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HYDRODYNAMICS OF FLUIDS WITH SPIN

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We discuss the possibility of a nonvanishing spin tensor in relativistic hydrodynamics and its relevance to the description of Quark–Gluon-Plasma evolution in relativistic heavy ion collisions. After a short historical introduction, we report on some recent theoretical results for fully equilibrated fluids.

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INTRODUCTION

The hydrodynamical description of the system created in the collisions of heavy nuclei at high energy has been remarkably successfull [1]. The hydrodynamical model is apparently able to account in some detail for the observed transverse momentum spectra of the various hadronic species and the large anisotropies of the azimuthal components of the transverse momentum, the so-called elliptic flow. While the first generation of hydro-based analyses did not include dissipative terms (ideal fluid description), shear and bulk viscosity terms of the stress-energy tensor are normally employed in current analyses. The success of such an approach and the evidence that dissipative terms seem to be crucial for the correct description of elliptic flow [2] rises one more question of whether possible further terms, thus far disregarded, could be present in the stress-energy tensor that may be able to influence the hydrodynamical evolution equations. In this paper we will discuss one possible addition to the familiar hydrodynamical scheme, namely terms related to the onset of a macroscopic spin density in the fluid. The paper is organized as follows: in Sec. 1 we will discuss the general features of fluids with finite spin density; in Sec. 2 we will briefly summarize the history of the subject, while in Sec. 3 we will present some recent results on the simplest instance of fluid with spin, the ideal Boltzmann gas with macroscopic angular momentum; in Sec. 4 we will discuss possible developments.

1. FLUIDS WITH SPIN

A fluid with spin is a fluid which needs a spin tensor $\mathcal{S}^{\lambda,\mu\nu}$ (the indices μ,ν being antisymmetric) to be described in addition to the familiar stress-energy tensor $T^{\mu\nu}$. The equations of motion are two coupled partial differential equations, the continuity equations for energy-momentum and angular momentum:

$$\begin{aligned} &\partial_\mu T^{\mu\nu} = 0, \\ &\partial_\lambda \mathcal{J}^{\lambda,\mu\nu} = \partial_\lambda(\mathcal{S}^{\lambda,\mu\nu} + x^\mu T^{\lambda\nu} - x^\nu T^{\lambda\mu}) = 0. \end{aligned} \tag{1}$$

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These equations enjoy a peculiar «gauge» invariance insofar as the two tensors can be transformed according to

$$
\begin{aligned}
T'^{\mu\nu} &= T^{\mu\nu} + \frac{1}{2}\partial_\lambda \left(\Phi^{\lambda,\mu\nu} - \Phi^{\mu,\lambda\nu} - \Phi^{\nu,\lambda\mu}\right), \\
\mathcal{S}'^{\lambda,\mu\nu} &= \mathcal{S}^{\lambda,\mu\nu} - \Phi^{\lambda,\mu\nu},
\end{aligned}
\tag{2}
$$

the (1) being unchanged in replacing original tensors with primed ones. Also unchanged under (2) are the spacial integrals yielding total four-momentum and angular momentum; this holds provided that the rank-3 tensor Φ fulfills suitable boundary conditions.

The transformation (2) is used to get rid of the spin tensor $\mathcal{S}^{\lambda,\mu\nu}$ by setting $\Phi = \mathcal{S}$ in the second of (2), a procedure leading to the well-known Belinfante symmetrized stress-energy tensor. By invoking this «gauge» invariance, the spin tensor is usually disregarded because in the particular Belinfante «gauge» it vanishes. However, the Belinfante procedure can be applied provided that the original spin tensor to be eliminated vanishes at the boundary, or at least fulfills boundary conditions such that total energy-momentum and angular momentum do not change, and that local entropy density is not affected. Both these conditions are indeed not trivial and, for instance, they are not fulfilled in the case of an ideal relativistic rotating gas [3]. This problem rises in turn some interesting issues about the possibility of eliminating the spin tensor at a microscopic level (in Quantum Field Theory) and the relation between the microscopic quantum stress-energy, spin tensors and their macroscopic correspondents. These fundamental problems are beyond the scope of the present paper; for the present, we will confine ourselves to the observation that a nonvanishing, noneliminable, spin tensor exists for the simple case of an ideal relativistic rotating gas, as shown in [3].

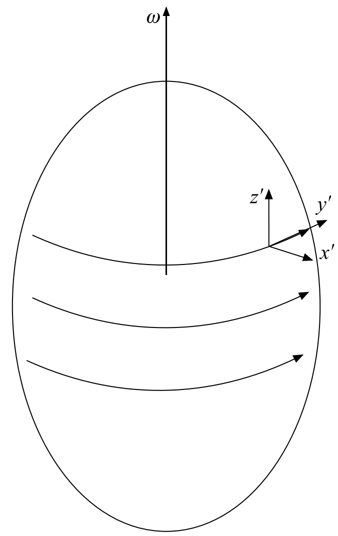
Anyhow, the need of including a spin tensor in the dynamical description of a fluid shall ultimately rest on experimental evidence rather than on theoretical arguments. Indeed, there is a clear demonstration of the need of a nonvanishing spin tensor, namely the Barnett effect [4]. This occurs when an uncharged body initially spun around its axis slows down and, at the same time, develops a small magnetization [5, 6]:

$$
M = \frac{\chi}{g}\omega,
\tag{3}
$$

where χ is the magnetic susceptibility, g the gyromagnetic factor and ω the angular velocity of the body. The onset of a magnetization is the result of a dissipative transformation of orbital angular momentum (= the total angular momentum at the very beginning) into spin angular momentum driven by spin-orbit coupling between the molecules of the body [6]. The converse effect is better known and it called Einstein–De Haas effect: an uncharged body initially put into an external magnetic field H starts rotating [7]. Since the Barnett effect is an irreversible process, it involves an increase of entropy and heating of the body. Altogether, it proves that at thermodynamical equilibrium a fraction of the original angular momentum must appear as polarization of the molecules along the rotation axis and that, therefore, a spin density and a spin tensor are needed to describe the thermodynamical state of the body.

2. A BRIEF HISTORY OF FLUIDS WITH SPIN

The first attempt to formulate a relativistic theory of fluids with internal spin dates back to Mathissen [8] and Weyssenhoff [9]. The latter formulated a model enforcing the condition of vanishing projection of the spin density tensor [1] over the four-velocity field, namely $t_\mu \equiv \sigma_{\mu\nu} u^\nu = 0$. This condition is commonly known as Frenkel condition and the fluid fulfilling it is also known as Weyssenhoff fluid. The Frenkel condition was critically examined in a later work by Bohm and Vigier [11] where they proposed to extend the Weyssenhoff theory to a more general motion allowing the four vector $t \neq 0$, yet without giving a definite quantitative formulation. A major step in formulating a theory of fluids with spin was made in 1960 by F. Halbwachs [12] who took a variational Lagrangian approach, though keeping the Frenkel condition as auxiliary external constraint. The hydrodynamics of relativistic fluids with spin has since become the subject of specialized literature in general relativity and cosmology [13], especially in the framework of Einstein–Cartan gravity theory [14].



A macroscopic fluid at thermodynamical equilibrium has a rigid velocity field. Also shown is a comoving frame with Frenet–Serret axes

Indeed, none of the thus far proposed general theories of (relativistic) fluids endowed with a macroscopic spin density appears to be really convincing. In all studies quoted above, at some point simplifying assumptions (especially the Frenkel condition) are taken in order to make the problem manageable, what implies a loss of generality and makes the validity of the approach eventually questionable.

Instead of trying to reformulate or to amend a previous general theory of fluids with spin, we have studied in two recent papers [3, 15] the simplest instance of fluids with spin, that is the ideal rotating Boltzmann gas made of massive particles with spin. This is a system at full thermodynamical equilibrium and can be therefore entirely studied within statistical mechanics, independently of any previous formulation of a hydrodynamical theory and freely from the danger of begging the question.

3. THE ROTATING GAS AT FULL THERMODYNAMICAL EQUILIBRIUM

The requirement of full thermodynamical equilibrium is a stringent condition: it implies that a macroscopic system must be rigidly rotating, a classic nonrelativistic result [16] which holds in relativistic mechanics (see [15] for the generalization). For a classical (nonquantum) system this means that the velocity field is rigid, i.e.:

$$\mathbf{v} = \boldsymbol{\omega} \times \mathbf{x} \tag{4}$$

[1] In special relativity, the spin density tensor $\sigma_{\mu\nu}$ is defined as the projection of the spin tensor $\mathcal{S}^{\lambda,\mu\nu}$ over the four-velocity field, i.e., $\sigma^{\mu\nu} = \mathcal{S}^{\lambda,\mu\nu} u_\lambda$.

with $\boldsymbol{\omega}$ a constant angular velocity vector, whereas for a quantum system this is most properly formulated as

$$\widehat{\rho} = \frac{1}{Z_\omega} \exp\left[-\frac{\widehat{H}}{T} + \boldsymbol{\omega} \cdot \frac{\widehat{\mathbf{J}}}{T} \right] \mathsf{P}_V, \tag{5}$$

$\widehat{\rho}$ being the density operator and P_V the projector onto the set of quantum states pertaining to the finite region V; such an additional operator is needed because, for a rigid rotation, the thermodynamical limit $V \to \infty$ is, strictly speaking, forbidden not to violate the light speed limit at a sufficiently large distance from the rotation axis. In Eq. (5) $\widehat{H}$ is the total energy, $\widehat{\mathbf{J}}$ the total angular momentum, T the temperature, $\boldsymbol{\omega}$ the constant angular velocity vector and Z_ω the normalizing factor also known as rotational grand-canonical partition function.

Among the many consequences of equations (4) and (5) (see [3, 15] for a detailed derivation), there is a remarkable modification of the expression of local entropy density:

$$T_0 s = \rho - \mu_0 q + p - \frac{1}{2} \Omega_{\mu\nu} \sigma^{\mu\nu}, \tag{6}$$

where T_0 and μ_0 are the proper (comoving) temperature and chemical potential, respectively. The relation between the proper temperature, measured by a comoving frame in the fluid, and the global temperature T, measured by the external inertial frame (which sees the fluid rotating around its axis) reads

$$T_0 = T\gamma = T(1 - v^2)^{-1/2},$$

as pointed out by Israel [17]; the same for the chemical potential. The most striking feature of Eq. (6) is that local entropy gets an additional term proportional to the double-contracted product between the acceleration tensor Ω, relevant to the Frenet–Serret tetrad of the velocity field (see [3] for a detailed discussion), and the spin density tensor σ. The acceleration tensor Ω is indeed a tetrad-dependent object, generally defined as

$$\Omega^{\mu\nu} = \sum_i \dot{e}^{i\mu} e_i^\nu \tag{7}$$

($e_i \; i = 0, 1, 2, 3$ being the tetrad four-vectors). While in general this can be written as a function of the acceleration vector $\mathbf{a}$ and the angular velocity vector $\boldsymbol{\omega}$ as

$$\Omega = \left(\begin{array}{c|c} 0 & \gamma^3 \mathbf{a} - \gamma^3 (\boldsymbol{\omega} \times \mathbf{v}) \\ \hline \gamma^3 \mathbf{a} - \gamma^3 (\boldsymbol{\omega} \times \mathbf{v}) & \gamma^3 \boldsymbol{\omega} - \gamma^3 \boldsymbol{\omega} \cdot \mathbf{v}\mathbf{v} + \gamma^3 \mathbf{a} \times \mathbf{v} \end{array} \right) \tag{8}$$

for the Frenet–Serret tetrad of a rigid velocity field becomes simply

$$\Omega = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & \gamma\omega & 0 \\ 0 & -\gamma\omega & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}. \tag{9}$$

For an ideal Boltzmann gas the single-particle polarization (that is the expectation value of the Pauli–Lubanski four-vector) [15] can be calculated explicitly by factorizing the density operator $\widehat{\rho}$ into single-particle density operators and turns out to be aligned with the vector $\boldsymbol{\omega}$ and essentially proportional to $\hbar\omega/KT$, a tiny value for most macroscopic systems [1]. The ensuing spin density tensor turns out to be proportional to Ω itself through a scalar coefficient ι [3]:

$$\sigma_{\mu\nu} = \iota\Omega_{\mu\nu}, \quad \iota = S(S+1)n/3T, \tag{10}$$

n being the particle density. It should be pointed out that the additional spin-dependent term in entropy expression has been used in literature in the context of relativistic fluids with spin [13] and the Einstein–Cartan theory [14]. It is a correction to the entropy density of quantum origin that should be applied to off-equilibrium situations as well, that is in dissipative fluid motion. It is interesting to note that, assuming that the spin tensor is simply given by

$$\mathcal{S}^{\lambda,\mu\nu} = \sigma^{\mu\nu}u^{\lambda}, \tag{11}$$

it is then impossible to eliminate it with the Belinfante procedure without violating the global angular momentum conservation and the invariance of local entropy [3].

The phase space distribution for this kind of gas can also be obtained from the distribution (5) and reads

$$f\,(\mathbf{x},\mathbf{p})_{\tau\sigma} = \lambda\,\mathrm{e}^{-\beta\cdot p}\frac{1}{2}\left(D^S\left([p]^{-1}\mathsf{R}_{\hat{\boldsymbol{\omega}}}\frac{i\omega}{T}[p]\right) + D^S\left([p]^{\dagger}\mathsf{R}_{\hat{\boldsymbol{\omega}}}\frac{i\omega}{T}[p]^{\dagger-1}\right)\right)_{\tau\sigma} \tag{12}$$

for a (Boltzmann) particle with spin S. In Eq. (12) λ is the fugacity, β is the temperature four-vector, R is a rotation and $[p]$ is the (arbitrary) Lorentz transformation taking the unit vector of the time axis into the unit vector along the particle four-momentum. The appearance of Wigner matrices D of the irreducible representations $(0,S)$ and $(S,0)$ of the Lorentz group is a remarkable consequence of the term proportional to the angula rmomentum operator $\widehat{\mathbf{J}}$ in the density operator. Therefore, for rotating systems, the phase space distribution is a nondiagonal 2×2 matrix rather than a scalar function, as pointed out in [18]. In the more familiar Dirac spinor formalism, formula (12) can be rewritten as

$$f(\mathbf{x},\mathbf{p})_{\sigma,\rho} = \lambda\,\mathrm{e}^{-\beta\cdot p}\frac{1}{2}\bar{u}_{\sigma}(p)D^{(0,1/2)\oplus(1/2,0)}\left(R_3\frac{i\omega}{T}\right)u_{\rho}(p) \tag{13}$$

for a particle with spin $1/2$; u is a four-component spinor and D now denotes the 4×4 matrix of the irreducible representation $(0,1/2)\oplus(1/2,0)$ of the Lorentz group. From the phase space distribution (12), the stress-energy tensor can be derived and takes the form required by Israel [17] at the complete thermodynamical equilibrium, namely

$$T^{\mu\nu} = (\rho+p)u^{\mu}u^{\nu} - pg^{\mu\nu}. \tag{14}$$

Another consequence of Eq. (9) is the nonvanishing of the four-vector t, that is a violation of the Frenkel condition:

$$t^{\mu} = \iota\Omega^{\mu\nu}u_{\nu} = \iota\,A^{\mu} = \iota(0,\gamma^2\boldsymbol{\omega}\times\mathbf{v}) \neq 0 \tag{15}$$

with $A^{\mu} = (0,\gamma^2\boldsymbol{\omega}\times\mathbf{v})$ the four-acceleration relevant to the velocity field (4). Since the Frenkel condition is not fulfilled even for the simplest instance of a fluid with spin, it turns out that such a constraint is not appropriate for a realistic general theory of such fluids and should be released right from the outset.

[1] It should be pointed out though that this tiny value is responsible for the Barnett effect.

4. OUTLOOK

We are just at the outset of a complete formulation of the theory of fluids with spin. Our finding [3] that Frenkel condition is violated even for an ideal gas at full thermodynamical equilibrium demands a revision of the theories proposed in literature. In a general theory, the stress-energy tensor will most likely be no longer symmetric and additional dissipative terms shall appear which depend on the spin density tensor. Our goal is to find these terms starting from an extension of the (relativistic) kinetic theory [18] including spin degrees of freedom. A new generation of kinetic coefficients governing the transformation of polarization of particles into orbital angular momentum and vice versa, as well as the spin transport, will eventually turn up.

The relevance of these new kinetic processes for the Quark–Gluon Plasma as a relativistic fluid is difficult to assess for the present. It is possible that they will turn out to be negligible corrections to the main hydrodynamical scheme, as they are in most macroscopic phenomena involving electromagnetic interactions (with the exception of peculiar effects like Barnett's and Einstein–De Haas'). However, the microscopic driving forces determining these new kinetic coefficients are spin-orbit and spin-tensor couplings which are notoriously larger for strong interactions than for electromagnetic ones, hence their effect could be quantitatively more important.

It is also possible that such, perhaps tiny, effects could be studied more effectively with cold atom clouds; such systems can be handled under fully controlled experimental conditions and can be spun at various angular velocities [19].

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