Письма в ЭЧАЯ. 2011. Т. 8, № 3(166). С. 394–407

ФИЗИКА ЭЛЕМЕНТАРНЫХ ЧАСТИЦ И АТОМНОГО ЯДРА. ТЕОРИЯ

JUCYS–MURPHY ELEMENTS FOR BIRMAN–MURAKAMI–WENZL ALGEBRAS

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The Birman–Murakami–Wenzl algebra, considered as the quotient of the braid group algebra, possesses the commutative set of Jucys–Murphy elements. We show that the set of Jucys–Murphy elements is maximal commutative for the generic Birman–Murakami–Wenzl algebra and reconstruct the representation theory of the tower of Birman–Murakami–Wenzl algebras.

PACS: 02.20.-a

INTRODUCTION

Let G be either the orthogonal group $SO(N)$ or, for N even, the symplectic group $Sp(N)$. Let $\mathbf{g}$ be its Lie algebra and $U_q(\mathbf{g})$ the corresponding quantum universal enveloping algebra. We denote the space of the defining irreducible representation (irrep) of G or $U_q(\mathbf{g})$ by V. Let $v_i \in V$ $(i = 1, \ldots, N)$ be a basis of V. Denote by K the G-invariant pairing in V, $K(v_i \otimes v_j) = K_{ij} \in \mathbb{C}$.

In 1937 R. Brauer [1] introduced a 1-parametric family of algebras $Br_n(x)$ to describe the centralizer of the action of G on the tensor powers $V^{\otimes n}$. More precisely, fix the value of the parameter x, $x = N$. The algebra $Br_n(N)$ has the representation τ: $Br_n(N) \to \mathrm{End}(V^{\otimes n})$; the image of $Br_n(N)$ in this representation coincides with the commutant of the action of G on $V^{\otimes n}$. The generators of the algebra $Br_n(N)$ are expressed in terms of the permutation P and the operator $\mathbf{K}$ related to the G-invariant pairing K:

$$P(v_i \otimes v_j) = v_j \otimes v_i, \quad \mathbf{K}(v_i \otimes v_j) = K_{ij} K^{kl} v_k \otimes v_l.$$

Here K^{kl} is inverse to K_{ij}, $K^{kl} K_{lj} = \delta^k_j$. The Brauer algebras play the same role in the representation theory of $SO(N)$ and $Sp(N)$ groups as the symmetric groups in the theory of representations of linear groups. The Brauer–Schur–Weyl duality establishes the correspondence between the finite-dimensional irreps of $SO(N)$, $Sp(N)$ and the irreps of $Br_n(N)$.

For quantum deformations $U_q(\mathbf{g})$, the Brauer algebras $Br_n(N)$ get q-deformed as well; instead of $Br_n(x)$ one now has a 2-parametric family of algebras $BMW_n(q, \nu)$. These algebras were introduced independently by J. Murakami [2] and by J. Birman and H. Wenzl [3].

The centralizers $\mathrm{End}_{U_q(\mathbf{g})}(V^{\otimes n})$ are realized by specific representations τ of the Birman–Murakami–Wenzl algebras. The value $\nu_{U_q(g)}$ of the parameter ν depends on q and $\mathbf{g}$. In the representation τ: $BMW_n(q, \nu_{U_q(g)}) \to \mathrm{End}(V^{\otimes n})$ the generators of BMW algebras are built with the help of the Yang–Baxter R-operator, the q-analogue of the permutation P (see [4] for the functorial construction of R-matrices of BMW type from the R-matrices of GL type). In contrast to the classical case, the q-analogue of $\mathbf{K}$ is a certain combination of Yang–Baxter R-operators, see, e.g., [5, 6].

For generic values of the parameters q and $\nu_{U_q(g)}$ the BMW algebra has the same representations as the Brauer algebra. Different aspects of the representation theory were extensively studied in the literature (see, e.g., [7–11] and references therein).

Here we generalize to the BMW algebras the approach of Vershik and Okounkov [12] developed for the representation theory of symmetric groups and adopted to the Hecke algebra case in [13]. The details of the proofs will be published in our forthcoming publication [14].

1. BIRMAN–MURAKAMI–WENZL (BMW) ALGEBRAS

1.1. Braid Group and Its Quotients. The braid group $\mathcal{B}_{M+1}$ is generated by elements σ_i, $i = 1, \ldots, M$, subject to relations:

$$\text{Braid}: \quad \sigma_i\, \sigma_{i+1}\, \sigma_i = \sigma_{i+1}\, \sigma_i\, \sigma_{i+1}, \tag{1}$$

$$\text{Locality}: \quad \sigma_i\, \sigma_j = \sigma_j\, \sigma_i \quad \text{if} \quad |i-j| > 1. \tag{2}$$

The braid group $\mathcal{B}_{M+1}$ is infinite. We shall discuss certain finite-dimensional quotients of $\mathbb{C}\mathcal{B}_{M+1}$.

1. The *Hecke algebra* $H_{M+1}(q)$ is defined by relations

$$(\sigma_i - q)(\sigma_i + q^{-1}) = 0, \tag{3}$$

where q is a parameter; $\dim(H_M(q)) = M!$.

2. The *Birman–Murakami–Wenzl algebra* $BMW_{M+1}(q, \nu)$ is defined by relations

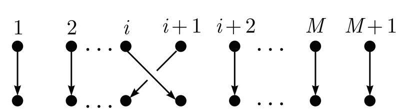
$$\left\{ \begin{array}{l} (\sigma_i - q)(\sigma_i + q^{-1})(\sigma_i - \nu) = 0, \\ \kappa_i \sigma_{i+1}^{\pm 1} \kappa_i = \nu^{\mp 1} \kappa_i, \end{array} \right. \tag{4}$$

where

$$\kappa_i := \frac{(q - \sigma_i)(\sigma_i + q^{-1})}{\nu(q - q^{-1})} \quad (i = 1, \ldots, M), \tag{5}$$

q and ν are parameters; $\dim(BMW_M(q, \nu)) = (2M-1)!!$.

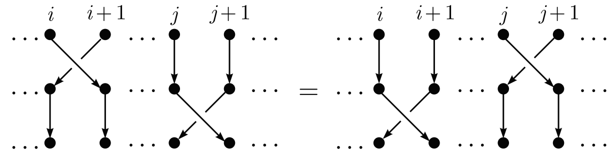
There is a beautiful graphical presentation of the braid group and its finite-dimensional quotients. The generators $\sigma_i \in \mathcal{B}_{M+1}$ are depicted by

$\sigma_i =$ 

.

For the locality relation (2) we have $(i+1<j<M)$

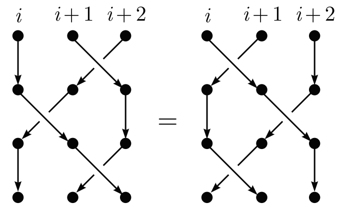
$$\sigma_i\sigma_j =$$



$$= \sigma_j\sigma_i.$$

The braid relation is

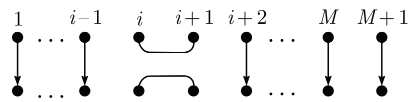
$$\sigma_{i+1}\sigma_i\sigma_{i+1} =$$



$$= \sigma_i\sigma_{i+1}\sigma_i.$$

It is sometimes convenient to depict the element (5) by

$$\kappa_i =$$



Below we shall omit the reference to the parameters in the notation $H_M(q)$ and $BMW_M(q,\nu)$ and write simply H_M and BMW_M.

1.2. Affine BMW Algebras αBMW_{M+1}. Affine Birman–Murakami–Wenzl algebras αBMW_{M+1} are extensions of the algebras BMW_{M+1}. The algebras αBMW_{M+1} are generated by the elements $\{\sigma_1,\ldots,\sigma_M\}$ with relations (1), (2), (4) and the affine element y_1 which satisfies

$$\begin{gathered}\sigma_1 y_1 \sigma_1 y_1 = y_1\sigma_1 y_1 \sigma_1, \quad [\sigma_k, y_1] = 0 \quad \text{for} \quad k>1,\\ \kappa_1 y_1\sigma_1 y_1 \sigma_1 = c\kappa_1 = \sigma_1 y_1 \sigma_1 y_1 \kappa_1,\\ \kappa_1 y_1^n \kappa_1 = \hat{z}^{(n)}\kappa_1, \quad n = 1,2,3,\ldots\end{gathered} \tag{6}$$

where c, $\hat{z}^{(n)}$ are central elements. Initially, for the Brauer algebras, the affine version was introduced by M. Nazarov [15].

Consider the set of affine elements

$$y_{k+1} = \sigma_k y_k \sigma_k, \quad k = 1,2,\ldots,M.$$

The elements y_k $(k = 1,2,\ldots,M+1)$ generate a commutative subalgebra Y_{M+1} in αBMW_{M+1}.

1.3. Central Elements in αBMW Algebra. We need some information about the center of αBMW.

Proposition 1. *The elements*

$$\hat{\mathcal{Z}} = y_1\cdot y_2 \cdots y_M, \quad \hat{\mathcal{Z}}_M^{(n)} = \sum_{k=1}^{M}(y_k^n - c^n y_k^{-n}), \quad n\in\mathbb{N},$$

are central in the αBMW_M algebra.

Remark. The set of «power sums» $\hat{\mathcal{Z}}^{(n)} = \sum_k (y_k^n - c^n y_k^{-n})$ has the generating function

$$\mathcal{Z}(t) = \sum_{n=1} \hat{\mathcal{Z}}^{(n)} t^{n-1} = \frac{d}{dt} \log \left(\prod_{k=1} \frac{y_k - c\,t}{1 - y_k t} \right).$$

Consider an ascending chain of subalgebras

$$\alpha BMW_0 \subset \alpha BMW_1 \subset \alpha BMW_2 \subset \ldots \subset \alpha BMW_M \subset \alpha BMW_{M+1},$$

where $\alpha BMW_0, \alpha BMW_1$ and αBMW_j $(j > 1)$ are generated by $\{c, \hat{z}^{(n)}\}$, $\{c, \hat{z}^{(n)}, y_1\}$ and $\{c, \hat{z}^{(n)}, y_1, \sigma_1, \sigma_2, \ldots, \sigma_{j-1}\}$, respectively. For the corresponding commutative subalgebras we have $Y_1 \subset Y_2 \subset \cdots \subset Y_M \subset Y_{M+1}$.

Proposition 2. *Let $\hat{Z}_k^{(n)}$ be the central elements in the algebra αBMW_k, $\alpha BMW_k \subset \alpha BMW_{k+2}$, defined by the generating function*

$$\sum_{n=0}^{\infty} \hat{Z}_k^{(n)} t^n = -\frac{\nu}{(q-q^{-1})} + \frac{1}{(1-ct^2)} + \left(\sum_{n=0}^{\infty} t^n \hat{z}^{(n)} + \frac{\nu}{(q-q^{-1})} - \frac{1}{(1-ct^2)} \right) \times \\ \times \prod_{r=1}^{k} \frac{(1-y_r t)^2 (q^2 - c y_r^{-1} t)(q^{-2} - c y_r^{-1} t)}{(1 - c y_r^{-1} t)^2 (q^2 - y_r t)(q^{-2} - y_r t)}. \quad (7)$$

The following relations hold:

$$\kappa_{k+1} y_{k+1}^n \kappa_{k+1} = \hat{Z}_k^{(n)} \kappa_{k+1} \in \alpha BMW_{k+2} \quad (\hat{Z}_0^{(n)} \equiv \hat{z}^{(n)}).$$

Remark. The evaluation map $\alpha BMW_M \to BMW_M$ is defined by

$$y_1 \mapsto 1 \Rightarrow c \mapsto \nu^2, \quad \hat{z}^{(n)} \mapsto 1 + \frac{\nu^{-1} - \nu}{q - q^{-1}}. \quad (8)$$

Under this map the function (7) transforms into the generating function presented in [9].

1.4. Intertwining Operators in αBMW Algebra. Introduce the *intertwining* elements $U_{k+1} \in \alpha BMW_{M+1}$ $(k = 1, \ldots, M)$:

$$U_{k+1} = [\sigma_k, y_k - c y_{k+1}^{-1}]. \quad (9)$$

Proposition 3. *The elements U_k satisfy*

$$U_{k+1} y_k = y_{k+1} U_{k+1}, \quad U_{k+1} y_{k+1} = y_k U_{k+1}, \quad U_{k+1} y_i = y_i U_{k+1} \quad \textit{for} \quad i \neq k, k+1,$$

$$U_{k+1} [\sigma_k, y_k] = (q y_k - q^{-1} y_{k+1})(q y_{k+1} - q^{-1} y_k) \left(1 - \frac{c}{y_k y_{k+1}} \right), \quad (10)$$

$$U_{k+1} U_k U_{k+1} = U_k U_{k+1} U_k, \quad \kappa_k U_{k+1} = U_{k+1} \kappa_k = 0.$$

The elements U_k provide an important information about the spectrum of the affine elements $\{y_j\}$.

Lemma 1. *The spectrum of the elements $y_j \in \alpha BMW_{M+1}$ satisfies*

$$\mathrm{Spec}\,(y_j) \subset \{q^{2\mathbb{Z}} \cdot \mathrm{Spec}\,(y_1),\ cq^{2\mathbb{Z}} \cdot \mathrm{Spec}\,(y_1^{-1})\}, \tag{11}$$

where $\mathbb{Z}$ is the set of integer numbers.

Proof. Induction in j. Equation (11) obviously holds for y_1. Assume that

$$\mathrm{Spec}\,(y_{j-1}) \subset \{q^{2\mathbb{Z}} \cdot \mathrm{Spec}\,(y_1), cq^{2\mathbb{Z}} \cdot \mathrm{Spec}\,(y_1^{-1})\}, \quad j > 1.$$

Let f be the characteristic polynomial of y_{j-1}, $f(y_{j-1}) = 0$. Then

$$0 = U_j f(y_{j-1})[\sigma_{j-1}, y_{j-1}] = f(y_j) U_j [\sigma_{j-1}, y_{j-1}] = \\ = f(y_j)(q^2 y_{j-1} - y_j)(y_j - q^{-2} y_{j-1})\left(y_j - c y_{j-1}^{-1}\right) y_j^{-1}.$$

Here we used (10). Thus, $\mathrm{Spec}\,(y_j) \subset \mathrm{Spec}\,(y_{j-1}) \cup q^{\pm 2} \cdot \mathrm{Spec}\,(y_{j-1}) \cup c \cdot \mathrm{Spec}\,(y_{j-1}^{-1})$. ■

We denote the image of $w \in \alpha BMW_M$ under the evaluation map (8) by $\tilde{w}$, e.g., $y_j \mapsto \tilde{y}_j$. The Jucys–Murphy (JM) elements $\tilde{y}_j$ $(j = 2, \ldots, M)$ are the images of y_j:

$$\tilde{y}_j = \sigma_{j-1} \ldots \sigma_2\, \sigma_1^2\, \sigma_2 \ldots \sigma_{j-1} \in BMW_M.$$

Lemma 1 provides the information about the spectrum of JM elements $\tilde{y}$'s.

Corollary. Since $\tilde{y}_1 = 1$ and $\tilde{c} = \nu^2$, it follows from (11) that

$$\mathrm{Spec}\,(\tilde{y}_j) \subset \{q^{2\mathbb{Z}}, \nu^2 q^{2\mathbb{Z}}\}. \tag{12}$$

2. REPRESENTATIONS OF AFFINE ALGEBRA αBMW_2

2.1. αBMW_2 Algebra and Its Modules V_D. The elements $\{y_i, y_{i+1}, \sigma_i, \kappa_i\} \in \alpha BMW_M$ (for fixed $i < M$) satisfy

$$(q - q^{-1})\kappa_i = \sigma_i^{-1} - \sigma_i + (q - q^{-1}), \tag{13}$$

$$y_{i+1} = \sigma_i y_i \sigma_i, \quad y_i y_{i+1} = y_{i+1} y_i, \quad \kappa_i y_i^n \kappa_i = \hat{Z}_{i-1}^{(n)} \kappa_i, \tag{14}$$

$$y_i y_{i+1} \kappa_i = c\kappa_i = \kappa_i y_{i+1} y_i. \tag{15}$$

The elements c and $\hat{Z}_{i-1}^{(n)}$ commute with $\{y_i, y_{i+1}, \sigma_i, \kappa_i\}$. The elements $\{y_i, y_{i+1}, \sigma_i, \kappa_i\} \in \alpha BMW_M$ generate a subalgebra isomorphic to αBMW_2.

Below we investigate representations ρ of αBMW_2 for which the generators $\rho(y_i)$ and $\rho(y_{i+1})$ are diagonalizable and $\rho(c) = \nu^2 \cdot \mathrm{Id}$. Let ψ be a common eigenvector of $\rho(y_i)$ and $\rho(y_{i+1})$ with some eigenvalues a and b:

$$\rho(y_i)\psi = a\psi, \quad \rho(y_{i+1})\psi = b\psi.$$

The element $\hat{z} = y_i y_{i+1}$ is central in αBMW_2. There are two possibilities:

$$\begin{array}{l} 1.\ \ \rho(\kappa_i) \neq 0 \overset{\text{Eq. (15)}}{\Longrightarrow} \rho(y_i y_{i+1}) = \nu^2 \cdot \mathrm{Id} \Rightarrow \underline{ab = \nu^2}; \\ 2.\ \ \rho(\kappa_i) = 0, \ \ \textit{the product } ab \textit{ is not fixed.} \end{array} \tag{16}$$

To save space we shall often omit the symbol ρ and denote, slightly abusively, the operator $\rho(x)$ for $x \in \alpha BMW$ by the same letter x; this should not lead to a confusion.

Applying the operators from αBMW_2 to the vector ψ, we produce, in general infinite-dimensional, αBMW_2-module V_∞ spanned by

$$\begin{array}{ll} e_2 = \psi, & \\ e_1 = \kappa_i \psi, & e_3 = \sigma_i \psi, \\ e_4 = y_i \kappa_i \psi, & e_5 = \sigma_i y_i \kappa_i \psi, \\ e_6 = y_i^2 \kappa_i \psi, & e_7 = \sigma_i y_i^2 \kappa_i \psi, \\ \quad \dots\dots\dots, & \quad \dots\dots\dots, \\ e_{2k+2} = y_i^k \kappa_i \psi, & e_{2k+3} = \sigma_i y_i^k \kappa_i \psi \ \ (k \geqslant 1), \dots\dots \end{array}$$

Using relations (13)–(15) for αBMW_2, one can write down the left action of elements $\{y_i, y_{i+1},\ \sigma_i, \kappa_i\}$ on V_∞. Our aim is to understand when the sequence e_j can terminate, giving therefore rise to a finite-dimensional module V_D (of dimension D) of αBMW_2, and investigate the (ir)reducibility of V_D.

We distinguish 3 cases for the module V_D:

(i) $\kappa_i V_D = 0$ (i.e., $\kappa_i\, e = 0 \ \ \forall\, e \in V_D$) and in particular $\kappa_i\, \psi = 0$. Therefore, $e_j = 0$ for all $j \neq 2, 3$ and V_∞ reduces to a 2-dim module with the basis $\{e_2, e_3\}$. In view of (16), the product $a\, b$ is not fixed and the irreps coincide with the irreps of the affine Hecke algebra αH_2 considered in [13].

(ii) $\kappa_i V_D \neq 0$ (i.e., $\exists\ e \in V_D$: $\kappa_i e \neq 0$). The module V_D is extracted from V_∞ by constraints

$$e_{2k+4} = \sum_{m=1}^{2k+3} \alpha_m\, e_m \quad (k \geqslant -1), \ \ ab = \nu^2, \tag{17}$$

with some parameters α_m. The independent basis vectors are $(e_1, e_2, \dots, e_{2k+3})$. The module V_D has *odd dimension*.

(iii) $\kappa_i V_D \neq 0$ and additional constraints are

$$e_{2k+3} = \sum_{m=1}^{2k+2} \alpha_m\, e_m \quad (k \geqslant 0), \ \ ab = \nu^2. \tag{18}$$

The independent basis vectors are $(e_1, e_2, \dots, e_{2k+2})$. The module V_D has *even dimension*.

Below we consider a version $\alpha' BMW_2$ of the affine BMW algebra. The additional requirement for this algebra concerns the spectrum of $y_i, y_{i+1} \in \alpha' BMW_2$:

$$\mathrm{Spec}\,(y_j) \subset \{q^{2\mathbb{Z}},\ \nu^2 q^{2\mathbb{Z}}\}.$$

The evaluation map (8) descends to the algebra $\alpha' BMW$ (cf. Corollary after Lemma 1). In particular, for the cases (ii) and (iii) we have

$$a = \nu^2 q^{2z}, \quad b = q^{-2z} \quad \text{or} \quad a = q^{2z}, \quad b = \nu^2 q^{-2z}$$

for some $z \in \mathbb{Z}$.

2.2. The Case $\kappa_i V_D = 0$: Hecke Algebra Case [13]. Representations of αBMW_2 with $\kappa_i V_D = 0$ reduce to representations of the affine Hecke algebra αH_2. In the basis $(e_2, e_3) = (\psi, \sigma_i \psi)$ the matrices of the generators are

$$\sigma_i = \begin{pmatrix} 0 & 1 \\ 1 & q - q^{-1} \end{pmatrix}, \ y_i = \begin{pmatrix} a & -(q-q^{-1})b \\ 0 & b \end{pmatrix}, \ \ y_{i+1} = \begin{pmatrix} b & (q-q^{-1})b \\ 0 & a \end{pmatrix}, \tag{19}$$

where $a \neq b$ (otherwise y_i, y_{i+1} are not diagonalizable). By Lemma 1, we have for $y_i, y_{i+1} \in \alpha' BMW_2$ the eigenvalues $a, b \in \{q^{2\mathbb{Z}}, \nu^2 q^{2\mathbb{Z}}\}$. The 2-dimensional representation (19) contains a 1-dimensional subrepresentation iff $a = q^{\pm 2} b$. Graphically these 1- and 2-dimensional irreps of $\alpha' BMW_2$ are visualized in Figs. 1 and 2.

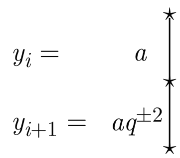


Fig. 1

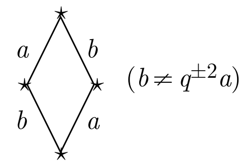


Fig. 2

Different paths going from the upper vertex to the lower vertex correspond to different eigenvectors of y_i, y_{i+1}. The indices on the edges are eigenvalues of y_i, y_{i+1}.

2.3. $\kappa_i V_D \neq 0$**: Odd-Dimensional Representations for** $\alpha' BMW_2$**.** Using the condition (17) for the reduction V_∞ to V_{2m+1}, one can describe odd-dimensional representations of $\alpha' BMW_2$, determine matrices for the action of y_i, y_{i+1} on V_{2m+1} and calculate

$$\det(y_i) = \prod_{r=1}^{2m+1} y_i^{(r)} = \nu^{2m}, \quad \det(y_{i+1}) = \prod_{r=1}^{2m+1} y_{i+1}^{(r)} = \nu^{2m+2}. \tag{20}$$

Here for eigenvalues $y_i^{(r)}$, $y_{i+1}^{(r)}$ $(r = 1, 2, \ldots, 2m+1)$ of y_i and y_{i+1} we have constraints

$$y_i^{(r)} y_{i+1}^{(r)} = \nu^2, \quad r = 1, \ldots, 2m+1.$$

and (see Eq. (12))

$$y_i^{(r)} \in \{q^{2\mathbb{Z}}, \nu^2 q^{2\mathbb{Z}}\}, \quad r = 1, \ldots, 2m+1.$$

These odd-dimensional irreps are visualized as graphs presented in Fig. 3, where $z_r \in \mathbb{Z}$ and $\sum\limits_{r=1}^{2m+1} z_r = 0$ as it follows from (20). Different paths going from the top vertex to the bottom vertex correspond to different common eigenvectors of y_i, y_{i+1}. Indices on upper and lower edges of these paths are the eigenvalues of y_i and y_{i+1}, respectively.

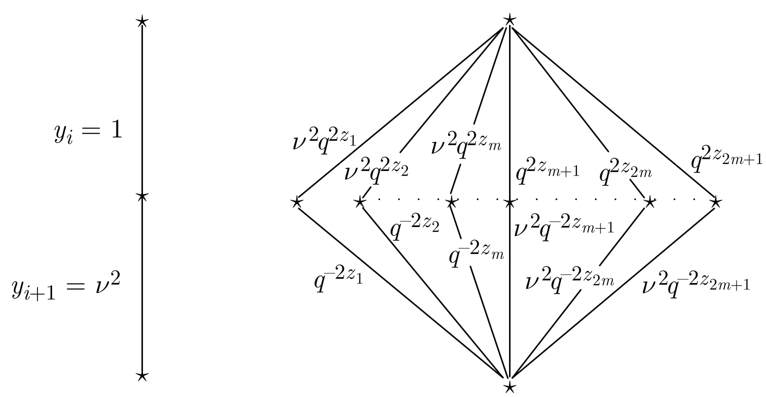
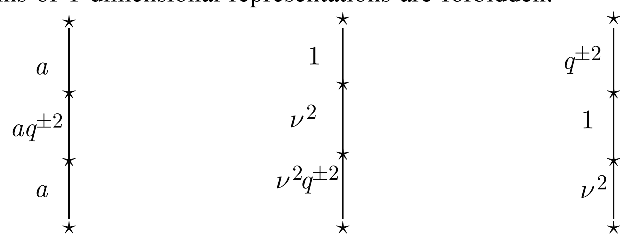


Fig. 3

Remark. In view of the braid relations $\sigma_i \sigma_{i\pm1} \sigma_i = \sigma_{i\pm1} \sigma_i \sigma_{i\pm1}$ and possible eigenvalues of σ's for 1-dimensional representations (described in Subsecs. 2.2 and 2.3), we conclude that the following chains of 1-dimensional representations are forbidden:



where $a = q^{2z}$ or $a = \nu^2 q^{2z}$ $(z \in \mathbb{Z})$.

2.4. $\kappa_i V_D \neq 0$**: Even-Dimensional Representations of** $\alpha' BMW_2$**.** With the help of the conditions (18) we reduce V_∞ to V_{2m}, then explicitly construct $(2m) \times (2m)$ matrices for the operators y_i, y_{i+1} and calculate their determinants

$$\det(y_i) = \prod_{r=1}^{2m} y_i^{(r)} = \epsilon q^\epsilon \nu^{2m-1}, \quad \det(y_{i+1}) = \prod_{r=1}^{2m} y_{i+1}^{(r)} = -\epsilon q^\epsilon \nu^{2m+1}, \tag{21}$$

where $y_i^{(r)}$, $y_{i+1}^{(r)}$ are eigenvalues of y_i, y_{i+1} (we have two possibilities: $\epsilon = \pm 1$). We see from (21) that all $(2m)$ eigenvalues of y_i, y_{i+1} cannot belong to the spectrum (12). More precisely, there is at least one eigenvalue $y_i^{(r)}$ of y_i (and the eigenvalue $y_{i+1}^{(r)}$ of y_{i+1}) such that

$$y_i^{(r)}, y_{i+1}^{(r)} \notin \{q^{2\mathbb{Z}}, \ \nu^2 q^{2\mathbb{Z}}\}.$$

Thus, even-dimensional irreps of αBMW_2 subject to the conditions (18) are not admissible for $\alpha' BMW_2$.

3. REPRESENTATIONS OF BMW ALGEBRAS

3.1. Spec $(y_1, \dots, y_n)$ **and Rules for Strings of Eigenvalues.** Now we reconstruct the representation theory of BMW algebras using an approach which generalizes the approach of Okounkov–Vershik [12] for symmetric groups.

The JM elements $\{\tilde{y}_1, \dots, \tilde{y}_n\}$ generate a commutative subalgebra in BMW_n. The basis in the space of an irrep of BMW_n can be chosen to be the common eigenbasis of all $\tilde{y}_i$. Each common eigenvector v of $\tilde{y}_i$,

$$\tilde{y}_i v = a_i v, \quad i = 1, \dots, n,$$

defines a string $(a_1, \dots, a_n) \in \mathbb{C}^n$. Denote by $\mathrm{Spec}(\tilde{y}_1, \dots, \tilde{y}_n)$ the set of such strings.

We summarize our results about representations of $\alpha' BMW_2$ and the spectrum of the JM elements $\tilde{y}_i$ in the following Proposition.

Proposition 4. *Consider the string*

$$\alpha = (a_1, \dots, a_i, a_{i+1}, \dots, a_n) \in \mathrm{Spec}(\tilde{y}_1, \dots, \tilde{y}_i, \tilde{y}_{i+1}, \dots, \tilde{y}_n).$$

Let v_α be the corresponding eigenvector of $\tilde{y}_i$: $\tilde{y}_i v_\alpha = a_i v_\alpha$. Then

$$
\begin{array}{ll}
(1) & a_i \in \{q^{2\mathbb{Z}},\ \nu^2 q^{2\mathbb{Z}}\};\\
(2) & a_i \neq a_{i+1},\ \ i=1,\dots,n-1;\\
(3a) & a_i a_{i+1} \neq \nu^2,\ \ a_{i+1} = q^{\pm 2} a_i \Rightarrow \sigma_i \cdot v_\alpha = \pm q^{\pm 1} v_\alpha,\ \ \kappa_i \cdot v_\alpha = 0;\\
(3b) & a_i a_{i+1} \neq \nu^2,\ \ a_{i+1} \neq q^{\pm 2} a_i \Rightarrow\\
& \alpha'=(a_1,\dots,a_{i+1},a_i,\dots,a_n) \in \mathrm{Spec}\,(\tilde{y}_1,\dots,\tilde{y}_i,\tilde{y}_{i+1},\dots,\tilde{y}_n),\ \kappa_i \cdot v_\alpha = 0,\ \kappa_i \cdot v_{\alpha'} = 0;\\
(4) & a_i a_{i+1} = \nu^2 \Rightarrow \exists\ \textit{ odd number of strings }\ \alpha^{(k)}\ \ (k=1,2,\dots,2m+1):\\
& \alpha^{(k)} = (a_1,\dots,a_{i-1},a_i^{(k)},a_{i+1}^{(k)},a_{i+2},\dots,a_n) \in \mathrm{Spec}\,(\tilde{y}_1,\dots,\tilde{y}_n)\ \forall k,\\
& \alpha \in \{\alpha^{(k)}\},\ \ a_i^{(k)} a_{i+1}^{(k)} = \nu^2,\ \ \prod\limits_{k=1}^{2m+1} a_i^{(k)} = \nu^{2m},\ \ \prod\limits_{k=1}^{2m+1} a_{i+1}^{(k)} = \nu^{2m+2}.
\end{array}
$$

The necessary and sufficient conditions for a string to belong to the common spectrum of $\tilde{y}_i$ are formulated in the following way.

Proposition 5. *The string $\alpha = (a_1, a_2, \dots, a_n)$, where $a_i \in (q^{2\mathbb{Z}}, \nu^2 q^{2\mathbb{Z}})$, belongs to the set* $\mathrm{Spec}\,(\tilde{y}_1, \tilde{y}_2, \dots, \tilde{y}_n)$ *iff α satisfies the following conditions ($z \in \mathbb{Z}$):*

$$
\begin{array}{ll}
(1) & a_1 = 1;\\
(2) & a_i = \nu^2 q^{-2z} \Rightarrow q^{2z} \in \{a_1,\dots,a_{i-1}\};\\
(3) & a_i = q^{2z} \Rightarrow \{a_i q^2, a_i q^{-2}\} \cap \{a_1,\dots,a_{i-1}\} \neq \emptyset,\ \ z \neq 0;\\
(4a) & a_i = a_j = q^{2z}\ (i<j) \Rightarrow \begin{cases} \textit{either}\,\{q^{2(z+1)}, q^{2(z-1)}\} \subset \{a_{i+1},\dots,a_{j-1}\},\\ \textit{or}\ \nu^2 q^{-2z} \in \{a_{i+1},\dots,a_{j-1}\}; \end{cases}\\
(4b) & a_i = a_j = \nu^2 q^{2z}\ (i<j) \Rightarrow \begin{cases} \textit{either}\ \{\nu^2 q^{2(z+1)}, \nu^2 q^{2(z-1)}\} \subset \{a_{i+1},\dots,a_{j-1}\},\\ \textit{or}\ q^{-2z} \in \{a_{i+1},\dots,a_{j-1}\}; \end{cases}\\
(5a) & a_i = \nu^2 q^{-2z},\ \ a_j = q^{2z'}\ (i<j) \Rightarrow q^{2z} \text{ or } \nu^2 q^{-2z'} \in \{a_{i+1},\dots,a_{j-1}\};\\
(5b) & a_i = q^{2z},\ \ a_j = \nu^2 q^{-2z'}\ (i<j) \Rightarrow \nu^2 q^{-2z} \text{ or } q^{2z'} \in \{a_{i+1},\dots,a_{j-1}\}.
\end{array}
$$

where in (5a) and (5b) we set $z' = z \pm 1$.

3.2. Young Graph for BMW Algebras. We illustrate the above considerations on the example of the colored (in the sense of [13]) Young graph for the algebra BMW_5 (see Fig. 4). This graph contains the whole information about the irreps of BMW_5 and the branching rules $BMW_5 \downarrow BMW_4$.

A vertex $\{\lambda; 5\}$ on the lowest level of this graph is labeled by some Young diagram λ; this vertex corresponds to the irrep $W_{\{\lambda;5\}}$ of BMW_5 (the notation $\{\lambda; 5\}$ is designed to encode the diagram λ and the level on which this diagram is located; the levels are counted starting from 0). Paths going down from the top vertex $\emptyset$ to the lowest level (that is, paths of length 5) correspond to common eigenvectors of the JM elements $\tilde{y}_1, \dots, \tilde{y}_5$. Paths ending at $\{\lambda; 5\}$ label the basis in $W_{\{\lambda;5\}}$. In particular, the number of different paths going down from the top $\emptyset$ to $\{\lambda; 5\}$ is equal to the dimension of the irrep $W_{\{\lambda;5\}}$.

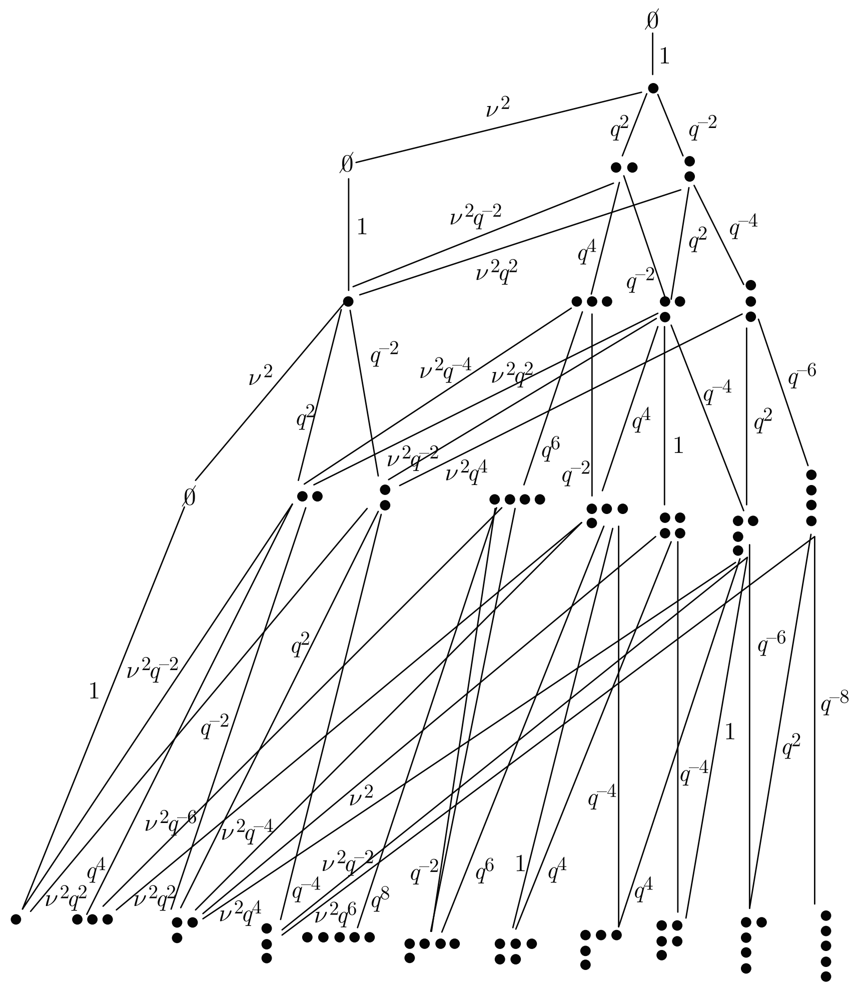
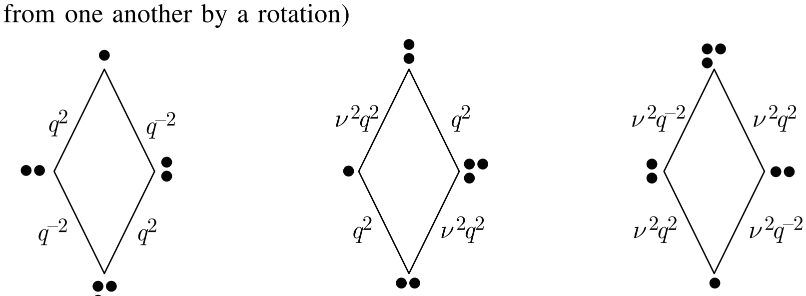


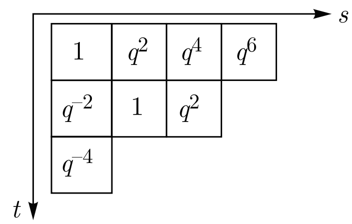
Fig. 4

Note that the colored Young graph in Fig. 4 contains subgraphs presented in Figs. 1–3. For example, in Fig. 4 one recognizes rhombic subgraphs (the vertices on the subgraphs are obtained from one another by a rotation)



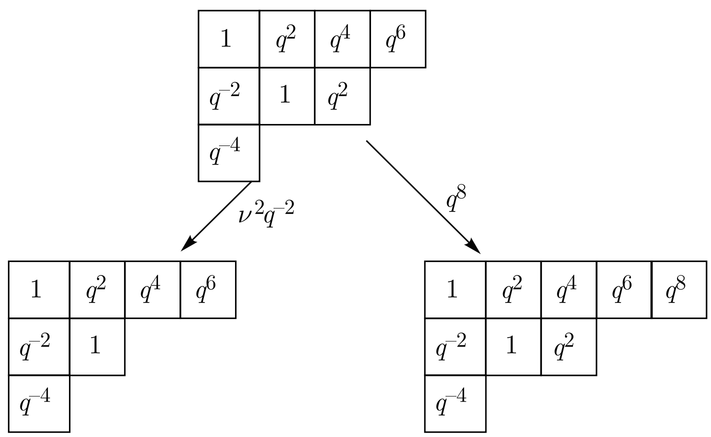
of the type presented in Fig. 2.

Let (s,t) be coordinates of a node in the Young diagram λ. To the node (s,t) of the diagram λ we associate a number $q^{2(s-t)}$ which is called «content»:



Then according to the colored Young graph in Fig. 4, at each step down along the path one can add or remove one node (therefore this graph is called the «oscillating» Young graph) and the eigenvalue of the corresponding JM element is determined by the content of the node:

$y_9 =$



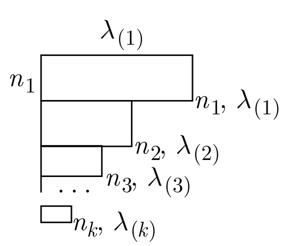
The eigenvalue corresponding to the addition or removal of the (s,t) node is $q^{2(s-t)}$ or $\nu^2 q^{-2(s-t)}$, respectively.

Let $X(n)$ be the set of paths of length n starting from the top vertex $\emptyset$ and going down in the Young graph of oscillating Young diagrams. Now we formulate the following Proposition.

Proposition 6. *There is a bijection between the set* $\mathrm{Spec}\,(\tilde{y}_1, \ldots, \tilde{y}_n)$ *and the set* $X(n)$.

3.3. Primitive Idempotents. The colored Young graph (as in Fig. 4) gives also the rule of construction of a complete set of orthogonal primitive idempotents for the BMW algebra. The completeness of the set of orthogonal primitive idempotents is equivalent to the maximality of the commutative set of JM elements. Let $\{\lambda; n\}$ be a vertex in the Young graph with

$\lambda =$



$(n_i, \lambda_{(i)})$ are coordinates of the nodes which are in the corners of $\lambda = [\lambda_{(1)}^{n_1}, \lambda_{(2)}^{n_2-n_1}, \ldots, \lambda_{(k)}^{n_k-n_{k-1}}]$.

Consider any path $T_{\{\lambda;n\}}$ going down from the top $\emptyset$ to this vertex. Let $E_{T_{\{\lambda;n\}}} \in BMW_n$ be the primitive idempotent corresponding to $T_{\{\lambda;n\}}$. Using the branching rule implied by the Young graph for BMW_{n+1}, we know all possible eigenvalues of the element $\tilde{y}_{n+1}$ and, therefore, obtain the identity

$$E_{T_{\{\lambda;n\}}} \prod_{r=1}^{k+1} \left(\tilde{y}_{n+1} - q^{2(\lambda_{(r)}-n_{r-1})}\right) \prod_{r=1}^{k} \left(\tilde{y}_{n+1} - \nu^2 q^{2(n_r-\lambda_{(r)})}\right) = 0,$$

where $\lambda_{(k+1)} = n_0 = 0$. So, for a new diagram λ' obtained by adding to λ a new node with coordinates $(n_{j-1}+1, \lambda_{(j)}+1)$, the corresponding primitive idempotent (after an appropriate normalization) reads

$$E_{T_{\{\lambda';n+1\}}} = E_{T_{\{\lambda;n\}}} \prod_{\substack{r=1\\ r\neq j}}^{k+1} \frac{\left(\tilde{y}_{n+1} - q^{2(\lambda_{(r)}-n_{r-1})}\right)}{\left(q^{2(\lambda_{(j)}-n_{j-1})} - q^{2(\lambda_{(r)}-n_{r-1})}\right)} \prod_{r=1}^{k} \frac{\left(\tilde{y}_{n+1} - \nu^2 q^{2(n_r-\lambda_{(r)})}\right)}{\left(q^{2(\lambda_{(j)}-n_{j-1})} - \nu^2 q^{2(n_r-\lambda_{(r)})}\right)}.$$

For a new diagram λ'' which is obtained from λ by removing a node with coordinates $(n_j, \lambda_{(j)})$, we construct the primitive idempotent

$$E_{T_{\{\lambda'';n+1\}}} = E_{T_{\{\lambda;n\}}} \prod_{r=1}^{k+1} \frac{\left(\tilde{y}_{n+1} - q^{2(\lambda_{(r)}-n_{r-1})}\right)}{\left(\nu^2 q^{2(n_j-\lambda_{(j)})} - q^{2(\lambda_{(r)}-n_{r-1})}\right)} \prod_{\substack{r=1\\ r\neq j}}^{k} \frac{\left(\tilde{y}_{n+1} - \nu^2 q^{2(n_r-\lambda_{(r)})}\right)}{\left(\nu^2 q^{2(n_j-\lambda_{(j)})} - \nu^2 q^{2(n_r-\lambda_{(r)})}\right)}.$$

Using these formulas and the «initial data» $E_{T_{\{\emptyset;0\}}} = 1$, one can deduce step by step explicit expressions for the primitive orthogonal idempotents related to the paths in the BMW Young graph.

4. OUTLOOK

In this paper we reconstructed the representation theory of the tower of the BMW algebras, using the properties of the commutative subalgebras, generated by the Jucys–Murphy elements, in the BMW algebras. This representation theory is of use in the representation theory of the quantum groups $U_q(osp(N|K))$ due to the Brauer–Schur–Weyl duality, but also finds applications in physical models. Recently [16] we have formulated integrable chain models with nontrivial boundary conditions in terms of the affine Hecke algebras αH_n and the affine BMW algebras αBMW_n. The Hamiltonians for these models are special elements of the algebras αH_n and αBMW_n. For example, for the αBMW_n algebra we have deduced [16] the Hamiltonians

$$\mathcal{H} = \sum_{m=1}^{n-1} \left(\sigma_m + \frac{(q-q^{-1})\nu}{\nu + a}\kappa_m\right) + \frac{(q-q^{-1})\xi}{y_1 - \xi}, \tag{22}$$

where $\xi^2 = -ac/\nu$ and the parameter a can take one of two values $a = \pm q^{\pm 1}$. Now different local representations ρ of the algebra αBMW_n give different integrable spin chain models with Hamiltonians $\rho(\mathcal{H})$ which in particular possess $U_q(osp(N|K))$ symmetries for some N and K. So, representations ρ of the algebra αBMW_n are related to the spin chain models of

osp type with n sites and nontrivial boundary conditions. BMW chains (chains based on the BMW algebras) describe in a unified way spin chains with $U_q(osp(N|K))$ symmetries.

The Hamiltonians for Hecke chain models are obtained from Hamiltonians for BMW chain models by taking the quotient $\kappa_j = 0$. These models were considered in [17, 18]. The Hecke chains (chain models based on the Hecke algebras) describe in a unified way spin chains with $U_q(sl(N|K))$ symmetries. In [17, 18] we investigated the integrable open chain models formulated in terms of generators of the Hecke algebra (nonaffine case, $y_1 = 1$). For the open Hecke chains of finite size, the spectrum of the Hamiltonians with free boundary conditions is determined [17] for special (corner-type) irreducible representations of the Hecke algebra. In [18] we investigated the functional equations for the transfer-matrix-type elements of the Hecke algebra that appeared in the theory of Hecke chains.

We postpone to future publications a construction of the algebra which extends the BMW algebra by the free algebra with generators labeled by the oscillating Young tableaux (as is done for the Hecke algebras in [19]).

Acknowledgements. The work of A. P. Isaev was partially supported by the grants RFBR 08-0100392-a, RFBR-CNRS 07-02-92166-a and RF President Grant N.Sh. 195.2008.2.

REFERENCES

1. *Brauer R.* On Algebras Which Are Connected with the Semisimple Continuous Groups // Ann. Math. 1937. V. 38. P. 854–872.
2. *Murakami J.* The Kauffman Polynomial of Links and Representation Theory // Osaka J. Math. 1987. V. 24. P. 745–758.
3. *Birman J. S., Wenzl H.* Braids, Link Polynomials and a New Algebra // Trans. Am. Math. Soc. 1989. V. 313, No. 1. P. 249–273.
4. *Ogievetsky O.* Uses of Quantum Spaces. Quantum Symmetries in Theoretical Physics and Mathematics // Contemp. Math. 2002. V. 294. P. 161–231.
5. *Faddeev L. D., Reshetikhin N. Yu., Takhtajan L. A.* Quantization of Lie Groups and Lie Algebras // Leningrad Math. J. 1990. V. 1, No. 1. P. 193–225.
6. *Isaev A. P.* Quantum Groups and Yang–Baxter Equations // Sov. J. Part. Nucl. 1995. V. 26. P. 501.
7. *Isaev A. P., Ogievetsky O. V., Pyatov P. N.* On R-matrix Representations of Birman–Murakami–Wenzl Algebras // Trudy Mat. Inst. Steklova. 2004. V. 246. P. 147–153; math.QA/0509251.
8. *Wenzl H.* Quantum Groups and Subfactors of Type B, C and D // Commun. Math. Phys. 1990. V. 133. P. 383–432.
9. *Beliakova A., Blanchet Ch.* Skein Construction of Idempotents in Birman–Murakami–Wenzl Algebras // Math. Ann. 2001. V. 321, No. 2. P. 347–373; math.QA/0006143.
10. *Tuba I., Wenzl H.* On Braided Tensor Categories of Type BCD // J. Reine Angew. Math. 2005. V. 581. P. 31–69; math.QA/0301142.
11. *Orellana R., Wenzl H.* q-Centralizer Algebras for Spin Groups // J. Algebra. 2002. V. 253, No. 2. P. 237–275.
12. *Okounkov A., Vershik A.* A New Approach to Representation Theory of Symmetric Groups // Selecta Mathematica. New Ser. 1996. V. 2, No. 4. P. 581–605.
13. *Isaev A. P., Ogievetsky O. V.* On Representations of Hecke Algebras // Czech. J. Phys. 2005. V. 55, No. 11. P. 1433–1441;
Isaev A. P., Ogievetsky O. V. Representations of A-type Hecke Algebras // Proc. of Intern. Workshop «Supersymmetries and Quantum Symmetries», Dubna, 2006; math.QA/0912.3701.

14. *Isaev A. P., Ogievetsky O. V.* Representations of Birman–Murakami–Wenzl Algebras. In preparation.
15. *Nazarov M.* Young's Orthogonal Form for Brauer's Centralizer Algebra // J. Algebra. 1996. V. 182. P. 664–693.
16. *Isaev A. P., Ogievetsky O. V.* On Baxterized Solutions of Reflection Equation and Integrable Chain Models // Nucl. Phys. B. 2007. V. 760. P. 167–183; math-ph/0510078.
17. *Isaev A. P., Ogievetsky O. V., Os'kin A. F.* Chain Models on Hecke Algebra for Corner-Type Representations // Rep Math. Phys. 2008. V. 61, No. 2. P. 309–315; math.QA/0710.0261.
18. *Isaev A. P.* Functional Equations for Transfer-Matrix Operators in Open Hecke Chain Models // Theor. Math. Phys. 2007. V. 150, No. 2. P. 187–202.
19. *Ogievetsky O., Pyatov P.* Lecture on Hecke Algebras // Proc. of the Intern. School «Symmetries and Integrable Systems», Dubna, 1999.