lambda) + D_1(\lambda)C_2(\lambda) & C_1(\lambda)B_2(\lambda) + D_1(\lambda)D_2(\lambda) \end{pmatrix}, \end{aligned} \tag{3.5}$$

and the M-magnon state is represented in the form

$$\begin{aligned} |\lambda_1, \ldots, \lambda_M\rangle = \prod_{k=1}^{M} B(\lambda_k)|0\rangle &= \\ &= \prod_{k=1}^{M} \big(A_1(\lambda_k)B_2(\lambda_k) + B_1(\lambda_k)D_2(\lambda_k)\big)|0\rangle_1 \otimes |0\rangle_2. \end{aligned} \tag{3.6}$$

The beautiful result of Izergin and Korepin [3] states that the Bethe vectors of the full model can be expressed in terms of the Bethe vectors of its components. To obtain this expression, we should commute in (3.6) all operators $A_1(\lambda_k)$ and $D_2(\lambda_k)$ to the right with the help of (2.61) and (2.62) and then use (3.3). Finally, we obtain the following result [3].

Proposition 1. *An arbitrary Bethe vector corresponding to the full system can be expressed in terms of the Bethe vectors of the first and second component.*

Let $I = \{\lambda_1, \dots, \lambda_M\}$ be a finite set of spectral parameters. To concise notation below, we will consider the set I as a finite set of indices $I = \{1, \dots, M\}$, then

$$\prod_{k\in I} B(\lambda_k)|0\rangle = \sum_{I_1\cup I_2} \prod_{k_1\in I_1} \big(\delta_2(\lambda_{k_1})B_1(\lambda_{k_1})\big) \prod_{k_2\in I_2} \big(\alpha_1(\lambda_{k_2})B_2(\lambda_{k_2})\big)\times \\ \times |0\rangle_1 \otimes |0\rangle_2 \prod_{k_1\in I_1}\prod_{k_2\in I_2} f(\lambda_{k_1},\lambda_{k_2}), \quad (3.7)$$

where $f(\lambda_{k_1},\lambda_{k_2})$ is defined in (2.65) *resp.* (2.66) *and the summation is performed over all divisions of the index set I into two disjoint subsets I_1 and I_2, where $I = I_1 \cup I_2$.*

Proof. The proof is just a matter of commutation relations (2.15) resp. (2.61)–(2.65). We use induction on the number of elements M of the index set I. We see that

$$B(\lambda)|0\rangle = \big(A_1(\lambda)B_2(\lambda) + B_1(\lambda)D_2(\lambda)\big)|0\rangle_1 \otimes |0\rangle_2 = \\ = \big(\alpha_1(\lambda)B_2(\lambda) + \delta_2(\lambda)B_1(\lambda)\big)|0\rangle_1 \otimes |0\rangle_2, \quad (3.8)$$

which is exactly formula (3.7) for $M = 1$. Let us suppose that (3.7) is valid for the index set $I = \{1, \dots, M-1\}$. Then we have

$$B(\lambda)\prod_{k\in I} B(\lambda_k)|0\rangle = \big(A_1(\lambda)B_2(\lambda) + B_1(\lambda)D_2(\lambda)\big)\times \\ \times \sum_{\substack{I_1,I_2\\ I=I_1\cup I_2}} \prod_{k_1\in I_1} \big(\delta_2(\lambda_{k_1})B_1(\lambda_{k_1})\big) \prod_{k_2\in I_2} \big(\alpha_1(\lambda_{k_2})B_2(\lambda_{k_2})\big)\times \\ \times |0\rangle_1 \otimes |0\rangle_2 \prod_{k_1\in I_1}\prod_{k_2\in I_2} f(\lambda_{k_1},\lambda_{k_2}) = \sum_{\substack{I_1,I_2\\ I=I_1\cup I_2}} \left(A_1(\lambda)\prod_{k_1\in I_1}\delta_2(\lambda_{k_1})B_1(\lambda_{k_1})\right)\times \\ \times \left(B_2(\lambda)\prod_{k_2\in I_2}\alpha_1(\lambda_{k_2})B_2(\lambda_{k_2})\right)|0\rangle_1 \otimes |0\rangle_2 \prod_{k_1\in I_1}\prod_{k_2\in I_2} f(\lambda_{k_1},\lambda_{k_2}) + \\ + \sum_{\substack{I_1,I_2\\ I=I_1\cup I_2}} \left(B_1(\lambda)\prod_{k_1\in I_1}\delta_2(\lambda_{k_1})B_1(\lambda_{k_1})\right)\left(D_2(\lambda)\prod_{k_2\in I_2}\alpha_1(\lambda_{k_2})B_2(\lambda_{k_2})\right)\times \\ \times |0\rangle_1 \otimes |0\rangle_2 \prod_{k_1\in I_1}\prod_{k_2\in I_2} f(\lambda_{k_1},\lambda_{k_2}). \quad (3.9)$$

In the first sum we use (2.77) to commute $A_1(\lambda)$ with $\prod_{k_1\in I_1} B_1(\lambda_{k_1})$ resp. (2.80), to commute $D_2(\lambda)$ with $\prod_{k_2\in I_2} B_2(\lambda_{k_2})$ in the second sum. Using just the second

term in (2.77) we get for the first sum:

$$\sum_{\substack{I_1,I_2\\ I=I_1\cup I_2}}\sum_{k\in I_1} g(\lambda,\lambda_k)\alpha_1(\lambda_k)\delta_2(\lambda_k)B_1(\lambda)B_2(\lambda)\prod_{\substack{j\in I_1\\ j\neq k}}\delta_2(\lambda_j)B_1(\lambda_j)\times$$
$$\times\prod_{i\in I_2}\alpha_1(\lambda_i)B_2(\lambda_i)|0\rangle_1\otimes|0\rangle_2\prod_{\substack{l\in I_1\\ l\neq k}}f(\lambda_l,\lambda_k)\prod_{k_1\in I_1}\prod_{k_2\in I_2}f(\lambda_{k_1},\lambda_{k_2}). \quad (3.10)$$

Similarly, using just the second term in (2.80), we get for the second sum:

$$\sum_{\substack{I_1,I_2\\ I=I_1\cup I_2}}\sum_{k\in I_2} g(\lambda_k,\lambda)\alpha_1(\lambda_k)\delta_2(\lambda_k)B_1(\lambda)B_2(\lambda)\prod_{j\in I_1}\delta_2(\lambda_j)B_1(\lambda_j)\times$$
$$\times\prod_{\substack{i\in I_2\\ i\neq k}}\alpha_1(\lambda_i)B_2(\lambda_i)|0\rangle_1\otimes|0\rangle_2\prod_{\substack{l\in I_2\\ l\neq k}}f(\lambda_k,\lambda_l)\prod_{k_1\in I_1}\prod_{k_2\in I_2}f(\lambda_{k_1},\lambda_{k_2})=$$
$$=\sum_{\substack{I_1',I_2'\\ I=I_1'\cup I_2'}}\sum_{k\in I_1'} g(\lambda_k,\lambda)\alpha_1(\lambda_k)\delta_2(\lambda_k)B_1(\lambda)B_2(\lambda)\prod_{\substack{j\in I_1'\\ j\neq k}}\delta_2(\lambda_j)B_1(\lambda_j)\times$$
$$\times\prod_{i\in I_2'}\alpha_1(\lambda_i)B_2(\lambda_i)|0\rangle_1\otimes|0\rangle_2\prod_{k_1\in I_1'}\prod_{k_2\in I_2'}f(\lambda_{k_1},\lambda_{k_2})\prod_{\substack{m\in I_1'\\ m\neq k}}f(\lambda_m,\lambda_k), \quad (3.11)$$

where we introduced new partition $I_1'=I_1\cup\{k\}$ and $I_2'=I_2\backslash\{k\}$. We see that (3.11) is almost the same as (3.10) with only one difference. In (3.10), there appears a factor $g(\lambda,\lambda_k)$ and in (3.11) there appears $g(\lambda_k,\lambda)$. Using the fact that $g(\lambda,\lambda_k)=-g(\lambda_k,\lambda)$, cf. (2.65), we see that these two sums cancel each other. Therefore, only the first parts of (2.77) and (2.80) contribute to (3.9). We get

$$B(\lambda)\prod_{k\in I}B(\lambda_k)|0\rangle=\sum_{\substack{I_1,I_2\\ I=I_1\cup I_2}}\left(\alpha_1(\lambda)\prod_{k_1\in I_1}f(\lambda_{k_1},\lambda)\delta_2(\lambda_{k_1})B_1(\lambda_{k_1})\right)\times$$
$$\times\left(B_2(\lambda)\prod_{k_2\in I_2}\alpha_1(\lambda_{k_2})B_2(\lambda_{k_2})\right)|0\rangle_1\otimes|0\rangle_2\prod_{k_1\in I_1}\prod_{k_2\in I_2}f(\lambda_{k_1},\lambda_{k_2})+$$
$$+\sum_{\substack{I_1,I_2\\ I=I_1\cup I_2}}\left(B_1(\lambda)\prod_{k_1\in I_1}\delta_2(\lambda_{k_1})B_1(\lambda_{k_1})\right)\left(\delta_2(\lambda)\prod_{k_2\in I_2}f(\lambda,\lambda_{k_2})\alpha_1(\lambda_{k_2})B_2(\lambda_{k_2})\right)\times$$
$$\times|0\rangle_1\otimes|0\rangle_2\prod_{k_1\in I_1}\prod_{k_2\in I_2}f(\lambda_{k_1},\lambda_{k_2}) \quad (3.12)$$

which proves the induction.

This result can be straightforwardly generalized to an arbitrary number of components $N\leqslant L$.

Proposition 2. *An arbitrary Bethe vector of the full system can be expressed in terms of the Bethe vectors of its components. For $N \leqslant L$ components, the Bethe vector is of the form*

$$\prod_{k\in I} B(\lambda_k)|0\rangle = \sum_{I_1\cup I_2\cup\ldots\cup I_N} \prod_{k_1\in I_1}\prod_{k_2\in I_2}\cdots\prod_{k_N\in I_N}\prod_{1\leqslant i<j\leqslant N}\times$$
$$\times(\alpha_i(\lambda_{k_j})\delta_j(\lambda_{k_i})f(\lambda_{k_i},\lambda_{k_j}))\times$$
$$\times B_1(\lambda_{k_1})|0\rangle_1\otimes B_2(\lambda_{k_2})|0\rangle_2\otimes\ldots\otimes B_N(\lambda_{k_N})|0\rangle_N, \quad (3.13)$$

where summation is performed over all divisions of the set I into its N mutually disjoint subsets $I_1, I_2, \ldots, I_N$.

Proof. The proof is simply performed by induction on the number of components N and by using (3.7). For $N = 2$ is (3.13) just (3.7). Let us suppose that (3.13) is valid for some $N < L$ and make induction step to $N+1$. The chain $[1,\ldots,L]$ is divided into N subchains $[1,\ldots,x_1]$, $[x_1+1,\ldots,x_2]$, etc., up to $[x_{N-1}+1,\ldots,L]$. Let us divide the last interval, if possible, into two subchains $[x_{N-1}+1,\ldots,x_N]$ and $[x_N+1,\ldots,L]$ and apply (3.7) to a set of B operators $\prod_{k_N\in I_N} B_N(\lambda_{k_N})|0\rangle_N$. We get

$$\prod_{k_N\in I_N} B_N(\lambda_{k_N})|0\rangle_N = \sum_{I'_N\cup I'_{N+1}}\prod_{k_N\in I'_N}\prod_{k_{N+1}\in I'_{N+1}}\times$$
$$\times\delta'_{N+1}(\lambda_{k_N})\alpha'_N(\lambda_{k_{N+1}})f(\lambda_{k_N},\lambda_{k_{N+1}})B'_N(\lambda_{k_N})B'_{N+1}(\lambda_{k_{N+1}})|0\rangle'_N\otimes|0\rangle'_{N+1}, \quad (3.14)$$

where the sum goes over all divisions of I_N into its two disjoint subsets I'_N and I'_{N+1} such that $I_N = I'_N\cup I'_{N+1}$ and operators $B'_N(\lambda)$ and $B'_{N+1}(\lambda)$ act on the new subchains $[x_{N-1}+1,\ldots,x_N]$ resp. $[x_N+1,\ldots,L]$; the same for $\alpha'_N(\lambda)$ resp. $\delta'_{N+1}(\lambda)$ and the pseudovacuum vectors $|0\rangle'_N$ resp. $|0\rangle'_{N+1}$. Let us remind that

$$\prod_{k_N\in I_N} = \prod_{k_N\in I'_N}\prod_{k_{N+1}\in I'_{N+1}}. \quad (3.15)$$

Inserting (3.14) into induction assumption (3.13), we get

$$\prod_{k\in I} B(\lambda_k)|0\rangle = \sum_{I_1\cup I_2\cup\ldots\cup I'_N\cup I'_{N+1}}\prod_{k_1\in I_1}\prod_{k_2\in I_2}\cdots\prod_{k_N\in I'_N}\prod_{k_{N+1}\in I'_{N+1}}\times$$
$$\times\prod_{1\leqslant i<j\leqslant N+1}(\alpha_i(\lambda_{k_j})\delta_j(\lambda_{k_i})f(\lambda_{k_i},\lambda_{k_j}))\times$$
$$\times B_1(\lambda_{k_1})|0\rangle_1\otimes B_2(\lambda_{k_2})|0\rangle_2\otimes\ldots\otimes B'_N(\lambda_{k_N})|0\rangle'_N\otimes B'_{N+1}(\lambda_{k_{N+1}})|0\rangle'_{N+1}, \quad (3.16)$$

which proves the induction.

4. BETHE VECTORS

In this section, we will see that computation of the Bethe vectors in the algebraic Bethe ansatz is just a matter of using proposition 2. By assumption we have a chain of length L. Let us divide it into L components, i.e., into L subchains of length one (1-chains). Using proposition 2 we get for the M-magnon (Bethe vector) with $M \leqslant L$ the following:

$$\prod_{k=1}^{M} B(\lambda_k)|0\rangle = \sum_{I_1 \cup I_2 \cup \cdots \cup I_L} \prod_{k_1 \in I_1} \prod_{k_2 \in I_2} \cdots \prod_{k_L \in I_L} \prod_{1 \leqslant i < j \leqslant L} \times$$
$$\times (\alpha_i(\lambda_{k_j})\delta_j(\lambda_{k_i}) f(\lambda_{k_i}, \lambda_{k_j})) B_1(\lambda_{k_1})|0\rangle_1 \otimes B_2(\lambda_{k_2})|0\rangle_2 \otimes \ldots \otimes B_L(\lambda_{k_L})|0\rangle_L. \tag{4.1}$$

It can be easily seen that for 1-chain, i.e., for a chain with Hilbert space $h = \mathbb{C}^2$,

$$B(\lambda)B(\mu)|0\rangle = 0. \tag{4.2}$$

Therefore, the sum over all divisions of $\{1, \ldots, M\}$ into L subsets contains just divisions into subsets containing at most one element, i.e., $|I_j| = 0, 1$. Moreover, only M of them is nonempty, let us denote them $I_{n_1}, I_{n_2}, \ldots, I_{n_M}$. We have to sum over all possible combinations of such sets, i.e., over all M-tuples $n_1 < n_2 < \ldots < n_M$. Next, we have to sum over all distributions of the parameters $\lambda_1, \lambda_2, \ldots, \lambda_M$ into the sets $I_{n_1}, \ldots, I_{n_M}$. We can simplify our life assuming that $\lambda_j \in I_{n_j}$. Then, by summing over all permutations $\sigma_\lambda \in S_M$ of $\{\lambda_1, \ldots, \lambda_M\}$, we get exactly all the other distributions.

Let us study what happens to the coefficient

$$\prod_{1 \leqslant i < j \leqslant L} (\alpha_i(\lambda_{k_j})\delta_j(\lambda_{k_i}) f(\lambda_{k_i}, \lambda_{k_j})). \tag{4.3}$$

It is easy to see that

$$\prod_{1 \leqslant i < j \leqslant L} \alpha_i(\lambda_{k_j}) = \prod_{j=1}^{L} \prod_{i=1}^{j-1} \alpha_i(\lambda_{k_j}), \tag{4.4}$$

but only λ_{k_j} from the sets $I_{n_1}, \ldots, I_{n_M}$ are relevant and by assumption $\lambda_j \in I_{n_j}$. Therefore, we can replace

$$\prod_{1 \leqslant i < j \leqslant L} \alpha_i(\lambda_{k_j}) \to \prod_{j=1}^{M} \prod_{i=1}^{n_j - 1} \alpha_i(\lambda_j). \tag{4.5}$$

Similar considerations can be conducted for both $\delta_j(\lambda_{k_i})$ and $f(\lambda_{k_i},\lambda_{k_j})$. We get

$$\prod_{k=1}^{M} B(\lambda_k)|0\rangle = \sum_{1\leqslant n_1<n_2<\ldots<n_M\leqslant L} \times$$
$$\times \sum_{\sigma_\lambda\in S_M} \sigma_\lambda\left(\prod_{j=1}^{M}\left(\prod_{i=1}^{n_j-1}\alpha_i(\lambda_j)\prod_{i=n_j+1}^{L}\delta_i(\lambda_j)\prod_{i=1}^{j-1}f(\lambda_i,\lambda_j)\right)\times\right.$$
$$\left.\times B_{n_1}(\lambda_1)B_{n_2}(\lambda_2)\cdots B_{n_M}(\lambda_M)\right)|0\rangle_1\otimes|0\rangle_2\otimes\ldots\otimes|0\rangle_L. \quad (4.6)$$

For 1-chain, it holds that $B(\lambda)=B$ is parameter-independent. Moreover, eigenvalues $\alpha_i(\lambda)=a(\lambda)$, $\delta_i(\lambda)=d(\lambda)$ are still the same for all components $i=1,\ldots,L$, where $a(\lambda)$ and $d(\lambda)$ are defined in (2.69) resp. (2.70). We get

$$\prod_{k=1}^{M} B(\lambda_k)|0\rangle = \sum_{1\leqslant n_1<n_2<\ldots<n_M\leqslant L} \times$$
$$\times \sum_{\sigma\in S_M} \sigma_\lambda\left(\prod_{j=1}^{M} a(\lambda_j)^{n_j-1}d(\lambda_j)^{L-n_j}\prod_{i=1}^{j-1}f(\lambda_i,\lambda_j)\right)\times$$
$$\times B_{n_1}B_{n_2}\cdots B_{n_M}|0\rangle_1\otimes|0\rangle_2\otimes\ldots\otimes|0\rangle_L =$$
$$= \prod_{j=1}^{M}\frac{d(\lambda_j)^L}{a(\lambda_j)} \sum_{1\leqslant n_1<n_2<\ldots<n_M\leqslant L} \times$$
$$\times \sum_{\sigma\in S_M} \sigma_\lambda\left(\prod_{1\leqslant i<j\leqslant M} f(\lambda_i,\lambda_j)\prod_{j=1}^{M}\left(\frac{a(\lambda_j)}{d(\lambda_j)}\right)^{n_j}\right)\times$$
$$\times B_{n_1}B_{n_2}\cdots B_{n_M}|0\rangle_1\otimes|0\rangle_2\otimes\ldots\otimes|0\rangle_L. \quad (4.7)$$

5. INHOMOGENEOUS BETHE ANSATZ

We start with the inhomogeneous monodromy matrix

$$T_a^{\boldsymbol{\xi}}(\lambda) = L_{a,1}(\lambda+\xi_1)L_{a,2}(\lambda+\xi_2)\cdots L_{a,L}(\lambda+\xi_L), \quad (5.1)$$

where $L_{a,j}(\lambda)$ are the Lax operators defined in (2.3) resp. (2.46) depending on whether we consider XXX or XXZ spin chain. Let us remark that for the XXZ chain the monodromy matrix is of the form

$$T_a^{\boldsymbol{\xi}}(\lambda) = L_{a,1}(\lambda\cdot\xi_1)L_{a,2}(\lambda\cdot\xi_2)\cdots L_{a,L}(\lambda\cdot\xi_L). \quad (5.2)$$

In what follows, we will use the notation connected with the XXX chain, but we can do for the XXZ chain the same as well.

Expressing $T_a^{\boldsymbol{\xi}}(\lambda)$ in the auxiliary space V_a, we get

$$T_a^{\boldsymbol{\xi}}(\lambda) = \begin{pmatrix} A^{\boldsymbol{\xi}}(\lambda) & B^{\boldsymbol{\xi}}(\lambda) \\ C^{\boldsymbol{\xi}}(\lambda) & D^{\boldsymbol{\xi}}(\lambda) \end{pmatrix}, \tag{5.3}$$

where, again, the operators $A^{\boldsymbol{\xi}}(\lambda)$, $B^{\boldsymbol{\xi}}(\lambda)$, $C^{\boldsymbol{\xi}}(\lambda)$, and $D^{\boldsymbol{\xi}}(\lambda)$ act in $\mathscr{H} = h_1 \otimes \ldots \otimes h_L$. Acting on the pseudovacuum vector $|0\rangle \in \mathscr{H}$, we get

$$A^{\boldsymbol{\xi}}(\lambda)|0\rangle = \alpha^{\boldsymbol{\xi}}(\lambda)|0\rangle, \tag{5.4}$$

$$D^{\boldsymbol{\xi}}(\lambda)|0\rangle = \delta^{\boldsymbol{\xi}}(\lambda)|0\rangle, \tag{5.5}$$

$$C^{\boldsymbol{\xi}}(\lambda)|0\rangle = 0, \tag{5.6}$$

where

$$\alpha^{\boldsymbol{\xi}}(\lambda) = a(\lambda+\xi_1)a(\lambda+\xi_2)\cdots a(\lambda+\xi_L), \tag{5.7}$$

$$\delta^{\boldsymbol{\xi}}(\lambda) = d(\lambda+\xi_1)d(\lambda+\xi_2)\cdots d(\lambda+\xi_L). \tag{5.8}$$

Here, the functions $a(\lambda)$ and $d(\lambda)$ are defined in (2.69) for XXX resp. in (2.70) for XXZ.

For the inhomogeneous version we can introduce the same N-component model as for the homogeneous Bethe ansatz. For the 2-component model, for example, we have

$$T_a^{\boldsymbol{\xi}}(\lambda) = \underbrace{L_{a,1}(\lambda+\xi_1)\cdots L_{a,x}(\lambda+\xi_x)}_{\text{1st component}} \underbrace{L_{a,x+1}(\lambda+\xi_{x+1})\cdots L_{a,L}(\lambda+\xi_L)}_{\text{2nd component}} = \\ = T_a^{\boldsymbol{\xi}_1}(\lambda) T_a^{\boldsymbol{\xi}_2}(\lambda), \tag{5.9}$$

where $\boldsymbol{\xi}_1 = (\xi_1, \ldots, \xi_x)$ resp. $\boldsymbol{\xi}_2 = (\xi_{x+1}, \ldots, \xi_L)$. We have

$$A^{\boldsymbol{\xi}}(\lambda)|0\rangle = \alpha_1^{\boldsymbol{\xi}_1}(\lambda)\alpha_2^{\boldsymbol{\xi}_2}(\lambda)|0\rangle, \quad D^{\boldsymbol{\xi}}(\lambda)|0\rangle = \delta_1^{\boldsymbol{\xi}_1}(\lambda)\delta_2^{\boldsymbol{\xi}_2}(\lambda)|0\rangle. \tag{5.10}$$

A very important property of the inhomogeneous chain is that its operators $A^{\boldsymbol{\xi}}(\lambda)$, $B^{\boldsymbol{\xi}}(\lambda)$, $C^{\boldsymbol{\xi}}(\lambda)$, and $D^{\boldsymbol{\xi}}(\lambda)$ satisfy the same fundamental commutation relations as the homogeneous chain (2.58)–(2.65), i.e., commutation relations are independent of the inhomogeneity parameters $\boldsymbol{\xi}$. Therefore, an analogy of propositions 1 and 2 can be easily formulated.

Proposition 3. *Let $N \leqslant L$. An arbitrary Bethe vector of the full system can be expressed in terms of the Bethe vectors of its N components*

$$\prod_{k\in I} B^{\boldsymbol{\xi}}(\lambda_k)|0\rangle = \\ = \sum_{I_1\cup\ldots\cup I_N} \prod_{k_1\in I_1} \cdots \prod_{k_N\in I_N} \prod_{1\leqslant i<j\leqslant N} \bigl(\alpha_i^{\boldsymbol{\xi}_i}(\lambda_{k_j})\delta_j^{\boldsymbol{\xi}_j}(\lambda_{k_i}) f(\lambda_{k_i},\lambda_{k_j})\bigr)\times \\ \times B_1^{\boldsymbol{\xi}_1}(\lambda_{k_1}) B_2^{\boldsymbol{\xi}_2}(\lambda_{k_2})\cdots B_N^{\boldsymbol{\xi}_N}(\lambda_{k_N})|0\rangle. \tag{5.11}$$

To get an explicit formula for the Bethe vectors, we have to divide the chain into L components of length 1, as we did in the last section. We get for the M-magnon

$$\begin{aligned}
&\prod_{k=1}^{M} B^{\boldsymbol{\xi}}(\lambda_k)|0\rangle = \\
&= \sum_{1\leqslant n_1<\ldots<n_M\leqslant L}\ \sum_{\sigma_\lambda\in S_M} \sigma_\lambda\left(\prod_{j=1}^{M}\left(\prod_{i=1}^{n_j-1}\alpha_i^{\xi_i}(\lambda_j)\prod_{i=n_j+1}^{L}\delta_i^{\xi_i}(\lambda_j)\prod_{i=1}^{j-1}f(\lambda_i,\lambda_j)\right)\times\right. \\
&\qquad\left.\times B_{n_1}^{\xi_{n_1}}(\lambda_1)\cdots B_{n_M}^{\xi_{n_M}}(\lambda_M)\right)|0\rangle = \\
&= \sum_{1\leqslant n_1<\ldots<n_M\leqslant L}\ \sum_{\sigma_\lambda\in S_M} \sigma_\lambda\left(\prod_{j=1}^{M}\left(\prod_{i=1}^{n_j-1}\alpha_i^{\xi_i}(\lambda_j)\prod_{i=n_j+1}^{L}\delta_i^{\xi_i}(\lambda_j)\prod_{i=1}^{j-1}f(\lambda_i,\lambda_j)\right)\right)\times \\
&\qquad\times B_{n_1}\cdots B_{n_M}|0\rangle = \prod_{j=1}^{M}\prod_{i=1}^{L} d(\lambda_j+\xi_i)\sum_{1\leqslant n_1<\ldots<n_M\leqslant L}\ \times \\
&\times \sum_{\sigma_\lambda\in S_M}\sigma_\lambda\left(\prod_{j=1}^{M}\frac{1}{a(\lambda_j+\xi_{n_j})}\prod_{i=1}^{n_j}\frac{a(\lambda_j+\xi_i)}{d(\lambda_j+\xi_i)}\prod_{i=1}^{j-1}f(\lambda_i,\lambda_j)\right)B_{n_1}\cdots B_{n_M}|0\rangle,
\end{aligned} \tag{5.12}$$

where again the B-operators $B_{n_j}^{\xi_{n_j}}(\lambda) = B_{n_j}$ are parameter-independent for 1-chains.

6. FREE FERMIONS

In this Section, we recall the well-known construction [19] of L-dimensional free fermion algebra in terms of the Pauli matrices. First, in $\mathbb{C}^2$ one can easily define 1-dimensional fermions using the properties of the Pauli matrices. Let

$$\psi \equiv \sigma^+ = \frac{1}{2}(\sigma^x + i\sigma^y), \quad \bar\psi \equiv \sigma^- = \frac{1}{2}(\sigma^x - i\sigma^y). \tag{6.1}$$

Thus defined $\psi, \bar\psi$ satisfy the fermionic relations

$$[\bar\psi, \psi]_+ = \mathbb{I}, \quad \psi^2 = 0, \quad \bar\psi^2 = 0. \tag{6.2}$$

For a tensor product of L copies of $\mathbb{C}^2$ we can define fermions as

$$\psi_k \equiv \left(\prod_{j=1}^{k-1}\sigma_j^z\right)\sigma_k^+, \quad \bar\psi_k \equiv \left(\prod_{j=1}^{k-1}\sigma_j^z\right)\sigma_k^- \quad (k=1,\ldots,L), \tag{6.3}$$

where σ_j^α denotes the sigma matrix attached to the jth vector space, i.e.,

$$\sigma_j^\alpha = \mathbb{I}^{\otimes(j-1)} \otimes \sigma^\alpha \otimes \mathbb{I}^{\otimes(L-j)}. \tag{6.4}$$

This concise notation is used throughout the whole text. Commutation relations for the fermions (6.3) are of the form

$$[\bar\psi_i, \psi_j]_+ = \delta_{ij}\mathbb{I}, \quad [\bar\psi_i, \bar\psi_j]_+ = 0, \quad [\psi_i, \psi_j]_+ = 0. \tag{6.5}$$

It is a straightforward task to check the following identities:

$$\bar\psi_{k+1}\psi_k + \bar\psi_k\psi_{k+1} + \bar\psi_k\bar\psi_{k+1} + \psi_{k+1}\psi_k = \sigma_k^x\sigma_{k+1}^x, \tag{6.6}$$

$$\bar\psi_{k+1}\psi_k + \bar\psi_k\psi_{k+1} - \bar\psi_k\bar\psi_{k+1} - \psi_{k+1}\psi_k = \sigma_k^y\sigma_{k+1}^y, \tag{6.7}$$

$$[\psi_k, \bar\psi_k] = \sigma_k^z, \tag{6.8}$$

$$(1 - 2\bar\psi_k\psi_k)(1 - 2\bar\psi_{k+1}\psi_{k+1}) = \sigma_k^z\sigma_{k+1}^z. \tag{6.9}$$

7. FERMIONIC REALIZATION OF XXX

We have seen that our definition (2.3) of the Lax operator $L_{a,i}(\lambda)$ led to expression (2.4) which is in fact identical to the definition of the R-matrix (2.8). Let us remind that the identity operator $\mathbb{I}$ is a member of the algebra of fermions because of commutation relation (6.5). Therefore, from expression (2.4) for $L_{a,i}(\lambda)$ we see that it remains to know a fermionic realization only for the permutation operator $P_{a,i}$.

Let us start with the permutation operator $P_{k,k+1}$ which permutes the neighboring vector spaces h_k and h_{k+1}. Due to identities (6.6)–(6.9) and definition of permutation operator (2.5) it is straightforward to check that

$$P_{k,k+1} = \mathbb{I} + \bar\psi_{k+1}\psi_k + \bar\psi_k\psi_{k+1} - \bar\psi_k\psi_k - \bar\psi_{k+1}\psi_{k+1} + 2\bar\psi_k\psi_k\bar\psi_{k+1}\psi_{k+1}. \tag{7.1}$$

Problems appear when we try to find a fermionic realization of the permutation operator $P_{j,k}$ in non-neighboring vector spaces h_j, h_k, where $j < k-1$. It turns out that $P_{j,k}$ becomes nonlocal in terms of fermions. Using properties of the Pauli matrices, $P_{j,k}$ could be rewritten as

$$P_{j,k} = \frac{1}{2}(\mathbb{I} + \sigma_j^z\sigma_k^z) + (\sigma_j^+\sigma_k^- + \sigma_j^-\sigma_k^+). \tag{7.2}$$

The first part is local even in terms of fermions

$$\frac{1}{2}(\mathbb{I} + \sigma_j^z\sigma_k^z) = \mathbb{I} - \bar\psi_k\psi_k - \bar\psi_j\psi_j + 2\bar\psi_k\psi_k\bar\psi_j\psi_j, \tag{7.3}$$

but the second part is nonlocal

$$\sigma_j^+\sigma_k^- + \sigma_j^-\sigma_k^+ = (\psi_j\bar{\psi}_k + \bar{\psi}_j\psi_k)\prod_{l=j}^{k-1}\sigma_l^z = (\psi_j\bar{\psi}_k + \bar{\psi}_j\psi_k)\prod_{l=j}^{k-1}(\mathbb{I} - 2\bar{\psi}_l\psi_l). \quad (7.4)$$

Therefore, the fermionic realization of $P_{j,k}$ for $j < k-1$ is a nonlocal operator.

The nonlocality of $P_{j,k}$ resp. $R_{j,k}(\lambda)$ is a serious problem. There appear difficulties when we attempt to express the monodromy matrix (2.11) in terms of such nonlocal operators. We need to avoid the nonlocality.

Let us remind once again that $L_{a,i}(\lambda) = R_{a,i}(\lambda)$. For the R-matrix $R_{ab}(\lambda)$ satisfying the Yang–Baxter equation (2.10), we can define the matrix $\hat{R}_{ab}(\lambda) = R_{ab}(\lambda)P_{ab}$ which satisfies

$$\hat{R}_{ab}(\lambda)\hat{R}_{bc}(\lambda+\mu)\hat{R}_{ab}(\mu) = \hat{R}_{ab}(\mu)\hat{R}_{bc}(\lambda+\mu)\hat{R}_{ab}(\lambda). \quad (7.5)$$

We substitute $L_{a,i}(\lambda) = \hat{R}_{a,i}(\lambda)P_{a,i}$ in the monodromy matrix (2.11) and obtain a very convenient expression

$$\begin{aligned} T_a(\lambda) = L_{a,1}(\lambda)L_{a,2}(\lambda)\cdots L_{a,L}(\lambda) = \\ = \hat{R}_{a,1}(\lambda)P_{a,1}\hat{R}_{a,2}(\lambda)P_{a,2}\cdots\hat{R}_{a,L}(\lambda)P_{a,L} = \\ = \hat{R}_{a,1}(\lambda)\hat{R}_{1,2}(\lambda)\cdots\hat{R}_{L-1,L}(\lambda)P_{a,1}P_{a,2}\cdots P_{a,L} = \\ = \hat{R}_{a,1}(\lambda)\hat{R}_{1,2}(\lambda)\cdots\hat{R}_{L-1,L}(\lambda)P_{L-1,L}\cdots P_{1,2}P_{a,1}. \end{aligned} \quad (7.6)$$

It contains the operators $\hat{R}_{k,k+1}$ resp. $P_{k,k+1}$ acting only in the neighboring spaces $h_k \otimes h_{k+1}$. From (7.1) we know the fermionic realization of $P_{k,k+1}$, and the fermionic realization of the R-matrix $\hat{R}_{k,k+1}(\lambda)$ is

$$\begin{aligned} \hat{R}_{k,k+1}(\lambda) = \lambda P_{k,k+1} + \mathbb{I} = (\lambda+1)\mathbb{I} + \lambda(\bar{\psi}_{k+1}\psi_k + \\ + \bar{\psi}_k\psi_{k+1} - \bar{\psi}_k\psi_k - \bar{\psi}_{k+1}\psi_{k+1} + 2\bar{\psi}_k\psi_k\bar{\psi}_{k+1}\psi_{k+1}). \end{aligned} \quad (7.7)$$

The natural next step is to express the monodromy matrix (7.6) as the 2×2 matrix in the auxiliary space $V_a = \mathbb{C}^2$. For this purpose we rewrite (7.6) as

$$T_a(\lambda) = \hat{R}_{a,1}(\lambda)X(\lambda)P_{a,1}, \quad (7.8)$$

where the operator $X(\lambda)$,

$$X(\lambda) = \hat{R}_{1,2}(\lambda)\cdots\hat{R}_{L-1,L}(\lambda)P_{L-1,L}\cdots P_{1,2}, \quad (7.9)$$

acts nontrivially only in the quantum spaces $\mathscr{H} = h_1 \otimes \ldots \otimes h_L$ and is a scalar in the auxiliary space V_a. Moreover, we know, due to Eqs. (7.1) and (7.7), how to express $X(\lambda)$ in terms of fermions.

What remains is to express $\hat{R}_{a,1}$ and $P_{a,1}$ as the 2×2 matrix in the auxiliary space V_a. The permutation matrix (2.5) can be rewritten as

$$P_{a,1} = \frac{1}{2}(\mathbb{I}\otimes\mathbb{I} + \sigma^x\otimes\sigma^x + \sigma^y\otimes\sigma^y + \sigma^z\otimes\sigma^z) =$$
$$= \frac{1}{2}\left[\begin{pmatrix} \mathbb{I} & 0 \\ 0 & \mathbb{I}\end{pmatrix} + \begin{pmatrix} 0 & \sigma^x \\ \sigma^x & 0\end{pmatrix} + \begin{pmatrix} 0 & -i\sigma^y \\ i\sigma^y & 0\end{pmatrix} + \begin{pmatrix} \sigma^z & 0 \\ 0 & -\sigma^z\end{pmatrix}\right] =$$
$$= \begin{pmatrix} \frac{1}{2}(\mathbb{I}+\sigma^z) & \frac{1}{2}(\sigma^x - i\sigma^y) \\ \frac{1}{2}(\sigma^x + i\sigma^y) & \frac{1}{2}(\mathbb{I}-\sigma^z)\end{pmatrix} = \quad (7.10)$$

and using (6.3) and (6.8) we get

$$= \begin{pmatrix} \psi_1\bar{\psi}_1 & \bar{\psi}_1 \\ \psi_1 & \bar{\psi}_1\psi_1\end{pmatrix} = \begin{pmatrix} \mathbb{I}-N_1 & \bar{\psi}_1 \\ \psi_1 & N_1\end{pmatrix}, \quad (7.11)$$

where $N_1 = \bar{\psi}_1\psi_1$. For $\hat{R}_{a,1}(\lambda)$, we get

$$\hat{R}_{a,1}(\lambda) = \mathbb{I}_{a,i} + \lambda P_{a,1} = \begin{pmatrix} (\lambda+1)\mathbb{I} - \lambda N_1 & \lambda\bar{\psi}_1 \\ \lambda\psi_1 & \lambda N_1 + \mathbb{I}\end{pmatrix}. \quad (7.12)$$

Using (7.11) and (7.12), the monodromy matrix (7.8) can be written in the following form:

$$T_a(\lambda) = \begin{pmatrix} (\lambda+1)\mathbb{I} - \lambda N_1 & \lambda\bar{\psi}_1 \\ \lambda\psi_1 & \lambda N_1 + \mathbb{I}\end{pmatrix} X(\lambda) \begin{pmatrix} \mathbb{I}-N_1 & \bar{\psi}_1 \\ \psi_1 & N_1\end{pmatrix} =$$
$$= \begin{pmatrix} A(\lambda) & B(\lambda) \\ C(\lambda) & D(\lambda)\end{pmatrix}, \quad (7.13)$$

where

$$A(\lambda) = (\lambda+1-\lambda N_1)X(\lambda)(1-N_1) + \lambda\bar{\psi}_1 X(\lambda)\psi_1, \quad (7.14)$$
$$B(\lambda) = (\lambda+1-\lambda N_1)X(\lambda)\bar{\psi}_1 + \lambda\bar{\psi}_1 X(\lambda)N_1, \quad (7.15)$$
$$C(\lambda) = \lambda\psi_1 X(\lambda)(1-N_1) + (\lambda N_1+1)X(\lambda)\psi_1, \quad (7.16)$$
$$D(\lambda) = \lambda\psi_1 X(\lambda)\bar{\psi}_1 + (\lambda N_1+1)X(\lambda)N_1. \quad (7.17)$$

8. BETHE VECTORS OF XXX

The goal of our text is to find expression for the Bethe vectors (2.75)

$$|\lambda_1,\dots,\lambda_M\rangle = B(\lambda_1)\cdots B(\lambda_M)|0\rangle. \quad (8.1)$$

For this purpose, the fermionic realization (7.15) of the creation operator $B(\lambda)$ is very convenient. The operator $X(\lambda) = \hat{R}_{12}(\lambda)\cdots\hat{R}_{L-1,L}(\lambda)P_{L-1,L}\cdots P_{12}$ can be written in terms of fermions due to Eqs. (7.11) and (7.12). From Eq. (2.71), our special representation, where $|0\rangle_k = \begin{pmatrix}1\\0\end{pmatrix}$, and the definition of free fermions (6.3), we can see that

$$\psi_k|0\rangle = 0 \tag{8.2}$$

for all $k = 1, \ldots, L$.

If we were to write $B(\lambda)$ in the normal form, our work would be simple. Unfortunately, it seems a rather difficult task. Instead, we have to use the "weak approach", i.e., to apply $B(\lambda)$ to the pseudovacuum $|0\rangle$ and try to commute the fermions $\bar{\psi}_k$ to the left and see what happens.

The details of this section are postponed to Appendix A. Here, we only write down the results.

We get the 1-magnon simply by application of (7.15) to pseudovacuum (2.71)

$$B(\mu)|0\rangle = n(\mu)\sum_{k=1}^{L}[\mu]^k\bar{\psi}_k|0\rangle, \tag{8.3}$$

where we use the concise notation

$$[\mu] = \frac{\mu+1}{\mu} \quad \text{and} \quad n(\mu) = \frac{\mu^L}{\mu+1}. \tag{8.4}$$

The 2-magnon state is of the form

$$\begin{aligned} &B(\mu)B(\lambda)|0\rangle = \\ &\quad = n(\mu)n(\lambda)\sum_{1\leqslant r<s\leqslant L}\left([\lambda]^r[\mu]^s\frac{\lambda-\mu+1}{\lambda-\mu} + [\mu]^r[\lambda]^s\frac{\mu-\lambda+1}{\mu-\lambda}\right)\bar{\psi}_r\bar{\psi}_s|0\rangle, \end{aligned} \tag{8.5}$$

and the 3-magnon state is

$$\begin{aligned} &B(\nu)B(\mu)B(\lambda)|0\rangle = n(\nu)n(\mu)n(\lambda)\times \\ &\times\sum_{1\leqslant q<r<s\leqslant L}\sum_{\sigma\in S_3}\sigma\left([\nu]^q[\mu]^r[\lambda]^s\frac{\nu-\mu+1}{\nu-\mu}\frac{\nu-\lambda+1}{\nu-\lambda}\frac{\mu-\lambda+1}{\mu-\lambda}\right)\bar{\psi}_q\bar{\psi}_r\bar{\psi}_s|0\rangle. \end{aligned} \tag{8.6}$$

From the results (8.3), (8.5), and (8.6) we can conjecture that the general M-magnon state is of the form

$$|\lambda_1,\dots,\lambda_M\rangle \equiv B(\lambda_1)\cdots B(\lambda_M)|0\rangle = \left(\prod_{i=1}^{M} n(\lambda_i)\right)\times$$
$$\times \sum_{1\leqslant k_1<\ldots<k_M\leqslant L}\ \sum_{\sigma_\lambda\in S_M} \sigma_\lambda\left(\prod_{i<j}\frac{\lambda_i-\lambda_j+1}{\lambda_i-\lambda_j}\prod_{i=1}^{M}[\lambda_i]^{k_i}\right)\bar\psi_{k_1}\cdots\bar\psi_{k_M}|0\rangle \equiv$$
$$\equiv \left(\prod_{i=1}^{M} n(\lambda_i)\right)\prod_{i<j}^{M}\frac{1}{\lambda_i-\lambda_j}\sum_{1\leqslant k_1<\ldots<k_M\leqslant L}\ \sum_{\sigma_\lambda\in S_M}(-1)^{p(\sigma_\lambda)}\times$$
$$\times\,\sigma_\lambda\left(\prod_{i<j}^{M}(\lambda_i-\lambda_j+1)\prod_{i=1}^{M}[\lambda_i]^{k_i}\right)\bar\psi_{k_1}\cdots\bar\psi_{k_M}|0\rangle, \quad (8.7)$$

where σ_λ is a permutation of the parameters $\{\lambda_1,\dots,\lambda_M\}$; $p(\sigma_\lambda)=0,1(\mathrm{mod}\,2)$ is the parity of the permutation σ_λ, and $\sum\limits_{\sigma_\lambda\in S_M}$ is the sum over all such permutations. However, in the light of previous results this is no more a conjecture but a special representation of (4.7).

9. FERMIONIC REALIZATION OF XXZ

Substituting (6.6)–(6.9) into (2.30) gives a fermionic representation for the generators (2.32) of the Hecke algebra

$$\hat R^{(q)}_{kk+1} = \bar\psi_{k+1}\psi_k + \bar\psi_k\psi_{k+1} - q\bar\psi_k\psi_k - q^{-1}bar\psi_{k+1}\psi_{k+1}+$$
$$+\,(q+q^{-1})\bar\psi_k\psi_k\bar\psi_{k+1}\psi_{k+1} + q\mathbb{I}. \quad (9.1)$$

In the following, we will use the baxterized R-matrix (2.34) multiplied by $\mu^{1/2}$ for a simpler formula, which is of the form

$$\hat R_{k,k+1}(\mu) = (1-\mu)\big[\bar\psi_{k+1}\psi_k + \bar\psi_k\psi_{k+1} - q\bar\psi_k\psi_k - q^{-1}\bar\psi_{k+1}\psi_{k+1}+$$
$$+\,(q+q^{-1})\bar\psi_k\psi_k\bar\psi_{k+1}\psi_{k+1}\big] + (q-\mu q^{-1})\mathbb{I}. \quad (9.2)$$

We repeat the construction used in Sec. 7 with the R-matrix of the form (9.1) instead of (7.7) and the Yang–Baxter equation (2.35) instead of (2.10) resp. (7.5).

We recall the monodromy matrix of the form (7.6). Again, we write it in the form (7.8)

$$T_a(\mu) = \hat R_{a,1}(\mu)\cdot\underbrace{\hat R_{12}(\mu)\cdots\hat R_{L-1,L}(\mu)P_{L-1,L}\cdots P_{12}}_{X(\mu)}\cdot P_{a,1}. \quad (9.3)$$

The fermionic representation of $X(\mu)$ is obtained by (7.11) and (9.2).

As we have seen, we need to express the monodromy matrix (7.8) as a matrix in the auxiliary space V_a. The generator of the Hecke algebra (2.30) is of the form

$$\hat{R}^{(q)}_{a,1} = \begin{pmatrix} q - q^{-1}N_1 & \bar{\psi}_1 \\ \psi_1 & qN_1 \end{pmatrix}. \tag{9.4}$$

Then (2.34) is

$$\begin{aligned} \hat{R}_{a,1}(\mu) &= (1-\mu)\hat{R}^{(q)}_{a,1} + \mu(q-q^{-1})\mathbb{I} = \\ &= \begin{pmatrix} (q-\mu q^{-1})\mathbb{I} - (1-\mu)q^{-1}N_1 & (1-\mu)\bar{\psi}_1 \\ (1-\mu)\psi_1 & (1-\mu)qN_1 + \mu(q-q^{-1}) \end{pmatrix}. \end{aligned} \tag{9.5}$$

The form of $P_{a,1}$ is known from (7.11).

Using (7.11) and (9.5), we get for the matrix elements of $T_a(\mu)$

$${}_a(\mu) = \hat{R}_{a,1}(\mu)X(\mu)P_{a,1} = \begin{pmatrix} A(\mu) & B(\mu) \\ C(\mu) & D(\mu) \end{pmatrix}, \tag{9.6}$$

that is,

$$A(\mu) = \big[(q-\mu q^{-1})\mathbb{I} - (1-\mu)q^{-1}N_1\big]X(\mu)(\mathbb{I}-N_1) + (1-\mu)\bar{\psi}_1 X(\mu)\psi_1, \tag{9.7}$$

$$B(\mu) = \big[(q-\mu q^{-1})\mathbb{I} - (1-\mu)q^{-1}N_1\big]X(\mu)\bar{\psi}_1 + (1-\mu)\bar{\psi}_1 X(\mu)N_1, \tag{9.8}$$

$$C(\mu) = (1-\mu)\psi_1 X(\mu)(\mathbb{I}-N_1) + \big[(1-\mu)qN_1 + \mu(q-q^{-1})\big]X(\mu)\psi_1, \tag{9.9}$$

$$D(\mu) = (1-\mu)\psi_1 X(\mu)\bar{\psi}_1 + \big[(1-\mu)qN_1 + \mu(q-q^{-1})\big]X(\mu)N_1. \tag{9.10}$$

10. BETHE VECTORS FOR THE HOMOGENEOUS XXZ MODEL

As in Sec. 8, we are interested in the Bethe vectors (2.75)

$$|\lambda_1, \dots, \lambda_M\rangle = B(\lambda_1)\cdots B(\lambda_M)|0\rangle \tag{10.1}$$

with the opeartor $B(\mu)$ of the form (9.8). The details are postponed to Appendix B.

For the 1-magnon we get

$$|\mu\rangle \equiv B(\mu)|0\rangle = n_q(\mu)\sum_{k=1}^{L}[\mu]_q^{\,k}\ \bar{\psi}_k|0\rangle, \tag{10.2}$$

where we introduce

$$[\mu]_q = \frac{q-\mu q^{-1}}{1-\mu} \tag{10.3}$$

and the normalization

$$n_q(\mu) = \frac{(q-q^{-1})(1-\mu)^L}{q-\mu q^{-1}}. \tag{10.4}$$

The 2-magnon state is obtained in the following form:

$$|\lambda,\mu\rangle \equiv B(\lambda)B(\mu) = n_q(\lambda)n_q(\mu)\times \\ \times \sum_{1\leqslant r<s\leqslant L}\left\{\frac{\lambda q^{-1}-\mu q}{\lambda-\mu}[\lambda]_q^r[\mu]_q^s + \frac{\mu q^{-1}-\lambda q}{\mu-\lambda}[\mu]_q^r[\lambda]_q^s\right\}\bar\psi_r\bar\psi_s|0\rangle. \tag{10.5}$$

We can see that the situation is very similar to that in Sec. 8. Again, we propose that the general M-magnon state possesses the form

$$|\lambda_1,\dots,\lambda_M\rangle = \prod_{l=1}^M n_q(\lambda_l) \sum_{1\leqslant k_1<\dots<k_M\leqslant L} \times \\ \times \sum_{\sigma_\lambda\in S_M}\sigma_\lambda\left(\prod_{i<j}\frac{\lambda_i q^{-1}-\lambda_j q}{\lambda_i-\lambda_j}\prod_{i=1}^M[\lambda_i]_q^{k_i}\right)\bar\psi_{k_1}\cdots\bar\psi_{k_M}|0\rangle, \tag{10.6}$$

where S_M is the symmetric group of order M, and $\sigma_\lambda \in S_M$ permutes the parameters $\{\lambda_1,\dots,\lambda_M\}$. Again, this is just a special representation of (4.7). In the next section, we prove formula (10.6) by using the coordinate Bethe ansatz.

11. FERMIONIC MODELS AND COORDINATE BETHE ANSATZ

In this Section, we will use the coordinate Bethe ansatz method to construct Bethe vectors for the periodic chain models which are formulated in terms of free fermions. The coordinate Bethe ansatz method is named after the seminal work by Hans Bethe [20]. Bethe found eigenfunctions and spectrum of the one-dimensional spin-1/2 isotropic magnet (which we called above as XXX Heisenberg closed spin chain model). The review of the applications of the coordinate Bethe ansatz method can be found in the book [21] (see also [22] and references therein).

11.1. R-Matrix, Hamiltonian and a Vacuum State. Recall that the fermionic representation of the Hecke algebra (2.33) is based on the realization of the R-matrix in the form

$$\widehat{R}_{k,k+1} = \bar\psi_{k+1}\psi_k + \bar\psi_k\psi_{k+1} - \\ - q\bar\psi_k\psi_k - q^{-1}\bar\psi_{k+1}\psi_{k+1} + (q+q^{-1})\bar\psi_k\psi_k\bar\psi_{k+1}\psi_{k+1} + q.$$

Consider the Hamiltonian for the periodic fermionic chain model ("small polaron model", see [5] and references therein)

$$\mathcal{H} = \sum_{k=1}^{L-1} \widehat{R}_{k,k+1} + \widehat{R}_{L,1} - qL =$$

$$= \sum_{k=1}^{L-1} (\bar\psi_{k+1}\psi_k + \bar\psi_k\psi_{k+1} - q\bar\psi_k\psi_k - q^{-1}\bar\psi_{k+1}\psi_{k+1} + (q+q^{-1})\bar\psi_k\psi_k\bar\psi_{k+1}\psi_{k+1}) +$$

$$+ \bar\psi_1\psi_L + \bar\psi_L\psi_1 - q\bar\psi_L\psi_L - q^{-1}\bar\psi_1\psi_1 + (q+q^{-1})\bar\psi_L\psi_L\bar\psi_1\psi_1 =$$

$$= \sum_{k=1}^{L-1} (\bar\psi_{k+1}\psi_k + \bar\psi_k\psi_{k+1} + (q+q^{-1})\bar\psi_k\psi_k\bar\psi_{k+1}\psi_{k+1}) +$$

$$+ \bar\psi_1\psi_L + \bar\psi_L\psi_1 + (q+q^{-1})\bar\psi_L\psi_L\bar\psi_1\psi_1 - (q+q^{-1})\sum_{k=1}^{L} \bar\psi_k\psi_k. \quad (11.1)$$

This model is not coincident with the XXZ spin chain in view of the representation of the matrix $\widehat{R}_{L,1}$ given in (2.30) in terms of fermions (6.3). In the XXZ case the fermionic representation of $\widehat{R}_{L,1}$ is nonlocal.

The vacuum state $|0\rangle$ of the Hamiltonian is defined by the equations $\psi_k|0\rangle = 0$ for $k = 1, 2, \ldots, L$.

11.2. The 1-Magnon States. We look for the 1-magnon solution in the form

$$|1\rangle = \sum_{n=1}^{L} c_n\, \bar\psi_n|0\rangle. \quad (11.2)$$

Substitution of (11.1) and (11.2) in the eigenvalue problem $\mathcal{H}|1\rangle = E|1\rangle$ gives the following equation for the coefficients c_n (the 4-fermionic term in (11.1) does not contribute to the equations):

$$c_{n-1} + c_{n+1} = (E + (q+q^{-1}))\, c_n, \quad 1 \leqslant n \leqslant L, \quad (11.3)$$

where $c_{n+L} = c_n$, i.e., $c_0 = c_L$ and $c_{L+1} = c_1$. Since Eq. (11.3) is the discrete version of the ordinary differential equation of the second order with constant coefficients, one can solve (11.3) if we insert $c_n = X^n$. As a result, we obtain the condition

$$E + (q+q^{-1}) = X + X^{-1}, \quad (11.4)$$

which is symmetric under the exchange $X \leftrightarrow X^{-1}$. Thus, the general solution of (11.3) is

$$c_n = A_1 X^n + A_2 X^{-n}, \quad (11.5)$$

where arbitrary constants A_1, A_2 are independent of n. The boundary conditions $c_k = c_{L+k}$ lead to the equation for X:

$$X^L = 1. \tag{11.6}$$

However, in this case, we have $X^{-n} = X^{L-n}$, and linearly independent solutions are

$$c_n = X^n, \quad \text{where} \quad X^L = 1. \tag{11.7}$$

Thus, to each solution $X = X_k$ of equation (11.6),

$$X_k = \exp\left(\frac{2\pi i k}{L}\right) \quad (k = 0, \dots, L-1), \tag{11.8}$$

we have two 1-magnon states (orthogonal to each other)

$$|1\rangle_k = \sum_{n=1}^{L} X_k^n \bar{\psi}_n |0\rangle, \quad |1\rangle'_k = \sum_{n=1}^{L} X_k^{-n} \bar{\psi}_n |0\rangle \tag{11.9}$$

with the same energy

$$E = (q + q^{-1}) + (X_k + X_k^{-1}). \tag{11.10}$$

On the other hand, we have $X_k^{-1} = X_{L-k}$, and the set of vectors $|1\rangle_{L-k}$ coincides with the set of vectors $|1\rangle'_k$. All these solutions correspond to the spectrum of free fermions.

11.3. The 2-Magnon States. We write $|n_1, n_2\rangle = \bar{\psi}_{n_1}\bar{\psi}_{n_2}|0\rangle$, where $1 \leqslant n_1 < n_2 \leqslant L$. It is easy to find that the action of the Hamiltonian on the vector $|2\rangle = \sum\limits_{1 \leqslant n_1 < n_2 \leqslant L} c_{n_1,n_2}|n_1, n_2\rangle$ is

$$\begin{aligned}\mathcal{H}|2\rangle = \sum_{1 \leqslant n_1 < n_2 \leqslant L} \big((1-\delta_{n_1,1})c_{n_1-1,n_2} + (1-\delta_{n_1+1,n_2})(c_{n_1,n_2-1} + c_{n_1+1,n_2}) + \\ + (1-\delta_{n_2,L})c_{n_1,n_2+1} - \delta_{n_1,1}(1-\delta_{n_2,L})\, c_{n_2,L} - (1-\delta_{n_1,1})\, \delta_{n_2,L} c_{1,n_1} + \\ + (q+q^{-1})(\delta_{n_1+1,n_2} + \delta_{n_1,1}\delta_{n_2,L} - 2)\, c_{n_1,n_2}\big)|n_1, n_2\rangle.\end{aligned}$$

Equation $\mathcal{H}|2\rangle = \mathcal{E}|2\rangle$ is then equivalent to the system of equations

$$\begin{aligned}(1-\delta_{n_1,1})\, c_{n_1-1,n_2} + (1-\delta_{n_1+1,n_2})(c_{n_1,n_2-1} + c_{n_1+1,n_2}) + \\ + (1-\delta_{n_2,L})c_{n_1,n_2+1} - \delta_{n_1,1}\big(1-\delta_{n_2,L}\big)c_{n_2,L} - (1-\delta_{n_1,1})\delta_{n_2,L} c_{1,n_1} = \\ = \big(\mathcal{E} + 2(q+q^{-1}) - (q+q^{-1})(\delta_{n_1+1,n_2} + \delta_{n_1,1}\delta_{n_2,L})\big)c_{n_1,n_2}\end{aligned}$$

for any $1 \leqslant n_1 < n_2 \leqslant L$.

The coordinate Bethe ansatz is based on the idea to write

$$\mathcal{E} + 2(q + q^{-1}) = X_1 + X_1^{-1} + X_2 + X_2^{-1}$$

and to find solution of the system in the form

$$c_{n_1,n_2} = A_{12} X_1^{n_1} X_2^{n_2} + A_{21} X_2^{n_1} X_1^{n_2},$$

where A_{12} and A_{21} are independent of n_1 and n_2, but they can depend on X_1 and X_2.

Substituting this assumption into the equation, we obtain

$$\begin{aligned}
\delta_{n_1+1,n_2}\big(A_{12}(X_1X_2 - (q+q^{-1})X_2 + 1) + A_{21}(X_1X_2 - (q+q^{-1})X_1 + 1)\big)\times \\
\times (X_1X_2)_1^n + (A_{12} + X_1^L A_{21})(\delta_{n_1,1} X_2^{n_2} + \delta_{n_2,L} X_1 X_2^{n_1}) + \\
+ (A_{21} + X_2^L A_{12})(\delta_{n_1,1} X_1^{n_2} + \delta_{n_2,L} X_1^{n_1} X_2) = \delta_{n_1,1}\delta_{n_2,L} \times \\
\times \big(((X_1X_2)^L + X_1X_2)(A_{12} + A_{21}) + (q+q^{-1})(A_{12}X_1X_2^L + A_{21}X_1^L X_2)\big).
\end{aligned}$$

To fulfill these equations, we put

$$\begin{gathered}
A_{12}\big(X_1X_2 - (q+q^{-1})X_2 + 1\big) + A_{21}\big(X_1X_2 - (q+q^{-1})X_1 + 1\big) = 0, \\
A_{12} + X_1^L A_{21} = 0, \quad A_{21} + X_2^L A_{12} = 0,
\end{gathered}$$

or equivalently

$$\begin{aligned}
\frac{A_{21}}{A_{12}} &= -\frac{X_1X_2 - (q+q^{-1})X_2 + 1}{X_1X_2 - (q+q^{-1})X_1 + 1}, \\
X_1^L &= \frac{X_1X_2 - (q+q^{-1})X_1 + 1}{X_1X_2 - (q+q^{-1})X_2 + 1}, \\
X_2^L &= \frac{X_1X_2 - (q+q^{-1})X_2 + 1}{X_1X_2 - (q+q^{-1})X_1 + 1}.
\end{aligned}$$

11.4. The 3-Magnon States. For $1 \leqslant n_1 < n_2 < n_3 \leqslant L$, we put $|n_1, n_2, n_3\rangle = \bar\psi_{n_1}\bar\psi_{n_2}\bar\psi_{n_3}|0\rangle$. The action of the Hamiltonian $\mathcal{H}$ on a vector $|3\rangle = \sum\limits_{1\leqslant n_1<n_2<n_3\leqslant L} c_{n_1,n_2,n_3}|n_1,n_2,n_3\rangle$ is

$$\begin{aligned}
\mathcal{H}|3\rangle = \sum_{1\leqslant n_1<n_2<n_3\leqslant L} \big((1-\delta_{n_1,1})c_{n_1-1,n_2,n_3} + \\
+ (1-\delta_{n_1+1,n_2})(c_{n_1,n_2-1,n_3} + c_{n_1+1,n_2,n_3}) + \\
+ (1-\delta_{n_2+1,n_3})(c_{n_1,n_2,n_3-1} + c_{n_1,n_2+1,n_3}) + (1-\delta_{n_3,L})c_{n_1,n_2,n_3+1} + \\
+ \delta_{n_1,1}(1-\delta_{n_3,L})c_{n_2,n_3,L} + \delta_{n_3,L}(1-\delta_{n_1,1})c_{1,n_1,n_2} + \\
+ (q+q^{-1})(\delta_{n_1+1,n_2} + \delta_{n_2+1,n_3} + \delta_{n_1,1}\delta_{n_3,L} - 3)c_{n_1,n_2,n_3}\big)|n_1,n_2,n_3\rangle.
\end{aligned}$$

Equation $\mathcal{H}|3\rangle = \mathcal{E}|3\rangle$ is equivalent to the system of equations

$$\begin{aligned}(1-\delta_{n_1,1})c_{n_1-1,n_2,n_3} + (1-\delta_{n_1+1,n_2})(c_{n_1,n_2-1,n_3} + c_{n_1+1,n_2,n_3}) + \\ + (1-\delta_{n_2+1,n_3})(c_{n_1,n_2,n_3-1} + c_{n_1,n_2+1,n_3}) + (1-\delta_{n_3,L})c_{n_1,n_2,n_3+1} + \\ + \delta_{n_1,1}(1-\delta_{n_3,L})\, c_{n_2,n_3,L} + \delta_{n_3,L}(1-\delta_{n_1,1})\, c_{1,n_1,n_2} = \\ = \big(\mathcal{E} + 3(q+q^{-1}) - (q+q^{-1})(\delta_{n_1+1,n_2} + \delta_{n_2+1,n_3} + \delta_{n_1,1}\delta_{n_3,L})\big)c_{n_1,n_2,n_3},\end{aligned}$$

where $1 \leqslant n_1 < n_2 < n_3 \leqslant L$. When we put

$$\mathcal{E} + 3(q+q^{-1}) = X_1 + X_1^{-1} + X_2 + X_2^{-1} + X_3 + X_3^{-1}$$

and look for solution of c_{n_1,n_2,n_3} in the form

$$c_{n_1,n_2,n_3} = \sum_{\sigma\in S_3} A_\sigma X_{\sigma(1)}^{n_1} X_{\sigma(2)}^{n_2} X_{\sigma(3)}^{n_3},$$

we obtain the following system of equations:

$$\begin{aligned}\delta_{n_1+1,n_2} \sum_{\sigma\in S_3} A_\sigma \big(X_{\sigma(1)}X_{\sigma(2)} - (q+q^{-1})X_{\sigma(2)} + 1\big)(X_{\sigma(1)}X_{\sigma(2)})^{n_1} X_{\sigma(3)}^{n_3} + \\ + \delta_{n_2+1,n_3} \sum_{\sigma\in S_3} A_\sigma \big(X_{\sigma(2)}X_{\sigma(3)} - (q+q^{-1})X_{\sigma(3)} + 1\big) X_{\sigma(1)}^{n_1}(X_{\sigma(2)}X_{\sigma(3)})^{n_2} + \\ + \delta_{n_1,1} \sum_{\sigma\in S_3} A_\sigma (X_{\sigma(2)}^{n_2} X_{\sigma(3)}^{n_3} - X_{\sigma(1)}^{n_2} X_{\sigma(2)}^{n_3} X_{\sigma(3)}^{L}) + \\ + \delta_{n_3,L} \sum_{\sigma\in S_3} A_\sigma (X_{\sigma(1)}^{n_1} X_{\sigma(2)}^{n_2} X_{\sigma(3)}^{L+1} - X_{\sigma(1)} X_{\sigma(2)}^{n_1} X_{\sigma(3)}^{n_2}) + \\ + \delta_{n_1,1}\delta_{n_3,L} \sum_{\sigma\in S_3} A_\sigma \big(X_{\sigma(1)}^{n_2} X_{\sigma(2)}^{L} X_{\sigma(3)}^{L} + X_{\sigma(1)}X_{\sigma(2)} X_{\sigma(3)}^{n_2} - \\ - (q+q^{-1}) X_{\sigma(1)} X_{\sigma(2)}^{n_2} X_{\sigma(3)}^{L}\big) = 0.\end{aligned}$$

Let π_1 be the transposition $1 \leftrightarrow 2$; and π_2, the transposition $2 \leftrightarrow 3$. To cancel the terms at δ_{n_1+1,n_2} and δ_{n_2+1,n_3}, it is sufficient for any $\sigma \in S_3$ to put

$$\begin{aligned}A_\sigma \big(X_{\sigma(1)}X_{\sigma(2)} - (q+q^{-1})X_{\sigma(2)} + 1\big) + \\ + A_{\sigma\circ\pi_1}\big(X_{\sigma(1)}X_{\sigma(2)} - (q+q^{-1})X_{\sigma(1)} + 1\big) = 0,\end{aligned}$$

$$\begin{aligned}A_\sigma \big(X_{\sigma(2)}X_{\sigma(3)} - (q+q^{-1})X_{\sigma(3)} + 1\big) + \\ + A_{\sigma\circ\pi_2}\big(X_{\sigma(2)}X_{\sigma(3)} - (q+q^{-1})X_{\sigma(2)} + 1\big) = 0.\end{aligned}$$

If we consider the element $\epsilon \in S_3$ defined by the relations $\epsilon(1) = 3$, $\epsilon(2) = 1$, $\epsilon(3) = 2$, we obtain

$$\sum_{\sigma\in S_3} A_\sigma(X^{n_2}_{\sigma(2)}X^{n_3}_{\sigma(3)} - X^{n_2}_{\sigma(1)}X^{n_3}_{\sigma(2)}X^{L}_{\sigma(3)}) =$$
$$= \sum_{\sigma\in S_3} (A_\sigma X^{n_2}_{\sigma(2)}X^{n_3}_{\sigma(3)} - A_\sigma X^{n_2}_{\sigma\circ\epsilon(2)}X^{n_3}_{\sigma\circ\epsilon(3)}X^{L}_{\sigma\circ\epsilon(1)}) =$$
$$= \sum_{\sigma\in S_3} (A_\sigma - A_{\sigma\circ\epsilon^{-1}}X^{L}_{\sigma(1)})X^{n_2}_{\sigma(2)}X^{n_3}_{\sigma(3)}.$$

So, we put for any $\sigma \in S_3$

$$A_\sigma - A_{\sigma\circ\epsilon^{-1}}X^{L}_{\sigma(1)} = 0, \quad \text{i.e.,} \quad A_{\sigma\circ\epsilon} = A_\sigma X^{L}_{\sigma(3)}.$$

It is easy to show that these three assumptions solve the whole system for c_{n_1,n_2,n_3}.

We obtained for the A_σ conditions

$$\begin{aligned} A_{\sigma\circ\pi_1} &= -\frac{X_{\sigma(1)}X_{\sigma(2)} - (q+q^{-1})X_{\sigma(2)} + 1}{X_{\sigma(1)}X_{\sigma(2)} - (q+q^{-1})X_{\sigma(1)} + 1}\, A_\sigma, \\ A_{\sigma\circ\pi_2} &= -\frac{X_{\sigma(2)}X_{\sigma(3)} - (q+q^{-1})X_{\sigma(3)} + 1}{X_{\sigma(2)}X_{\sigma(3)} - (q+q^{-1})X_{\sigma(2)} + 1}\, A_\sigma. \end{aligned} \tag{11.11}$$

It follows from these relations that for any $\sigma \in S_3$

$$A_{(\sigma\circ\pi_1)\circ\pi_1} = -\frac{X_{\sigma\circ\pi_1(1)}X_{\sigma\circ\pi_1(2)} - (q+q^{-1})X_{\sigma\circ\pi_1(2)} + 1}{X_{\sigma\circ\pi_1(1)}X_{\sigma\circ\pi_1(2)} - (q+q^{-1})X_{\sigma\circ\pi_1(1)} + 1}\, A_{\sigma\circ\pi_1} = A_\sigma$$

holds. Similarly, we can show that $A_{(\sigma\circ\pi_2)\circ\pi_2} = A_\sigma$.

Moreover, we have

$$A_{((\sigma\circ\pi_1)\circ\pi_2)\circ\pi_1} = -\frac{X_{\sigma(2)}X_{\sigma(3)} - (q+q^{-1})A_{\sigma(3)} + 1}{X_{\sigma(2)}X_{\sigma(3)} - (q+q^{-1})A_{\sigma(2)} + 1}\, A_{(\sigma\circ\pi_1)\circ\pi_2} =$$
$$= \frac{X_{\sigma(2)}X_{\sigma(3)} - (q+q^{-1})A_{\sigma(3)}+1}{X_{\sigma(2)}X_{\sigma(3)}-(q+q^{-1})A_{\sigma(2)} + 1}\frac{X_{\sigma(1)}X_{\sigma(3)} - (q+q^{-1})A_{\sigma(3)} + 1}{X_{\sigma(1)}X_{\sigma(3)} - (q+q^{-1})A_{\sigma(1)}+1}\, A_{\sigma\circ\pi_1} =$$
$$= -\frac{X_{\sigma(2)}X_{\sigma(3)} - (q+q^{-1})A_{\sigma(3)} + 1}{X_{\sigma(2)}X_{\sigma(3)} - (q+q^{-1})A_{\sigma(2)} + 1}\frac{X_{\sigma(1)}X_{\sigma(3)} - (q+q^{-1})A_{\sigma(3)} + 1}{X_{\sigma(1)}X_{\sigma(3)} - (q+q^{-1})A_{\sigma(1)} + 1}\times$$
$$\times \frac{X_{\sigma(1)}X_{\sigma(2)} - (q+q^{-1})X_{\sigma(2)} + 1}{X_{\sigma(1)}X_{\sigma(2)} - (q+q^{-1})X_{\sigma(1)} + 1}\, A_\sigma,$$

$$A_{((\sigma\circ\pi_2)\circ\pi_1)\circ\pi_2} = -\frac{X_{\sigma(1)}X_{\sigma(2)} - (q+q^{-1})A_{\sigma(2)} + 1}{X_{\sigma(1)}X_{\sigma(2)} - (q+q^{-1})A_{\sigma(1)} + 1} A_{(\sigma\circ\pi_2)\circ\pi_1} =$$

$$= \frac{X_{\sigma(1)}X_{\sigma(2)} - (q+q^{-1})A_{\sigma(2)}+1}{X_{\sigma(1)}X_{\sigma(2)} - (q+q^{-1})A_{\sigma(1)}+1} \frac{X_{\sigma(1)}X_{\sigma(3)} - (q+q^{-1})A_{\sigma(3)}+1}{X_{\sigma(1)}X_{\sigma(3)} - (q+q^{-1})A_{\sigma(1)}+1} A_{\sigma\circ\pi_2} =$$

$$= -\frac{X_{\sigma(1)}X_{\sigma(2)} - (q+q^{-1})A_{\sigma(2)} + 1}{X_{\sigma(1)}X_{\sigma(2)} - (q+q^{-1})A_{\sigma(1)} + 1} \frac{X_{\sigma(1)}X_{\sigma(3)} - (q+q^{-1})A_{\sigma(3)} + 1}{X_{\sigma(1)}X_{\sigma(3)} - (q+q^{-1})A_{\sigma(1)} + 1} \times$$

$$\times \frac{X_{\sigma(2)}X_{\sigma(3)} - (q+q^{-1})X_{\sigma(3)} + 1}{X_{\sigma(2)}X_{\sigma(3)} - (q+q^{-1})X_{\sigma(2)} + 1} A_{\sigma}.$$

So for any $\sigma \in S_3$ the relation $A_{((\sigma\circ\pi_1)\circ\pi_2)\circ\pi_1} = A_{((\sigma\circ\pi_2)\circ\pi_1)\circ\pi_2}$ holds. Therefore, A_σ is really a function of the symmetric group S_3.

Since $\epsilon = \pi_2 \circ \pi_1$, the equality $A_{\sigma\circ\epsilon} = X^L_{\sigma(3)} A_\sigma$ leads to the relation

$$A_{\sigma\circ\epsilon} = A_{(\sigma\circ\pi_2)\circ\pi_1} = -\frac{X_{\sigma(1)}X_{\sigma(3)} - (q+q^{-1})X_{\sigma(3)} + 1}{X_{\sigma(1)}X_{\sigma(3)} - (q+q^{-1})X_{\sigma(1)} + 1} A_{\sigma\circ\pi_2} =$$

$$= \frac{X_{\sigma(1)}X_{\sigma(3)} - (q+q^{-1})X_{\sigma(3)} + 1}{X_{\sigma(1)}X_{\sigma(3)} - (q+q^{-1})X_{\sigma(1)} + 1} \frac{X_{\sigma(2)}X_{\sigma(3)} - (q+q^{-1})X_{\sigma(3)} + 1}{X_{\sigma(2)}X_{\sigma(3)} - (q+q^{-1})X_{\sigma(2)} + 1} A_\sigma =$$

$$= X^L_{\sigma(3)} A_\sigma.$$

So for any $\sigma \in S_3$, the relation

$$X^L_{\sigma(3)} = \frac{X_{\sigma(1)}X_{\sigma(3)} - (q+q^{-1})X_{\sigma(3)} + 1}{X_{\sigma(1)}X_{\sigma(3)} - (q+q^{-1})X_{\sigma(1)} + 1} \frac{X_{\sigma(2)}X_{\sigma(3)} - (q+q^{-1})X_{\sigma(3)} + 1}{X_{\sigma(2)}X_{\sigma(3)} - (q+q^{-1})X_{\sigma(2)} + 1}$$

has to hold. It is possible to rewrite these relations in the form

$$X^L_i = \prod_{k\neq i} \frac{X_iX_k - (q+q^{-1})X_i + 1}{X_iX_k - (q+q^{-1})X_k + 1}. \tag{11.12}$$

11.5. The M-Magnon States. For $1 \leqslant n_1 < n_2 < \ldots < n_{M-1} < n_M \leqslant L$ we denote

$$|\mathbf{n}\rangle = |n_1, n_2, \ldots, n_M\rangle = \bar\psi_{n_1}\bar\psi_{n_2}\cdots\bar\psi_{n_M}|0\rangle$$

and take the vector

$$|M\rangle = \sum_{\mathbf{n}} c_{\mathbf{n}}|\mathbf{n}\rangle = \sum_{1\leqslant n_1<n_2<\ldots<n_M\leqslant L} c_{n_1,\ldots,n_M}|n_1,\ldots,n_M\rangle.$$

When we write

$$(\mathbf{n}\pm\mathbf{e}_k) = (n_1,\ldots,n_{k-1},n_k\pm1,n_{k+1},\ldots,n_M),$$

it is possible to show that

$$\mathcal{H}|M\rangle = \sum_{\mathbf{n}} \Bigg((1-\delta_{n_1,1})\, c_{\mathbf{n}-\mathbf{e}_1} + \sum_{k=1}^{M-1} (1-\delta_{n_k+1,n_{k+1}})(c_{\mathbf{n}+\mathbf{e}_k} + c_{\mathbf{n}-\mathbf{e}_{k+1}}) + \\ + (1-\delta_{n_M,L})\, c_{\mathbf{n}+\mathbf{e}_M} + (-1)^{M-1}\delta_{n_1,1}(1-\delta_{n_M,L})\, c_{n_2,\ldots,n_M,L} + \\ + (-1)^{M-1}\delta_{n_M,L}(1-\delta_{n_1,1})\, c_{1,n_1,\ldots,n_{M-1}} + \\ + (q+q^{-1}) \sum_{k=1}^{M-1} \delta_{n_k+1,n_{k+1}} c_{\mathbf{n}} + (q+q^{-1})\delta_{n_1,1}\delta_{n_M,L} c_{\mathbf{n}} - M(q+q^{-1})\, c_{\mathbf{n}} \Bigg) |\mathbf{n}\rangle.$$

Equation $\mathcal{H}|M\rangle = \mathcal{E}|M\rangle$ is then equivalent to the system

$$(1-\delta_{n_1,1})\, c_{\mathbf{n}-\mathbf{e}_1} + \sum_{k=1}^{M-1} (1-\delta_{n_k+1,n_{k+1}})(c_{\mathbf{n}+\mathbf{e}_k} + c_{\mathbf{n}-\mathbf{e}_{k+1}}) + \\ + (1-\delta_{n_M,L})\, c_{\mathbf{n}+\mathbf{e}_M} + (-1)^{M-1}\delta_{n_1,1}(1-\delta_{n_M,L})\, c_{n_2,\ldots,n_M,L} + \\ + (-1)^{M-1}\delta_{n_M,L}(1-\delta_{n_1,1})\, c_{1,n_1,\ldots,n_{M-1}} = \\ = \left(\mathcal{E} + M(q+q^{-1}) - (q+q^{-1}) \sum_{k=1}^{M-1} \delta_{n_k+1,n_{k+1}} - (q+q^{-1})\delta_{n_1,1}\delta_{n_M,L} \right) c_{\mathbf{n}}.$$

When we write the eigenvalue of the Hamiltonian as

$$\mathcal{E} = \sum_{k=1}^{M} (X_k + X_k^{-1}) - M(q+q^{-1}), \tag{11.13}$$

look for the solution in the form

$$c_{\mathbf{n}} = \sum_{\sigma \in S_M} A_\sigma X_{\sigma(1)}^{n_1} X_{\sigma(2)}^{n_2} \cdots X_{\sigma M}^{n_M}$$

and substitute these assumptions into the system, we obtain

$$\sum_{k=1}^{M-1} \delta_{n_k+1,n_{k+1}} \sum_{\sigma \in S_M} A_\sigma (1 + X_{\sigma(k)} X_{\sigma(k+1)} - (q+q^{-1}) X_{\sigma(k+1)}) \times \\ \times X_{\sigma(1)}^{n_1} \cdots (X_{\sigma(k)} X_{\sigma(k+1)})^{n_k} \cdots X_{\sigma(M)}^{n_M} + \delta_{n_1,1} \sum_{\sigma \in S_M} A_\sigma (X_{\sigma(2)}^{n_2} \cdots X_{\sigma(M)}^{n_M} + \\ + (-1)^M X_{\sigma(1)}^{n_2} X_{\sigma(2)}^{n_3} \cdots X_{\sigma(M-1)}^{n_M} X_{\sigma(M)}^{L}) + \\ + \delta_{n_M,L} \sum_{\sigma \in S_M} A_\sigma (X_{\sigma(1)}^{n_1} \cdots X_{\sigma(M-1)}^{n_M-1} X_{\sigma(M)}^{L+1} +$$

$$+(-1)^M X_{\sigma(1)} X_{\sigma(2)}^{n_1} X_{\sigma(3)}^{n_2} \cdots X_{\sigma(M)}^{n_{M-1}}) -$$
$$-(-1)^M \delta_{n_1,1}\delta_{n_M,L} \sum_{\sigma\in S_M} A_\sigma \big(X_{\sigma(1)}^{n_2} X_{\sigma(2)}^{n_3} \cdots X_{\sigma(M-1)}^{L} X_{\sigma(M)}^{L} +$$
$$+ X_{\sigma(1)} X_{\sigma(2)} X_{\sigma(3)}^{n_2} \cdots X_{\sigma(M)}^{n_{M-1}} + (-1)^M (q+q^{-1}) \times$$
$$\times X_{\sigma(1)} X_{\sigma(2)}^{n_2} \cdots X_{\sigma(M-1)}^{n_{M-1}} X_{\sigma(M)}^{L} \big) = 0.$$

Let π_k, $k = 1, \ldots, M-1$, be transpositions $k \leftrightarrow k+1$. When the relation

$$(X_{\sigma(k)} X_{\sigma(k+1)} - (q+q^{-1}) X_{\sigma(k+1)} + 1) A_\sigma +$$
$$+ (X_{\sigma(k)} X_{\sigma(k+1)} - (q+q^{-1}) X_{\sigma(k)} + 1) A_{\sigma\circ\pi_k} = 0 \quad (11.14)$$

is true for any $\sigma \in S_M$ and $k = 1, \ldots, M-1$, the terms at $\delta_{n_k+1,n_{k+1}}$ vanish.

Let $\epsilon \in S_M$ be defined by the relations $\epsilon(k) = k-1$ for $k = 2, \ldots, M$ and $\epsilon(1) = M$. If we require

$$A_\sigma + (-1)^M X_{\sigma(1)}^L A_{\sigma\circ\epsilon^{-1}} = 0, \quad \text{i.e.,} \quad A_{\sigma\circ\epsilon} = (-1)^{M-1} X_{\sigma(M)}^L A_\sigma \quad (11.15)$$

for any $\sigma \in S_M$, the terms at $\delta_{n_1,1}$ and $\delta_{n_M,L}$ are annulled.

Combining (11.14) and (11.15) we get

$$\sum_{\sigma\in S_M} A_\sigma \big(X_{\sigma(1)}^{n_2} X_{\sigma(2)}^{n_3} \cdots X_{\sigma(M-1)}^{L} X_{\sigma(M)}^{L} + X_{\sigma(1)} X_{\sigma(2)} X_{\sigma(3)}^{n_2} \cdots X_{\sigma(M)}^{n_{M-1}} +$$
$$+ (-1)^M (q+q^{-1}) X_{\sigma(1)} X_{\sigma(2)}^{n_2} \cdots X_{\sigma(M-1)}^{n_{M-1}} X_{\sigma(M)}^{L} \big) =$$
$$= \sum_{\sigma\in S_M} A_\sigma \big(1 + X_{\sigma(1)} X_{\sigma(2)} - (q+q^{-1}) X_{\sigma(2)} \big) X_{\sigma(3)}^{n_2} X_{\sigma(4)}^{n_3} \cdots X_{\sigma(M)}^{n_{M-1}} = 0.$$

Therefore, the assumptions (11.14) and (11.15) solve the system for $c_{\mathbf{n}}$.

We rewrite relation (11.14) as

$$A_{\sigma\circ\pi_k} = -\frac{X_{\sigma(k)} X_{\sigma(k+1)} - (q+q^{-1}) X_{\sigma(k+1)} + 1}{X_{\sigma(k)} X_{\sigma(k+1)} - (q+q^{-1}) X_{\sigma(k)} + 1} A_\sigma. \quad (11.16)$$

From this relation it is easy to show that for any $\sigma \in S_M$ and $k = 1, \ldots, M-1$, the relations

$$A_{(\sigma\circ\pi_k)\circ\pi_k} = A_\sigma, \quad A_{((\sigma\circ\pi_k)\circ\pi_{k+1})\circ\pi_k} = A_{((\sigma\circ\pi_{k+1})\circ\pi_k)\circ\pi_{k+1}}$$

are valid. Therefore, A_σ is really a function on symmetry group S_M.

If we write $\epsilon = \pi_{M-1} \circ \pi_{M-2} \circ \ldots \circ \pi_2 \circ \pi_1$ and use (11.16), it is possible to rewrite (11.15) as

$$\begin{aligned} A_{\sigma\circ\epsilon} &= -\frac{X_{\sigma(1)}X_{\sigma(M)} - (q+q^{-1})X_{\sigma(M)} + 1}{X_{\sigma(1)}X_{\sigma(M)} - (q+q^{-1})X_{\sigma(1)} + 1} A_{\sigma\circ\pi_{M-1}\circ\ldots\circ\pi_2} = \\ &= (-1)^2 \frac{X_{\sigma(1)}X_{\sigma(M)} - (q+q^{-1})X_{\sigma(M)} + 1}{X_{\sigma(1)}X_{\sigma(M)} - (q+q^{-1})X_{\sigma(1)} + 1} \times \\ &\times \frac{X_{\sigma(2)}X_{\sigma(M)} - (q+q^{-1})X_{\sigma(M)} + 1}{X_{\sigma(2)}X_{\sigma(M)} - (q+q^{-1})X_{\sigma(2)} + 1} A_{\sigma\circ\pi_{M-1}\circ\ldots\circ\pi_3} = \\ &= (-1)^{M-1} \prod_{k=1}^{M-1} \frac{X_{\sigma(k)}X_{\sigma(M)} - (q+q^{-1})X_{\sigma(M)} + 1}{X_{\sigma(k)}X_{\sigma(M)} - (q+q^{-1})X_{\sigma(k)} + 1} A_\sigma = \\ &= (-1)^{M-1} X^L_{\sigma(M)} A_\sigma. \end{aligned}$$

This implies that for any $i = 1, 2, \ldots, M$, the relation

$$X_i^L = \prod_{k\neq i} \frac{X_iX_k - (q+q^{-1})X_i + 1}{X_iX_k - (q+q^{-1})X_k + 1} \tag{11.17}$$

has to be true.

11.6. Comparison with the Standard XXZ Model. In the standard XXZ model, the eigenvalues of the Hamiltonian are also given by relation (11.13). Moreover, relations (11.16) are also of the same form, i.e., the relations

$$A_{\sigma\circ\pi_k} = -\frac{X_{\sigma(k)}X_{\sigma(k+1)} - (q+q^{-1})X_{\sigma(k+1)} + 1}{X_{\sigma(k)}X_{\sigma(k+1)} - (q+q^{-1})X_{\sigma(k)} + 1} A_\sigma$$

are valid also for the XXZ model.

However, there is one important difference. In the relation corresponding to (11.15) the multiplier $(-1)^{M-1}$ is missing, i.e., for XXZ, the relation

$$A_{\sigma\circ\epsilon} = X^L_{\sigma(M)} A_\sigma$$

is valid. Therefore, we obtain in the XXZ model the relation

$$X_i^L = (-1)^{M-1} \prod_{k\neq i} \frac{X_iX_k - (q+q^{-1})X_i + 1}{X_iX_k - (q+q^{-1})X_k + 1} \tag{11.18}$$

instead of (11.17). Comparing (11.17) and (11.18), we conclude that the spectrum of the fermion (soft polaron) and the standard XXZ model are the same for odd M, but if M is even, the spectrum of these models can be different.

The Bethe equations (11.18) are equivalent to the Bethe equations (2.86) if we substitute

$$X_k = \frac{q - q^{-1}\lambda_k}{1 - \lambda_k}.$$

For this substitution, the right-hand side of (11.18) is simplified and coincides with the right-hand side of (2.86).

12. SOME REMARKS ON THE OPEN HECKE CHAIN

Let $H_n(q)$ be the Hecke algebra generated by the invertible elements T_k $(k = 1, \ldots, n-1)$ subject to relations (1.1) and (1.2). For future convenience, instead of (1.6), we will consider the following form of the Hamiltonian:

$$\mathcal{H}_n = \sum_{k=1}^{n-1} T_k - \frac{(n-1)}{2}(q - q^{-1}) = \sum_{k=1}^{n-1} s_k \in H_n(q), \tag{12.1}$$

where we have introduced the new generators of the A-type Hecke algebra $H_n(q)$

$$s_k = T_k - \frac{q - q^{-1}}{2} = \frac{i}{2}\, T_k(x)|_{x^{1/2}=i}, \tag{12.2}$$

and $T_k(x)|_{x^{1/2}=i}$ are the baxterized elements (1.3) taken at the point $x^{1/2} = i$.

Remark. The representation theory of the Hecke algebras $H_n(q)$ is well known. For the details of this representation theory, see, e.g., [24–27, 30–33, 36, 41] and references therein. Each irreducible representation (irrep) of the Hecke algebra $H_n(q)$ (q is a generic parameter) corresponds to the Young diagram Λ with n nodes. The dimension of the irrep Λ is given by the hook formula (see, e.g., [28] and [32])

$$\dim(\Lambda) = \frac{n!}{\prod\limits_{\alpha\in\Lambda} h_\alpha}, \tag{12.3}$$

where h_α is a hook length of the nod $\alpha \in \Lambda$. Recall, that the Young diagram Λ with m rows of the lengths $(\lambda_1, \lambda_2, \ldots, \lambda_m)$,

$$\lambda_1 \geqslant \lambda_2 \geqslant \cdots \geqslant \lambda_m, \quad \sum_{k=1}^{m} \lambda_k = n,$$

is called dual to the diagram Λ' if $(\lambda_1, \lambda_2, \ldots, \lambda_m)$ are the lengths of the columns of Λ'. It is clear that $\dim(\Lambda) = \dim(\Lambda')$.

The quantum integrable systems with the Hamiltonians (12.1) were considered in [2, 23]. In the next subsection, we list the characteristic identities for the Hamiltonians $\mathcal{H}_n$ for the cases $n = 2, \ldots, 6$. These identities define the whole energy spectrum of the Hecke chains of the length $n = 2, \ldots, 6$.

12.1. Characteristic Identities for $\mathcal{H}_n$. Here, we use the notation

$$\lambda = q - q^{-1}, \quad \bar{q} = q + q^{-1}, \quad v = \frac{1}{2}(q + q^{-1}).$$

12.1.1. The Case $n = 2$. The characteristic identity for the Hamiltonian $\mathcal{H}_2 = T_1 - \lambda/2$ is

$$\left(\mathcal{H}_2 - \frac{1}{2}\overline{q}\right)\left(\mathcal{H}_2 + \frac{1}{2}\overline{q}\right) = 0.$$

Two eigenvalues $(1/2)\overline{q}$ and $-(1/2)\overline{q}$ correspond to the one-dimensional irreps $T_1 = q$ and $T_1 = -q^{-1}$ labeled, respectively, by the Young diagrams (2) and (1^2).

12.1.2. The Case $n = 3$. Here, we have the set of commuting elements [2]

$$j_1 = T_1 + T_2, \quad j_2 = T_1T_2 + T_2T_1, \quad j_3 = T_2T_1T_2 + T_1 + T_2. \tag{12.4}$$

Note that the elements j_2, j_3 are expressed in terms of j_1:

$$j_2 = j_1^2 - \lambda j_1 - 2, \quad j_3 = \frac{1}{2}(j_1^3 - 2\lambda j_1^2 + (\lambda^2 - 1)j_1 + 2\lambda).$$

The element $\mathcal{H}_3 = j_1 - \lambda$ is the Hamiltonian (12.1) for the open Hecke chain and j_3 is a central element in H_3. The characteristic identity for the Hamiltonian $\mathcal{H}_3$ is

$$(\mathcal{H}_3 + \overline{q})(\mathcal{H}_3 - \overline{q})(\mathcal{H}_3 - 1)(\mathcal{H}_3 + 1) = 0. \tag{12.5}$$

This means that $\mathrm{Spec}(\mathcal{H}_3) = \{\pm\overline{q}, \pm 1\}$. The first two eigenvalues $\pm\overline{q}$ correspond to the one-dimensional representations $T_i = \pm q^{\pm 1}$ $(i = 1, 2)$ of $H_3(q)$, which are related to the Young diagrams (3), (1^3). The eigenvalues (± 1) correspond to the two-dimensional irrep $(2, 1)$ of $H_3(q)$.

12.1.3. The Case $n = 4$. In this case, we have the following set of commuting elements:

$$j_1 = \sum_{i=1}^{3} T_i, \quad j_2 = \{T_1, T_2\}_+ + \{T_2, T_3\}_+ + 2T_3T_1,$$

$$j_3 = \{T_1T_3, T_2\}_+ + (T_1 + T_3)T_2(T_1 + T_3) + \lambda T_3T_1 + 2\sum_{i=1}^{3} T_i,$$

$$j_4 = \{T_2T_3T_2, T_1\}_+ + \{T_2T_1T_2, T_3\}_+ + \{T_2, T_3\}_+ + \{T_2, T_1\}_+,$$

$$j_5 = T_1T_2T_3T_2T_1 + T_2T_3T_2 + T_1T_2T_1 + T_1 + T_2 + T_3.$$

The element j_5 is a central element in $H_4(q)$. Therefore, the longest element in $H_4(q)$: $j = T_1T_2T_3T_1T_2T_1 = (j_5 - j_1)(j_1 - \lambda) - j_4$ commutes with the

Hamiltonian $\mathcal{H}_4 = j_1 - (3/2)\lambda$ (12.1). This Hamiltonian satisfies the characteristic identity

$$\left(\mathcal{H}_4 + \frac{3}{2}\,\overline{q}\right)\left(\mathcal{H}_4 - \frac{3}{2}\,\overline{q}\right)\left(\mathcal{H}_4 + \frac{1}{2}\,\overline{q}\right)\left(\left(\mathcal{H}_4 + \frac{1}{2}\,\overline{q}\right)^2 - 2\right)\times$$
$$\times\left(\mathcal{H}_4 - \frac{1}{2}\,\overline{q}\right)\left(\left(\mathcal{H}_4 - \frac{1}{2}\,\overline{q}\right)^2 - 2\right)\left(\mathcal{H}_4^2 - \frac{1}{4}\overline{q}^2 - 2\right) = 0. \quad (12.6)$$

Thus, the spectrum of $\mathcal{H}_4$ consists of the eigenvalues:

$((3/2)\overline{q}, -(3/2)\overline{q})$ for the two dual one-dimensional irreps (4), (1^4) of the Hecke algebra $H_4(q)$;

$((1/2)\overline{q}, (1/2)\overline{q} \pm \sqrt{2})$ and $(-(1/2)\overline{q}, -(1/2)\overline{q} \pm \sqrt{2})$ for the two dual three-dimensional irreps $(3,1)$, $(2,1^2)$;

$(\pm(1/2)\sqrt{\overline{q}^2 + 8})$ for the two-dimensional irrep (2^2) of $H_4(q)$.

12.1.4. The Case $n = 5$. For the Hamiltonian $\mathcal{H}_5 = \sum_{i=1}^{4} T_i - 2\lambda$ the characteristic polynomial is an odd function of order 25, and the characteristic identity is

$$\begin{aligned}
(1^5)\cdot(5):&\quad (\mathcal{H}_5 + 2\,\overline{q})(\mathcal{H}_5 - 2\,\overline{q})\times\\
(3,1^2):&\quad \times\mathcal{H}_5\,(\mathcal{H}_5^2 - 1)(\mathcal{H}_5^2 - 5)\times\\
(2,1^3):&\quad \times\left((\mathcal{H}_5 + \overline{q})^2 - \frac{(\sqrt{5}-1)^2}{4}\right)\left((\mathcal{H}_5 + \overline{q})^2 - \frac{(\sqrt{5}+1)^2}{4}\right)\times\\
(4,1):&\quad \times\left((\mathcal{H}_5 - \overline{q})^2 - \frac{(\sqrt{5}-1)^2}{4}\right)\left((\mathcal{H}_5 - \overline{q})^2 - \frac{(\sqrt{5}+1)^2}{4}\right)\times\\
(2^2,1):&\quad \times(\mathcal{H}_5^2 + \overline{q}\mathcal{H}_5 - 1)\left(\mathcal{H}_5^3 + \overline{q}\mathcal{H}_5^2 - 5\mathcal{H}_5 - 2\overline{q}\right)\times\\
(3,2):&\quad \times(\mathcal{H}_5^2 - \overline{q}\mathcal{H}_5 - 1)\left(\mathcal{H}_5^3 - \overline{q}\mathcal{H}_5^2 - 5\mathcal{H}_5 + 2\overline{q}\right) = 0. \qquad (12.7)
\end{aligned}$$

The last two lines give the eigenvalues of $\mathcal{H}_5$, which correspond to the five-dimensional representations labeled by two dual Young diagrams $(2^2,1)$, $(3,2)$. The sum of the dimensions of the irreps for $H_5(q)$ is equal to 26. We obtain the 25th order of the characteristic identity since the eigenvalue 0 has multiplicity 2. This eigenvalue appears in the self-dual irrep $(3,1^2)$.

12.1.5. The Case $n = 6$. For the Hamiltonian $\mathcal{H}_6 = \sum_{i=1}^{5} T_i - \frac{5}{2}\lambda$ the characteristic polynomial is an even function of $\mathcal{H}_6$ of order 72. The characteristic

polynomial is much more complicated:

$$
\begin{aligned}
(1^6)\cdot(6): \quad & \left(\mathcal{H}_6+\frac{5}{2}\,\overline{q}\right)\left(\mathcal{H}_6-\frac{5}{2}\overline{q}\right)\times\\
(2,1^4): \quad & \times\left(\mathcal{H}_6+\frac{3}{2}\overline{q}\right)\left(\left(\mathcal{H}_6+\frac{3}{2}\overline{q}\right)^2-1\right)\left(\left(\mathcal{H}_6+\frac{3}{2}\overline{q}\right)^2-3\right)\times\\
(5,1): \quad & \times\left(\mathcal{H}_6-\frac{3}{2}\overline{q}\right)\left(\left(\mathcal{H}_6-\frac{3}{2}\overline{q}\right)^2-1\right)\left(\left(\mathcal{H}_6-\frac{3}{2}\overline{q}\right)^2-3\right)\times\\
& \times\left(\mathcal{H}_6+\frac{1}{2}\overline{q}\right)\left(\left(\mathcal{H}_6+\frac{1}{2}\overline{q}\right)^2-1\right)\left(\left(\mathcal{H}_6+\frac{1}{2}\overline{q}\right)^2-3\right)\times\\
(3,1^3): \quad & \times\left(\left(\mathcal{H}_6+\frac{1}{2}\overline{q}\right)^2-(\sqrt{3}+1)^2\right)\left(\left(\mathcal{H}_6+\frac{1}{2}\overline{q}\right)^2-(\sqrt{3}-1)^2\right)\times\\
& \times\left(\mathcal{H}_6-\frac{1}{2}\overline{q}\right)\left(\left(\mathcal{H}_6-\frac{1}{2}\overline{q}\right)^2-1\right)\left(\left(\mathcal{H}_6-\frac{1}{2}\overline{q}\right)^2-3\right)\times\\
(4,1^2): \quad & \times\left(\left(\mathcal{H}_6-\frac{1}{2}\overline{q}\right)^2-(\sqrt{3}+1)^2\right)\left(\left(\mathcal{H}_6-\frac{1}{2}\overline{q}\right)^2-(\sqrt{3}-1)^2\right).
\end{aligned}
\tag{12.8}
$$

$$
\begin{aligned}
(4,2): \quad & \big[(3\overline{q}^3-20\overline{q}+(24-14\overline{q}^2)\mathcal{H}_6+20\overline{q}\mathcal{H}_6^2-8\mathcal{H}_6^3)\times\\
& \times\big\{9\overline{q}^6-228\overline{q}^4+512\overline{q}^2-256-(1408\overline{q}-1280\overline{q}^3+84\overline{q}^5)\mathcal{H}_6+\\
& +(768-2528\overline{q}^2+316\overline{q}^4)\mathcal{H}_6^2+(2048\overline{q}-608\overline{q}^3)\mathcal{H}_6^3+\\
& +(624\overline{q}^2-576)\mathcal{H}_6^4-320\overline{q}\mathcal{H}_6^5+64\mathcal{H}_6^6\big\}\times\\
(2^3): \quad & \times\big(3\overline{q}^4+16\overline{q}^2-64+(8\overline{q}^3+160\overline{q})\mathcal{H}_6+\\
& +(-8\overline{q}^2+128)\mathcal{H}_6^2-32\overline{q}\mathcal{H}_6^3-16\mathcal{H}_6^4\big)\times\\
(3,2,1): \quad & \times\big(\overline{q}^8+16\overline{q}^4-256\overline{q}^2-256+(2560+1024\overline{q}^2+64\overline{q}^4)\mathcal{H}_6-\\
& -(5120+1152\overline{q}^2+192\overline{q}^4+16\overline{q}^6)\mathcal{H}_6^2-(2048+512\overline{q}^2)\mathcal{H}_6^3+\\
& +(8448+1536\overline{q}^2+96\overline{q}^4)\mathcal{H}_6^4+1024\mathcal{H}_6^5-\\
& -(3072+256\overline{q}^2)\mathcal{H}_6^6+256\mathcal{H}_6^8)\big][\mathcal{H}_6\to-\mathcal{H}_6],
\end{aligned}
\tag{12.9}
$$

where in the left-hand side we indicate the corresponding representations which have dimensions

$$\dim(6) = \dim(1^6) = 1, \quad \dim(3,1^3) = \dim(4,1^2) = 10,$$
$$\dim(5,1) = \dim(2,1^4) = 5.$$

The factors in (12.9) correspond to the representation (2^3) with $\dim = 5$, the representation $(3,2,1)$ with $\dim = 16$, the representation $(4,2)$ with $\dim = 9$ and their dual irreps which can be obtained from the previous ones by substitution $\mathcal{H}_6 \to -\mathcal{H}_6$. The sum of the dimensions (12.3) for all these representations of H_6 is equal to 76. Since the order of the characteristic polynomial is equal to 72, we conclude that some of these eigenvalues are degenerated. Two of such eigenvalues appear in the dual hook-type irreps $(3,1^3)$, $(4,1^2)$ and other two appear in the dual nontrivial irreps $(3,3)$, (2^3) (see below). It is clear that the degenerated eigenvalues are $\pm(1/2)\overline{q}$ (with the multiplicities 3). The important problem is to find an additional operator j_k which commutes with the Hamiltonian $\mathcal{H}_6$ and removes this degeneracy.

Remark. The factor, which corresponds to the self-dual representation $(3,2,1)$ with $\dim = 8$ presented in (12.9), can also be written in the concise form (we remove the common factor 2^8)

$$\begin{aligned}\{v^8 + v^4 - 4v^2 - 1 + 10x + 16xv^2 + 4xv^4 - 20x^2 - 18x^2v^2 - 12x^2v^4 -\\ - 4x^2v^6 - 8x^3 - 8x^3v^2 + 33x^4 + 24x^4v^2 + 6x^4v^4 + 4x^5 - 12x^6 - 4x^6v^2 + x^8\} =\\ = Z^4Y^4 - Z^2Y^2(6x+1)(2x-1) + [4ZY + (4x^2+6x-1)](2x-1)^2,\end{aligned} \tag{12.10}$$

where $x = \mathcal{H}_6$, $v = \overline{q}/2$, $Z = x + v$, $Y = x - v$.

12.2. Characteristic Polynomials for $\mathcal{H}_n$ in the Representations $(n-2,2)$. Now we impose additional relations on the generators T_k of the Hecke algebra (1.1), (1.2):

$$T_k(q^2)\,T_{k-1}(q^4)\,T_k(q^2) = 0, \quad T_k(q^2)\,T_{k+1}(q^4)\,T_k(q^2) = 0, \tag{12.11}$$

where $T_k(x)$ are the baxterized elements (1.3). The factor of the Hecke algebra over the relations (12.11) is called the Temperley–Lieb algebra TL_n. It is known that all irreps of the algebra TL_n coincide with irreps $\rho_{(n-k,k)}$ (here $n \geqslant 2k$) of the Hecke algebra numerated by the Young diagrams $(n-k,k)$ with only two rows. The spectrum of all Hamiltonians $\rho_{(n-k,k)}(\mathcal{H}_n)$ $(k = 0,1,\ldots,[n/2])$ gives the energy spectrum of the XXZ Heisenbegr spin chain of the length n (see, e.g., [34] and references therein). In this subsection, we present the characteristic polynomials for the Hamiltonians $\rho_{(n-2,2)}(\mathcal{H}_n)$. The dimensions of

the representations $(n-2,2)$ are $\dim \rho_{(n-2,2)} = \dfrac{n(n-3)}{2}$. Our method of the calculation is the following. We construct explicitly matrix representations of $\rho_{(n-k,k)}(\mathcal{H}_n)$ (see the next subsection) and then use the Mathematica to evaluate the characteristic polynomials of $\rho_{(n-k,k)}(\mathcal{H}_n)$.

12.2.1. The Case $n = 4$ *and Representation* $(2,2)$ *with* $\dim = 2$. The Hamiltonian $\mathcal{H}_4 = x$ (see (12.1)) has the characteristic polynomial (see (12.6))

$$(x^2 - v^2 - 2) = YZ - 2, \tag{12.12}$$

where $Z = x + v$, $Y = x - v$.

12.2.2. The Case $n = 5$ *and Representation* $(3,2)$ *with* $\dim = 5$. The Hamiltonian $\mathcal{H}_5 = x$ has the characteristic polynomial as a product of two factors of orders 2 and 3 (see (12.7))

$$(x^2 - \overline{q}x - 1)(x^3 - \overline{q}x^2 - 5x + 2\overline{q}) = (YZ - 1)\{YZ^2 - (2Y + 3Z)\}, \tag{12.13}$$

where $Z = x$, $Y = x - 2v$.

12.2.3. The Case $n = 6$ *and Representation* $(4,2)$ *with* $\dim = 9$. The Hamiltonian $\mathcal{H}_6 = x$ has the characteristic polynomial (see (12.9)) which is factorized into two factors of the 3rd and 6th orders

$$\begin{aligned}&\left\{-3v^3 + 5v + (7v^2 - 3)x - 5vx^2 + x^3\right\}\times\\ &\times \left\{9v^6 - 57v^4 + 32v^2 - 4 + (-44v + 160v^3 - 42v^5)x + (12 - 158v^2 + 79v^4)x^2 + \right.\\ &\left. + (64v - 76v^3)x^3 + (39v^2 - 9)x^4 - 10vx^5 + x^6\right\}. \end{aligned}\tag{12.14}$$

In terms of the new variables $Z = x - v$, $Y = x - 3v$ the factors in (12.14) are simplified to be

$$\{YZ^2 - (Y + 2Z)\}\{Y^2Z^4 - (5Y + 4Z)YZ^2 + 2(5Y + Z)Z - 4\}.$$

12.2.4. The Case $n = 7$ *and the Representation* $(5,2)$ *with* $\dim = 14$. For the Hamiltonian $\mathcal{H}_7 = x$, the characteristic polynomial in this representation is factorized into two factors of the 6th and 8th orders. In terms of the new variables $Z = x - 2v$, $Y = x - 4v$ it reads

$$\begin{aligned}&\{Z^4Y^2 - 3Z^2Y(Y + Z) + Z(Z + 4Y) - 1\}\times\\ &\quad\times \{Z^6Y^2 - (9Y + 5Z)Z^4Y + Z^2(Z + 6Y)(5Z + 2Y) - (5Z + 2Y)^2\}.\end{aligned}$$

12.2.5. The Case $n = 8$ *and the Representation* $(6,2)$ *with* $\dim = 20$. For the Hamiltonian $\mathcal{H}_8 = x$ the characteristic polynomial in this representation is

factorized into two factors of the 8th and 12th orders

$$\{Z^6Y^2 - 2Z^4Y(3Y+2Z) + Z^2(Z+5Y)(3Z+Y) - (3Z+Y)^2\}\times$$
$$\times\{Y^3Z^9 - Z^7Y^2(14Y+6Z) + Z^5Y(9Z^2+68ZY+49Y^2) -$$
$$- Z^3(2Z^3+85Z^2Y+168ZY^2+49Y^3) + 2Z^2(9Z^2+68ZY+49Y^2) -$$
$$- 8Z(7Y+3Z) + 8\},$$

where $Z = x - 3v$ and $Y = x - 5v$.

12.2.6. The Case $n = 9$ and the Representation $(7,2)$ with $\dim = 27$. For the Hamiltonian $\mathcal{H}_9 = x$, the characteristic polynomial in this representation is factorized into two factors of the 12th and 15th orders

$$\{Y^3Z^9 - 5Y^2Z^7(2Y+Z) + 3YZ^5(8Y^2+13ZY+2Z^2) -$$
$$- Z^3(16Y^3+64ZY^2+38Z^2Y+Z^3) + 3Z^2(8Y^2+13ZY+2Z^2) -$$
$$-5Z(2Y+Z)+1\}\{Y^3Z^{12} - Y^2Z^{10}(20Y+7Z) + YZ^8(126Y^2+121ZY+14Z^2) -$$
$$- Z^6(304Y^3+620ZY^2+212Z^2Y+7Z^3) + Z^4(2Y+7Z)\times$$
$$\times(126Y^2+121ZY+14Z^2) - Z^2(2Y+7Z)^2(20Y+7Z) + (2Y+7Z)^3\},$$

where $Z = x - 4v$, $Y = x - 6v$.

12.2.7. The Case $n = 10$ and the Representation $(8,2)$ with $\dim = 35$. The characteristic polynomial in this representation is factorized into two factors of the 15th and 20th orders

$$P_{(8,2)} = \{Z^{12}Y^3 - 3Z^{10}Y^2(5Y+2Z) + Z^8Y(69Y^2+76ZY+10Z^2) -$$
$$- Z^6(119Y^3+278ZY^2+109Z^2Y+4Z^3) + Z^4(Y+4Z)(69Y^2+76ZY+10Z^2) -$$
$$- 3Z^2(5Y+2Z)(Y+4Z)^2 + (Y+4Z)^3\}\times$$
$$\times\{Z^{16}Y^4 - Z^{14}Y^3(27Y+8Z) + Z^{12}Y^2(261Y^2+194ZY+20Z^2) -$$
$$- Z^{10}Y(1143Y^3+1632ZY^2+439Z^2Y+16Z^3) +$$
$$+ Z^8(2349Y^4+5982ZY^3+3216Z^2Y^2+326Z^3Y+2Z^4) -$$
$$- Z^6(2187Y^4+9720ZY^3+9812Z^2Y^2+2124Z^3Y+40Z^4) +$$
$$+ 3Z^4(243Y^4+2214ZY^3+4098Z^2Y^2+1816Z^3Y+84Z^4) -$$
$$- 2Z^3(729Y^3+3033ZY^2+2584Z^2Y+304Z^3) +$$
$$+ 36Z^2(27Y^2+54ZY+14Z^2) - 80Z(3Y+2Z) + 16\}, \quad (12.15)$$

where $x = \mathcal{H}_{10}$, $Z = x - 5v$, $Y = x - 7v$.

12.2.8. The Case $n = 11$ *and the Representation* $(9, 2)$ *with* $\dim = 44$. The characteristic polynomial in this representation is factorized into two factors of the 20th and 24th order

$$\begin{aligned}
P_{(9,2)} = \{&Z^{16}Y^4 - 7Z^{14}Y^3(3Y + Z) + 5Z^{12}Y^2(31Y^2 + 26ZY + 3Z^2) - \\
&- Z^{10}Y(510Y^3 + 822ZY^2 + 249Z^2Y + 10Z^3) + \\
&+ Z^8(775Y^4 + 2228ZY^3 + 1351Z^2Y^2 + 153Z^3Y + Z^4) - \\
&- Z^6(525Y^4 + 2635ZY^3 + 3002Z^2Y^2 + 730Z^3Y + 15Z^4) + \\
&+ Z^4(125Y^4 + 1290ZY^3 + 2697Z^2Y^2 + 1346Z^3Y + 69Z^4) - \\
&- Z^3(200Y^3 + 941ZY^2 + 904Z^2Y + 119Z^3) + \\
&+ 3Z^2(35Y^2 + 79ZY + 23Z^2) - 5Z(4Y + 3Z) + 1\} \times \\
\times \{&Z^{20}Y^4 - Z^{18}Y^3(35Y + 9Z) + Z^{16}Y^2(475Y^2 + 290ZY + 27Z^2) - \\
&- Z^{14}Y(3230Y^3 + 3558ZY^2 + 805Z^2Y + 30Z^3) + \\
&+ Z^{12}(11875Y^4 + 21404ZY^3 + 8949Z^2Y^2 + 839Z^3Y + 9Z^4) - \\
&- Z^{10}(23883Y^4 + 67717ZY^3 + 47590Z^2Y^2 + 8550Z^3Y + 243Z^4) + \\
&+ Z^8(25365Y^4 + 113066ZY^3 + 128825Z^2Y^2 + 40518Z^3Y + 2349Z^4) - \\
&- Z^6(2Y + 9Z)(6650Y^3 + 17345ZY^2 + 10194Z^2Y + 1143Z^3) + \\
&+ Z^4(2Y + 9Z)^2(855Y^2 + 1183ZY + 261Z^2) - \\
&- Z^2(2Y + 9Z)^3(50Y + 27Z) + (2Y + 9Z)^4\}, \quad (12.16)
\end{aligned}$$

where $x = \mathcal{H}_{11}$, $Z = x - 6v$, $Y = x - 8v$.

12.2.9. The Case $n = 12$ *and the Representation* $(10, 2)$ *with* $\dim = 54$. The characteristic polynomial in this representation is factorized into two factors of the 24th and 30th order

$$\begin{aligned}
P_{(10,2)} = \{&Z^{20}Y^4 - 4Z^{18}Y^3(7Y + 2Z) + 3Z^{16}Y^2(100Y^2 + 68ZY + 7Z^2) - \\
&- Z^{14}Y(1591Y^3 + 1954ZY^2 + 491Z^2Y + 20Z^3) + \\
&+ Z^{12}(4508Y^4 + 9064ZY^3 + 4218Z^2Y^2 + 436Z^3Y + 5Z^4) - \\
&- Z^{10}(6907Y^4 + 21850ZY^3 + 17112Z^2Y^2 + 3406Z^3Y + 105Z^4) + \\
&+ Z^8(5527Y^4 + 27480ZY^3 + 34909Z^2Y^2 + 12200Z^3Y + 775Z^4) - \\
&- 2Z^6(Y + 5Z)(1082Y^3 + 3150ZY^2 + 2061Z^2Y + 255Z^3) + \\
&+ Z^4(Y + 5Z)^2(411Y^2 + 634ZY + 155Z^2) - \\
&- 7Z^2(5Y + 3Z)(Y + 5Z)^3 + (Y + 5Z)^4\} \times \\
\times \{&Z^{25}Y^5 - 2Z^{23}Y^4(22Y + 5Z) + Z^{21}Y^3(792Y^2 + 412ZY + 35Z^2) - \\
&- Z^{19}Y^2(7623Y^3 + 6866ZY^2 + 1355Z^2Y + 50Z^3) +
\end{aligned}$$

$$+Z^{17}Y(43076Y^4+60390ZY^3+20954Z^2Y^2+1834Z^3Y+25Z^4)-$$
$$-Z^{15}(147983Y^5+307010ZY^4+168558Z^2Y^3+26510Z^3Y^2+883Z^4Y+2Z^5)+$$
$$+Z^{13}(310123Y^5+930974ZY^4+770091Z^2Y^3+196172Z^3Y^2+$$
$$+12139Z^4Y+70Z^5)-2Z^{11}(194326Y^5+840587ZY^4+1026993Z^2Y^3+$$
$$+404211Z^3Y^2+42041Z^4Y+475Z^5)+Z^9(278179Y^5+1759824ZY^4+$$
$$+3173478Z^2Y^3+1898244Z^3Y^2+317599Z^4Y+6460Z^5)-$$
$$-Z^7(102487Y^5+1011560ZY^4+2742586Z^2Y^3+2500438Z^3Y^2+$$
$$+665275Z^4Y+23750Z^5)+Z^5(14641Y^5+284834ZY^4+1251866Z^2Y^3+$$
$$+1769240Z^3Y^2+753437Z^4Y+47766Z^5)-$$
$$-2Z^4(14641Y^4+135520ZY^3+320474Z^2Y^2+219128Z^3Y+25365Z^4)+$$
$$+8Z^3(2662Y^3+13673ZY^2+16106Z^2Y+3325Z^3)-$$
$$-8Z^2(847Y^2+2222ZY+855Z^2)+80Z(11Y+10Z)-32\}, \quad (12.17)$$

where $x=\mathcal{H}_{12}$, $Z=x-7v$ and $Y=x-9v$.

12.2.10. The Case $n=13$ and the Representation $(11,2)$ with $\dim=65$. The characteristic polynomial in this representation is factorized into two factors of the 30th and 35th order

$$Y^5Z^{25}-9Y^4(4Y+Z)Z^{23}+7Y^3(75Y^2+43ZY+4Z^2)Z^{21}-$$
$$-Y^2(4056Y^3+4031ZY^2+874Z^2Y+35Z^3)Z^{19}+$$
$$+Y(18231Y^4+28222ZY^3+10781Z^2Y^2+1030Z^3Y+15Z^4)Z^{17}-$$
$$-(49380Y^5+113163ZY^4+68496Z^2Y^3+11805Z^3Y^2+424Z^4Y+Z^5)Z^{15}+$$
$$+(80891Y^5+268257ZY^4+244835Z^2Y^3+68531Z^3Y^2+4605Z^4Y+28Z^5)Z^{13}-$$
$$-(78576Y^5+375429ZY^4+506270Z^2Y^3+219334Z^3Y^2+$$
$$+24905Z^4Y+300Z^5)Z^{11}+$$
$$+(43200Y^5+301984ZY^4+601090Z^2Y^3+396150Z^3Y^2+$$
$$+72634Z^4Y+1591Z^5)Z^9-$$
$$-2(6048Y^5+66096ZY^4+197920Z^2Y^3+198881Z^3Y^2+$$
$$+58122Z^4Y+2254Z^5)Z^7+$$
$$+(1296Y^5+28080ZY^4+136554Z^2Y^3+212848Z^3Y^2+$$
$$+99644Z^4Y+6907Z^5)Z^5-$$
$$-(2160Y^4+22176ZY^3+57906Z^2Y^2+43570Z^3Y+5527Z^4)Z^4+$$
$$+(1296Y^3+7360ZY^2+9551Z^2Y+2164Z^3)Z^3-$$
$$-3(112Y^2+324ZY+137Z^2)Z^2+35(Y+Z)Z-1; \quad (12.18)$$

$$Y^5Z^{30} - Y^4(54Y + 11Z)Z^{28} + Y^3(1239Y^2 + 563ZY + 44Z^2)Z^{26} -$$
$$- Y^2(15894Y^3 + 12153ZY^2 + 2140Z^2Y + 77Z^3)Z^{24} +$$
$$+ Y(126279Y^4 + 145446ZY^3 + 43545Z^2Y^2 + 3578Z^3Y + 55Z^4)Z^{22} -$$
$$- (650946Y^5 + 1067749ZY^4 + 486798Z^2Y^3 + 68967Z^3Y^2 +$$
$$+ 2466Z^4Y + 11Z^5)Z^{20} + (2219569Y^5 + 5028863ZY^4 + 3303181Z^2Y^3 +$$
$$+ 723221Z^3Y^2 + 45485Z^4Y + 484Z^5)Z^{18} - (5017266Y^5 + 15459111ZY^4 +$$
$$+ 14203130Z^2Y^3 + 4550870Z^3Y^2 + 451913Z^4Y + 8712Z^5)Z^{16} +$$
$$+ (7433784Y^5 + 30999936ZY^4 + 39276278Z^2Y^3 + 17901642Z^3Y^2 +$$
$$+ 2662000Z^4Y + 83853Z^5)Z^{14} - 2(3523048Y^5 + 19967988ZY^4 +$$
$$+ 34790550Z^2Y^3 + 22275099Z^3Y^2 + 4830078Z^4Y + 236918Z^5)Z^{12} +$$
$$+ (4121784Y^5 + 32057560ZY^4 + 77399690Z^2Y^3 + 69595592Z^3Y^2 +$$
$$+ 21779274Z^4Y + 1627813Z^5)Z^{10} - (2Y + 11Z)(715128Y^4 +$$
$$+ 3714176ZY^3 + 5556166Z^2Y^2 + +2682086Z^3Y + 310123Z^4)Z^8 +$$
$$+ (2Y + 11Z)^2(71532Y^3 + 233700ZY^2 + 191279Z^2Y + 35332Z^3)Z^6 -$$
$$- (2Y + 11Z)^3(3924Y^2 + 7128ZY + 2299Z^2) \times$$
$$\times Z^4 + 7(2Y + 11Z)^4(15Y + 11Z)Z^2 - (2Y + 11Z)^5,$$

where $x = \mathcal{H}_{13}$, $Z = x - 8v$ and $Y = x - 10v$.

In all examples considered above, the characteristic polynomials for $\rho_{(n-2,2)}(\mathcal{H}_n)$ are factorized into two factors with integer coefficients. So we formulate the following Conjecture.

Conjecture. *Let $n \geqslant 4$. For irrep of the Hecke algebra $H_n(q)$ with Young diagram $(n-2,2)$ and dimension $\dfrac{n(n-3)}{2} = p_{n-1} + p_n$, the characteristic polynomial for the Hamiltonian $\rho_{(n-2,2)}(\mathcal{H}_n) = x$ is represented as the product of two polynomial factors: the short factor* short_n *of the order p_{n-1} and the long factor* long_n *of the order p_n with integer coefficients, where*

$$p_n = \frac{1}{8}(((-1)^n - 1)3 - 4n + 2n^2) = \begin{cases} \dfrac{1}{4}n(n-2) \ for \ \text{ even } n, \\ \dfrac{1}{4}(n+1)(n-3)n \ for \ \text{ odd } n, \end{cases}$$

that is,

$$p_3 = 0, \ \ p_4 = 2, \ \ p_5 = 3, \ \ p_6 = 6, \ \ p_7 = 8, \ \ p_8 = 12, \ \ p_9 = 15,$$
$$p_{10} = 20, \ \ p_{11} = 24, \ \ p_{12} = 30, \ \ p_{13} = 35, \ \ p_{14} = 42, \ldots$$

These polynomial factors are $(p_n = k_n + \overline{k}_n)$

$$\begin{aligned}
\mathrm{short}_n = Z^{\overline{k}_{n-1}} Y^{k_{n-1}} \Big\{ 1 - (n-4) Z^{-1} Y^{-1} - \frac{(n-4)(n-5)}{2} Z^{-2} + \\
+ \frac{(n-5)(n-6)}{2} Z^{-2} Y^{-2} + \frac{(n-6)(n^2-7n+8)}{2} Z^{-3} Y^{-1} + \\
+ \frac{(n-6)(n-7)(n^2-5n-4)}{8} Z^{-4} - - \frac{(n-6)(n-7)(n-8)}{6} Z^{-3} Y^{-3} - \\
- \frac{(n^4 - 20n^3 + 137n^2 - 338n + 116)}{4} Z^{-4} Y^{-2} - (\dots) Z^{-5} Y^{-1} \Big\} + \\
\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots \\
+ (-1)^{[(n-2)/4]} \frac{(1+(-1)^n)}{2} \left\{ \left(\frac{n}{2} - 1 \right) Z + Y \right\}^{k_{n-1}} + \\
+ (-1)^{[(n-1)/4]} \frac{(1-(-1)^n)}{2}, \quad (12.19)
\end{aligned}$$

$$\begin{aligned}
\mathrm{long}_n = \\
= Z^{\overline{k}_n} Y^{k_n} \Big\{ 1 - (n-2) Z^{-1} Y^{-1} - \frac{(n-1)(n-4)}{2} Z^{-2} + \frac{(n-2)(n-5)}{2} Z^{-2} Y^{-2} + \\
+ \left(\frac{(n-4)(n-5)(n+9)}{3} + \frac{(n-6)(n-7)(n-8)}{6} \right) Z^{-3} Y^{-1} + \\
+ \frac{(n-6)(n-1)(n^2-3n-12)}{8} Z^{-4} - \frac{(n-2)(n-6)(n-7)}{6} Z^{-3} Y^{-3} - \\
\frac{(n^4 - 12n^3 + 37n^2 + 18n - 124)}{4} Z^{-4} Y^{-2} - \\
- \frac{(n^5 - 12n^4 + 23n^3 + 128n^2 - 252n - 224)}{8} Z^{-5} Y^{-1} + \dots\dots\dots\dots \Big\} + \\
+ (-1)^{[(n-1)/4]} \frac{(1-(-1)^n)}{2} \{ (n-2) Z + 2Y \}^{k_n} + \\
+ (-1)^{[n/4]} \frac{(1+(-1)^n)}{2} 2^{k_n}, \quad (12.20)
\end{aligned}$$

where $[x]$ *is the integer part of* x *(e.g.,* $[n/4]$ *— integer part of* $n/4$*),*

$$Z = x - (n-5)v, \quad Y = x - (n-3)v, \quad v = \frac{q + q^{-1}}{2},$$

$$k_n = \left[\frac{n}{2} \right] - 1 = \begin{cases} \frac{1}{2}(n-2) & \text{even } n, \\ \frac{1}{2}(n-3) & \text{odd } n \end{cases}$$

and

$$\overline{k}_n = \frac{1}{8}((-1)^n + 7 - 8n + 2n^2) = \begin{cases} \frac{1}{4}(n-2)^2 = k_n^2 & \text{even } n, \\ \frac{1}{4}(n-1)(n-3) = k_n^2 + k_n & \text{odd } n. \end{cases}$$

For n-odd, one can write (12.19) and (12.20) as the series

$$\begin{aligned} &\mathrm{long}_n \sim \mathrm{short}_{n+1} \sim \\ &\sim Z^{\overline{k}_n} Y^{k_n} \left(1 - Z^{-2}\left(C_{2,1}\frac{Z}{Y} + C_{2,0}\right) + Z^{-4}\left(C_{4,2}\frac{Z^2}{Y^2} + C_{4,1}\frac{Z}{Y} + C_{4,0}\right) - \right. \\ &\left. - Z^{-6}\left(C_{6,3}\frac{Z^3}{Y^3} + C_{6,2}\frac{Z^2}{Y^2} + C_{6,1}\frac{Z}{Y} + C_{6,0}\right) + Z^{-8}\sum_{j=0}^{4} C_{8,j}\left(\frac{Z}{Y}\right)^j - \dots\right) = \\ &= Z^{\overline{k}_n} Y^{k_n} \left(\sum_{m=0}^{\frac{\overline{k}_n}{2}} (-Z^{-2})^m \sum_{j=0}^{m} C_{2m,j}\left(\frac{Z}{Y}\right)^j\right). \end{aligned}$$

For n-even, Eqs. (12.19) and (12.20) also can be written as the series over Z/Y. But we do not present it explicitly here.

Remark 1. For the hook-type representations $(n-1,1)$ we have the following spectrum for the Hamiltonian (see [35,36]) $\mathcal{H}_n = \sum\limits_{k=1}^{n-1}(T_k - q)$:

$$\mathrm{Spec}\,(\mathcal{H}_n) = \mathrm{Spec}\left(\sum_{k=1}^{n-1}(T_k - q)\right) = 2\cos\left(\frac{\pi m}{n}\right) - (q + q^{-1}),\ m = 1, \dots, n-1.$$

If, as usual (cf. (11.4)), we substitute $2\cos(\pi m/n) = X^{1/2} + X^{-1/2}$, then for X we will have the characteristic identity $X^n - 1 = 0$, $X \neq 1$. We see that the spectrum of the open XXZ spin chain (for even m) contains the spectrum of one-magnon states (except for the case $X = 1$) for closed XXZ spin chain (see (11.10)).

Remark 2. We have calculated the characteristic polynomials $P_{(n-3,3)}$ for the Hamiltonian (12.1) in the irreps $(n-3,3)$ ($n = 6,7,8,9$) and observed the same factorization of $P_{(n-3,3)}$ into two factors, which are the polynomials with the integer coefficients.

Remark 3. The quantum inverse scattering (R-matrix) method and the algebraic Bethe ansatz method for the open XXZ spin chain were elaborated by Sklyanin in [37] (about analytical Bethe ansatz approach see [38,39] and references therein). The quantum group symmetry in the open XXZ spin chain was discovered in [40].

12.3. Method of Calculation. In this subsection, we explain the method of construction of the explicit matrix irreps for the Hecke algebra $H_{M+1}(q)$, related to the fixed Young diagram Λ. A similar method was also considered in [36,41]).

First of all, we define the affine extension $\hat{H}_{M+1}(q)$ of the Hecke algebra $H_{M+1}(q)$. The affine Hecke algebra $\hat{H}_{M+1}(q)$ (see, e.g., Ch. 12.3 in [1]) is an extension of the Hecke algebra $H_{M+1}(q)$ by the additional affine elements y_k $(k = 1, \ldots, M+1)$ subjected to the relations:

$$y_{k+1} = T_k y_k T_k, \quad y_k\, y_j = y_j y_k, \quad y_j T_i = T_i y_j \quad (j \neq i, i+1). \tag{12.21}$$

The elements $\{y_k\}$ form a commutative subalgebra in $\hat{H}_{M+1}$, while the symmetric functions in y_k form the center in $\hat{H}_{M+1}$. Let us introduce the intertwining elements [29] (presented in another form in [27])

$$U_{n+1} = (\sigma_n y_n - y_n \sigma_n) \frac{1}{f(y_n, y_{n+1})} \in \hat{H}_{M+1}(q) \quad (1 \leqslant n \leqslant M), \tag{12.22}$$

where $f(y_n, y_{n+1})$ is an arbitrary function of the two variables y_n, y_{n+1}. The elements U_i satisfy the relations

$$U_n\, U_{n+1} U_n = U_{n+1} U_n U_{n+1}, \tag{12.23}$$

$$U_{n+1}^2 = \frac{(q y_n - q^{-1} y_{n+1})(q y_{n+1} - q^{-1} y_n)}{f(y_n, y_{n+1}) f(y_{n+1}, y_n)}, \tag{12.24}$$

$$\begin{gathered} U_{n+1} y_n = y_{n+1} U_{n+1}, \quad U_{n+1} y_{n+1} = y_n U_{n+1}, \\ [U_{n+1},\, y_k] = 0 \quad (k \neq n, n+1). \end{gathered} \tag{12.25}$$

As is seen from (12.25), the operators U_{n+1} "permute" the elements y_n and y_{n+1}, and this confirms the statement that the center of the Hecke algebra $\hat{H}_{M+1}(q)$ is generated by the symmetric functions in $\{y_i\}$ $(i = 2, \ldots, M+1)$.

One may check that the Hamiltonian (12.1) satisfies

$$\begin{aligned} [\mathcal{H}_{M+1},\, y_k] &= U_{k+1} f_{k+1} - U_k f_k, \\ [\mathcal{H}_{M+1},\, y_k^{-1}] &= \overline{U}_{k+1} f_{k+1} - \overline{U}_k f_{k+1}, \end{aligned} \tag{12.26}$$

where $f_{k+1} = f(y_k, y_{k+1})$, $\overline{U}_{k+1} = U_{k+1}(y_k y_{k+1})^{-1}$ and $U_1 = U_{M+2} = 0$. From (12.26) follows that

$$\left[\mathcal{H}_{M+1},\, \sum_{i=1}^{k} y_k\right] = U_{k+1} f_{k+1}, \quad \left[\mathcal{H}_{M+1},\, \sum_{i=1}^{k} y_i^{-1}\right] = \overline{U}_{k+1} f_{k+1}. \tag{12.27}$$

Further, it is convenient to fix

$$f(y_k, y_{k+1}) = y_k - y_{k+1}.$$

Now we have

$$U_{n+1}^2 = \frac{(qy_{n+1} - q^{-1}y_n)(q^{-1}y_{n+1} - qy_n)}{(y_{n+1} - y_n)^2},$$
$$U_{n+1} = \sigma_n + \frac{\lambda y_{n+1}}{(y_n - y_{n+1})} = \left(\sigma_n - \frac{\lambda}{2}\right) + \frac{\lambda}{2}\frac{(y_n + y_{n+1})}{(y_n - y_{n+1})}, \tag{12.28}$$

and, therefore,

$$s_n = U_{n+1} + v_{n+1}, \tag{12.29}$$

where

$$v_{n+1} = \frac{\lambda}{2}\frac{(y_{n+1} + y_n)}{(y_{n+1} - y_n)}. \tag{12.30}$$

Due to the relations $s_n^2 = \overline{q}^2/4$, we conclude that

$$U_{n+1}^2 + v_{n+1}^2 = \frac{(q + q^{-1})^2}{4}, \quad U_{n+1}v_{n+1} + v_{n+1}U_{n+1} = 0.$$

Finally, for the Hamiltonian (12.1) we obtain

$$\mathcal{H}_{M+1} = \sum_{n=1}^{M} s_n = \sum_{n=1}^{M} (U_{n+1} + v_{n+1}) = \sum_{n=1}^{M} \left(U_{n+1} + \frac{\lambda}{2}\frac{(y_{n+1} + y_n)}{(y_{n+1} - y_n)}\right) =$$
$$= \sum_{n=1}^{M} \left(\sqrt{U_{n+1}^2}\cdot \widetilde{U}_{n+1} + \frac{\lambda}{2}\frac{(y_{n+1} + y_n)}{(y_{n+1} - y_n)}\right), \tag{12.31}$$

where the operators $\widetilde{U}_{n+1}$ permute indices in the Young diagram Λ, or put Λ equal to zero.

Example 1. Consider the basis for the representation $(2, 1)$, which is related to the Young diagram:

1	q^2
q^{-2}	

. (12.32)

We have 2 standard tableaux

$$\psi_0 = \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 \\ \cline{1-1}\end{array}, \quad \psi_1 = U_3\psi_0 = \begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 \\ \cline{1-1}\end{array}. \tag{12.33}$$

Using (12.29), we obtain the action of s_i on the vectors ψ_0, ψ_1 (12.33)

$$s_1\psi_0 = \frac{1}{2}\overline{q}\psi_0, \quad s_1\psi_1 = -\frac{1}{2}\overline{q}\psi_1,$$
$$s_2\psi_0 = \psi_1 + \frac{\lambda}{2}\frac{(q^{-2}+q^2)}{(q^{-2}-q^2)}\psi_0, \quad s_2\psi_1 = U_3^2\psi_0 + \frac{\lambda}{2}\frac{(q^2+q^{-2})}{(q^2-q^{-2})}\psi_1,$$

where $U_3^2\psi_0 = \dfrac{(q^3-q^{-3})(q-q^{-1})}{(q^2-q^{-2})^2}\psi_0$.

Example 2. Consider the basis for the representation (3^2) which is related to the Young diagram

1	q^2	q^4
q^{-2}	1	q^2

. (12.34)

We have 5 standard tableaux (the operators U_{k+1} permute numbers k and $k+1$ in the standard tableaux)

$$\psi_0 = \begin{array}{|c|c|c|}\hline 1&2&3\\\hline 4&5&6\\\hline\end{array}, \quad \psi_1 = U_4\psi_0 = \begin{array}{|c|c|c|}\hline 1&2&4\\\hline 3&5&6\\\hline\end{array}, \quad \psi_2 = U_5U_4\psi_0 = \begin{array}{|c|c|c|}\hline 1&2&5\\\hline 3&4&6\\\hline\end{array},$$
$$\psi_3 = U_3U_4\psi_0 = \begin{array}{|c|c|c|}\hline 1&3&4\\\hline 2&5&6\\\hline\end{array}, \quad \psi_4 = U_5U_3U_4\psi_0 = \begin{array}{|c|c|c|}\hline 1&3&5\\\hline 2&4&6\\\hline\end{array}. \tag{12.35}$$

Using (12.29), we find the action of the operators s_n to the basis vectors ψ_i (12.35)

$$s_1\psi_0 = \frac{1}{2}\overline{q}\psi_0, \;\; s_1\psi_1 = \frac{1}{2}\overline{q}\psi_1, \;\; s_1\psi_2 = \frac{1}{2}\overline{q}\psi_2, \;\; s_1\psi_3 = -\frac{1}{2}\overline{q}\psi_3, \;\; s_1\psi_4 = -\frac{1}{2}\overline{q}\psi_4,$$
$$s_2\psi_0 = \frac{1}{2}\overline{q}\psi_0, \;\; s_2\psi_1 = \psi_3 - \frac{\lambda}{2}\frac{(q^2+q^{-2})}{(q^2-q^{-2})}\psi_1, \;\; s_2\psi_2 = \psi_4 - \frac{\lambda}{2}\frac{(q^2+q^{-2})}{(q^2-q^{-2})}\psi_2,$$
$$s_2\psi_3 = U_3^2\psi_1 - \frac{\lambda}{2}\frac{(q^{-2}+q^2)}{(q^{-2}-q^2)}\psi_3, \;\; s_2\psi_4 = U_3^2\psi_2 - \frac{\lambda}{2}\frac{(q^{-2}+q^2)}{(q^{-2}-q^2)}\psi_4,$$
$$s_3\psi_0 = \psi_1 - \frac{\lambda}{2}\frac{(q^4+q^{-2})}{(q^4-q^{-2})}\psi_0, \;\; s_3\psi_1 = U_4^2\psi_0 - \frac{\lambda}{2}\frac{(q^{-2}+q^4)}{(q^{-2}-q^4)}\psi_1,$$
$$s_3\psi_2 = \frac{1}{2}\overline{q}\psi_2, \;\; s_3\psi_3 = \frac{1}{2}\overline{q}\psi_3, \;\; s_3\psi_4 = -\frac{1}{2}\overline{q}\psi_4,$$

$$s_4\psi_0 = \frac{1}{2}\overline{q}\psi_0,\ \ s_4\psi_1 = \psi_2 - \frac{\lambda}{2}\frac{(q^4+1)}{(q^4-1)}\psi_1,\ \ s_4\psi_2 = U_5^2\psi_1 - \frac{\lambda}{2}\frac{(1+q^4)}{(1-q^4)}\psi_2,$$

$$s_4\psi_3 = \psi_4 - \frac{\lambda}{2}\frac{(q^4+1)}{(q^4-1)}\psi_3,\ \ s_4\psi_4 = U_5^2\psi_3 - \frac{\lambda}{2}\frac{(1+q^4)}{(1-q^4)}\psi_4,$$

$$s_5\psi_0 = \frac{1}{2}\overline{q}\psi_0,\ \ s_5\psi_1 = \frac{1}{2}\overline{q}\psi_1,\ \ s_5\psi_2 = -\frac{1}{2}\overline{q}\psi_2,\ \ s_5\psi_3 = \frac{1}{2}\overline{q}\psi_3,\ \ s_5\psi_4 = -\frac{1}{2}\overline{q}\psi_4,$$

where

$$U_3^2\psi_i = \frac{(q^3-q^{-3})(q-q^{-1})}{(q^2-q^{-2})^2}\psi_i\ \ (i=1,2),\ \ U_4^2\psi_0 = \frac{(q^5-q^{-3})(q^3-q^{-1})}{(q^4-q^{-2})^2}\psi_0,$$

$$U_5^2\psi_i = \frac{(q^5-q^{-1})(q^3-q)}{(q^4-1)^2}\psi_i (i=1,3).$$

Then the equation for eigenvalues ν of $\mathcal{H}_6$ and eigenvectors in the space of the irreducible representation (12.34) is given as follows:

$$\left(\sum_{i=1}^{5} s_i - \nu\right)(\psi_0 + a_1\psi_1 + a_2\psi_2 + a_3\psi_3 + a_4\psi_4) = 0,$$

which is equivalent to the characteristic identity $\nu = \mathcal{H}_6$:

$$\left(\nu - \frac{\overline{q}}{2}\right)\{3\overline{q}^4+16\overline{q}^2-64-(8\overline{q}^3+160\overline{q})\nu+(-8\overline{q}^2+128)\nu^2+32\overline{q}\nu^3-16\nu^4\} = 0. \tag{12.36}$$

Note that eigenvalue $\mathcal{H}_6 = 1/2\overline{q}$ has multiplicity 2 since it has already been presented in $(4,1^2)$ (12.8). The second factor in the characteristic identity (12.36) is related to the Young diagram (3^2) and dual to the factor presented in (12.9).

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Appendix A
DETAILS IN XXX

Equations (7.11) and (7.12) result in the following useful identities:

$$P_{k,k+1}\ \bar{\psi}_k = \bar{\psi}_{k+1} - \bar{\psi}_{k+1}N_k + \bar{\psi}_k N_{k+1}, \tag{A.1}$$

$$\hat{R}_{k,k+1}(\lambda)\bar{\psi}_{k+1} = \bar{\psi}_{k+1} + \lambda\bar{\psi}_k + \lambda\bar{\psi}_{k+1}N_k - \bar{\psi}_k N_{k+1}, \tag{A.2}$$

where $N_k = \bar{\psi}_k \psi_k$, again. We can see that

$$P_{k,k+1}\bar{\psi}_k|0\rangle = \bar{\psi}_{k+1}|0\rangle, \tag{A.3}$$

$$\hat{R}_{k,k+1}(\mu)\bar{\psi}_{k+1}|0\rangle = \bar{\psi}_{k+1}|0\rangle + \mu\bar{\psi}_k|0\rangle \tag{A.4}$$

and

$$\hat{R}_{k,k+1}(\mu)|0\rangle = (\mu+1)|0\rangle, \tag{A.5}$$

$$P_{k,k+1}|0\rangle = |0\rangle. \tag{A.6}$$

For higher magnons we will also need

$$\hat{R}_{k,k+1}(\lambda)\bar{\psi}_k\bar{\psi}_{k+1} = (\lambda+1)\bar{\psi}_k\bar{\psi}_{k+1} \tag{A.7}$$

and

$$\hat{R}_{l,l+1}(\lambda)\cdots\hat{R}_{k-1,k}(\lambda)\bar{\psi}_k|0\rangle = \\ = \lambda^{k-l}\bar{\psi}_l|0\rangle + \frac{\lambda^{k-l}}{\lambda+1}\sum_{j=1}^{k-l}\left(\frac{\lambda+1}{\lambda}\right)^j\bar{\psi}_{l+j}|0\rangle, \tag{A.8}$$

$$\hat{R}_{l,l+1}(\lambda)\cdots\hat{R}_{k,k+1}(\lambda)\bar{\psi}_k|0\rangle = \\ = \hat{R}_{l,l+1}(\lambda)\cdots\hat{R}_{k-1,k}(\lambda)\bar{\psi}_k|0\rangle + \lambda(\lambda+1)^{k-l}\bar{\psi}_{k+1}|0\rangle. \tag{A.9}$$

It is obvious that the second term in (7.15) annihilates vacuum state. Then, using (A.8), we get for the 1-magnon state

$$|\mu\rangle \equiv B(\mu)|0\rangle = (\mu+1-\mu N_1)\hat{R}_{12}(\mu)\cdots\hat{R}_{L-1,L}(\mu)P_{L-1,L}\cdots P_{12}\bar{\psi}_1|0\rangle = \\ = (\mu+1-\mu N_1)\hat{R}_{12}(\mu)\cdots\hat{R}_{L-1,L}(\mu)\bar{\psi}_L|0\rangle = \\ = (\mu+1-\mu N_1)\left[\mu^{L-1}\bar{\psi}_1|0\rangle + \frac{\mu^{L-1}}{\mu+1}\sum_{j=1}^{L-1}\Big(\frac{\mu+1}{\mu}\Big)^j\bar{\psi}_{j+1}\right]|0\rangle = \\ = \frac{\mu^L}{\mu+1}\sum_{k=1}^{L}\left(\frac{\mu+1}{\mu}\right)^k\bar{\psi}_k|0\rangle = n(\mu)\sum_{k=1}^{L}[\mu]^k\bar{\psi}_k|0\rangle \tag{A.10}$$

if we use the notation (8.4).

Using (A.8) and (A.9), we obtain

$$B(\lambda)\bar{\psi}_k|0\rangle = (\lambda+1)\lambda^{L-2}\sum_{m=0}^{k-2}[\lambda]^m\bar{\psi}_{m+1}\bar{\psi}_k|0\rangle + \\ + \frac{\lambda^{L-2}}{\lambda+1}\sum_{m=0}^{k-2}\sum_{j=1}^{L-k}[\lambda]^{j+m}\bar{\psi}_{m+1}\bar{\psi}_{k+j}|0\rangle + \frac{\lambda^L}{\lambda+1}[\lambda]^{k-1}\sum_{j=1}^{L-k}[\lambda]^j\bar{\psi}_k\bar{\psi}_{k+j}|0\rangle. \tag{A.11}$$

We get for the 2-magnon state using (A.11)

$$|\lambda,\mu\rangle \equiv B(\lambda)B(\mu)|0\rangle = n(\mu)\sum_{k=1}^{L}[\mu]^k B(\lambda)\bar{\psi}_k|0\rangle =$$

$$= n(\mu)n(\lambda)\left[\sum_{k=2}^{L}\sum_{m=0}^{k-2}[\mu]^k\,[\lambda]^{m+2}\bar{\psi}_{m+1}\bar{\psi}_k + \frac{1}{\lambda(\lambda+1)}\times\right.$$

$$\left.\times\sum_{k=2}^{L-1}\sum_{m=0}^{k-2}\sum_{j=1}^{L-k}[\mu]^k\,[\lambda]^{j+m+1}\bar{\psi}_{m+1}\bar{\psi}_{k+j} + \sum_{k=1}^{L-1}\sum_{j=1}^{L-k}[\mu]^k\,[\lambda]^{k+j-1}\bar{\psi}_k\bar{\psi}_{k+j}\right]|0\rangle =$$

$$= n(\mu)n(\lambda)\left\{\sum_{1\leqslant r<s\leqslant L}\left[[\mu]^s[\lambda]^{r+1}[\mu]^r[\lambda]^{s-1}\right]\bar{\psi}_r\bar{\psi}_s|0\rangle + \right.$$

$$\left. + \frac{1}{\lambda(\lambda+1)}\sum_{s=3}^{L}\sum_{r=1}^{s-2}\sum_{k=r+1}^{s-1}[\mu]^k[\lambda]^{s+r-k}\bar{\psi}_r\bar{\psi}_s|0\rangle\right\} =$$

$$= n(\mu)n(\lambda)\sum_{1\leqslant r<s\leqslant L}\left[[\mu]^s[\lambda]^{r+1} + [\mu]^r[\lambda]^{s-1} + \right.$$

$$\left. + \frac{1}{\lambda(\lambda+1)}\sum_{k=r+1}^{s-1}[\mu]^k[\lambda]^{s+r-k}\right]\bar{\psi}_r\bar{\psi}_s|0\rangle. \quad \text{(A.12)}$$

The finite sum in (A.12) can be calculated explicitly by means of geometric progression

$$\frac{1}{\lambda(\lambda+1)}\sum_{k=r+1}^{s-1}[\mu]^k[\lambda]^{s+r-k} = \frac{\mu}{\lambda(\lambda-\mu)}[\mu]^{r+1}[\lambda]^{s-1}\left(\left(\frac{[\mu]}{[\lambda]}\right)^{s-r-1} - 1\right) =$$

$$= \frac{\mu}{\lambda(\lambda-\mu)}\left([\mu]^s[\lambda]^r - [\mu]^{r+1}[\lambda]^{s-1}\right). \quad \text{(A.13)}$$

Substitution of (A.13) into (A.12) gives

$$B(\lambda)B(\mu)|0\rangle = n(\mu)n(\lambda)\sum_{1\leqslant r<s\leqslant L}\left[[\mu]^s[\lambda]^{r+1} + [\mu]^r[\lambda]^{s-1} + \right.$$

$$\left. + \frac{\mu}{\lambda(\lambda-\mu)}\left([\mu]^s[\lambda]^r - [\mu]^{r+1}[\lambda]^{s-1}\right)\right]\bar{\psi}_r\bar{\psi}_s|0\rangle =$$

$$= n(\mu)n(\lambda)\sum_{1\leqslant r<s\leqslant L}\left[[\lambda]^r[\mu]^s\frac{\lambda-\mu+1}{\lambda-\mu} + [\mu]^r[\lambda]^s\frac{\mu-\lambda+1}{\mu-\lambda}\right]\bar{\psi}_r\bar{\psi}_s|0\rangle. \quad \text{(A.14)}$$

For the 3-magnon, we need at first

$$
\begin{aligned}
B(\nu)\bar{\psi}_r\bar{\psi}_s|0\rangle = (\nu+1-\nu N_1)X_{12\dots L}(\nu)\bar{\psi}_1\bar{\psi}_r\bar{\psi}_s = \\
= \nu^{L-3}(\nu+1)^2\sum_{m=0}^{r-2}[\nu]^m\bar{\psi}_{m+1}\bar{\psi}_r\bar{\psi}_s|0\rangle + \\
+\nu^{L-3}\sum_{l=1}^{s-r-1}\sum_{m=0}^{r-2}[\nu]^{l+m}\bar{\psi}_{m+1}\bar{\psi}_{r+l}\bar{\psi}_s|0\rangle + \\
+\nu^{L-3}\sum_{j=1}^{L-s}\sum_{m=0}^{r-2}[\nu]^{j+m}\bar{\psi}_{m+1}\bar{\psi}_r\bar{\psi}_{s+j}|0\rangle + \\
+\nu^{L-3}(\nu+1)^{-2}\sum_{j=1}^{L-s}\sum_{l=1}^{s-r-1}\sum_{m=0}^{r-2}[\nu]^{j+l+m}\bar{\psi}_{m+1}\bar{\psi}_{r+l}\bar{\psi}_{s+j}|0\rangle + \\
+\nu^{L-s+r-1}(\nu+1)^{s-r-2}\sum_{j=1}^{L-s}\sum_{m=0}^{r-2}[\nu]^{j+m}\bar{\psi}_{m+1}\bar{\psi}_s\bar{\psi}_{s+j}|0\rangle + \\
+\nu^{L-s+2}(\nu+1)^{s-3}\sum_{j=1}^{L-s}[\nu]^j\bar{\psi}_r\bar{\psi}_s\bar{\psi}_{s+j}|0\rangle + \\
+\nu^{L-r}(\nu+1)^{r-1}\sum_{l=1}^{s-r-1}[\nu]^l\bar{\psi}_r\bar{\psi}_{r+l}\bar{\psi}_s|0\rangle + \\
+\nu^{L-r}(\nu+1)^{r-3}\sum_{j=1}^{L-s}\sum_{l=1}^{s-r-1}[\nu]^{j+l}\bar{\psi}_r\bar{\psi}_{r+l}\bar{\psi}_{s+j}|0\rangle. \quad \text{(A.15)}
\end{aligned}
$$

The 3-magnon state is obtained from the 2-magnon state (A.14)

$$
\begin{aligned}
|\nu,\mu,\lambda\rangle \equiv B(\nu)B(\mu)B(\lambda)|0\rangle = \\
= n(\mu)n(\lambda)\sum_{1\leqslant r<s\leqslant L}K_2(s,r)B(\nu)\bar{\psi}_r\bar{\psi}_s|0\rangle, \quad \text{(A.16)}
\end{aligned}
$$

where we denote for more comfort

$$
K_2(s,r) = [\mu]^s[\lambda]^r\frac{\lambda-\mu+1}{\lambda-\mu} + [\mu]^r[\lambda]^s\frac{\mu-\lambda+1}{\mu-\lambda}. \quad \text{(A.17)}
$$

Using (A.15), we get

$$|\nu,\mu,\lambda\rangle = n(\nu)n(\mu)n(\lambda)\sum_{1\leqslant q<r<s\leqslant L}\Bigg[[\nu]^{q+2}K_2(s,r)+ + \frac{1}{\nu^2}\sum_{l=1}^{r-q-1}[\nu]^{l+q}K_2(s,r-l)+\frac{1}{\nu^2}\sum_{j=1}^{s-r-1}[\nu]^{j+q}K_2(s-j,r)+ + \frac{1}{\nu^2(\nu+1)^2}\sum_{j=1}^{s-r-1}\sum_{l=1}^{r-q-1}[\nu]^{j+l+q}K_2(s-j,r-l)+ + \frac{1}{(\nu+1)^2}\sum_{l=q+1}^{r-1}[\nu]^{s-l+q}K_2(r,l)+[\nu]^{s-2}K_2(r,q)+[\nu]^{r}K_2(s,q)+ + \frac{1}{(\nu+1)^2}\sum_{l=r+1}^{s-1}[\nu]^{s+r-l}K_2(l,q)\Bigg]\bar\psi_q\bar\psi_r\bar\psi_s|0\rangle = n(\nu)n(\mu)n(\lambda)\times$$

$$\times\sum_{1\leqslant q<r<s\leqslant L}\sum_{T\in S_3}T\left([\nu]^q[\mu]^r[\lambda]^s\frac{\nu-\mu+1}{\nu-\mu}\frac{\nu-\lambda+1}{\nu-\lambda}\frac{\mu-\lambda+1}{\mu-\lambda}\right)\bar\psi_q\bar\psi_r\bar\psi_s|0\rangle. \tag{A.18}$$

Appendix B

DETAILS IN XXZ

It is convenient to introduce the following notation:

$$d(\lambda)=1-\lambda,\quad b(\lambda)=q-q^{-1},\quad a(\lambda)=q-\lambda q^{-1}. \tag{B.1}$$

We see that coefficient (10.3) resp. normalization (10.4) can be written as

$$[\lambda]_q=\frac{a(\lambda)}{d(\lambda)},\qquad n_q(\lambda)=\frac{d(\lambda)^L b(\lambda)}{a(\lambda)}. \tag{B.2}$$

The R-matrix $\hat R_{k,k+1}(\lambda)$ is of the form (9.2) and the operator $B(\lambda)$ is of the form (9.8). For computing of Bethe vectors, the following set of identities seems to be very useful:

$$\hat R_{k,k+1}(\lambda)\bar\psi_{k+1}|0\rangle = d(\lambda)\psi_k|0\rangle + b(\lambda)\bar\psi_{k+1}|0\rangle, \tag{B.3}$$

$$\hat R_{k,k+1}(\lambda)\bar\psi_{k}|0\rangle = \lambda b(\lambda)\psi_k|0\rangle + d(\lambda)\bar\psi_{k+1}|0\rangle, \tag{B.4}$$

$$\hat R_{k,k+1}(\lambda)\bar\psi_{j} = a(\lambda)\psi_j \quad \text{for} \quad j\notin\{k,k+1\} \tag{B.5}$$

and

$$\hat{R}_{l,l+1}(\lambda)\cdots\hat{R}_{k-1,k}(\lambda)\bar{\psi}_k|0\rangle = \\ = d(\lambda)^{k-l}\bar{\psi}_l|0\rangle + b(\lambda)\sum_{j=1}^{k-l} d(\lambda)^{k-l-j}a(\lambda)^{j-1}\bar{\psi}_{l+j}|0\rangle, \qquad \text{(B.6)}$$

$$\hat{R}_{l,l+1}(\lambda)\cdots\hat{R}_{k,k+1}(\lambda)\bar{\psi}_k|0\rangle = \\ = \lambda b(\lambda)\hat{R}_{l,l+1}(\lambda)\cdots\hat{R}_{k-1,k}(\lambda)\bar{\psi}_k|0\rangle + d(\lambda)a(\lambda)^{k-l}\bar{\psi}_{k+1}|0\rangle. \qquad \text{(B.7)}$$

Using (B.6), we can straightforwardly calculate the q-deformed 1-magnon state for $B(\mu)$ defined in (9.8)

$$|\mu\rangle \equiv B(\mu)|0\rangle = \frac{d(\mu)^L b(\mu)}{a(\mu)}\sum_{k=1}^{L}\left(\frac{a(\mu)}{d(\mu)}\right)^k \bar{\psi}_k|0\rangle = n_q(\mu)\sum_{k=1}^{L}[\mu]_q^k\bar{\psi}_k|0\rangle \qquad \text{(B.8)}$$

recalling (10.3) and (10.4).

Using the formulas mentioned (B.3)–(B.7), we can calculate the q-deformed 2-magnon state. First of all, we need $B(\lambda)\bar{\psi}_k|0\rangle$

$$B(\lambda)\bar{\psi}_k|0\rangle = b(\lambda)a(\lambda)d(\lambda)^{L-2}\sum_{i=0}^{k-2}[\lambda]_q^i\bar{\psi}_{1+i}\bar{\psi}_k|0\rangle + \\ + b(\lambda)a(\lambda)^{k-2}d(\lambda)^{L-k+1}\sum_{j=1}^{L-k}[\lambda]_q^j\bar{\psi}_k\bar{\psi}_{k+j}|0\rangle + \\ + \lambda b(\lambda)^3 a(\lambda)^{-1}d(\lambda)^{L-2}\sum_{i=0}^{k-2}\sum_{j=1}^{L-k}[\lambda]_q^{i+j}\bar{\psi}_{1+i}\bar{\psi}_{k+j}|0\rangle. \qquad \text{(B.9)}$$

Hence, we get the q-deformed 2-magnon state

$$|\lambda,\mu\rangle \equiv B(\lambda)B(\mu)|0\rangle = n_q(\mu)\sum_{k=1}^{L}[\mu]_q^k B(\lambda)\bar{\psi}_k|0\rangle = \\ = n_q(\mu)n_q(\lambda)\sum_{1\leqslant r<s\leqslant L}\left\{\frac{a(\lambda)}{d(\lambda)}\left(1+\frac{\lambda b(\lambda)^2 a(\lambda)^{-1}d(\mu)}{a(\mu)d(\lambda)-a(\lambda)d(\mu)}\right)[\lambda]_q^r[\mu]_q^s + \right. \\ \left. + \left(\frac{d(\lambda)}{a(\lambda)}-\frac{a(\mu)}{d(\mu)}\frac{\lambda b(\lambda)^2 a(\lambda)^{-1}d(\mu)}{a(\mu)d(\lambda)-a(\lambda)d(\mu)}\right)[\lambda]_q^s[\mu]_q^r\right\}\bar{\psi}_r\bar{\psi}_s|0\rangle = \\ = n_q(\mu)n_q(\lambda)\sum_{1\leqslant r<s\leqslant L}\left\{\frac{\lambda q^{-1}-\mu q}{\lambda-\mu}[\lambda]_q^r[\mu]_q^s + \frac{\mu q^{-1}-\lambda q}{\mu-\lambda}[\mu]_q^r[\lambda]_q^s\right\}\bar{\psi}_r\bar{\psi}_s|0\rangle. \qquad \text{(B.10)}$$

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