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ON FERMIONIC TILDE-CONJUGATION RULES AND THERMAL BOSONIZATION. HOT AND COLD THERMOFIELDS

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A generalization of Ojima tilde-conjugation rules is suggested, which is useful for the thermofield bosonization. The notion of hot and cold thermofields is introduced to distinguish different thermofield representattions giving the correct normal form of thermofield solution for the finite-temperature Thirring model with correct renormalization and anticommutation properties.

Предложено обобщение правил тильда-сопряжения Ожимы, удобное для термополевой бозонизации. Введение понятия «горячих» и «холодных» термополей позволяет различать их представления и приводит к корректной нормальной форме термополевых решений модели Тирринга при конечной температуре с правильными свойствами антикоммутативности и перенормировки.

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1. THERMODYNAMICS OF IDEAL 1-DIMENSION GASES

From the standard course [1] it is known that equilibrium thermodynamics of the free massless bosons in the 1-dimension box L coincides with that of the free massless spin-$1/2$ fermions at the same temperature $k_B T = 1/\varsigma$ only for both zero chemical potentials $\mu_{(B),(F)} = 0$ [2], giving a simplest example of thermal bosonization for pressure $\mathcal{P}$, internal energy $\mathcal{U}$ and entropy S:

$$\mathcal{P}_{(B),(F)} = \frac{\mathcal{U}_{(B),(F)}}{L} = \frac{\pi^2}{3\varsigma^2 hc}, \quad \frac{S_{(B),(F)}}{k_B L} = \frac{2\pi^2}{3\varsigma hc}, \tag{1}$$

$$\text{however, for } \overline{n}_{(B),(F)} = \frac{N_{(B),(F)}}{L}, \quad h = 2\pi\hbar, \quad c \text{ — speed of light:} \tag{2}$$

$$\mu_{(B)} = \frac{1}{\varsigma}\ln\left(1 - \mathrm{e}^{-\overline{n}_{(B)}\varsigma hc/2}\right), \quad \mu_{(F)} = \frac{1}{\varsigma}\ln\left(\mathrm{e}^{\overline{n}_{(F)}\varsigma hc/4} - 1\right). \tag{3}$$

Thus, «equilibrium» here means also that both systems for the same ς, L have the same $\mathcal{P}, \mathcal{U}, S$ and other thermodynamic potentials. The condition $\mu_{(B)} = 0$ for arbitrary temperature implies an infinite boson density, $\overline{n}_{(B)} \mapsto \infty$, corresponding to specific case of thermodynamic limit $N_{(B)} \to \infty$, $L \to \infty$ for bosonic «picture». The «equilibrium» fermion pressure (1) actually

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is a sum of partial ones for left fermions and right antifermions with opposite values of chemical potentials $\mu^{\pm} = \pm\mu_{(F)}$ [11]:

$$\mathcal{P}_{(F)} = \mathcal{P}^{+} + \mathcal{P}^{-} = \frac{\pi^2}{3\varsigma^2 hc} + \frac{\mu_{(F)}^2}{hc}, \text{ and for } N_{(F)}^{+} + N_{(F)}^{-} = N_{(F)}, \tag{4}$$

the «equilibrium» for Gibbs potentials reads as

$$N_{(F)}^{+}\mu^{+} + N_{(F)}^{-}\mu^{-} = \left(N_{(F)}^{+} - N_{(F)}^{-}\right)\mu_{(F)} = N_{(B)}\mu_{(B)} \Longrightarrow 0, \tag{5}$$

$$\text{which takes place for } \mu_{(F)} \neq 0, \text{ only if } N_{(F)}^{+} = N_{(F)}^{-}. \tag{6}$$

Nevertheless, $\mu_{(F)} \mapsto 0$ for $\overline{n}_{(F)} = 4\ln 2/(\varsigma hc)$. We want to point out that for nonzero temperature the usual infrared regularization parameter L acquires physical meaning as thermodynamical parameter of the box size, so corresponding dependence requires additional care.

2. ON FERMIONIC TILDE-CONJUGATION RULES

Following Ojima [10], let us start with simplest fermionic oscillator, which has only two normalized states $|0\rangle$, $|1\rangle$, with energy 0 and ω, annihilated or created by fermionic operators $b, b^{\dagger}$: $b|0\rangle = 0$, and $|1\rangle = b^{\dagger}|0\rangle$, $\{b, b^{\dagger}\} = 1$, $\{b, b\} = 0$. The temperature vacuum appears as a normalized sum of tensor products of two independent copies of these states $|0\widetilde{0}\rangle$, $|1\widetilde{1}\rangle$, weighted with corresponding Gibbs and relative phase exponential factors [10], so that for $\{b, \widetilde{b}^{\#}\} = 0$ ($\widetilde{b}^{\#} = \widetilde{b}, \widetilde{b}^{\dagger}$):

$$|0(\varsigma)\rangle_{(F)} = \frac{|0\widetilde{0}\rangle + \mathrm{e}^{i\Phi}\mathrm{e}^{-\varsigma\omega/2}|1\widetilde{1}\rangle}{\left[\langle 0\widetilde{0}|0\widetilde{0}\rangle + \mathrm{e}^{-\varsigma\omega}\langle 1\widetilde{1}|1\widetilde{1}\rangle\right]^{1/2}} \equiv \cos\vartheta\left(1 + \mathrm{e}^{i\Phi}\tan\vartheta\, b^{\dagger}\widetilde{b}^{\dagger}\right)|0\widetilde{0}\rangle = \mathcal{V}_{\vartheta(F)}^{-1}|0\widetilde{0}\rangle, \tag{7}$$

where

$$\tan^2\vartheta(k^1, \varsigma) = \mathrm{e}^{-\varsigma\omega}, \quad \omega = \omega(k^1), \tag{8}$$

$$\mathcal{V}_{\vartheta(F)}^{-1} = \exp\left\{\mathrm{e}^{i\Phi}\tan\vartheta\, G_{+}\right\}\exp\left\{-\ln(\cos^2\vartheta)\, G_3\right\}\exp\left\{-\mathrm{e}^{-i\Phi}\tan\vartheta\, G_{-}\right\}, \tag{9}$$

$$G_{+} = b^{\dagger}\widetilde{b}^{\dagger}, \quad G_{-} = \widetilde{b}b = (G_{+})^{\dagger}, \quad G_3 = \frac{1}{2}\left(b^{\dagger}b + \widetilde{b}^{\dagger}\widetilde{b} - 1\right), \tag{10}$$

$$[G_{+}, G_{-}] = 2G_3, \quad [G_3, G_{\pm}] = \pm G_{\pm}, \tag{11}$$

thus,

$$\mathcal{V}_{\vartheta(F)}^{-1} = \exp\left\{\vartheta\left[\mathrm{e}^{i\Phi}G_{+} - \mathrm{e}^{-i\Phi}G_{-}\right]\right\} = \mathcal{V}_{-\vartheta(F)} \tag{12}$$

is a standard form of operator of the coherent state for group $SU(2)$ [3]. This observation allows one to identify the algebra (12) as «quasi-spin» algebra [4], with the cold vacuum $|0\widetilde{0}\rangle$ as its lowest state for representation with «quasi-spin» 1/2, and the state $|1\widetilde{1}\rangle$ as the highest one:

$$|0\widetilde{0}\rangle = \left|\frac{1}{2}, -\frac{1}{2}\right\rangle, \quad |1\widetilde{1}\rangle = \left|\frac{1}{2}, \frac{1}{2}\right\rangle, \tag{13}$$

$$G_3\left|\frac{1}{2}, \pm\frac{1}{2}\right\rangle = \pm\frac{1}{2}\left|\frac{1}{2}, \frac{1}{2}\right\rangle, \quad G_{\pm}\left|\frac{1}{2}, \pm\frac{1}{2}\right\rangle = 0. \tag{14}$$

The unique appearing arbitrary relative phase Φ reflects now the fact that the quantum state is not a vector, rather a ray. Thus, the thermal vacuum (7), as a coherent state, is annihilated by operator G_{-}, as well as by operators:

$$\begin{aligned} b(\varsigma) &= \mathcal{V}_{\vartheta(F)}^{-1}\, b\, \mathcal{V}_{\vartheta(F)} = b\cos\vartheta - \widetilde{b}^{\dagger}\mathrm{e}^{i\Phi}\sin\vartheta, \\ \underset{\sim}{b}(\varsigma) &= \mathcal{V}_{\vartheta(F)}^{-1}\, \widetilde{b}\, \mathcal{V}_{\vartheta(F)} = \widetilde{b}\cos\vartheta + b^{\dagger}\mathrm{e}^{i\Phi}\sin\vartheta. \end{aligned} \tag{15}$$

Up to now the $\widetilde{b}^{\#}$ is only notation that does not define any operation. To fix it as an operation: $\underset{\sim}{b}(\varsigma) \mapsto \widetilde{b}(\varsigma)$, one should choose the value of Φ. The popular choice $\Phi = 0$ leads to complicated tilde-conjugation rules for the fermionic case, different from the bosonic one [9]. The Ojima choice $\Phi = -\pi/2$ gives the same rules for both bosonic and fermionic cases [10]. We see now that the choice $\Phi = \pi/2$ is also good and, as well as the original Ojima's one, satisfies the properties of antilinear homomorphism and the condition $\widetilde{\widetilde{b}}(\varsigma) = b(\varsigma)$. It seems very convenient for the purposes of bosonization that the tilde operation has the same properties for both Fermi and Bose cases. As a byproduct, we observe a useful interpretation of the thermal vacuum, defined by Bogolubov transformation (7), as a coherent state, obtained by coherent $SU(2)$ rotation of vacuum states for all Fermi oscillators as a lowest quasi-spin states, around the unit vector $\mathbf{u} = (\sin\Phi, \cos\Phi, 0)$, on the angle -2ϑ: $\mathcal{V}_{\vartheta(F)}^{-1} = \exp\left[i2\vartheta\,(\mathbf{u}\cdot\mathbf{G})\right]$ [3].

Analogous picture may be obtained for bosonic temperature Bogolubov transformation $\mathcal{V}_{\vartheta(B)}$ leading to connection between the bosonic thermal vacuum and coherent state for the discrete series representation of group $SU(1,1)$ [3]. However, for this case the numerator in (7) contains a countable number of terms with countable number of arbitrary phases Φ_n [10]. The coherent state of the type (9), (12) would be obtained only for countable number of coherent choices: $\Phi_n \mapsto n\Phi$, $n = 0, 1, 2, \ldots$ We did not find a reason to prefer this choice instead of the usual one $\Phi_n = 0$ [9, 10].

3. HOT AND COLD THERMOFIELDS

So, at finite temperature, in the framework of thermofield dynamics [9] it is necessary to double the number of degrees of freedom by providing all the fields Ψ with their tilde partners $\widetilde{\Psi}$. According to [9], the resulting theory will be determined by the Hamiltonian $\widehat{H}[\Psi, \widetilde{\Psi}] = H[\Psi] - \widetilde{H}[\widetilde{\Psi}]$, where $\widetilde{H}[\widetilde{\Psi}] = H^{*}[\widetilde{\Psi}^{*}]$, with $H[\Psi] = H_{0[\Psi]}(x^0) + H_{I[\Psi]}(x^0)$, so that for Thirring model [7] $\widetilde{H}_{I[\widetilde{\Psi}]} = H_{I[\widetilde{\Psi}]}$ and $\widetilde{H}_{0[\widetilde{\Psi}]} = -H_{0[\widetilde{\Psi}]}$. Though the substitution like (15), for the free massless Dirac thermofields, $\chi(x) \mapsto \chi(x, \varsigma)$, also does not change [9] the form of the free operator: $\widehat{H}_0[\chi, \widetilde{\chi}] = H_0[\chi] - \widetilde{H}_0[\widetilde{\chi}]$, these free fields, generally speaking, are not now the physical fields of this QFT model [5, 11], and, as is well known [5, 9], each term $H[\Psi]$ in $\widehat{H}[\Psi, \widetilde{\Psi}]$ must be equivalent in a weak sense to the free Hamiltonian of massless (pseudo) scalar fields $(\phi(x))$, $\varphi(x)$, at least at $T = 0$.

For any functional $\mathcal{F}[\Psi]$ of Heisenberg fields (HF) in the given representation of physical fields $\psi(x)$, — dynamical mapping (DM), $\Psi(x) = \Upsilon[\psi(x)]$ [9] for the zero temperature, to be interesting in the matrix elements for the thermal vacuum of the type

$$\langle 0(\varsigma)|\mathcal{F}[\Psi(x)]|0(\varsigma)\rangle = \langle 0\widetilde{0}|\mathcal{V}_{\vartheta}\mathcal{F}[\Psi(x)]\mathcal{V}_{\vartheta}^{-1}|0\widetilde{0}\rangle = \langle 0\widetilde{0}|\mathcal{F}\left[\mathcal{V}_{\vartheta}\Psi(x)\mathcal{V}_{\vartheta}^{-1}\right]|0\widetilde{0}\rangle, \tag{16}$$

we come to formal mapping:

$$\mathcal{V}_\vartheta \Psi(x) \mathcal{V}_\vartheta^{-1} = \Psi(x,\varsigma) = \Upsilon\left[\mathcal{V}_\vartheta \psi(x) \mathcal{V}_\vartheta^{-1}\right] = \Upsilon\left[\psi(x,[-]\varsigma)\right] \tag{17}$$

onto the «cold» physical thermofield:

$$\psi(x,[-]\varsigma) = \mathcal{V}_\vartheta \psi(x) \mathcal{V}_\vartheta^{-1}, \tag{18}$$

essentially with the same coefficient functions as for the initial DM $\Psi(x) = \Upsilon[\psi(x)]$, contrary to [9, 10], thus transferring so all the temperature dependence from the state (7) onto these «cold» physical thermofields. However, to compute the matrix element (16), it is necessary to substitute into the r.h.s. of (16), (17) the cold physical thermofields (18) again in terms of the initial physical fields $\psi(x)$ via obtained from (18) their linear combinations, analogous (but not the same!) to Eqs. (15), and reordering again the so obtained operator with respect to the initial physical fields $\psi(x)$. The same operations also convert the formal mapping (17) into temperature-dependent DM over the cold vacuum $|0\widetilde{0}\rangle$, and precisely in such sense we call further the r.h.s. of (17) again as DM.

On the contrary, the standard computation way [9, 10] implies the substitution into the l.h.s. of (16) of the inverse to (15) linear expressions of physical fields $\psi(x)$ in terms of the «hot» physical thermofields, $\psi(x,[+]\varsigma) = \mathcal{V}_\vartheta^{-1}\psi(x)\mathcal{V}_\vartheta$, given by (15), and reordering the so obtained operator with respect to *this* hot physical thermofield over the thermal («hot») vacuum (7), (8). Of course, such operations give DM for the initial HF $\Psi(x)$ over this thermal vacuum. To avoid some ambiguities [12–14], one should carefully distinguish the hot and cold physical thermofields $\psi(x,[\pm]\varsigma)$.

The kinematical independence of tilde-conjugate fields $\widetilde{\Psi}$ means

$$\left\{\Psi_\xi(x), \widetilde{\Psi}^{\#}_{\xi'}(y)\right\}\Big|_{x^0=y^0} = 0, \qquad \left\{\Psi_\xi(x), \widetilde{\Psi}^{\#}_{\xi'}(y)\right\}\Big|_{(x-y)^2<0} = 0 \tag{19}$$

and corresponds to the above independence of their Hamiltonians and their HEqs. This allows one to consider a solution only for one of them. Since the thermal transformations $\mathcal{V}_{\vartheta(F)}$, $\mathcal{V}_{\vartheta(B)}$ do not depend on coordinates and time, they can be applied directly to zero temperature HEq of Thirring model [8], resulting[1] in the same HEqs for the new HF (17):

$$i\partial_0 \Psi(x,\varsigma) = \left[\Psi(x,\varsigma), \widehat{H}[\Psi,\widetilde{\Psi}]\right] = \left[E(P^1) + g\gamma^0\gamma_\nu J^\nu_{(\Psi)}(x,\varsigma)\right]\Psi(x,\varsigma), \tag{20}$$

or

$$2\partial_\xi \Psi_\xi(x,\varsigma) = -igJ^{-\xi}_{(\Psi)}(x,\varsigma)\Psi_\xi(x,\varsigma), \quad \xi = \pm \tag{21}$$

for each ξ-component of the field that are also related to the corresponding current components as

$$J^\xi_{(\Psi)}(x,\varsigma) = J^0_{(\Psi)}(x,\varsigma) + \xi J^1_{(\Psi)}(x;\varsigma) \longmapsto 2\Psi^\dagger_\xi(x,\varsigma)\Psi_\xi(x,\varsigma), \quad \xi = \pm. \tag{22}$$

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[1]Here: $x^\mu = (x^0, x^1)$; $x^0 = t$; $\hbar = c = 1$; $\partial_\mu = (\partial_0, \partial_1)$; for $g^{\mu\nu}$: $g^{00} = -g^{11} = 1$; for $\epsilon^{\mu\nu}$: $\epsilon^{01} = -\epsilon^{10} = 1$; $\overline{\Psi}(x) = \Psi^\dagger(x)\gamma^0$; $\gamma^0 = \sigma_1$, $\gamma^1 = -i\sigma_2$, $\gamma^5 = \gamma^0\gamma^1 = \sigma_3$, $\gamma^\mu\gamma^5 = -\epsilon^{\mu\nu}\gamma_\nu$, where σ_i — Pauli matrices, and I — unit matrix; $x^\xi = x^0 + \xi x^1$, $2\partial_\xi = 2\partial/\partial x^\xi = \partial_0 + \xi\partial_1$, $P^1 = -i\partial_1$, $E(P^1) = \gamma^5 P^1$; summation over ξ is nowhere implied.

Thus, to integrate these HEqs, we can consequently repeat all the steps of our previous works [8] with the same linearization, renormalization and bosonization conditions (here w means week equality):

$$\gamma^0\gamma_\nu J^\nu_{(\Psi)}(x,\varsigma) \xmapsto{w} \frac{\beta}{2\sqrt{\pi}}\gamma^0\gamma_\nu \widehat{J}^\nu_{(\chi)}(x,\varsigma), \tag{23}$$

$$\widehat{J}^\nu_{(\chi)}(x,\varsigma) = \lim_{\varepsilon,(\widetilde{\varepsilon})\to 0} \widehat{J}^\nu_{(\chi)}\left(x;\varepsilon(\widetilde{\varepsilon}),\varsigma\right) \equiv : J^\nu_{(\chi)}(x,\varsigma) :, \tag{24}$$

that for the same subsequently renormalized (normal ordered) current

$$J^0_{(\Psi)}(x,\varsigma) \longmapsto \lim_{\widetilde{\varepsilon}\to 0} \widehat{J}^0_{(\Psi)}(x;\widetilde{\varepsilon},\varsigma) = \widehat{J}^0_{(\Psi)}(x,\varsigma), \tag{25}$$

$$J^1_{(\Psi)}(x,\varsigma) \longmapsto \lim_{\varepsilon\to 0} \widehat{J}^1_{(\Psi)}(x;\varepsilon,\varsigma) = \widehat{J}^1_{(\Psi)}(x,\varsigma), \tag{26}$$

$$\text{where at first } \ \widetilde{\varepsilon}^0 = \varepsilon^1 \to 0, \ \text{ when } \ \widetilde{\varepsilon}^1 = \varepsilon^0, \ \ \varepsilon^2 = -\widetilde{\varepsilon}^2 > 0, \ \text{ for} \tag{27}$$

$$\widehat{J}^\nu_{(\Psi)}(x;a,\varsigma) = Z^{-1}_{(\Psi)}(a)\left[\overline{\Psi}(x+a,\varsigma)\gamma^\nu\Psi(x,\varsigma) - \langle 0|\overline{\Psi}(x+a,\varsigma)\gamma^\nu\Psi(x,\varsigma)|0\rangle\right], \tag{28}$$

leads again to the linearization of both equations (20) and (21) in the representation of these free fields $\chi(x,\varsigma)$ with the same free bosonization rules:

$$\widehat{J}^\mu_{(\chi)}(x,\varsigma) = \frac{1}{\sqrt{\pi}}\partial^\mu\varphi(x,\varsigma) = -\frac{1}{\sqrt{\pi}}\epsilon^{\mu\nu}\partial_\nu\phi(x,\varsigma), \tag{29}$$

$$\widehat{J}^{-\xi}_{(\chi)}(x,\varsigma) = \frac{2}{\sqrt{\pi}}\partial_\xi\varphi^\xi\left(x^\xi,\varsigma\right). \tag{30}$$

The thermofields $\varphi(x,\varsigma)$ and $\phi(x,\varsigma)$ are defined in (40) below as unitarily inequivalent representations of the massless scalar and pseudoscalar Klein–Gordon fields: $\partial_\mu\partial^\mu\varphi(x,\varsigma) = 0$, and $\partial_\mu\partial^\mu\phi(x,\varsigma) = 0$, and are taking again mutually dual and coupled by the symmetric integral relations:

$$\left.\begin{array}{c}\phi(x,\varsigma)\\ \varphi(x,\varsigma)\end{array}\right\} = -\frac{1}{2}\int\limits_{-\infty}^{\infty} dy^1\varepsilon\left(x^1-y^1\right)\partial_0\left\{\begin{array}{l}\varphi\left(y^1,x^0,\varsigma\right),\\ \phi\left(y^1,x^0,\varsigma\right),\end{array}\right. \tag{31}$$

with corresponding charges

$$\left.\begin{array}{c}O(\varsigma)\\ O_5(\varsigma)\end{array}\right\} = \lim_{L\to\infty}\int\limits_{-\infty}^{\infty} dy^1\Delta\left(\frac{y^1}{L}\right)\partial_0\left\{\begin{array}{c}\varphi\left(y^1,x^0,\varsigma\right)\\ \phi\left(y^1,x^0,\varsigma\right)\end{array}\right\} \underset{\Delta=1}{\Longrightarrow} \tag{32}$$

$$\underset{\Delta=1}{\Longrightarrow}\left\{\begin{array}{l}\phi(-\infty,x^0,\varsigma) - \phi(\infty,x^0,\varsigma)\\ \varphi(-\infty,x^0,\varsigma) - \varphi(\infty,x^0,\varsigma).\end{array}\right. \tag{33}$$

Right and left fields $\varphi^\xi\left(x^\xi,\varsigma\right)$ and their charges $\mathcal{Q}^\xi(\varsigma)$ are defined again by linear combinations [5]:

$$\varphi^\xi\left(x^\xi,\varsigma\right) = \frac{1}{2}\left[\varphi(x,\varsigma) - \xi\phi(x,\varsigma)\right] \ \text{ for } \ \xi = \pm, \tag{34}$$

$$\mathcal{Q}^\xi(\varsigma) = \frac{1}{2}\left[O(\varsigma) - \xi O_5(\varsigma)\right] = \pm 2\varphi^\xi\left(x^0 \pm \infty,\varsigma\right). \tag{35}$$

The fields $\varphi(x,\varsigma)$, $\phi(x,\varsigma)$, $\varphi^{\xi}(x^{\xi},\varsigma)$ and their charges obey the commutation relations that do not depend on temperature, for example:

$$[\varphi(x,\varsigma),\partial_0\varphi(y,\varsigma)]\big|_{x^0=y^0}=[\phi(x,\varsigma),\partial_0\phi(y,\varsigma)]\big|_{x^0=y^0}=i\delta(x^1-y^1), \tag{36}$$

$$[\varphi(x,\varsigma),\varphi(y,\varsigma)]=[\phi(x,\varsigma),\phi(y,\varsigma)]=-i\frac{\varepsilon(x^0-y^0)}{2}\theta\left((x-y)^2\right), \tag{37}$$

$$\left[\varphi^{\xi}(s,\varsigma),\varphi^{\xi'}(\tau,\varsigma)\right]=-\frac{i}{4}\varepsilon(s-\tau)\delta_{\xi,\xi'},\quad \left[\varphi^{\xi}(s,\varsigma),\mathcal{Q}^{\xi'}(\varsigma)\right]=\frac{i}{2}\delta_{\xi,\xi'}. \tag{38}$$

Moreover, the same commutation relations take place for their tilde partner, that remain kinematically independent also at finite temperature. So, up to now we cannot distinguish the hot and cold physical thermofields.

The kinematic independence of the tilde partners is breaking and the difference between the hot and cold physical thermofields appears on going to the «frequency» parts of corresponding fields $\varphi^{\xi(\pm)}\left(x^{\xi},\varsigma\right)$ and their charges $\mathcal{Q}^{\xi(\pm)}(\varsigma)$. It manifests itself in the commutators of annihilation $(+)$ and creation $(-)$ (frequency) parts, defined by annihilation and creation operators over the initial cold vacuum $|0\widetilde{0}\rangle$ for the pseudoscalar fields [8]: $c(k^1)|0\widetilde{0}\rangle=\widetilde{c}(k^1)|0\widetilde{0}\rangle=0$, for both hot $[+]$ and cold $[-]$ thermofields, in the form

$$|0(\varsigma)\rangle=\mathcal{V}^{-1}_{\vartheta(B)}|0\widetilde{0}\rangle\equiv\mathcal{V}_{(B)}[-\vartheta]|0\widetilde{0}\rangle,\ \ \tanh^2\vartheta=\mathrm{e}^{-\varsigma k^0},\ \ \vartheta=\vartheta(k^1;\varsigma), \tag{39}$$

$$\varphi(x;[\pm]\varsigma)=\mathcal{V}^{\mp1}_{\vartheta(B)}\varphi(x)\mathcal{V}^{\pm1}_{\vartheta(B)}\Longrightarrow\varphi^{(+)}(x;[\pm]\varsigma)+\varphi^{(-)}(x;[\pm]\varsigma), \tag{40}$$

and so on for all other fields, with corresponding Fourier expansions and commutators, where we put corresponding $\pm$ into respective braces, $k^0=|k^1|$:

$$\varphi^{\xi(+)}\left(x^{\xi};[\pm]\varsigma\right)=-\frac{\xi}{2\pi}\int\limits_{-\infty}^{\infty}\frac{dk^1}{2k^0}\theta\left(-\xi k^1\right)\Big[\cosh\vartheta c\left(k^1\right)\mathrm{e}^{-ik^0x^{\xi}}\mp \tag{41}$$

$$\mp\sinh\vartheta\widetilde{c}\left(k^1\right)\mathrm{e}^{ik^0x^{\xi}}\Big],\quad \varphi^{\xi(-)}\left(x^{\xi};[\pm]\varsigma\right)=\Big\{\varphi^{\xi(+)}\left(x^{\xi};[\pm]\varsigma\right)\Big\}^{\dagger}, \tag{42}$$

$$\widetilde{\varphi}^{\xi(+)}\left(x^{\xi};[\pm]\varsigma\right)=-\frac{\xi}{2\pi}\int\limits_{-\infty}^{\infty}\frac{dk^1}{2k^0}\theta\left(-\xi k^1\right)\Big[\cosh\vartheta\widetilde{c}\left(k^1\right)\mathrm{e}^{ik^0x^{\xi}}\mp \tag{43}$$

$$\mp\sinh\vartheta c\left(k^1\right)\mathrm{e}^{-ik^0x^{\xi}}\Big],\quad \widetilde{\varphi}^{\xi(-)}\left(x^{\xi};[\pm]\varsigma\right)=\Big\{\widetilde{\varphi}^{\xi(+)}\left(x^{\xi};[\pm]\varsigma\right)\Big\}^{\dagger}, \tag{44}$$

$$Q^{\xi(+)}([\pm]\varsigma)=\lim_{L\to\infty}i\frac{\xi}{2}\int\limits_{-\infty}^{\infty}dk^1\theta\left(-\xi k^1\right)\Big[\cosh\vartheta c\left(k^1\right)\mathrm{e}^{-ik^0\widehat{x}^0}\pm \tag{45}$$

$$\pm\sinh\vartheta\widetilde{c}\left(k^1\right)\mathrm{e}^{ik^0\widehat{x}^0}\Big]\delta_L\left(k^1\right),\quad Q^{\xi(-)}([\pm]\varsigma)=\Big\{Q^{\xi(+)}([\pm]\varsigma)\Big\}^{\dagger}, \tag{46}$$

$$\widetilde{Q}^{\xi(+)}([\pm]\varsigma)=\lim_{L\to\infty}-i\frac{\xi}{2}\int\limits_{-\infty}^{\infty}dk^1\theta\left(-\xi k^1\right)\Big[\cosh\vartheta\widetilde{c}\left(k^1\right)\mathrm{e}^{ik^0\widehat{x}^0}\pm \tag{47}$$

$$\pm\sinh\vartheta c\left(k^1\right)\mathrm{e}^{-ik^0\widehat{x}^0}\Big]\delta_L\left(k^1\right),\quad \widetilde{Q}^{\xi(-)}([\pm]\varsigma)=\Big\{\widetilde{Q}^{\xi(+)}([\pm]\varsigma)\Big\}^{\dagger}. \tag{48}$$

Here the $\widehat{x}^0$-dependence of charge frequence parts is fictitious and unphysical. It is the artefact of space regularization (32) and should be removed at the end of calculation.

Only for hot $[+]$ thermofields one has

$$\langle 0(\varsigma)|\varphi^{\xi}(s;[+]\varsigma)\varphi^{\xi'}(\tau;[+]\varsigma)|0(\varsigma)\rangle = \langle 0|\varphi^{\xi}(s),\varphi^{\xi'}(\tau)|0\rangle = \tag{49}$$

$$= \left[\varphi^{\xi(+)}(s),\varphi^{\xi'(-)}(\tau)\right] = \frac{\delta_{\xi,\xi'}}{i}D^{(-)}(s-\tau). \tag{50}$$

But for both of them (here $D^{(-)}(s) = \lim\limits_{\varsigma\to\infty} D^{(-)}(\pm s,\varsigma;\mu_1)$)

$$\langle 0\widetilde{0}|\varphi^{\xi}(s;[\pm]\varsigma)\varphi^{\xi'}(\tau;[\pm]\varsigma)|0\widetilde{0}\rangle = \left[\varphi^{\xi(+)}(s;[\pm]\varsigma),\varphi^{\xi'(-)}(\tau;[\pm]\varsigma)\right], \tag{51}$$

$$\begin{gathered}\left[\varphi^{\xi(\pm)}(s;[\pm]\varsigma),\varphi^{\xi'(\mp)}(\tau;[\pm]\varsigma)\right] = (\pm 1)\frac{\delta_{\xi,\xi'}}{i}D^{(-)}(\pm(s-\tau),\varsigma;\mu_1) = \\ = (\mp 1)\frac{1}{4\pi}\delta_{\xi,\xi'}\left\{\ln\left(i\mu\frac{\varsigma}{\pi}\sinh\left(\frac{\pi}{\varsigma}(\pm(s-\tau)-i0)\right)\right) - g(\varsigma,\mu_1)\right\},\end{gathered} \tag{52}$$

$$\begin{gathered}\left[\widetilde{\varphi}^{\xi(\pm)}(s;[\pm]\varsigma),\widetilde{\varphi}^{\xi'(\mp)}(\tau;[\pm]\varsigma)\right] = (\pm 1)\frac{\delta_{\xi,\xi'}}{i}\widetilde{D}^{(-)}(\pm(s-\tau),\varsigma;\mu_1) = \\ = (\mp 1)\frac{1}{4\pi}\delta_{\xi,\xi'}\left\{\ln\left(i\mu\frac{\varsigma}{\pi}\sinh\left(\frac{\pi}{\varsigma}(\mp(s-\tau)-i0)\right)\right) - g(\varsigma,\mu_1)\right\},\end{gathered} \tag{53}$$

$$\left[\varphi^{\xi(\pm)}(s;[\pm]\varsigma),\widetilde{\varphi}^{\xi'(\mp)}(\tau;[\pm]\varsigma)\right] = \tag{54}$$

$$= (\pm 1)[\pm 1]\frac{1}{4\pi}\delta_{\xi,\xi'}\left\{\ln\left(\cosh\left(\frac{\pi}{\varsigma}(s-\tau)\right)\right) - f(\varsigma,\mu_2)\right\}, \tag{55}$$

$$\left[\varphi^{\xi(\pm)}(s;[\pm]\varsigma),Q^{\xi'(\mp)}([\pm]\varsigma)\right] = \frac{i}{4}\delta_{\xi,\xi'} = -\left[\widetilde{\varphi}^{\xi(\pm)}(s;[\pm]\varsigma),\widetilde{Q}^{\xi'(\mp)}([\pm]\varsigma)\right], \tag{56}$$

$$\left[\varphi^{\xi(\pm)}(s;[\pm]\varsigma),\widetilde{Q}^{\xi'(\mp)}([\pm]\varsigma)\right] = (\pm 1)[\pm 1]\delta_{\xi,\xi'}\left(\frac{\widehat{x}^0 - s}{2\varsigma}\right), \tag{57}$$

$$\left[Q^{\xi(\pm)}([\pm]\varsigma),Q^{\xi'(\mp)}([\pm]\varsigma)\right] = (\pm 1)a_1\delta_{\xi,\xi'} = \left[\widetilde{Q}^{\xi(\pm)}([\pm]\varsigma),\widetilde{Q}^{\xi'(\mp)}([\pm]\varsigma)\right], \tag{58}$$

$$\left[Q^{\xi(\pm)}([\pm]\varsigma),\widetilde{Q}^{\xi'(\mp)}([\pm]\varsigma)\right] = (\pm 1)[\mp 1]a_2\delta_{\xi,\xi'} = \left[\widetilde{Q}^{\xi(\pm)}([\pm]\varsigma),Q^{\xi'(\mp)}([\pm]\varsigma)\right]. \tag{59}$$

Here the following quantities are defined:

$$g(\varsigma,\mu_1) = \int\limits_{\mu_1}^{\infty}\frac{dk^1}{k^0}\frac{2}{\mathrm{e}^{\varsigma k^0}-1} \Longrightarrow \frac{2}{\varsigma\mu_1} - \ln\left(\frac{2\pi}{\varsigma\overline{\mu}_1}\right), \quad \overline{\mu}_1 = \mu_1\mathrm{e}^{C_{\ni}} \to 0, \tag{60}$$

$$\lim_{\varsigma\to\infty} g(\varsigma,\mu_1) = 0, \tag{61}$$

$$f(\varsigma,\mu_2) = \int\limits_{\mu_2}^{\infty}\frac{dk^1}{k^0}\frac{1}{\sinh(\varsigma k^0/2)} \Longrightarrow \frac{2}{\varsigma\mu_2} - \ln 2, \quad \mu_2 \to 0, \tag{62}$$

$$\lim_{\varsigma\to\infty} f(\varsigma,\mu_2) = 0, \tag{63}$$

$$\delta_L\left(k^1\right) = \int\limits_{-\infty}^{\infty}\frac{dx^1}{2\pi}\Delta\left(\frac{x^1}{L}\right)\mathrm{e}^{\pm ik^1x^1} = L\overline{\Delta}\left(k^1L\right), \quad \lim_{L\to\infty}\delta_L\left(k^1\right) = \delta\left(k^1\right), \tag{64}$$

where

$$a_0 = \pi \int_0^\infty dk^1\, k^1 \left(\delta_L(k^1)\right)^2 = \pi \int_0^\infty dt\, t(\overline{\Delta}(t))^2 \equiv \pi I_1^\Delta, \;\; I_n^\Delta \equiv \int_0^\infty dt\, t^n \left(\overline{\Delta}(t)\right)^2, \tag{65}$$

$$a_1 = a_0 + 2\pi \int_0^\infty dk^1\, k^1 \frac{\left(\delta_L(k^1)\right)^2}{\mathrm{e}^{\varsigma k^0} - 1} = a_0 + 2\pi \int_0^\infty dt\, t \frac{(\overline{\Delta}(t))^2}{\mathrm{e}^{\varsigma t/L} - 1} \Longrightarrow \tag{66}$$

$$\Longrightarrow 2\pi I_0^\Delta \frac{L}{\varsigma} + \frac{\pi}{6} I_2^\Delta \frac{\varsigma}{L} + O\left(\left(\frac{\varsigma}{L}\right)^3\right), \quad L \to \infty, \tag{67}$$

$$a_2 = \pi \int_0^\infty dk^1\, k^1 \frac{\left(\delta_L(k^1)\right)^2}{\sinh(\varsigma k^0/2)} = \pi \int_0^\infty dt\, t \frac{(\overline{\Delta}(t))^2}{\sinh\left(t\varsigma/2L\right)} \Longrightarrow \tag{68}$$

$$\Longrightarrow 2\pi I_0^\Delta \frac{L}{\varsigma} + \left(\frac{\pi}{6} - \frac{\pi}{4}\right) I_2^\Delta \frac{\varsigma}{L} + O\left(\left(\frac{\varsigma}{L}\right)^3\right), \quad L \to \infty, \tag{69}$$

where $C_\ni$ is the Euler–Mascheroni constant. The value of commutator (57) has no physical meaning and below is chosen to be equal to 0.

Following [5], by the use of the fields given above, one can construct a variety of different inequivalent representations of solutions of the Dirac equation for a free massless trial field at finite temperature, $\partial_\xi \chi_\xi\left(x^{-\xi}, \varsigma\right) = 0$ in the form of local normal-ordered exponentials of the left and right bosonic thermofields $\varphi^\xi(x^\xi, \varsigma)$ and their charges $\mathcal{Q}^\xi(\varsigma)$ (34), (35). However, the kinematic independence (19) of the tilde partners can be achieved only «by mixing» in the same field both the charges $\mathcal{Q}^\xi(\varsigma)$ and $\widetilde{\mathcal{Q}}^\xi(\varsigma)$. Let us choose the most simple of them, which leads to the bosonization relations (30) for the currents (25)–(28) of trial fields $\chi(x, \varsigma)$ with $Z_{(\chi)}(a) = 1$ (here ϖ and Θ are arbitrary initial overal and relative phases):

$$\chi_\xi(x^{-\xi}; [\pm]\varsigma) = \mathcal{N}_\varphi \left\{\mathrm{e}^{R_\xi(x^{-\xi}; [\pm]\varsigma)}\right\} u_\xi\left(\mu_1, \varsigma\right), \tag{70}$$

$$R_\xi(x^{-\xi}; [\pm]\varsigma) = -i2\sqrt{\pi}\left[\varphi^{-\xi}\left(x^{-\xi}; [\pm]\varsigma\right) + \frac{1}{4}\sigma_1^\xi \mathcal{Q}^\xi([\pm]\varsigma) + \right.$$
$$\left. + \frac{1}{4}\sigma_2^\xi \widetilde{\mathcal{Q}}^\xi([\pm]\varsigma) + \frac{1}{4}\sigma_3^\xi \widetilde{\mathcal{Q}}^{-\xi}([\pm]\varsigma)\right], \tag{71}$$

$$u_\xi\left(\mu_1, \varsigma\right) = \left(\frac{\mu}{2\pi}\right)^{1/2} \exp\left[-\frac{1}{2} g\left(\varsigma, \mu_1\right)\right] \exp\left(-\frac{\pi}{8} a_1\right) \exp\left(i\varpi - i\xi \frac{\Theta}{2}\right). \tag{72}$$

Thus, following [8], we obtain the normal exponent of the DM for Thirring field in the form analogous to [5] (Λ is ultraviolet cut-off):

$$\Psi_\xi(x; [\pm]\varsigma) = \mathcal{N}_\varphi \left\{\mathrm{e}^{\Xi_\xi(x; [\pm]\varsigma)}\right\} w_\xi\left(\mu_1, \varsigma\right), \tag{73}$$

$$\Xi_\xi(x; [\pm]\varsigma) = -i\left[\overline{\alpha}\varphi^{-\xi}\left(x^{-\xi}; [\pm]\varsigma\right) + \overline{\beta}\varphi^\xi\left(x^\xi; [\pm]\varsigma\right) + \right.$$
$$\left. + \frac{1}{4}\overline{\alpha}\sigma_1^\xi \mathcal{Q}^\xi([\pm]\varsigma) - \frac{1}{4}\overline{\beta}\sigma_1^\xi \mathcal{Q}^{-\xi}([\pm]\varsigma) + \frac{1}{4}\Sigma_2^\xi \widetilde{\mathcal{Q}}^\xi([\pm]\varsigma) + \frac{1}{4}\Sigma_3^\xi \widetilde{\mathcal{Q}}^{-\xi}([\pm]\varsigma)\right], \tag{74}$$

$$w_\xi\left(\mu_1,\varsigma\right)=\left(\frac{\mu}{2\pi}\right)^{1/2}\left(\frac{\mu}{\Lambda}\right)^{\frac{1}{4\pi}\overline{\beta}^2}\exp\left[-g\left(\varsigma,\mu_1\right)\left(\frac{1}{2}+\frac{1}{4\pi}\overline{\beta}^2\right)\right]\times$$
$$\times\exp\left[-\frac{\pi}{4}a_1\left(\frac{1}{2}+\frac{1}{4\pi}\overline{\beta}^2\right)\right]\exp\left(i\varpi-i\xi\frac{\Theta}{2}\right),\tag{75}$$

(76)

where

$$\Sigma_2^\xi=\overline{\alpha}\sigma_2^\xi-\overline{\beta}\sigma_3^\xi,\quad \Sigma_3^\xi=\overline{\alpha}\sigma_3^\xi-\overline{\beta}\sigma_2^\xi,\tag{77}$$

by imposing again the conditions on the parameters that are necessary to have correct Lorentz-transformation properties corresponding to the spin $1/2$, and correct canonical anticommutation relations, respectively [8]:

$$\overline{\alpha}^2-\overline{\beta}^2=4\pi,\quad \overline{\beta}-\frac{\beta g}{2\pi}=0.\tag{78}$$

The following conditions provide all the anticommutation relations for both the free and Thirring fields and their tilde partners:

$$\sigma_1^\xi=[\pm]\left(1+2n\right)+\xi,\quad \sigma_2^\xi=\left(1+2n\right)+[\pm]\xi,\quad \sigma_3^\xi=\xi\ell\tag{79}$$

for $\xi=\pm$, n — integer, and $\ell=\pm1$. Contrary to [12], the so obtained fields possess the correct symmetry properties [9, 10] under the «tilde»-operation and correct anticommutation relations, including (19).

Straightforward calculation of the vector current operators (25)–(28) for the solution (73) with $Z_{(\Psi)}(a)=(-\mu^2a^2)^{-\overline{\beta}^2/4\pi}$ by means of Eqs. (38)–(59) and (78), under the conditions [8]

$$\overline{\alpha}=\left(\frac{2\pi}{\beta}+\frac{\beta}{2}\right),\quad \overline{\beta}=\left(\frac{2\pi}{\beta}-\frac{\beta}{2}\right),\quad\text{or}\quad \frac{2\sqrt{\pi}}{\beta}=\sqrt{1+\frac{g}{\pi}},\tag{80}$$

reproduces the bosonization relations (23)–(30), demonstrating self-consistency of all the above calculations. The obtained normal form of Thirring thermofields has a correct renormalization properties:

$$\left\{\Psi_\xi(x,\varsigma),\Psi^\dagger_{\xi'}(y,\varsigma)\right\}\Big|_{x^0=y^0}=\widehat{\mathcal{Z}}_{(\Psi)}\delta_{\xi,\xi'}\delta\left(x^1-y^1\right),\tag{81}$$

$$\widehat{\mathcal{Z}}_{(\Psi)}=\left[-\Lambda^2(x-y)^2\right]^{-\overline{\beta}^2/4\pi}\Big|_{x^0=y^0}=\left[\Lambda^2(x^1-y^1)^2\right]^{-\overline{\beta}^2/4\pi}\to1,\tag{82}$$

for $x^1-y^1\simeq1/\Lambda$. It also satisfies «correspondence principle» at $T\to0$, ($\varsigma\to\infty$), because the addition $[\pm]\left(1+2n\right)$ in σ_1^ξ becomes irrelevant for this limit.

CONCLUSION

The main lesson of this work is very simple: the true HF should be a fully normal-ordered operator in the sense of DM, including also all Klein factors. Only this form assures its correct renormalization, commutation and symmetry properties.

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