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ФИЗИКА ЭЛЕМЕНТАРНЫХ ЧАСТИЦ И АТОМНОГО ЯДРА. ТЕОРИЯ

NONCOMPACT HOPF MAPS, QUANTUM HALL EFFECT, AND TWISTOR THEORY

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Introducing a noncompact version of the Hopf maps, we develop a noncompact formulation of the quantum Hall effect. In particular, we focus on a construction of quantum Hall effect on a 4D hyperboloid. It is pointed that the noncompact 4D quantum Hall effect shares remarkably analogous structures with twistor theory. The contents are based on the recent papers arXiv:0902.2523 and arXiv:0905.2792.

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INTRODUCTION

The notion of twistor theory was introduced by Penrose [1], and the twistor theory has been developed in mathematical physics. Meanwhile, the quantum Hall effect has mainly been developed in condensed matter physics. Though it is less well known, there are close mathematical structures between twistor and quantum Hall effect. Their analogies are pointed out in [2–5] in the development of higher dimensional quantum Hall effect [6] (for earlier discussions about analogies between monopole system and twistor, see [7]). In recent years, the construction of quantum Hall systems has successfully been generalized in higher dimensional spaces such as higher dimensional spheres [8–10] and complex projective spaces [2]. For the construction of the higher dimensional quantum Hall effect, such compact spaces are mainly utilized [2]. However, the twistor usually describes a massless particle in Minkowski space-time which is a noncompact space. Then, it would be reasonable to formulate the quantum Hall effect on a noncompact manifold to explore analogies between quantum Hall effect and twistor theory [11]. With such motivations, in this work, we develop a noncompact formulation of quantum Hall effect on a 4D hyperboloid [12]. For this purpose, we first introduce a noncompact version of the Hopf maps [13].

The contents are organized as follows. In Sec. 1, we give a brief introduction to the twistor theory. In Sec. 2, the noncompact Hopf maps are introduced and the corresponding monopole system is clarified. In Sec. 3, with the use of the (2nd) noncompact Hopf map, we explore geometrical structures of the lowest Landau level on a 4D hyperboloid and discuss relations to twistor theory. In Sec. 4, we develop Hamiltonian formalism, and investigate Landau problem and many-body states. Section 5 is devoted to summary and discussion.

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[2]On a noncompact manifold $SU(n,1)/U(1)$, quantum Hall effect is formulated in [14]. In particular, for 2D hyperboloid $SU(1,1)/U(1)$ and its supersymmetric generalization, one may consult [15].

1. BRIEF INTRODUCTION TO TWISTORS

The twistor space was introduced as a more fundamental space than space-time itself [1]. In this section, we briefly review the twistor description of massless particle based on Shirafuji [16]. The Lagrangian of a free particle is given by

$$L = \frac{M}{2}\eta_{\mu\nu}\dot{x}^\mu\dot{x}^\nu, \tag{1}$$

where «·» indicates the proper-time derivative and $\eta_{\mu\nu} = \mathrm{diag}\,(+,-,-,-,)$. By introducing the momentum p^μ, the free-particle Lagrangian is rewritten as

$$L = \eta_{\mu\nu}p^\mu\dot{x}^\nu + \frac{1}{2M}\eta_{\mu\nu}p^\mu p^\nu. \tag{2}$$

The equation of motion of p^μ is derived as

$$p^\mu = M\dot{x}^\mu. \tag{3}$$

In the massless case, p^μ satisfies the massless condition $\eta_{\mu\nu}p^\mu p^\nu = 0$. Provided p^μ is given by a bilinear form of $SL(2,C)$ spinor π

$$p^\mu = \pi^\dagger\sigma^\mu\pi, \tag{4}$$

the momentum p^μ automatically satisfies the massless condition. Then, the $SL(2,C)$ spinor π could be regarded as a more fundamental variable to describe massless particle rather than p^μ. By inserting (4) into (2), the Lagrangian takes the form of the first derivative of proper time:

$$L = \dot{x}^\mu\pi^\dagger\sigma_\mu\pi. \tag{5}$$

The coordinates x^μ are not gauge-independent quantities, in the sense that Lagrangian is invariant under the following transformation:

$$x^\mu \to x^\mu + \lambda(x)\,\mathrm{tr}\,(\sigma^\mu\pi\pi^\dagger). \tag{6}$$

Instead of the coordinates x^μ, we introduce the gauge-invariant quantity ω^α:

$$\omega^\alpha = i(x^\mu\sigma_\mu)^{\alpha\beta}\pi_\beta. \tag{7}$$

The twistor is a four-component spinor that consists of π_α and ω^α:

$$Z^a = (\omega^\alpha, \pi_\beta). \tag{8}$$

From (7), one may see its upper and lower two components are related by

$$\begin{pmatrix} Z^1 \\ Z^2 \end{pmatrix} = ix^\mu\sigma_\mu \begin{pmatrix} Z^3 \\ Z^4 \end{pmatrix}. \tag{9}$$

This relation is known as the incidence relation that connects the space-time and twistor variables. With twistor variables, the Lagrangian (2) can simply be rewritten as

$$L = -iZ_a^*\frac{d}{d\tau}Z^a, \tag{10}$$

where Z_a is the dual twistor defined as $Z_a = (\pi_\alpha, \omega^\beta)$. With the new definition of the twistor variables

$$\begin{pmatrix} \mathcal{Z}^1 \\ \mathcal{Z}^2 \\ \mathcal{Z}^3 \\ \mathcal{Z}^4 \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} Z^1 + Z^3 \\ Z^1 - Z^3 \\ Z^2 + Z^4 \\ Z^2 - Z^4 \end{pmatrix}, \tag{11}$$

the Lagrangian (10) is expressed as

$$L = -i\mathcal{Z}^\dagger k \frac{d}{d\tau}\mathcal{Z}, \tag{12}$$

where k denotes the $SU(2,2)$ invariant matrix:

$$k = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}. \tag{13}$$

Physically, the norm of twistor represents the helicity of massless particle [16][1]:

$$\lambda = \frac{1}{2} Z_a^* Z^a = \frac{1}{2}\mathcal{Z}^\dagger k \mathcal{Z}. \tag{15}$$

At the massless limit of free particle, the system enjoys the conformal symmetry $SO(4,2)$ rather than Poincare symmetry, and indeed such $SO(4,2) \simeq SU(2,2)$ symmetry is manifest in the action (12) and the constraint (15) of the twistors. This is an advantage to use twistor variables, since the $SU(2,2)$ conformal symmetry is manifest in the formalism. Following the original motivation of introduction of twistor, we treat twistors as fundamental variables and impose the canonical quantization condition on them. From the action (12), the canonical conjugation of the twistor is derived as $-ik\mathcal{Z}^*$, which is regarded as derivative by the canonical quantization condition:

$$[\mathcal{Z}_a, (k\mathcal{Z}^*)_b] = \delta_{ab}. \tag{16}$$

This suggests the noncommutative geometry in twistor space since the twistor variables are no longer commutative.

2. THE NONCOMPACT HOPF MAPS

The mathematical structure of the original 2D quantum Hall effect is based on the first compact Hopf map [17, 18], and so the quantum Hall effect may be generalized into higher dimensions by utilizing the higher dimensional Hopf maps. Indeed, based on the second

[1] By (9), Z^a should satisfy the constraint

$$Z_a^* Z^a = \mathcal{Z}^\dagger k \mathcal{Z} = 0. \tag{14}$$

Namely, zero helicity $\lambda = 0$. For the description of nonzero helicity, a complexified Minkowski space-time has to be introduced [16].

or quaternionic Hopf map, the 4D generalization of the quantum Hall effect was introduced by Zhang and Hu [8]. In the following, we explore a noncompact formulation of their model. We utilize the split-quaternionic Hopf map that is regarded as a natural noncompact counterpart of the quaternionic Hopf map. Before discussing the details of construction of noncompact quantum Hall effect, we briefly review the original Hopf maps and their noncompact generalization. The (original) Hopf maps are introduced as topological map from sphere to sphere in different dimensions [19, 20]:

$$\begin{array}{ccccc} & & S^1 & & \\ & S^3 & \longrightarrow & S^2 & \quad \text{(1st)} \\ S^7 & \longrightarrow & S^4 & & \quad \text{(2nd)} \\ S^{15} \longrightarrow & S^8 & & & \quad \text{(3rd).} \end{array} \tag{17}$$

These maps are deeply related to the division algebras. The first Hopf map is related to the complex numbers, the second one to the quaternions, and the third one to the octonions [21]. Physically, bundle structures of three Hopf maps are related to monopole bundles of the gauge group $U(1)$, $SU(2)$ and $SO(8)$, respectively [22–24]. Interestingly, the division algebras have a noncompact cousins known as split algebras introduced by James Cockle [25, 26]. The split algebras also consist of three species: split complex numbers, split quaternions and split octonions. By simply replacing the role of the division algebra with the split algebra, the noncompact version of the Hopf maps is introduced as [12]

$$\begin{array}{ccccc} & & H^{1,0} & & \\ & H^{2,1} & \longrightarrow & H^{1,1} & \quad \text{(1st)} \\ H^{4,3} & \longrightarrow & H^{2,2} & & \quad \text{(2nd)} \\ H^{8,7} \longrightarrow & H^{4,4}, & & & \quad \text{(3rd),} \end{array} \tag{18}$$

where $H^{p,q}$ denotes a higher dimensional hyperboloid defined by

$$\sum_{i=1}^{p} x^i x^i - \sum_{j=p+1}^{p+q+1} x^j x^j = -1. \tag{19}$$

With the $SO(3,2)$ four-component spinor subject to the «normalization» condition

$$\psi^\dagger k \psi = 1, \tag{20}$$

the split-quaternionic Hopf map, i.e., the 2nd noncompact Hopf map, is realized as

$$\psi \rightarrow x^a = \psi^\dagger k \gamma^a \psi, \tag{21}$$

where γ^a are the $SO(3,2)$ gamma matrices that satisfy $\{\gamma^a, \gamma^b\} = -2\eta^{ab}$ ($\eta_{ab} = \eta^{ab} = \mathrm{diag}\,(1,1,-1,-1,-1)$), or more explicitly,

$$\begin{aligned} &\gamma^i = \begin{pmatrix} 0 & -i\tau^i \\ i\tau^i & 0 \end{pmatrix}, \quad \gamma^4 = \begin{pmatrix} 0 & 1_{2\times2} \\ 1_{2\times2} & 0 \end{pmatrix}, \\ &\gamma^5 = \begin{pmatrix} 1_{2\times2} & 0 \\ 0 & -1_{2\times2} \end{pmatrix}, \quad k = \begin{pmatrix} \sigma^3 & 0 \\ 0 & \sigma^3 \end{pmatrix}, \end{aligned} \tag{22}$$

where τ^i are $SU(1,1)$ Pauli matrices given by $\tau^i = i\sigma^1, i\sigma^2, \sigma^3$. The quantities x^a defined by (21) automatically satisfy the normalization condition

$$\eta_{ab}x^a x^b = -(\psi^\dagger k \psi)^2 = -1, \tag{23}$$

and they are regarded as coordinates on a four-dimensional hyperboloid $H^{2,2}$. Inverting the split-quaternionic Hopf map, we obtain

$$\psi = \frac{1}{\sqrt{2(1+x^5)}} \begin{pmatrix} (1+x^5)\phi \\ (x^4 + i\kappa^i x_i)\phi \end{pmatrix}, \tag{24}$$

where ϕ denotes a two-component spinor subject to the condition

$$\phi^\dagger \sigma^3 \phi = 1. \tag{25}$$

The gauge field is induced as

$$A = -i\psi^\dagger k \, d\psi = dx^a \phi^\dagger \sigma^3 A_a \phi, \tag{26}$$

where

$$A_\mu = -\frac{1}{2}\eta_{\mu\nu i}\frac{x^\nu}{1+x^5}\tau^i, \quad A_5 = 0. \tag{27}$$

Here, $\mu, \nu = 1,2,3,4$ and $\eta_{\mu\nu i}$ is a $SO(2,2)$ version of the 't Hooft tensor. The gauge field may naturally be interpreted as $SU(1,1)$ gauge monopole field since the gauge fields are given by $SU(1,1)$ generators. Therefore, the corresponding physical setup is given by the four-dimensional hyperboloid $H^{2,2}$ in the $SU(1,1)$ monopole background.

3. LOWEST LANDAU LEVEL AND TWISTOR-LIKE GEOMETRY

Next, we explore one-particle mechanics on a hyperboloid $H^{2,2}$ in $SU(1,1)$ monopole background. The one-particle Lagrangian is given by

$$L = \frac{M}{2}\eta_{ab}\dot{x}^a \dot{x}^b + \dot{x}^a A_a, \tag{28}$$

where the coordinates on hyperboloid x^a should satisfy the constraint (23). In the lowest Landau level (LLL) limit, the kinetic term is quenched and the Lagrangian is simply reduced to $L_{\mathrm{LLL}} = \dot{x}^a A_a = -iI\psi^\dagger k (d/dt)\psi$, with monopole charge $I/2$. By changing the scale of ψ as $\psi \to (1/\sqrt{I})\psi$, the Lagrangian is expressed as

$$L_{\mathrm{LLL}} = -i\psi^\dagger k \frac{d}{dt}\psi, \tag{29}$$

and the constraint (20) becomes

$$\psi^\dagger k \psi = I. \tag{30}$$

The constraint of the coordinates x^a (23) can be restated as that on the spinor ψ (30). As we see from (29) and (30), the LLL physics enjoys the larger $SU(2,2)$ symmetry than the original $SO(3,2)$ symmetry. Apparent analogies can be found between the LLL physics and

Table 1. **Analogies between the noncompact QHE and twistor: The original setups are different; the base manifold of the QHE is $H^{2,2}$ whose isometry is $SO(3,2)$, while that of twistor is Minkowski space whose isometry is Poincare. However, once «massless limit» is taken, both systems enjoy the enlarged $SU(2,2)$ conformal symmetry and everything goes parallel**

	QHE	Twistor
Fundamental quantity	Hopf spinor	Twistor
Quantized value	Monopole charge	Helicity
Base manifold	Hyperboloid $H^{2,2}$	Minkowski space
Original symmetry	$SO(3,2)$	Poincare
Special limit	LLL ($M \to 0$)	Zero mass ($M \to 0$)
Enhanced symmetry	$SU(2,2)$	$SU(2,2)$
Emergent manifold	CP^3	CP^3
Fuzzy manifold	Fuzzy hyperboloid	Fuzzy twistor space

the twistors, for their actions (12) and (29), and for their constraints (15) and (30). (Further analogies are summarized in Table 1.) The analogies between the twistor and the LLL physics could also be observed in their incidence relations. With stereographic coordinates $x_L^\mu = \frac{1}{1+x^5}x^\mu$, the upper and lower two-components of the noncompact second Hopf spinor (24) are related as

$$\begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} = (x_L^4 - i x_L^i \kappa_i) \begin{pmatrix} \psi_3 \\ \psi_4 \end{pmatrix}. \tag{31}$$

Obviously, this corresponds to the incidence relation of the twistors (9).

In the LLL, from (29), the canonical momentum of ψ will be $k\psi^*$. By imposing the canonical quantization condition on ψ and $k\psi^*$, ψ^* is regarded as a derivative

$$\psi^* = -k\frac{\partial}{\partial \psi}. \tag{32}$$

Inserting this relation into the split-quaternionic Hopf map (21), we obtain

$$X_a = -\psi^t \gamma_a^t \frac{\partial}{\partial \psi}. \tag{33}$$

Similarly, the $SO(3,2)$ generators L_{ab} are represented as

$$X_{ab} = -\psi^t \sigma_{ab}^t \frac{\partial}{\partial \psi}, \tag{34}$$

since $L_{ab}\psi = -\sigma_{ab}\psi$ with $\sigma_{ab} = -i(1/4)[\gamma_a, \gamma_b]$. The operators X^a and X^{ab} satisfy the following algebra:

$$\begin{aligned} &[X_a, X_b] = 4iX_{ab}, \quad [X_a, X_{bc}] = i(\eta_{ab}X_c - \eta_{ac}X_b), \\ &[X_{ab}, X_{cd}] = -i(\eta_{ac}X_{bd} - \eta_{ad}X_{bc} + \eta_{bd}X_{ac} - \eta_{bc}X_{ad}). \end{aligned} \tag{35}$$

With definition X_{AB} $(A, B = 1, 2, \ldots, 6)$; $X_{a6} = -(1/2)X_a$ and $X_{ab} = X_{ab}$, Eq. (35) represents the $SO(4,2) \simeq SU(2,2)$ algebra of X_{AB}. The $SU(2,2)$ noncommutative algebra naturally defines the fuzzy manifold of $CP^{2,1}$, which is the projective twistor space. Such fuzzy $CP^{2,1}$ is the manifold behind the LLL physics, and the emergence of $CP^{2,1}$ could also be understood as follows. We adopted the Hopf spinor ψ, which is the coordinates of $H^{4,3}$, and the modulo $U(1)$ phase of ψ gives rise to a manifold $H^{4,3}/S^1$ which is $CP^{2,1}$. Here, we add some crucial comments. To derive the noncommutative algebras (35), we did *not* quantize the original space-time coordinates, but quantized the more fundamental (Hopf spinor) variables, and the fuzziness in the original space-time was induced by that of the more fundamental space. Indeed, this realizes the original philosophy of twistor; the space-time fuzziness should come from the more fundamental (twistor) space. Around the north pole of the hyperboloid, the noncommutative relation becomes

$$[X_\mu, X_\nu] = i\ell_B^2 \eta_{\mu\nu i}\tau^i. \tag{36}$$

This is the fundamental relation for the LLL geometry unifying the space-time fuzziness and the internal «spin» structure as first pointed out in the original setup of 4D quantum Hall effect [8].

4. NONCOMPACT 4D QUANTUM HALL EFFECT

Here, we analyze the $SO(3,2)$ symmetric Landau problem on a 4D hyperboloid $H^{2,2}$ in $SU(1,1)$ monopole background[1]. The $SO(3,2)$ covariant angular momentum is defined as $\Lambda_{ab} = -ix_a(\partial_b + iA_b) + ix_b(\partial_a + iA_a)$ that satisfies $[\Lambda_{ab}, \Lambda_{cd}] = i(\eta_{ac}\Lambda_{bd} - \eta_{ad}\Lambda_{bc} + \eta_{bd}\Lambda_{ac} - \eta_{bc}\Lambda_{ad}) - i(x_a x_c F_{bd} - x_a x_d F_{bd} + x_b x_d F_{ac} - x_b x_c F_{ad})$ with monopole field strength $F_{ab} = \partial_a A_b - \partial_b A_a + i[A_a, A_b]$. On $H^{2,2}$, the Landau Hamiltonian is given by

$$H = -\frac{1}{2MR^2}\sum_{a<b}\Lambda_{ab}^2, \tag{37}$$

and, in the discrete series, the energy eigenvalues are derived as

$$E_n = \frac{1}{2MR^2}(I(n+1) - n(n+3)), \tag{38}$$

where n represents Landau level index. In the thermodynamic limit $R, I \to \infty$ with magnetic length $\ell_B = R\sqrt{2/I}$ fixed, the energy eigenvalues reproduce the planar Landau levels $E_n \to \frac{I}{2MR^2}(n+1)$.

Next, we move to investigation of many-body problem on $H^{2,2}$. Since the original Laughlin–Haldane wave function is constructed so as to respect the $SU(2)$ symmetry of the base manifold S^2 [17], it may be natural to adopt the $SO(3,2)$ singlet ground-state wave function (made by the Hopf spinors) as the many-body ground state on $H^{2,2}$. Such a wave function is derived as

$$\Psi = \prod_{i<j}(\psi_i^t rk\psi_j)^m, \tag{39}$$

[1]A $SO(1,4)$ symmetric oscillator-like model on $H^{1,3}$ was analyzed in [27].

where r is the charge conjugation matrix

$$r = \begin{pmatrix} \sigma^1 & 0 \\ 0 & \sigma^1 \end{pmatrix}. \tag{40}$$

Topological excitations are induced by flux penetrations. Their annihilation and creation operators are respectively given by

$$A(\chi) = \prod_i^N \chi^\dagger r \frac{\partial}{\partial \psi_i}, \qquad A^\dagger(\chi) = \prod_i^N \psi_i^t r k \chi, \tag{41}$$

where χ denotes a flux penetration point on $H^{2,2}$ by the relation $\chi^\dagger k \gamma^a \chi = \Omega^a(\chi)$. The operators (41) satisfy the creation and annihilation relations, $[A(\chi), A^\dagger(\chi)] = 1$, $[A(\chi), A(\chi')] = 0$, and $[A^\dagger(\chi), A^\dagger(\chi')] = 0$. With fuzzy hyperboloid coordinates $X_a = -\psi^t \gamma_a^t \frac{\partial}{\partial \psi}$, the creation operator satisfies $[\Omega_a(\chi) X^a, A^\dagger(\chi)] = N A^\dagger(\chi)$. This implies that particles on $H^{2,2}$ are pushed «outward» from the flux penetration point, and a charge deficit is generated at the point. The constant spinor χ carries «extra degrees» of AdS^3-fibre as its phase, and, up to $U(1)$ phase, such extra degrees account for a hyperbolic membrane of the form $H^{2,0} \simeq AdS^3/U(1)$.

5. SUMMARY AND DISCUSSION

We developed a noncompact formulation of 4D quantum Hall effect based on the split-quaternionic Hopf map. The striking resemblances to twistor formalism were clarified. The present construction is related to many exotic mathematics and physical ideas, such as split algebras, noncompact Hopf maps, higher dimensional quantum liquid, membrane-like excitations, noncommutative geometry. The most peculiar structure would be a relation to extra-time physics. By the use of the noncompact Hopf maps, we have to deal with the noncompact groups that naturally bring a notion of split-signature space-time [28,29] (see also Table 2). In the present construction, we encountered the noncompact manifold $H^{2,2}$, which has two-time dimensions. Thus, the present model gives a physical realization of the extra-time physics and may demonstrate particular properties speculated in [30–33]. The edge physics may also be an interesting subject to be pursued. As edge excitations of the original 4D quantum Hall effect, there appear higher spin spectra including photon and graviton [8]. However, a field theoretical description of such higher spin massless contents has not successfully been constructed in flat space-time. Meanwhile, in the present model, the base manifold is taken to be a hyperboloid and its edge manifold also has negative curvature. It is reported that

Table 2. **Lorentz and split-Lorentz groups from division and split algebras**

	Division algebras	Split algebra
Real numbers	$SO(2,1) \simeq SL(2,\mathcal{R})$	$SO(2,1) \simeq SL(2,\mathcal{R})$
Complex numbers	$SO(3,1) \simeq SL(2,\mathcal{C})$	$SO(2,2) \simeq SL(2,\mathcal{C}')$
Quaternions	$SO(5,1) \simeq SL(2,\mathcal{H})$	$SO(3,3) \simeq SL(2,\mathcal{H}')$
Octonions	$SO(9,1) \simeq SL(2,\mathcal{O})$	$SO(5,5) \simeq SL(2,\mathcal{O}')$

higher spin field theory will become consistent in AdS space [34], and present edge physics is expected to describe such a higher spin theory in negative curvature space.

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