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ФИЗИКА ЭЛЕМЕНТАРНЫХ ЧАСТИЦ И АТОМНОГО ЯДРА. ТЕОРИЯ

ON THE MOTION OF A THREE-BODY SYSTEM ON HYPERSURFACE OF PROPER ENERGY

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Based on the fact that for a Hamiltonian system there exists equivalence between phase trajectories and geodesic trajectories on the Riemannian manifold $\mathcal{M}$ (the Lagrangian surface of the body system), the classical three-body problem is formulated in the framework of six ordinary differential equations (ODEs) of the second order on the energy surface of body system. It is shown that in the case when the total interaction potential of the body system depends on the relative distances between particles, the three of six geodesic equations describing rotations of formed by three bodies triangle are solved exactly. Using this fact, it is shown that the three-body problem can be described in the limits of three nonlinear ODEs of canonical form, which in phase space is equivalent to the autonomous sixth-order system. The equations of geodesic deviations on the manifold $\mathbb{R}^3$ (the space of relative distances between particles) are derived in an explicit form. A system of algebraic equations for finding the homographic solutions of restricted three-body problem is obtained. The initial and asymptotic conditions for solution of the classical scattering problem are found.

Ввиду того, что для гамильтоновой системы существует эквивалентность между фазовыми траекториями и геодезическими траекториями на римановом многообразии $\mathcal{M}$ (поверхность Лагранжа системы тел), классическая задача трех тел сформулирована в рамках шести обыкновенных дифференциальных уравнений (ОДУ) второго порядка на тангенциальном расслоении многообразия $\mathcal{M}_t$. Показано, что в случае, когда общий потенциал взаимодействия системы тел зависит от относительных расстояний между частицами, три из шести геодезических уравнений, описывающих вращения образованного тремя телами треугольника, решаются точно. С помощью этого факта было доказано, что общая задача трех тел может быть описана в рамках трех нелинейных ОДУ канонического вида. Показано, что редуцированная задача описывает динамику системы трех тел на плоскости рассеяния с учетом полного углового момента вращающегося треугольника тел. На многообразии $\mathbb{R}^3$ (пространстве относительных расстояний между частицами) выведены явные виды уравнений геодезических отклонений. Для нахождения гомографических решений ограниченной задачи трех тел получена система алгебраических уравнений. Найдены начальные и асимптотические условия для решения классической задачи рассеяния.

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INTRODUCTION

The general three-body classical problem concerns the question of understanding the motions of three arbitrary point masses traveling in space according to Newton's laws of mechanics. Many works on analytical mechanics, celestial mechanics, stellar and molecular dynamics (see [1–9]) are devoted to the study of this problem. Note that most thoroughly, in particular theoretically and numerically, the three-body problem was studied in the case of a restricted problem when one of the masses is negligible compared to the other two masses. In this case the problem is naturally reduced to the two-body problem, which was first exactly solved by Newton in his *Principia* in 1687. For solution of the general problem different approaches based on series expansions methods have been proposed; however, due to the poor convergence of these expansions they are often used and are useful only for solving particular problems, where the system of three bodies is in a stable bound state [2, 10]. Moreover, the three-body problem is a typical example of a dynamic system, where on the large scales of the phase space we observe all features of a complex motion including the bifurcation and chaos. That makes the numerical simulation method a basic way of the researching of the mentioned problem.

Thus, we can say that despite the centuries of exploration, there is no solution to the general three-body problem as there are no coordinate transformations that can simplify the problem; unlike the two-body problem or the restricted three-body problem, the motion of each body has to be considered along with the motions of the other two bodies because the vectors of the mutual forces do not line up with the centre of mass.

Let us note that the general problem of three bodies with consideration of specifics of the multichannel scattering differs by additional complexities, which are associated primarily with the need for numerical simulation of the problem for an infinite number of initial data. In other words, for numerical investigation of the problem, if this is possible, it is important to reduce the dimensionality of the problem which allows one to reduce the volume of calculations and makes them true and accurate.

The problem of separating the vibration motions from the collective motions, i.e., the translational and rotational motions of the molecular system, has been under continuous attention both in classical and quantum mechanics. In any case, the problem of separation of motions in a molecular system makes one study what is meant by the Eckart condition of the translational and rotational motions [11–13]. In particular, in work [14], the vibration motions were defined rigorously, and it was thereby shown that the vibration motions cannot be separated from the rotation motions in the theory of connections in differential geometry. In paper [15], on the basis of the connection theory for the centre-of-mass coordinate system, it was proved that the Eckart frame exists for any configuration of the molecule but not uniquely. Moreover, as is shown in this work, one can choose a moving frame relative to which the molecule moves without rotation.

In Krylov's outstanding work [16], where statistical properties of a dynamical system consisting of N classical particles (gas relaxation) are studied, for the first time geodesic flow on the Lagrangian surface of a system of particles was used. Later, by means of this method, statistical properties of non-Abelian Yang–Mills gauge field [17, 18] and relaxation properties of stellar systems [19, 20] were studied in detail.

The main aim of this work is to find new opportunities to separation of the internal and collective motions in the general classical three-body problem, which will have key importance for the reducing of the dimensionality of the studied dynamical problem. Following aforementioned works [19, 20], we have used the geodesic trajectories approach on the Lagrangian surface of a three-body system to describe the coupling between the rotational and internal motions at the collision of bodies. We have shown the possibility of nontrivial separation of motions in the general three-body problem on the energy hypersurface of a body system. It should be noted that for the first time the reducing of the dimensionality of the classical three-body problem has been made on the basis of heuristic considerations at the investigation of the problem of quantum chaos in the three-body system [21].

1. FORMULATION OF THE PROBLEM

1.1. Reduction of the Problem of Multichannel Scattering to the Problem of Motion of Effective Mass in 6D Configuration Space. The 3D classical three-body problem in a most general formulation as the problem of multichannel scattering with several possible outcomes can be represented as

$$1+(23) \longrightarrow \begin{cases} 1+(23), \\ 1+2+3, \\ (12)+3, \\ (13)+2, \\ (123)^* \longrightarrow \begin{cases} 1+(23), \\ 1+2+3, \\ (12)+3, \\ (13)+2, \end{cases} \end{cases}$$

where the numbers 1, 2, and 3 denote colliding particles; the brackets $(\ldots)$ and $(\ldots)^*$ denote two particles in bound state and the short-living resonance, correspondingly. It is obvious that the investigation of short-living resonance $(123)^*$ is close to the restricted problem of three bodies.

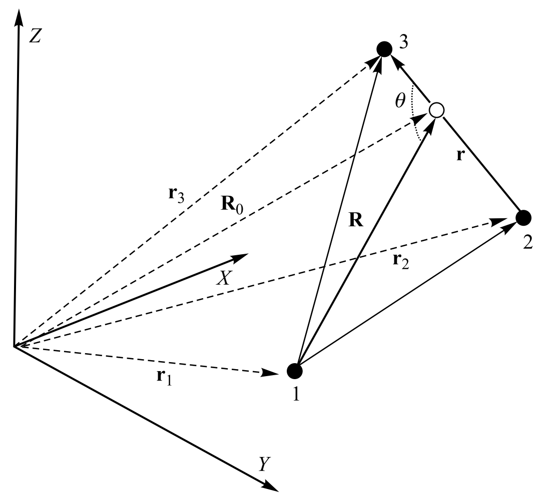
The classical Hamiltonian of a three-body system in the Cartesian coordinates system (see the Figure) has the following form:

$$H = \sum_{i=1}^{3} \frac{\mathbf{p}_i^2}{2m_i} + V(\mathbf{r}_1, \mathbf{r}_2, \mathbf{r}_3), \tag{1}$$

where $(\mathbf{r}_1, \mathbf{r}_2, \mathbf{r}_3)$ and $(\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3)$ are position vectors and momenta of the corresponding particles; (m_1, m_2, m_3) are their masses, and $V(\mathbf{r}_1, \mathbf{r}_2, \mathbf{r}_3)$ denotes the total interaction potential between the particles. It will be assumed that the total interaction potential of the body system depends on the relative distances between the particles:

$$V(\mathbf{r}_1, \mathbf{r}_2, \mathbf{r}_3) \equiv V(||\mathbf{r}_{12}||, ||\mathbf{r}_{13}||, ||\mathbf{r}_{23}||), \tag{2}$$

where $\mathbf{r}_{12} = \mathbf{r}_1 - \mathbf{r}_2$, $\mathbf{r}_{13} = \mathbf{r}_1 - \mathbf{r}_3$ and $\mathbf{r}_{23} = \mathbf{r}_2 - \mathbf{r}_3$ are relative distances between the particles; in the particular case, the total potential can consist of pair potentials $V(\mathbf{r}_1, \mathbf{r}_2, \mathbf{r}_3) = V_{12}(||\mathbf{r}_{12}||) + V_{13}(||\mathbf{r}_{13}||) + V_{23}(||\mathbf{r}_{23}||)$.



The Cartesian coordinates system where the set of vectors $\mathbf{r}_1$, $\mathbf{r}_2$, and $\mathbf{r}_3$ denotes coordinates of the 1, 2, and 3 particles, respectively. The ◯ is the centre of mass of pair (12) which in the Cartesian system is expressed by $\mathbf{R}_0$. The Jacobi coordinates system described by the radius vectors $\mathbf{R}$ and $\mathbf{r}$, in addition to θ, denotes scattering angle

After the Jacobi coordinates transformation, Hamiltonian (1) acquires the form

$$H = \sum_{k=1}^{3} \frac{\mathbf{P}_k^2}{2\mu_k} + V'(\mathbf{r}, \mathbf{R}), \quad V'(\mathbf{r}, \mathbf{R}) = V(||\mathbf{R} - \lambda_- \mathbf{r}||, ||\mathbf{R} + \lambda_+ \mathbf{r}||, ||\mathbf{r}||), \tag{3}$$

where $\mathbf{r} = \mathbf{r}_2 - \mathbf{r}_3$ is the relative position of particle 2 to particle 3; $\mathbf{R} = \mathbf{r}_1 - \mathbf{R}_0$ is the relative position of particle 1 to the centre of mass of pair (23) the radius vector of which is defined by the expression $\mathbf{R}_0 = (m_2\mathbf{r}_2 + m_3\mathbf{r}_3)/(m_2 + m_3)$. In addition, the following designations are made:

$$\mathbf{P}_1 = \mathbf{p}_1 + \mathbf{p}_2 + \mathbf{p}_3, \quad \mathbf{P}_2 = \frac{m_3\mathbf{p}_2 - m_2\mathbf{p}_3}{m_2 + m_3}, \quad \mathbf{P}_3 = \frac{(m_2 + m_3)\mathbf{p}_1 - m_1(\mathbf{p}_2 + \mathbf{p}_3)}{\mu_1},$$
$$\mu_1 = m_1 + m_2 + m_3, \quad \mu_2 = \frac{m_2 m_3}{m_2 + m_3}, \quad \mu_3 = \frac{m_1(m_2 + m_3)}{\mu_1},$$
$$\lambda_- = \frac{\mu_2}{m_2}, \quad \lambda_+ = \frac{\mu_2}{m_3},$$

where $\mathbf{P}_1$ describes the total momentum of the three-body system; $\mathbf{P}_2 = \mu_2\dot{\mathbf{r}} = \mu_2 d\mathbf{r}/dt$ is the momentum of the centre of mass of pair (23), and $\mathbf{P}_3 = \mu_3\dot{\mathbf{R}} = \mu_3 d\mathbf{R}/dt$ is correspondingly the momentum of the effective mass μ_3 which describes the three-body configuration $1 + (23)$.

After deleting the motion of the centre of mass of the three-body system (that is equivalent to the condition $\mathbf{P}_1 = 0$) [22, 23], we can find for the Hamiltonian the following expression:

$$\tilde{H} = \frac{1}{2\mu_0} \sum_{k=2}^{3} \tilde{\mathbf{P}}_k^2 + V'(\mathbf{r}, \mathbf{R}), \tag{4}$$

where

$$\mu_0=\sqrt{\frac{m_1m_2m_3}{\mu_1}},\quad \tilde{\mathbf{P}}_2=\sqrt{\mu_2\mu_0}\,\dot{\mathbf{r}},\quad \tilde{\mathbf{P}}_3=\sqrt{\mu_3\mu_0}\,\dot{\mathbf{R}}.$$

More clearly, Hamiltonian (4) can be represented as

$$\tilde{H}(\mathbf{P}_{x_1},\mathbf{P}_{x_2};\mathbf{x}_1,\mathbf{x}_2)=\frac{1}{2\mu_0}\big(\mathbf{P}^2_{x_1}+\mathbf{P}^2_{x_2}\big)+\tilde{V}(\mathbf{x}_1,\mathbf{x}_2)=\mathcal{H}(\mathbf{P}_x;\mathbf{x})=\frac{1}{2\mu_0}\mathbf{P}^2_x+U(\mathbf{x}),\quad (5)$$

where

$$\mathbf{x}_1=\sqrt{\frac{\mu_2}{\mu_0}}\,\mathbf{r},\quad \mathbf{x}_2=\sqrt{\frac{\mu_3}{\mu_0}}\,\mathbf{R},\quad \mathbf{x}=\mathbf{x}_1\bigoplus\mathbf{x}_2\in\mathbb{R}^6,$$

in addition $\mathbf{P}_{x_1}\equiv\tilde{\mathbf{P}}_2, \mathbf{P}_{x_2}\equiv\tilde{\mathbf{P}}_3$, and $U(\mathbf{x})=\tilde{V}(\mathbf{x}_1,\mathbf{x}_2)\equiv V'(\mathbf{r},\mathbf{R})$.

Thus, the three-body problem can be reduced to the problem of motion of imaginary point with the effective mass μ_0 in the six-dimensional configuration space $\mathbb{R}^6$ with the Euclidean metrics.

1.2. The Equation of Motion on the Energy Hypersurface of a Three-Body System. As is easy to see, the classical system of three bodies at their motion in the 3D Euclidean space permanently forms a triangle, and Newton's equations describe the dynamical system on the space of such triangles [24]. This means that we can formally consider the motion of a body system consisting of two parts. The first is the rotational motion of the body-triangle in the 3D Euclidian space and the second is the internal motion of bodies on the plane defined by the triangle. It follows that theoretically the aforementioned motions can be separated by introducing a nontrivial moving coordinate frame.

Mathematically, the configuration manifold of solid body $\mathbb{R}^6$ can be represented as a direct product of two subspaces [25]:

$$\mathbb{R}^3\times S^3:\Leftrightarrow\mathbb{R}^6,$$

where $\mathbb{R}^3$ is the manifold which is defined as an orthonormal space of relative distances between the bodies (the internal space) while S^3 denotes the space of rotation group $SO(3)$ (the external space). However, in the considered problem, the connections between the bodies are not holonomic and, correspondingly, the configuration manifold $\mathbb{M}$ must be different:

$$\mathbb{M}\cong\mathcal{M}_t\times S^3\subset\mathbb{R}^6,\quad \mathcal{M}_t\subset\mathbb{R}^3,$$

where the manifold $\mathcal{M}_t$ denotes a space of relative distances between moving bodies (see definition (7)).

Let us now introduce a local system of generalized coordinates:

$$(x^1,x^2,x^3,x^4,x^5,x^6)\in\mathbb{M},\tag{6}$$

where we assume that in the configuration space $\mathbb{M}$ the set of the first three coordinates (internal coordinates) $\{\bar{x}\}=(x^1,x^2,x^3)\in\mathcal{M}_t$, while the second set of three coordinates $(x^4,x^5,x^6)\in S^3$.

The coordinates which describe the internal motions are defined as follows:

$$x^1=||\mathbf{x}_1||\in\big[0,\infty\big),\quad x^2=||\mathbf{x}_2||\in\big[0,\infty\big),$$

$$x^3=||\mathbf{x}_1+\mathbf{x}_2||=\sqrt{(x^1)^2-2\,x^1x^2\cos\theta+(x^2)^2}\in\big[|x^1-x^2|,|x^1+x^2|\big],\tag{7}$$

where θ is the angle between the vectors $\mathbf{x}_1$ and $\mathbf{x}_2$ (see the Figure) which in the Jacobi coordinates system coincides with the scattering angle.

The set of external coordinates (x^4, x^5, x^6) describes the rotational motion of a triangle (plane) and is uniquely related with the Euler angles $(\omega_1, \omega_2, \omega_3)$, the changing ranges of which are correspondingly defined as $(\omega_1, \omega_2) \in \big(-\pi, \pi\big]$ and $\omega_3 \in \big[0, \pi\big]$.

Now, Hamiltonian (5) can be represented in the form of bilinear expansion in the system of local coordinates (7):

$$\mathcal{H}(\mathbf{P}_x; \mathbf{x}) = \frac{1}{2\mu_0}\, g^{ij}(\{x\}) p_{\,i} p_{\,j}, \quad i, j = 1, 2, \ldots, 6, \quad \{x\} = (x^1, \ldots, x^6), \tag{8}$$

where p_i are the components of decomposition of the 6D momentum in the local coordinate frame and g^{ij} is the symmetric matrix. Note that in (8) and below in the text by the dummy indexes summation is implied.

Using the representation (8), we can obtain equations of motion in the Hamiltonian form:

$$\frac{dx^i}{dt} = \frac{\partial \mathcal{H}}{\partial p_i} = \frac{1}{2\mu_0}\, g^{ij} p_j, \quad \frac{dp_i}{dt} = -\frac{\partial \mathcal{H}}{\partial x^i} = -\frac{1}{2\mu_0}\, g^{kl}_{;i} p_k p_l, \tag{9}$$

where $g^{lj}_{;i} = \partial_{x^i} g^{lj}$, and t is the time. Note that the linear infinitesimal element on the metric $g_{ij}(\{x\})$ is defined as follows: $(ds)^2 = g^{ij}(\{x\}) dx^i\, dx^j$. Further, in view of the physical reasoning, the metrics of the $\mathbb{R}^6$ manifold will be conveniently defined as the energy hypersurface of the three-body system:

$$g_{ij}(\{x\}) = \big[E - U(\{x\})\big]\delta_{ij} = g(\{x\})\delta_{ij}, \quad g^{ij} = g^{-1}\delta_{ij}, \tag{10}$$

where E is the total energy of the body system; in addition for the case when the full interaction potential depends on relative distances between particles, for the metrics we have the expression $g_{ij}(\{x\}) \equiv g_{ij}(\{\bar{x}\})$ (see Eq. (2)). In further, the surface which is defined by (10) will be called Lagrange surface of a body system.

Finally, we can write the explicit dependence of the total potential energy on the internal coordinates. Using Eqs. (3), (5), and (6), it is easy to obtain the following expression:

$$U\big(\{\bar{x}\}\big) = V\big(f_+(\{\bar{x}\}), f_-(\{\bar{x}\}), ax^1\big), \tag{11}$$

where $f_\pm(\{\bar{x}\}) = \sqrt{\sum\limits_{k=1}^{3} b_k^\pm (x^k)^2} > 0$; in addition, the following designations are made:

$$a = \sqrt{\frac{\mu_0}{\mu_2}}, \quad b_1^\pm = \frac{\mu_0}{\mu_2}\Big(1 \pm \lambda_\pm \sqrt{\frac{\mu_2}{\mu_3}}\Big),$$

$$b_2^\pm = \frac{\mu_0}{\mu_2}\Big(1 \pm \lambda_\pm \sqrt{\frac{\mu_3}{\mu_2}}\Big), \quad b_3^\pm = \mp\lambda_\pm \frac{\mu_0}{\sqrt{\mu_2\mu_3}}.$$

Let us note that the bilinear form under the square root is positive by definition regardless of coefficients' values $b_k^\pm$, where $k = 1, 2, 3$.

2. THE EQUATION OF MOTION FOR GEODESIC TRAJECTORIES

Since the Hamiltonian $\mathcal{H}$ describes a conservative system, the energy is an integral of motion; hence the equation $\mathcal{H}(\mathbf{P}_x; \mathbf{x}) = E = \text{const}$ determines the 11-dimensional energy hypersurface in the 12-dimensional phase space. As is well known, phase trajectories which describe the behavior of Hamiltonian system (9) may be presented as geodesic trajectories of the Riemannian manifold (see [27,28]) given generally in a subspace $\Xi \subset \mathcal{M}_t$ defined by the inequality $U(\{\bar{x}\} \in \Xi) < E$.

The geodesic equations on the Riemannian manifold can be derived using the variational principle of Maupertuis [25,26] and are equivalent to system of equations (9)

$$\ddot{x}^i + \Gamma^i_{jk} \dot{x}^j \dot{x}^k = 0, \quad i, j, k = 1, \dots, 6, \tag{12}$$

where $\dot{x}^i = dx^i/ds$ and $\ddot{x}^i = d^2x^i/ds^2$; in addition, Γ^i_{jk} designates Christoffel symbol

$$\Gamma^i_{jk}(\{x\}) = \frac{1}{2} g^{il} \big(\partial_k\, g_{lj} + \partial_j\, g_{kl} - \partial_l\, g_{jk}\big), \quad \partial_\alpha \equiv \partial_{x^\alpha}. \tag{13}$$

Taking into account (12) and (13), we can obtain the following system of six ordinary differential equations which describe the motion of the effective mass μ_0 on the configuration manifold $\mathbb{M} \subset \mathbb{R}^6$:

$$\begin{aligned}
\ddot{x}^1 &= -a_1 \left\{ (\dot{x}^1)^2 - \sum_{i \neq 1,\, i=2}^{6} (\dot{x}^i)^2 \right\} - 2\dot{x}^1 \left\{ a_2 \dot{x}^2 + a_3 \dot{x}^3 \right\}, \\
\ddot{x}^2 &= -a_2 \left\{ (\dot{x}^2)^2 - \sum_{i=1,\, i \neq 2}^{6} (\dot{x}^i)^2 \right\} - 2\dot{x}^2 \left\{ a_3 \dot{x}^3 + a_1 \dot{x}^1 \right\}, \\
\ddot{x}^3 &= -a_3 \left\{ (\dot{x}^3)^2 - \sum_{i=1,\, i \neq 3}^{6} (\dot{x}^i)^2 \right\} - 2\dot{x}^3 \left\{ a_1 \dot{x}^1 + a_2 \dot{x}^2 \right\}, \\
\ddot{x}^4 &= -2\dot{x}^4 \left\{ a_1 \dot{x}^1 + a_2 \dot{x}^2 + a_3 \dot{x}^3 \right\}, \\
\ddot{x}^5 &= -2\dot{x}^5 \left\{ a_1 \dot{x}^1 + a_2 \dot{x}^2 + a_3 \dot{x}^3 \right\}, \\
\ddot{x}^6 &= -2\dot{x}^6 \left\{ a_1 \dot{x}^1 + a_2 \dot{x}^2 + a_3 \dot{x}^3 \right\},
\end{aligned} \tag{14}$$

where the following denotations are made:

$$g(\{\bar{x}\}) = g_{11}(\{\bar{x}\}) = \ldots = g_{66}(\{\bar{x}\}), \quad a_k(\{\bar{x}\}) = \left(\frac{1}{2}\right) \partial_k \ln g(\{\bar{x}\}), \quad k = 1, 2, 3.$$

In system (14), the last three equations are solved exactly

$$\dot{x}^l = J_l / g(\{\bar{x}\}), \quad J_l = \text{const}_l, \quad l = 4, 5, 6. \tag{15}$$

Let us note that J_4, J_5, and J_6 are the integrals of motion. They can be interpreted as projections of the total angular momentum of the three-body system on corresponding axes which are defined by the initial conditions.

Finally, substituting (15) into Eq. (14), we obtain the following system of nonlinear second-order differential equations which describe dynamics of the three-body system on the *internal space* $\mathcal{M}_t$ taking into account rotations of the triangle on the external space S^3

$$\begin{aligned}\ddot{x}^1 &= -a_1\big\{(\dot{x}^1)^2 - (\dot{x}^2)^2 - (\dot{x}^3)^2 - (J/g)^2\big\} - 2\dot{x}^1\big\{a_2\dot{x}^2 + a_3\dot{x}^3\big\},\\ \ddot{x}^2 &= -a_2\big\{(\dot{x}^2)^2 - (\dot{x}^3)^2 - (\dot{x}^1)^2 - (J/g)^2\big\} - 2\dot{x}^2\big\{a_3\dot{x}^3 + a_1\dot{x}^1\big\},\\ \ddot{x}^3 &= -a_3\big\{(\dot{x}^3)^2 - (\dot{x}^1)^2 - (\dot{x}^2)^2 - (J/g)^2\big\} - 2\dot{x}^3\big\{a_1\dot{x}^1 + a_2\dot{x}^2\big\},\end{aligned} \tag{16}$$

where $J = \sqrt{J_4^2 + J_5^2 + J_6^2} = \mathrm{const}$ is the integral of motion of the total angular momentum of the three-body system.

Thus, we have proved that the general 3D classical three-body scattering problem can be reduced to the problem of solution of the three nonlinear second-order differential equations on the tangential bundle $\mathcal{M}_t$ of the Lagrange manifold $\mathcal{M} = \big[\{\bar{x}\} \equiv (x^1, x^2, x^3) \in \mathcal{M}_t;\ g_{ij} = (E - U\{\bar{x}\})\delta_{ij} > 0\big]$.

Many important properties of the dynamical system can be studied by means of investigation of the behavior of linear deviations $\eta^i = \tilde{x}^i - x^i$ between close geodesic trajectories $\tilde{\mathbf{l}} = \tilde{\mathbf{l}}(\tilde{x}^1, \tilde{x}^2, \tilde{x}^3)$ and $\mathbf{l} = \mathbf{l}(x^1, x^2, x^3)$. Recall that the linear deviation satisfies equation [25]

$$\mathcal{D}_s^2\eta^i = -R^i_{jkl}\big(\{\bar{x}\}\big)\dot{x}^j\,\eta^k\,\dot{x}^l, \quad i,j,k,l = 1,2,\ldots,6, \tag{17}$$

where $R^i_{jkl}\big(\{\bar{x}\}\big) = \partial_k\Gamma^i_{lj} - \partial_l\Gamma^i_{jk} + \Gamma^i_{k\lambda}\Gamma^\lambda_{lj} - \Gamma^i_{l\lambda}\Gamma^\lambda_{jk}$ is the Riemann tensor and $\mathcal{D}_sA^i \equiv \mathcal{D}A^i/\mathcal{D}s = \dot{A}^i + \Gamma^i_{jl}\big(\{\bar{x}\}\big)\dot{x}^jA^l$ denotes the covariant derivative.

The explicit form of the deviation equations is very difficult to derive from (17). However, this can be done easily on the way of expansion of equations system (16) on degrees of deviations keeping only the linear terms of deviation:

$$\begin{aligned}\ddot{\eta}^1 = &-\big\{c_{1l}[(\dot{x}^1)^2 - (\dot{x}^2)^2 - (\dot{x}^3)^2] - 2\dot{x}^1(c_{2l}\dot{x}^2 + c_{3l}\dot{x}^3) + (2a_1a_l - g^{-2})J^2\big\}\eta^l -\\ &- 2a_1\big\{\dot{x}^1\dot{\eta}^1 - \dot{x}^2\dot{\eta}^2 - \dot{x}^3\dot{\eta}^3\big\} - 2\dot{\eta}^1\big\{a_2\dot{x}^2 + a_3\dot{x}^3\big\} - 2\dot{x}^1\big\{a_2\dot{\eta}^2 + a_3\dot{\eta}^3\big\},\\ \ddot{\eta}^2 = &-\big\{c_{2l}[(\dot{x}^2)^2 - (\dot{x}^3)^2 - (\dot{x}^1)^2] - 2\dot{x}^2(c_{3l}\dot{x}^3 + c_{1l}\dot{x}^1) + (2a_2a_l - g^{-2})J^2\big\}\eta^l -\\ &- 2a_2\big\{\dot{x}^2\dot{\eta}^2 - \dot{x}^3\dot{\eta}^3 - \dot{x}^1\dot{\eta}^1\big\} - 2\dot{\eta}^2\big\{a_3\dot{x}^3 + a_1\dot{x}^1\big\} - 2\dot{x}^2\big\{a_3\dot{\eta}^3 + a_1\dot{\eta}^1\big\},\\ \ddot{\eta}^3 = &-\big\{c_{3l}[(\dot{x}^3)^2 - (\dot{x}^1)^2 - (\dot{x}^2)^2] - 2\dot{x}^3(c_{1l}\dot{x}^1 + c_{2l}\dot{x}^2) + (2a_3a_l - g^{-2})J^2\big\}\eta^l -\\ &- 2a_3\big\{\dot{x}^3\dot{\eta}^3 - \dot{x}^1\dot{\eta}^1 - \dot{x}^2\dot{\eta}^2\big\} - 2\dot{\eta}^3\big\{a_1\dot{x}^1 + a_2\dot{x}^2\big\} - 2\dot{x}^3\big\{a_1\dot{\eta}^1 + a_2\dot{\eta}^2\big\},\end{aligned} \tag{18}$$

where $c_{k,l} = \partial_la_k$ and $k,l = 1,2,3$.

While analyzing the system of Eqs. (18), we can construct the explicit form of the Riemann tensor. It is obvious that the system of linear equations for deviations (18) can be solved together with the nonlinear equations of geodesics (16).

3. THE RESTRICTED THREE-BODY PROBLEM

Using representation for the Hamiltonian (8) and solutions (15) we can find the reduced Hamiltonian on the *internal space* $\mathcal{M}_t$

$$\bar{\mathcal{H}}\big(\{\bar{x}\};\{\dot{\bar{x}}\}\big) = \frac{\mu_0}{2g(\{\bar{x}\})}\Big\{u^2 + v^2 + w^2 + \big(J/g(\{\bar{x}\})\big)^2\Big\}, \quad \{\dot{\bar{x}}\} = (u,v,w), \tag{19}$$

where $(\{\bar{x}\},\{\dot{\bar{x}}\}) \in \mathcal{M}_t$.

As is easy to see, equations system (16) can be transformed to the system of nonlinear differential equations of Riccati type:

$$\begin{aligned}
&\dot u + a_1\big\{u^2 - v^2 - w^2 - (J/g)^2\big\} + 2\big\{a_2 v + a_3 w\big\}u = 0, \quad u = \dot x^1,\\
&\dot v + a_2\big\{v^2 - w^2 - u^2 - (J/g)^2\big\} + 2\big\{a_3 w + a_1 u\big\}v = 0, \quad v = \dot x^2,\\
&\dot w + a_3\big\{w^2 - u^2 - v^2 - (J/g)^2\big\} + 2\big\{a_1 u + a_2 v\big\}w = 0, \quad w = \dot x^3.
\end{aligned} \tag{20}$$

The system of the sixth-order Eqs. (20) describes the dynamics of a three-body system satisfying condition (11) on the Lagrange manifold $\mathcal{M}$. Obviously, their solutions must satisfy the energy conservation law: $\bar{\mathcal{H}}\big(\{\bar x\};\{\dot{\bar x}\}\big) = E = \mathrm{const}$. Note that this equation defines the 5-dimensional energy hypersurface in the reduced 6-dimensional phase space.

An important class of solutions for the three-body problem is the restricted problem $(123)^*$ (see scheme of scattering, Subsec. 1.1.). Some basic properties of this problem can be studied without solving equations of motion (16) (or (20)).

Using Eqs. (20) we can derive conditions at which formation of stable configurations for a three-body system is possible.

The first condition which must be satisfied for stable configuration of a body system is obviously the condition of absence of external forces:

$$\boldsymbol{\nabla}\bar{\mathcal{H}}\big(\{\bar x\};\{\dot{\bar x}\}\big) = 0, \quad \boldsymbol{\nabla} = g^{ij}\partial_j = g^{-1}\sum_{j=1}^{3}\partial_j, \quad \partial_j = \frac{\partial}{\partial x^j}. \tag{21}$$

Substituting (19) into (21) with the account of the definition of coefficients a_i (see (15)), we can find the following system of algebraic equations:

$$a_1(\{\bar x\}) = 0, \quad a_2(\{\bar x\}) = 0, \quad a_3(\{\bar x\}) = 0. \tag{22}$$

Solving the system (22) we can find sets of stationary points $\{\bar x\}_i$, where $i = 0, 1\ldots$

It is obvious that from these sets of points stable configurations will form only those for which the following conditions are satisfied:

$$\begin{aligned}
&\partial^2_{11}\bar{\mathcal{H}}(\{\bar x\}_{0i};\{\dot{\bar x}\}_{0i}) > 0,\\
&\det\big(\partial^2_{ij}\bar{\mathcal{H}}(\{\bar x\}_{0i};\{\dot{\bar x}\}_{0i})\big) > 0,\\
&\det\big(\partial^2_{kl}\bar{\mathcal{H}}(\{\bar x\}_{0i};\{\dot{\bar x}\}_{0i})\big) > 0,
\end{aligned} \tag{23}$$

where $i, j = 1, 2$ and $k, l = 1, 2, 3$; in addition, in the (23) designation $\partial^2_{kl} = \partial^2/\partial x^k \partial x^l$ is made. However, system of Eqs. (22) together with conditions (23) defines stable configurations $(\{\bar x\}_{0i};\ \{\dot{\bar x}\}_{0i} = 0)$ of motionless bodies. Note that these stable stationary configurations are interesting in that they can serve as bases for constructing homographic solutions (the solutions which conserve the configuration of bodies during the time). In other words, near the stationary points $\{\bar x\}_i \approx \{\bar x\}_{0i}$ configuration of bodies should be moving freely. The latter means that we can ignore the first derivatives in Eqs. (20) and write them in the form of algebraic equations:

$$\begin{aligned}
&a_1\big\{u^2 - v^2 - w^2 - (J/g)^2\big\} + 2\big\{a_2 v + a_3 w\big\}u = 0,\\
&a_2\big\{v^2 - w^2 - u^2 - (J/g)^2\big\} + 2\big\{a_3 w + a_1 u\big\}v = 0,\\
&a_3\big\{w^2 - u^2 - v^2 - (J/g)^2\big\} + 2\big\{a_1 u + a_2 v\big\}w = 0.
\end{aligned} \tag{24}$$

If we assume that coefficients satisfy the following limit transitions:

$$\lambda_{ij} = \lim_{\{\bar{x}\}_i \to \{\bar{x}\}_{0i}} \frac{a_j}{a_i}, \quad \lambda_{ij} = \lambda_{ji}^{-1}, \quad i,j = 1,2,3,$$

in this case system of equations (24) at the stationary point $\{\bar{x}\}_{0i}$ can be written in the form

$$\begin{aligned} &u^2 - v^2 - w^2 - \lambda_0 + 2\big(\lambda_{12}v + \lambda_{13}w\big)u = 0,\\ &v^2 - w^2 - u^2 - \lambda_0 + 2\big(\lambda_{23}w + \lambda_{21}u\big)v = 0,\\ &w^2 - u^2 - v^2 - \lambda_0 + 2\big(\lambda_{31}u + \lambda_{32}v\big)w = 0, \end{aligned} \tag{25}$$

where $\lambda_0 = \big(J/g(\{\bar{x}\}_{0i})\big)^2 = \mathrm{const}_i \geqslant 0$.

Solving system of equations (25), we can find in the general case eight sets of solutions for velocities $\{\dot{\bar{x}}\}_{0i}^k$, where $k = 1,\ldots,8$. The existence of sets of real solutions will mean that for the body system with account of rotations on Euler angles, there are homographic solutions. In the case when there is at least one set of solutions for system of Eqs. (25), it is important to seek solutions to (24) near a stationary point with consideration of conditions (23). By these computations, we can find a region in the phase space, where the coupled three-body system $(123)^*$ depending on specific conditions can be in the stable or quasi-stable equilibrium state.

4. THE INITIAL AND ASYMPTOTIC CONDITIONS OF THE THREE-BODY SCATTERING PROBLEM

For the solution of equations system (16) and interpretation of its results from the point of view of multichannel scattering, it is necessary to define the initial conditions of the problem and analyze the asymptotic behaviors of these solutions. Obviously, for the solution of the scattering problem on the example of a specific system, the initial position and velocity of the imaginary mass μ_0 on the configuration space $\mathbb{M}$ must be defined. However, for the qualitative investigation of the scattering problem, the initial conditions can be defined from the physical considerations on the reduced space more clearly on the bundle $\mathcal{M}_t$ of the Lagrange manifold $\mathcal{M}$.

As is seen from the scheme of multichannel scattering given at the beginning, the system of three bodies is located in the subspace (in) where particle 1 is in the free state while the other two particles 2 and 3 form the bound state (23). In terms of coordinates of the *internal space* $\mathcal{M}_t$ this asymptotic state is defined by the following characteristic distances:

$$x^1 \leqslant l_{23} = \mathrm{const} > 0, \quad x^2 \to \infty, \quad x^3 \to \infty, \tag{26}$$

where l_{23} denotes the oscillation amplitude of imaginary point with the mass μ_0, which is proportional to the distance between particles 2 and 3, when pair (23) is in the equilibrium. In addition, projections of the initial velocity of the imaginary point will be defined as follows:

$$\dot{x}^1 = |v_1| \leqslant v_{01} = \mathrm{const} > 0, \quad \dot{x}^2 = \dot{x}^3 = v_0, \tag{27}$$

where v_{01} and v_0 are the maximum velocity at oscillations by the coordinate x^1 and the velocity of translational motion of the imaginary point, respectively. The total energy of the body system is an important integral of motion which can be written with the help of initial conditions:

$$E = \frac{\mu_0}{2}\left[(v_{01}^2 + 2v_0^2) + 2J^2\right] + V_{23}(l_{23}), \tag{28}$$

where V_{23} denotes the interaction potential between bodies 2 and 3.

Thus, (26) and (28) are the necessary initial conditions for the solution of the system of nonlinear equations (16) (equivalent to system (20)) which describes multichannel scattering in a three-body system.

Let us note that we can analyze behaviors of geodesic trajectories in the limit $s \to \infty$ and find full information on the outcome of the collision.

In particular, after the collision of particles, the geodesic trajectory comes to one of (out) asymptotic subspaces which are characterized by specific configurations of particles (see *the scheme of scattering*, Subsec. 1.1):

a) In the case when $f_-(\{x\}) \to \infty$, $f_+(\{x\}) \to \infty$ and $x^1 \leqslant l_{23}$, we have the outcome $1 + (23)$ (the excitation).

b) When $\big(f_-(\{x\}), f_+(\{x\})\big) \to \infty$ and $x^1 \to \infty$, the outcome of the process is the dissociation $1 + 2 + 3$ (all the particles are free).

c) When $f_-(\{x\}) \leqslant l_{12} = \text{const}$, $f_+(\{x\}) \to \infty$ and $x^1 \to \infty$, the outcome is the regrouping of particles with formation of the new bound state $(12) + 3$, where l_{12} is the scaled distance between the particles of pair (12) at the equilibrium.

d) When $f_-(\{x\}) \to \infty$, $f_+(\{x\}) \leqslant l_{13} = \text{const}$ and $x^1 \to \infty$, the outcome is new regrouping of the particles $(13) + 2$, where l_{13} is the scaled distance between the particles of pair (13).

Note that all processes which go across a phase of formation of the transition complex $(123)^*$ eventually come to one of the four aforementioned asymptotic subspaces.

CONCLUSION

As was shown by Poincare, the three-body problem is generally a nonintegrable system where the system of bodies in the phase space often demonstrates chaotic behavior. It means that the small differences in the initial conditions produce very significant changes in the motion of the system on relatively smallish intervals of time, which makes practically impossible the prediction of evolution of bodies system in the phase space. The latter in turn means that any small error at calculations of the three-body problem can develop in a short time into an enormous mistake. The reduction of the dimensionality of the general classical three-body problem is a mathematical problem of great importance. It should be noted that for the solution of this problem a lot of effort has been made, but the maximal possible reduction of dimensionality of the three-body and N-body problem is achieved only when the motion of bodies is constrained on a plane [29].

As is shown by this study, the reducing of three-body problem can be successfully solved if the dynamical problem is formulated as a geodesic trajectories problem on the energy hypersurface of body system. Note that in this case the dynamics of the three-body system is described in the internal space $\mathcal{M}_t$ by the system of three nonlinear autonomy ODEs of canonical form (16). The system of equations (16) can be represented to a system of Riccati equations (sixth-order system), which is in stationary points correspondingly transformed into the system of algebraic equations (25). If the total interaction potential is defined, we can solve the system of equations (25) and find all homographic solutions of a moving three-body system. In the paper, the initial and asymptotic conditions of the multichannel scattering problem are also discussed in detail.

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