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RARITA–SCHWINGER FIELD AND MULTICOMPONENT WAVE EQUATION

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We suggest a simple method to solve a wave equation for Rarita–Schwinger field without additional constraints. This method based on the use of off-shell projection operators allows one to diagonalize spin-$1/2$ sector of the field.

Мы предлагаем простой метод решения волнового уравнения для поля Рариты–Швингера без дополнительных связей. Метод основан на использовании внемассовых проекционных операторов для диагонализации компоненты поля со спином $1/2$.

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INTRODUCTION

It is well known that standard approach to description of free higher spin fields in quantum field theory leads to constraints on field components (see, e.g., [1])[1]. But for interacting fields such constraints can generate serious problems and contradictions. For Rarita–Schwinger field [4], which is used for description of spin-$3/2$ particles, there are well-known issues of such type, see [5, 6] and a more general discussion in [7].

One of the ways to avoid these problems with constrained fields is to choose a special form of interaction to preserve constraints imposed on free field (if it is possible).

Another way is not to use the constraints at all and to eliminate the redundant components only after calculating of the observables. It may be considered as some regularization.

Such an approach was used in [8, 9] for the free Rarita–Schwinger field with all components: besides $s = 3/2$ this field contains two extra $s = 1/2$ components. It was shown that it is possible to regularly quantize free field [8, 9] and that in this case the inclusion of electromagnetic interaction does not generate the old contradictions [5, 6]. Following this line in [10, 11], we constructed the dressed propagator of Rarita–Schwinger field with all components and found the renormalization procedure, which guarantees the absence of redundant spin-1/2 poles. The essential point of our method is the use of the off-shell projection operators.

[1]Lagrange formulation of such theories needs to introduce auxiliary fields, see, e.g., [2, 3].

Here we use this technique to solve the wave equation for multicomponent Rarita–Schwinger field. It really gives a simple and transparent method to solve multicomponent equation as compared with cumbersome calculations in [8, 9]. Moreover, in spin-1/2 sector it allows one to find explicit form of projectors onto mass states, which solves the diagonalization problem.

1. WAVE EQUATION FOR RARITA–SCHWINGER FIELD

We use the action principle to get the wave equations for Ψ_μ field components. The action for RaritaT–Schwinger field is written in the form

$$\mathscr{A} = \int \mathscr{L} d^4x, \quad \mathscr{L} = \bar{\Psi}^\mu S_{\mu\nu} \Psi^\nu, \tag{1}$$

where $S_{\mu\nu}$ is a tensor-spinor operator. We write it down in the form of Λ-basis decomposition in momentum representation (see details in Appendix):

$$S_{\mu\nu} = \sum_{i=1}^{10} \bar{S}_i \mathcal{P}^i_{\mu\nu}, \tag{2}$$

with arbitrary coefficients $\bar{S}_i$, where bar is used to denote coefficients of Λ basis.

The wave function Ψ_μ can be expressed as sum of orthogonal components:

$$\Psi_\mu = n_{1\mu}\Psi_1 + n_{2\mu}\Psi_2 + \chi_\mu, \tag{3}$$

where $n_{i\mu} n^\mu_j = \delta_{ij}$, $n_{i\mu}\chi^\mu = 0$, see Appendix.

The variation of the field may be written in the same form $\delta\Psi_\mu = n_{1\mu}\delta\Psi_1 + n_{2\mu}\delta\Psi_2 + \delta\chi_\mu$ and $\delta\mathscr{L}$ is

$$\delta\mathscr{L} = \delta\bar{\Psi}_1 n_{1\mu}(S^{\mu\nu}\Psi_\nu) + \delta\bar{\Psi}_2 n_{2\mu}(S^{\mu\nu}\Psi_\nu) + \delta\bar{\chi}_\mu(S^{\mu\nu}\Psi_\nu) + \text{h.c.}$$

Considering the $\delta\Psi_i$, $\delta\chi_\mu$ to be independent, after some algebra we obtain a system of equations for Ψ_i, $i = 1, 2$:

$$\begin{cases} \left(\bar{S}_3\Lambda^- + \bar{S}_4\Lambda^+\right)\Psi_1 + \left(\bar{S}_7\Lambda^- + \bar{S}_8\Lambda^+\right)\Psi_2 = 0, \\ \left(\bar{S}_9\Lambda^+ + \bar{S}_{10}\Lambda^-\right)\Psi_1 + \left(\bar{S}_5\Lambda^+ + \bar{S}_6\Lambda^-\right)\Psi_2 = 0, \end{cases} \tag{4}$$

and a single equation for χ_μ

$$\left(\bar{S}_1\Lambda^+ + \bar{S}_2\Lambda^-\right)\chi_\mu = 0. \tag{5}$$

Here $\Lambda^\pm$ are off-shell projection operators $\Lambda^\pm = 1/2(1 \pm \hat{p}/W)$, $W = \sqrt{p^2}$.

Equation (5) is the usual Dirac equation for spin-3/2 particle, as will be seen below. The system for (4) for Ψ_i resembles the system of coupled Dirac equations. It has nontrival solutions only if $\Delta_1\Delta_2 = 0$, where $\Delta_1 = \bar{S}_3\bar{S}6 - \bar{S}_7\bar{S}_{10}$, $\Delta_2 = \bar{S}_4\bar{S}_5 - \bar{S}_8\bar{S}_9$.

2. MASS STATES IN SPIN-$1/2$ SECTOR

The system of equations (4) shows that Ψ_i do not have the definite mass. In order to diagonalize this system we use a projection representation of tensor-spinor operator

$$S_{\mu\nu} = \sum_i \lambda_i \Gamma_{i\,\mu\nu}, \quad (S^{-1})_{\mu\nu} = \sum_i \frac{1}{\lambda_i} \Gamma_{i\,\mu\nu}, \tag{6}$$

where $\Gamma_{i\mu\nu}$ are projection operators of $S_{\mu\nu}$ and λ_i corresponding eigenvalues.

In our case it is convenient to formulate the eigenstate problem in terms of projectors. So $\Gamma_{\mu\nu}$ should satisfy the following requirements:

$$S_\mu^\alpha \Gamma_{\alpha\nu} = \lambda \Gamma_{\mu\nu}, \quad \Gamma_{i\mu}^\alpha \Gamma_{j\alpha\nu} = \delta_{ij} \Gamma_{i\mu\nu}. \tag{7}$$

It is convenient to use Λ-basis decomposition for $\Gamma_{\mu\nu}$ to solve this problem, see (2).

For spin-$3/2$ sector it is obvious that $\mathcal{P}_1$ and $\mathcal{P}_2$ are projection operators and the corresponding eigenvalues are $\bar{S}_1$ and $\bar{S}_2$.

For spin-$1/2$ terms we obtain the following equations on eigenvalues (here E is a unit matrix):

$$\begin{aligned} &\det(S_1 - \lambda_i E) = 0, \quad i = 1,2; \quad S_1 = \begin{pmatrix} \bar{S}_3 & \bar{S}_7 \\ \bar{S}_{10} & \bar{S}_6 \end{pmatrix}, \\ &\det(S_2 - \lambda_j E) = 0, \quad j = 3,4; \quad S_2 = \begin{pmatrix} \bar{S}_4 & \bar{S}_8 \\ \bar{S}_9 & \bar{S}_4 \end{pmatrix}. \end{aligned} \tag{8}$$

The eigenvalues are W-dependent and are exchanged under the transformation $W \to -W$, namely $1,2 \leftrightarrow 3,4$, that follows from the property of Λ-basis coefficients.

After some algebra we obtain four projection operators for spin-$1/2$ terms (tensor indices are omitted)

$$\Gamma_1 = \frac{1}{\lambda_2 - \lambda_1} \left((\bar{S}_6 - \lambda_1)\mathcal{P}_3 - \bar{S}_{10}\mathcal{P}_{10}\right) + \frac{-1}{\lambda_2 - \lambda_1} \left((\bar{S}_3 - \lambda_1)\mathcal{P}_6 - \bar{S}_7\mathcal{P}_7\right), \tag{9}$$

$$\Gamma_3 = \frac{1}{\lambda_4 - \lambda_3} \left((\bar{S}_5 - \lambda_3)\mathcal{P}_4 - \bar{S}_9\mathcal{P}_9\right) + \frac{-1}{\lambda_4 - \lambda_3} \left((\bar{S}_4 - \lambda_3)\mathcal{P}_5 - \bar{S}_8\mathcal{P}_8\right), \tag{10}$$

$$\Gamma_2 = \frac{-1}{\lambda_2 - \lambda_1} \left((\bar{S}_6 - \lambda_2)\mathcal{P}_3 - \bar{S}_{10}\mathcal{P}_{10}\right) + \frac{1}{\lambda_2 - \lambda_1} \left((\bar{S}_3 - \lambda_2)\mathcal{P}_6 - \bar{S}_7\mathcal{P}_7\right), \tag{11}$$

$$\Gamma_4 = \frac{-1}{\lambda_4 - \lambda_3} \left((\bar{S}_5 - \lambda_4)\mathcal{P}_4 - \bar{S}_9\mathcal{P}_9\right) + \frac{1}{\lambda_4 - \lambda_3} \left((\bar{S}_4 - \lambda_4)\mathcal{P}_5 - \bar{S}_8\mathcal{P}_8\right), \tag{12}$$

corresponding to eigenvalues (8).

The found projection operators allow us to obtain corresponding eigenvectors $l_{i\mu}$ which have the following form:

$$l_{1\mu} = \sqrt{\frac{\bar{S}_7\bar{S}_{10}}{\bar{S}_7\bar{S}_{10} + |\bar{S}_3 - \lambda_1|^2}} \left(n_{1\mu} - \frac{\bar{S}_3 - \lambda_1}{\bar{S}_7} n_{2\mu} \right) \Lambda^-, \tag{13}$$

$$l_{2\mu} = \sqrt{\frac{\bar{S}_7\bar{S}_{10}}{\bar{S}_7\bar{S}_{10} + |\bar{S}_3 - \lambda_2|^2}} \left(n_{1\mu} - \frac{\bar{S}_3 - \lambda_2}{\bar{S}_7} n_{2\mu} \right) \Lambda^-, \tag{14}$$

$$l_{3\mu} = \sqrt{\frac{\bar{S}_8\bar{S}_9}{\bar{S}_8\bar{S}_9 + |\bar{S}_4 - \lambda_3|^2}} \left(n_{1\mu} - \frac{\bar{S}_4 - \lambda_3}{\bar{S}_8} n_{2\mu} \right) \Lambda^+, \tag{15}$$

$$l_{4\mu} = \sqrt{\frac{\bar{S}_8\bar{S}_9}{\bar{S}_8\bar{S}_9 + |\bar{S}_4 - \lambda_4|^2}} \left(n_{1\mu} - \frac{\bar{S}_4 - \lambda_3}{\bar{S}_8} n_{2\mu} \right) \Lambda^+. \tag{16}$$

As a result, the projection operators can be written as

$$\Gamma_{i\mu\nu} = l_{i\mu}\bar{l}_{i\nu}, \tag{17}$$

where $\bar{l}_\mu = \gamma^0 l_\mu^\dagger \gamma^0$. The main property of $l_{i\mu}$ is

$$\bar{l}_{i\mu} l_i^\mu = \Lambda^-, \quad i = 1, 2, \quad \bar{l}_{j\mu} l_j^\mu = \Lambda^+, \quad j = 3, 4. \tag{18}$$

As a result, the projection representation (6) for $(S_{\mu\nu})^{-1}$ now takes a more concrete form

$$(S^{-1})_{\mu\nu} = \frac{1}{\bar{S}_1}\mathcal{P}_{1\mu\nu} + \frac{1}{\bar{S}_2}\mathcal{P}_{2\mu\nu} + \sum_{i=1}^{4} \frac{1}{\lambda_i}\Gamma_{i\,\mu\nu}. \tag{19}$$

Now it is possible to write wave function Ψ_μ as a sum of orthogonal components with definite mass

$$\Psi_\mu(p) = (l_{1\mu} + l_{3\mu})\varphi_1(p) + (l_{2\mu} + l_{4\mu})\varphi_2(p) + \chi_\mu(p), \tag{20}$$

where $\chi_\mu l_i^\mu = 0$, $i = 1, \ldots, 4$, so Lagrangian (1) is expanden into

$$\mathscr{L} = \bar{\chi}_\mu \left(\bar{S}_1\Lambda^+ + \bar{S}_2\Lambda^- \right) \chi^\mu + \bar{\varphi}_1 \left(\lambda_3\Lambda^+ + \lambda_1\Lambda^- \right) \varphi_1 + \bar{\varphi}_2 \left(\lambda_4\Lambda^+ + \lambda_2\Lambda^- \right) \varphi_2. \tag{21}$$

Repeating the steps in deriving the wave equation but using instead of (3) the decomposition (20), we get independent motion equations for φ_1 and φ_2:

$$\begin{aligned} (\Lambda^+\lambda_3 + \Lambda^-\lambda_1)\varphi_1 = 0, \\ (\Lambda^+\lambda_4 + \Lambda^-\lambda_2)\varphi_2 = 0. \end{aligned} \tag{22}$$

3. LAGRANGIAN OF MASSIVE RARITA–SCHWINGER FIELD

To contretize the above general formulae, let us consider the most general form of Lagrangian of free Rarita–Schwinger field[1]

$$\begin{aligned} \mathscr{L} &= \bar{\Psi}_\mu S^{\mu\nu} \Psi_\nu, \\ S^{\mu\nu} &= g^{\mu\nu}(\hat{p} - M) - p^\mu \gamma^\nu - p^\nu \gamma^\mu + \gamma^\mu \gamma^\nu M(1+r) + \gamma^\mu \hat{p} \gamma^\nu \left(\frac{\delta}{2} + 1\right), \end{aligned} \tag{23}$$

which besides mass M has two real parameters r, δ. Eigenvalues (8) in this case takes the form

$$\begin{aligned} \lambda_{1,2} = M(1+2r) - W(1+\delta) \mp \\ \mp \left[4M^2(1+r)^2 - 2M(1+r)(1+2\delta)W + \left(1+\delta+\delta^2\right) W^2\right]^{1/2}, \end{aligned} \tag{24}$$

$$\begin{aligned} \lambda_{3,4} = M(1+2r) + W(1+\delta) \mp \\ \mp \left[4M^2(1+r)^2 + 2M(1+r)(1+2\delta)W + \left(1+\delta+\delta^2\right) W^2\right]^{1/2}. \end{aligned} \tag{25}$$

It is obvious that the values depend on W and are exchanged when $W \to -W$.

In the general case eigenvalues $\lambda_i(W)$ have specific dependence on W, so Eqs. (22) are not exactly Dirac equations. To see it, we can look again at the system of equations (4). Indeed, we can rewrite the system for Ψ_i as a system of equations resembling Dirac equation

$$[\hat{p}\mathbf{K} - \mathbf{M}] \begin{pmatrix} \Psi_1 \\ \Psi_2 \end{pmatrix} = 0 \tag{26}$$

with nondiagonal kinetic $\mathbf{K}$ and mass $\mathbf{M}$ matrices. The Lagrangian (23) leads to the following matrices:

$$\begin{aligned} \mathbf{K} &= \frac{M}{W} \begin{pmatrix} 0 & \sqrt{3}(1+r) \\ \sqrt{3}(1+r) & 0 \end{pmatrix} + \begin{pmatrix} (4+3\delta)/2 & 0 \\ 0 & \delta/2 \end{pmatrix}, \\ \mathbf{M} &= -M \begin{pmatrix} 2+3r & 0 \\ 0 & r \end{pmatrix} - W \begin{pmatrix} 0 & \sqrt{3}/2\delta \\ \sqrt{3}/2\delta & 0 \end{pmatrix}. \end{aligned}$$

Note that matrices $\mathbf{M}$ and $\mathbf{K}$ are W-dependent.

Zeroes of eigenvalues are poles for the propagator. The first two terms in (19) have poles in points M and $-M$, correspondingly. Denote the poles in spin-$1/2$ sector as m_1, m_2, i.e.,

$$\lambda_1(m_1) = 0, \quad \lambda_2(m_2) = 0. \tag{27}$$

The above-mentioned property of eigenvalues suggests that $-m_1$ and $-m_2$ are zeroes of λ_3 and λ_4, correspondingly. From the explicit form of eigenvalues (24) one derives that

$$m_{1,2} = M \frac{r - \delta \pm \sqrt{(\delta - r)^2 + (3+4r)\delta}}{\delta}. \tag{28}$$

[1]The most general form of Lagrangian depends on four parameters [8, 12, 13], but poles positions depend only on two parameters. So without loss of generality we use two-parameter Lagrangian. Details can be found in [12].

The Dirac case corresponds to particular choice of parameter $r = -1$, when eigenvectors are linear on W:

$$\lambda_{1,2} = -M - W(1 + \delta \pm \sqrt{1 + \delta + \delta^2}), \tag{29}$$

$$\lambda_{3,4} = -M + W(1 + \delta \pm \sqrt{1 + \delta + \delta^2}). \tag{30}$$

For this case Eqs. (22) have standard Dirac form and Lagrangian (21) is written as

$$\mathscr{L} = \bar{\chi}_\mu \left(\bar{S}_1 \Lambda^+ + \bar{S}_2 \Lambda^-\right) + \bar{\varphi}_1 \Big((1 + \delta + \sqrt{1 + \delta + \delta^2})\hat{p} - M\Big)\varphi_1 + \\ + \bar{\varphi}_2 \Big((1 + \delta - \sqrt{1 + \delta + \delta^2})\hat{p} - M\Big)\varphi_2. \tag{31}$$

The diagonal form of Lagrangian (31) allows us to see the sign of every contribution to the energy. Indeed, comparing with the Dirac case, we see that field φ_1 gives always positive contribution to Hamiltonian, while the contribution of φ_2 depends on the sign of δ; if δ is negative, then Hamiltonian has a negative contribution from φ_2. Tnis conclusion differs from the conclusion obtained in work [8]: according to it at least one contribution from spin-$1/2$ components to Hamiltonian is negative.

CONCLUSION

We have found that with the use of Λ basis and decomposition of field (3), it is easy to obtain the wave equations for different spin sectors of Rarita–Schwinger field. For spin-$3/2$ it is in fact a Dirac equation, as for spin-$1/2$ sector, we have two coupled Dirac-like equations with nondiagonal kinetic and mass matrices. In the general case the mass matrix in (26) is energy-dependent $M(W)$, only special choice of parameters leads to constant matrix.

We used the most general form of wave operator (2) because the above-presented Lagrangians (23) do not exhaust all possibilities. For instanse, there exist some examples of Rarita–Schwinger Lagrangians with higher derivatives [14, 15].

The Λ basis allows one also to find projectors onto the mass states for spin-$1/2$ sector. Convenient trick here is the use of projection representation of operator in the form of (6), (20).

We can conclude that the methods presented here allow us to work effectively with multicomponent field without additional constraints and may be useful for other higher spin fields and more complex Lagrangians.

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Appendix

NOTATIONS

Throughout the work we use the following notation. The elements of Λ basis are defined as

$$\begin{gathered}\mathcal{P}_1 = \Lambda^+\mathcal{P}^{3/2}, \quad \mathcal{P}_2 = \Lambda^-\mathcal{P}^{3/2}, \quad \mathcal{P}_3 = \Lambda^+\mathcal{P}^{1/2}_{11}, \quad \mathcal{P}_4 = \Lambda^-\mathcal{P}^{1/2}_{11}, \\ \mathcal{P}_5 = \Lambda^+\mathcal{P}^{1/2}_{22}, \quad \mathcal{P}_6 = \Lambda^-\mathcal{P}^{1/2}_{22}, \quad \mathcal{P}_7 = \Lambda^+\mathcal{P}^{1/2}_{21}, \quad \mathcal{P}_8 = \Lambda^-\mathcal{P}^{1/2}_{21}, \\ \mathcal{P}_9 = \Lambda^+\mathcal{P}^{1/2}_{12}, \quad \mathcal{P}_{10} = \Lambda^-\mathcal{P}^{1/2}_{12},\end{gathered} \tag{32}$$

where

$$\begin{gathered}(\mathcal{P}^{3/2})^{\mu\nu} = g^{\mu\nu} - n_1^\mu n_1^\nu - n_2^\mu n_2^\nu, \\ (\mathcal{P}^{1/2}_{11})^{\mu\nu} = n_1^\mu n_1^\nu, \quad (\mathcal{P}^{1/2}_{22})^{\mu\nu} = n_2^\mu n_2^\nu, \\ (\mathcal{P}^{1/2}_{21})^{\mu\nu} = n_1^\mu n_2^\nu, \quad (\mathcal{P}^{1/2}_{12})^{\mu\nu} = n_2^\mu n_1^\nu.\end{gathered} \tag{33}$$

The unit vectors $n_{1\mu}$ and $n_{2\mu}$ are defined as follows:

$$n_{1\mu} = \frac{1}{\sqrt{3}}\left(g_{\mu\alpha} - \frac{p_\mu p_\alpha}{p^2}\right)\gamma^\alpha, \quad n_{2\mu} = \frac{p_\mu}{\sqrt{p^2}}, \quad n_{i\mu}\cdot n_j^\mu = \delta_{ij}, \tag{34}$$

and

$$\Lambda^\pm = \frac{1}{2}\Big(1 \pm \frac{\hat{p}}{W}\Big), \quad W = \sqrt{p^2}. \tag{35}$$

The decomposition over the Λ basis of $S_{\mu\nu}$ is written as

$$S_{\mu\nu} = \sum_{i=1}^{10} \bar{S}_i \mathcal{P}^i_{\mu\nu}. \tag{36}$$

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