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ФИЗИКА ЭЛЕМЕНТАРНЫХ ЧАСТИЦ И АТОМНОГО ЯДРА. ТЕОРИЯ

INVARIANT DIFFERENTIAL OPERATORS FOR NONCOMPACT LIE GROUPS: THE REDUCED $su(3,3)$ MULTIPLETS

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In the present paper, we continue the project of systematic construction of invariant differential operators on the example of the noncompact algebras $su(n,n)$. Earlier were given the main multiplets of indecomposable elementary representations for $n \leqslant 4$, and the reduced ones for $n = 2$. Here, we give all reduced multiplets containing physically relevant representations including the minimal ones for the algebra $su(3,3)$. Due to the recently established parabolic relations, the results are valid also for the algebra $sl(6,\mathbb{R})$ with suitably chosen maximal parabolic subalgebra.

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INTRODUCTION

Invariant differential operators play very important role in the description of physical symmetries. In a recent paper [1], we started the systematic explicit construction of invariant differential operators. We gave an explicit description of the building blocks, namely, the parabolic subgroups and subalgebras, from which the necessary representations are induced. Thus, we have set the stage for study of different noncompact groups.

In the present paper, we focus on the algebra $su(3,3)$. The algebras $su(n,n)$ belong to a narrow class of algebras, which we call "conformal Lie algebras", which have very similar properties to the canonical conformal algebras of the Minkowski space-time. This class was identified from our point of view in [2]. The same class was identified independently from different considerations and under different names in [3, 4].

This paper is a sequel of [5], and due to the lack of space, we refer to it and to [6] for motivations and extensive list of literature on the subject.

1. PRELIMINARIES

Let G be a semisimple noncompact Lie group, and K a maximal compact subgroup of G. Then, we have an Iwasawa decomposition $G = KA_0N_0$, where A_0 is Abelian simply connected vector subgroup of G, N_0 is a nilpotent simply connected subgroup of G preserved by the action of A_0. Further, let M_0 be the centralizer of A_0 in K. Then, the subgroup $P_0 = M_0A_0N_0$ is a minimal parabolic subgroup of G. A parabolic subgroup $P = MAN$ is any subgroup of G, which contains a minimal parabolic subgroup.

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The importance of the parabolic subgroups comes from the fact that the representations induced from them generate all (admissible) irreducible representations of G [7–9].

Let ν be a (non-unitary) character of A, $\nu \in \mathcal{A}^*$, let μ fix an irreducible representation D^μ of M on a vector space V_μ.

We call the induced representation $\chi = \mathrm{Ind}_P^G(\mu \otimes \nu \otimes 1)$ an *elementary representation* of G [10]. Their spaces of functions are

$$\mathcal{C}_\chi = \{\mathcal{F} \in C^\infty(G, V_\mu) | \mathcal{F}(\mathrm{gman}) = \mathrm{e}^{-\nu(H)} D^\mu(m^{-1})\, \mathcal{F}(g)\}, \tag{1}$$

where $a = \exp(H) \in A$, $H \in \mathcal{A}$, $m \in M$, $n \in N$. The representation action is the *left* regular action:

$$(\mathcal{T}^\chi(g)\mathcal{F})(g') = \mathcal{F}(g^{-1}g'), \quad g, g' \in G. \tag{2}$$

For our purposes we need to restrict to *maximal* parabolic subgroups P, so that rank $A = 1$. Thus, for our representations the character ν is parameterized by a real number d, called the conformal weight or energy.

An important ingredient in our considerations are the *highest/lowest weight representations* of $\mathcal{G}$. These can be realized as (factor-modules of) the Verma modules V^Λ over $\mathcal{G}^\mathbb{C}$, where $\Lambda \in (\mathcal{H}^\mathbb{C})^*$, $\mathcal{H}^\mathbb{C}$ is a Cartan subalgebra of $\mathcal{G}^\mathbb{C}$, weight $\Lambda = \Lambda(\chi)$ is determined uniquely from χ [11, 12].

Actually, since our ERs will be induced from the finite-dimensional representations of $\mathcal{M}$ (or their limits), the Verma modules are always reducible. Thus, it is more convenient to use the *generalized Verma modules* $\tilde{V}^\Lambda$ such that the role of the highest/lowest weight vector v_0 is taken by the space $V_\mu v_0$. For the generalized Verma modules (GVMs) the reducibility is controlled only by the value of the conformal weight d. Relatedly, for the intertwining differential operators only the reducibility w.r.t. noncompact roots is essential.

One main ingredient of our approach is as follows. We group the (reducible) ERs with the same Casimirs in sets called *multiplets* [12, 13]. The multiplet corresponding to fixed values of the Casimirs may be depicted as a connected graph, the vertices of which correspond to the reducible ERs and the lines between the vertices correspond to intertwining operators. The explicit parameterization of the multiplets and of their ERs is important for understanding of the situation.

In fact, the multiplets contain explicitly all the data necessary to construct the intertwining differential operators. Actually, the data for each intertwining differential operator consist of the pair (β, m), where β is a (noncompact) positive root of $\mathcal{G}^\mathbb{C}$, $m \in I\!N$, such that the BGG [14] Verma module reducibility condition (for highest weight modules) is fulfilled:

$$(\Lambda + \rho, \beta^\vee) = m, \quad \beta^\vee \equiv 2\beta/(\beta, \beta). \tag{3}$$

When (3) holds, then the Verma module with shifted weight $V^{\Lambda - m\beta}$ (or $\tilde{V}^{\Lambda - m\beta}$ for GVM and β noncompact) is embedded in the Verma module V^Λ (or $\tilde{V}^\Lambda$). This embedding is realized by a singular vector v_s determined by a polynomial $\mathcal{P}_{m,\beta}(\mathcal{G}^-)$ in the universal enveloping algebra $(U(\mathcal{G}_-))\ v_0$, $\mathcal{G}^-$ is the subalgebra of $\mathcal{G}^\mathbb{C}$ generated by the negative root generators [15]. More explicitly, [12], $v^s_{m,\beta} = \mathcal{P}^m_\beta v_0$ (or $v^s_{m,\beta} = \mathcal{P}^m_\beta V_\mu v_0$ for GVMs). Then, there exists [12] an intertwining differential operator

$$\mathcal{D}^m_\beta : \mathcal{C}_{\chi(\Lambda)} \longrightarrow \mathcal{C}_{\chi(\Lambda - m\beta)} \tag{4}$$

given explicitly by

$$\mathcal{D}^m_\beta = \mathcal{P}^m_\beta(\widehat{\mathcal{G}^-}), \tag{5}$$

where $\widehat{\mathcal{G}^-}$ denotes the $right$ action on the functions $\mathcal{F}$, cf. (1).

2. THE NONCOMPACT LIE ALGEBRA $su(3,3)$

Let $\mathcal{G} = su(3,3)$. This algebra has discrete series representations and highest/lowest weight representations since the maximal compact subalgebra is $\mathcal{K} \cong u(1) \oplus su(3) \oplus su(3)$.

We choose a *maximal* parabolic $\mathcal{P} = \mathcal{MAN}$ such that $\mathcal{A} \cong so(1,1)$, $\mathcal{M} = sl(3,\mathbb{C})_{\mathbb{R}}$. We note also that $\mathcal{K}^{\mathbb{C}} \cong u(1)^{\mathbb{C}} \oplus sl(3,\mathbb{C}) \oplus sl(3,\mathbb{C}) \cong \mathcal{M}^{\mathbb{C}} \oplus \mathcal{A}^{\mathbb{C}}$. Thus, the factor $\mathcal{M}$ has the same finite-dimensional (nonunitary) representations as the finite-dimensional (unitary) representations of the semisimple subalgebra of $\mathcal{K}$.

We label the signature of the ERs of $\mathcal{G}$ as follows:

$$\chi = \{n_1, n_2, n_4, n_5; c\}, \quad n_j \in \mathbb{Z}_+, \quad c = d - 3, \tag{6}$$

where the last entry of χ labels the characters of $\mathcal{A}$, and the first four entries are labels of the finite-dimensional nonunitary irreps of $\mathcal{M}$, when all $n_j > 0$, or limits of the latter, when some $n_j = 0$.

Below we shall use the following conjugation on the finite-dimensional entries of the signature:

$$(n_1, n_2, n_4, n_5)^* \doteq (n_4, n_5, n_1, n_2). \tag{7}$$

The ERs in the multiplet are related also by intertwining integral operators introduced in [16]. These operators are defined for any ER, the general action being

$$\begin{gathered} G_{\mathrm{KS}} : \mathcal{C}_\chi \longrightarrow \mathcal{C}_{\chi'}, \\ \chi = \{n_1, n_2, n_4, n_5; c\}, \quad \chi' = \{(n_1, n_2, n_4, n_5)^*; -c\}. \end{gathered} \tag{8}$$

For the classification of the multiplets we shall need one more conjugation for the entries of the $\mathcal{M}$ representations:

$$(n_1, n_2, n_4, n_5)^\spadesuit \doteq (n_5, n_4, n_2, n_1). \tag{9}$$

Further, we need the root system of the complexification $\mathcal{G}^{\mathbb{C}} = sl(6,\mathbb{C})$. The positive roots in terms of the simple roots are given standardly as

$$\begin{aligned} &\alpha_{ij} = \alpha_i + \ldots + \alpha_j, \quad 1 \leqslant i < j \leqslant 5, \\ &\alpha_{jj} = \alpha_j, \quad 1 \leqslant j \leqslant 5. \end{aligned} \tag{10}$$

From these the compact roots are those that form (by restriction) the root system of the semisimple part of $\mathcal{K}^{\mathbb{C}}$, the rest are noncompact, i.e.,

$$\text{noncompact:} \quad \alpha_{ij}, \quad 1 \leqslant i \leqslant 3, \quad 3 \leqslant j \leqslant 5. \tag{11}$$

Further, we give the correspondence between the signatures χ and the highest weight Λ. The connection is through the Dynkin labels

$$m_i \equiv (\Lambda + \rho, \alpha_i^\vee) = (\Lambda + \rho, \alpha_i), \quad i = 1, \ldots, 5, \tag{12}$$

where $\Lambda = \Lambda(\chi)$, ρ is half the sum of the positive roots of $\mathcal{G}^{\mathbb{C}}$. The explicit connection is

$$n_i = m_i, \quad c = -\frac{1}{2}(m_{\tilde{\alpha}} + m_3) = -\frac{1}{2}(m_1 + m_2 + 2m_3 + m_4 + m_5), \tag{13}$$

where $\tilde{\alpha} = \alpha_1 + \ldots + \alpha_5$ is the highest root.

We shall also use the so-called Harish-Chandra parameters:

$$m_{jk} \equiv (\Lambda + \rho, \alpha_{jk}) = m_j + \ldots + m_k, \quad j < k, \quad m_{jj} \equiv m_j. \tag{14}$$

Finally, we remind that according to [6], the above results for $su(3,3)$ are valid also for the algebra $sl(6, I\!\!R)$ with parabolic $\mathcal{M}$-factor $sl(3, I\!\!R) \oplus sl(3, I\!\!R)$.

3. MULTIPLETS OF $su(3,3)$

3.1. Main Multiplets. There are two types of multiplets: main and reduced. The multiplets of the main type are in one-to-one correspondence with the finite-dimensional irreps of $su(3,3)$, i.e., they are labelled by the five positive Dynkin labels $m_i \in I\!\!N$. In [5], we have given explicitly the main multiplets for $n = 2, 3, 4$, and the reduced ones for $n = 2$.

A main multiplet contains 20 ERs/GVMs, whose signatures can be given in the following pair-wise manner [5]:

$$\begin{aligned}
\chi_0^{\pm} &= \{(m_1, m_2, m_4, m_5)^{\pm};\ \pm(m_{\tilde{\alpha}} + m_3)\},\\
\chi_a^{\pm} &= \{(m_1, m_{23}, m_{34}, m_5)^{\pm};\ \pm(m_{\tilde{\alpha}} - m_3)\},\\
\chi_b^{\pm} &= \{(m_{12}, m_3, m_{24}, m_5)^{\pm};\ \pm m_{1,45}\},\\
\chi_{b'}^{\pm} &= \{(m_1, m_{24}, m_3, m_{45})^{\pm};\ \pm m_{12,5}\},\\
\chi_c^{\pm} &= \{(m_2, m_3, m_{14}, m_5)^{\pm};\ \pm(m_{45} - m_1)\},\\
\chi_{c'}^{\pm} &= \{(m_{12}, m_{34}, m_{23}, m_{45})^{\pm};\ \pm(m_1 + m_5)\},\\
\chi_{c''}^{\pm} &= \{(m_1, m_{25}, m_3, m_4)^{\pm};\ \pm(m_{12} - m_5)\},\\
\chi_d^{\pm} &= \{(m_2, m_{34}, m_{13}, m_{45})^{\pm};\ \pm(m_5 - m_1)\},\\
\chi_{d'}^{\pm} &= \{(m_{12}, m_{35}, m_{23}, m_4)^{\pm};\ \pm(m_1 - m_5)\},\\
\chi_e^{\pm} &= \{(m_2, m_{35}, m_{13}, m_4)^{\pm};\ \pm(m_1 + m_5)\},
\end{aligned} \tag{15}$$

where $(k_1, k_2, k_3, k_4)^- = (k_1, k_2, k_3, k_4)$, $(k_1, k_2, k_3, k_4)^+ = (k_1, k_2, k_3, k_4)^*$. They are given explicitly in Fig. 1 (first in [5]). The pairs $\Lambda^{\pm}$ are symmetric w.r.t. to the bullet in the middle of the figure — this represents the Weyl symmetry realized by the Knapp–Stein operators (8): $G_{\rm KS} : \mathcal{C}_{\chi^{\mp}} \longleftrightarrow \mathcal{C}_{\chi^{\pm}}$.

Matters are arranged so that in every multiplet only the ER with signature χ_0^- contains a finite-dimensional nonunitary subrepresentation in a finite-dimensional subspace $\mathcal{E}$. The latter corresponds to the finite-dimensional irrep of $su(3,3)$ with signature $\{m_1, \ldots, m_5\}$. The subspace $\mathcal{E}$ is annihilated by the operator G^+, and is the image of the operator G^-. The subspace $\mathcal{E}$ is annihilated also by the intertwining differential operator acting from χ_0^- to χ_a^-. When all $m_i = 1$, then $\dim \mathcal{E} = 1$, and in that case, $\mathcal{E}$ is also the trivial one-dimensional UIR of the whole algebra $\mathcal{G}$. Furthermore, in that case, the conformal weight is zero: $d = 3 + c = 3 - (1/2)(m_1 + m_2 + 2m_3 + m_4 + m_5)_{|_{m_i=1}} = 0$.

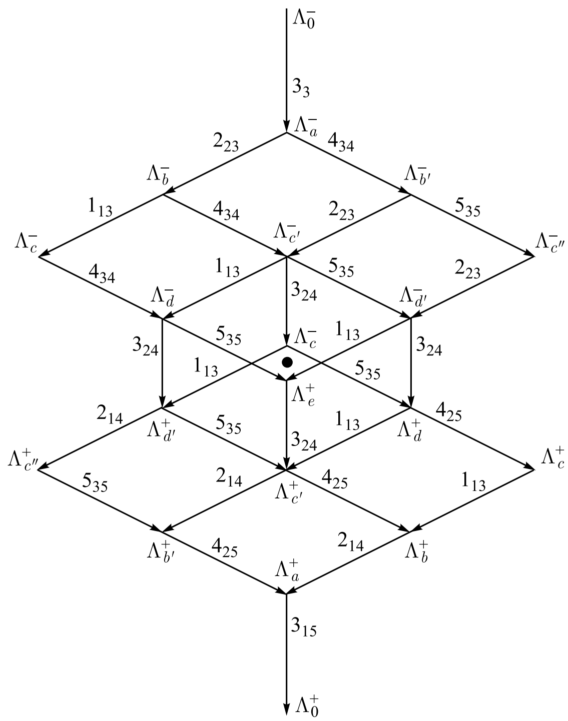

Fig. 1. Main multiplets for $su(3,3)$

Analogously, in every multiplet only the ER with signature χ_0^+ contains holomorphic discrete series representation. This is guaranteed by the criterion [11] that for such an ER all Harish-Chandra parameters for noncompact roots must be negative, i.e., in our situation, $m_\alpha < 0$. (That this holds for our χ^+ can be easily checked using the signatures (15).)

Note that the ER χ_0^+ contains also the conjugate antiholomorphic discrete series. The direct sum of the holomorphic and the antiholomorphic representations is realized in an invariant subspace $\mathcal{D}$ of the ER χ_0^+. That subspace is annihilated by the operator G^-, and is the image of the operator G^+. Note, that the corresponding lowest weight GVM is infinitesimally equivalent only to the holomorphic discrete series, while the conjugate highest weight GVM is infinitesimally equivalent to the antiholomorphic discrete series. The conformal weight of the ER χ_0^+ has the restriction $d = 3 + c = 3 + (1/2)(m_1 + m_2 + 2m_3 + m_4 + m_5) \geqslant 6$.

In Fig. 1 and below, we use the notation: $\Lambda^\pm = \Lambda(\chi^\pm)$. Each intertwining differential operator is represented by an arrow accompanied by a symbol i_{jk}, encoding the root α_{jk} and the number $m_{\alpha_{jk}}$, which is involved in the BGG criterion. This notation is used to save space, but it can be used due to the fact that only intertwining differential operators, which are noncomposite, are displayed, and that the data β, m_β, which are involved in the embedding $V^\Lambda \longleftrightarrow V^{\Lambda - m_\beta, \beta}$, turn out to involve only the m_i corresponding to simple roots, i.e., for each β, m_β there exists $i = i(\beta, m_\beta, \Lambda) \in \{1, \ldots, 5\}$, such that $m_\beta = m_i$. Hence, the data α_{jk}, $m_{\alpha_{jk}}$ are represented by i_{jk} on the arrows.

3.2. Reduced Multiplets. There are five types of reduced multiplets, R_a, $a = 1, \ldots, 5$, which may be obtained from the main multiplet by setting formally $m_a = 0$. Multiplets of type R_4, R_5, are conjugate to the multiplets of type R_2, R_1, respectively, as follows. First, we make the conjugation on the roots and exchange all indices: $1 \longleftrightarrow 5$, $2 \longleftrightarrow 4$. With this operation we obtain the diagrams of the conjugated cases from one another. For the entries of the $\mathcal{M}$ representation we have further to employ the conjugation (9). Then, we obtain the signatures of the conjugated cases from one another. Thus, we give explicitly only first three types.

The reduced multiplets of type R_3 contain 14 ERs/GVMs, whose signatures can be given in the following pair-wise manner:

$$\begin{aligned}
\chi_0^\pm &= \{(m_1, m_2, m_4, m_5)^\pm;\ \pm m_{12,45}\},\\
\chi_b^\pm &= \{(m_{12}, 0, m_{24}, m_5)^\pm;\ \pm m_{1,45}\},\\
\chi_{b'}^\pm &= \{(m_1, m_{24}, 0, m_{45})^\pm;\ \pm m_{12,5}\},\\
\chi_c^\pm &= \{(m_2, 0, m_{14}, m_5)^\pm;\ \pm(m_{45} - m_1)\},\\
\chi_{c''}^\pm &= \{(m_1, m_{25}, 0, m_4)^\pm;\ \pm(m_{12} - m_5)\},\\
\chi_d^\pm &= \{(m_2, m_4, m_{12}, m_{45})^\pm;\ \pm(m_5 - m_1)\},\\
\chi_e^\pm &= \{(m_2, m_{45}, m_{12}, m_4)^\pm;\ \pm(m_1 + m_5)\}.
\end{aligned} \tag{16}$$

These multiplets are given in Fig. 2. They may be called the main type of reduced multiplets since here in χ_0^+ are contained the limits of the (anti)holomorphic discrete series.

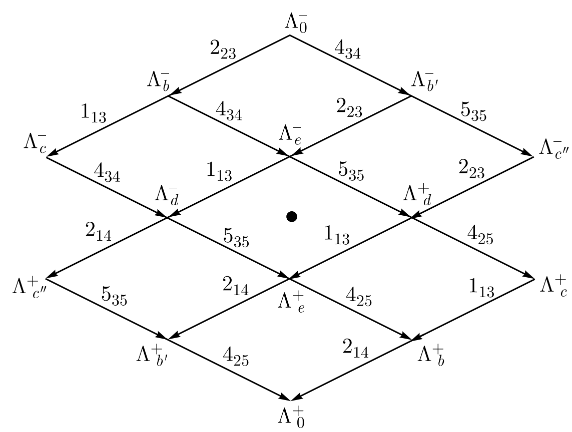

Fig. 2. $su(3,3)$ reduced multiplets of type R_3

The reduced multiplets of type R_2 contain 14 ERs/GVMs, whose signatures can be given in the following pair-wise manner:

$$\begin{aligned}
\chi_0^\pm &= \{(m_1, 0, m_4, m_5)^\pm;\ \pm(m_{\tilde\alpha} + m_3)\},\\
\chi_b^\pm &= \{(m_1, m_3, m_{34}, m_5)^\pm;\ \pm m_{1,45}\},\\
\chi_c^\pm &= \{(0, m_3, m_{14}, m_5)^\pm;\ \pm(m_{45} - m_1)\},\\
\chi_{c'}^\pm &= \{(m_1, m_{34}, m_3, m_{45})^\pm;\ \pm(m_1 + m_5)\},
\end{aligned} \tag{17}$$

$$\chi_d^\pm = \{(0, m_{34}, m_{13}, m_{45})^\pm;\ \pm(m_5 - m_1)\},$$
$$\chi_{d'}^\pm = \{(m_1, m_{35}, m_3, m_4)^\pm;\ \pm(m_1 - m_5)\},$$
$$\chi_e^\pm = \{(0, m_{35}, m_{13}, m_4)^\pm;\ \pm(m_1 + m_5)\}.$$

These multiplets are given in Fig. 3.

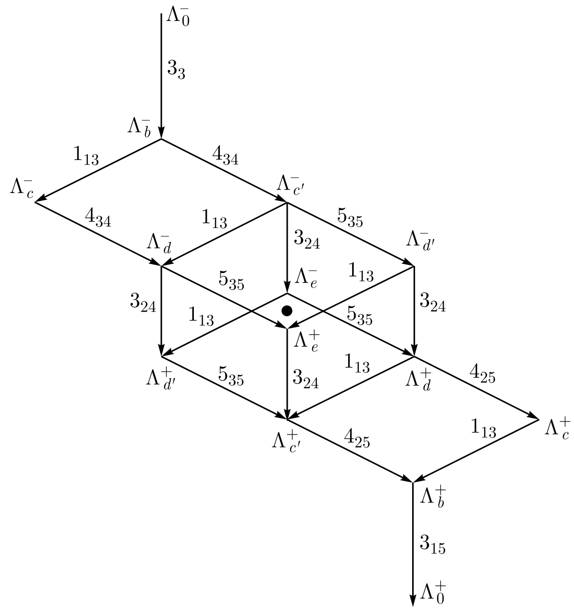

Fig. 3. $su(3,3)$ reduced multiplets of type R_2

The reduced multiplets of type R_1 contain 14 ERs/GVMs, whose signatures can be given in the following pair-wise manner:

$$\begin{aligned}
\chi_0^\pm &= \{(0, m_2, m_4, m_5)^\pm;\ \pm(m_{\tilde\alpha} + m_3)\},\\
\chi_a^\pm &= \{(0, m_{23}, m_{34}, m_5)^\pm;\ \pm(m_{\tilde\alpha} - m_3)\},\\
\chi_{b'}^\pm &= \{(0, m_{24}, m_3, m_{45})^\pm;\ \pm m_{2,5}\},\\
\chi_c^\pm &= \{(m_2, m_3, m_{24}, m_5)^\pm;\ \pm m_{45}\},\\
\chi_{c''}^\pm &= \{(0, m_{25}, m_3, m_4)^\pm;\ \pm(m_2 - m_5)\},\\
\chi_d^\pm &= \{(m_2, m_{34}, m_{23}, m_{45})^\pm;\ \pm m_5\},\\
\chi_e^\pm &= \{(m_2, m_{35}, m_{23}, m_4)^\pm;\ \pm m_5\}.
\end{aligned} \tag{18}$$

These multiplets are given in Fig. 4.

3.3. Further Reduction of Multiplets. There are further reductions of the multiplets denoted by R^3_{ab}, $a, b = 1, \ldots, 5$, $a < b$, which may be obtained from the main multiplet by setting formally $m_a = m_b = 0$. From these ten reductions four (for $(a, b) = (1, 2), (2, 3), (3, 4), (4, 5)$) do not contain representations of physical interest, i.e., induced from the finite-dimensional irreps of the $\mathcal{M}$ subalgebra. From the others, R^3_{35} and R^3_{25} are

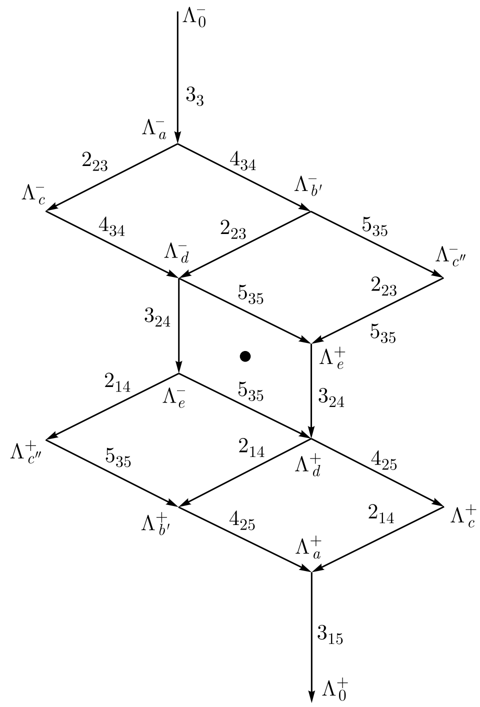

Fig. 4. $su(3,3)$ reduced multiplets of type R_1

conjugated to R_{13}^3 and R_{14}^3, respectively, as explained above. Thus, we present explicitly only four types of multiplets.

The reduced multiplets of type R_{13}^3 contain 10 ERs/GVMs, whose signatures can be given in the following pair-wise manner:

$$\begin{aligned}
\chi_0^\pm &= \left\{(0, m_2, m_4, m_5)^\pm;\ \pm\frac{1}{2}m_{2,45}\right\},\\
\chi_b^\pm &= \left\{(0, m_{2,4}, 0, m_{45})^\pm;\ \pm\frac{1}{2}m_{2,5}\right\},\\
\chi_{b'}^\pm &= \left\{(m_2, 0, m_{2,4}, m_5)^\pm;\ \pm\frac{1}{2}m_{45}\right\},\\
\chi_c^\pm &= \left\{(0, m_{2,45}, 0, m_4)^\pm;\ \pm\frac{1}{2}(m_2 - m_5)\right\},\\
\chi_d^\pm &= \left\{(m_2, m_4, m_2, m_{45})^\pm;\ \pm\frac{1}{2}m_5\right\}.
\end{aligned} \tag{19}$$

The multiplets are given in Fig. 5.

Note, that the differential operator from χ_d^- to χ_d^+ is a reduction of an integral Knapp–Stein operator.

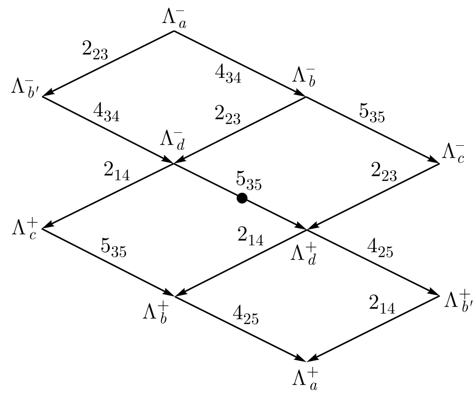

Fig. 5. $su(3,3)$ reduced multiplets of type R^3_{13}

The reduced multiplets of type R^3_{14} contain 10 ERs/GVMs, whose signatures can be given in the following pair-wise manner:

$$\begin{aligned}
\chi_0^\pm &= \left\{(0, m_2, 0, m_5)^\pm;\ \pm\frac{1}{2}(m_{23,5} + m_3)\right\},\\
\chi_b^\pm &= \left\{(0, m_{23}, m_3, m_5)^\pm;\ \pm\frac{1}{2}m_{2,5}\right\},\\
\chi_c^\pm &= \left\{(0, m_{23,5}, m_3, 0)^\pm;\ \pm\frac{1}{2}(m_2 - m_5)\right\},\\
\chi_{c'}^\pm &= \left\{(m_2, m_3, m_{23}, m_5)^\pm;\ \pm\frac{1}{2}m_5\right\},\\
\chi_d^\pm &= \left\{(m_2, m_{3,5}, m_{23}, 0)^\pm;\ \mp\frac{1}{2}m_5\right\}.
\end{aligned} \tag{20}$$

The multiplets are given in Fig. 6.

The reduced multiplets of type R^3_{15} contain 10 ERs/GVMs, whose signatures can be given in the following pair-wise manner:

$$\begin{aligned}
\chi_0^\pm &= \left\{(0, m_2, m_4, 0)^\pm;\ \pm\frac{1}{2}(m_{24} + m_3)\right\},\\
\chi_a^\pm &= \left\{(0, m_{23}, m_{34}, 0)^\pm;\ \pm\frac{1}{2}m_{24}\right\},\\
\chi_{b'}^\pm &= \left\{(0, m_{24}, m_3, m_4)^\pm;\ \pm\frac{1}{2}m_2\right\},\\
\chi_c^\pm &= \left\{(m_2, m_3, m_{24}, 0)^\pm;\ \pm\frac{1}{2}m_4\right\},\\
\chi_d^\pm &= \{(m_2, m_{34}, m_{23}, m_4)^\pm;\ 0\}.
\end{aligned} \tag{21}$$

The multiplets are given in Fig. 7.

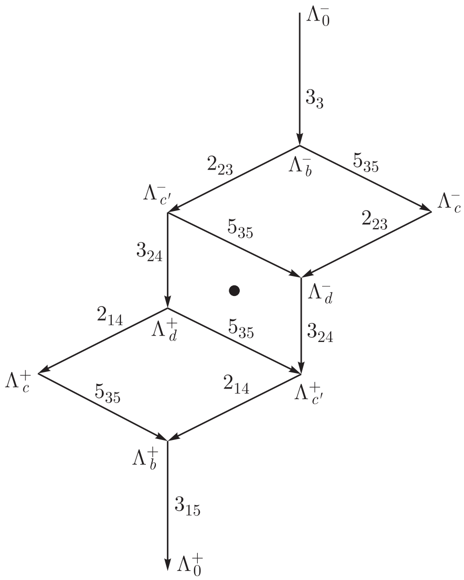

Fig. 6. $su(3,3)$ reduced multiplets of type R^3_{14}

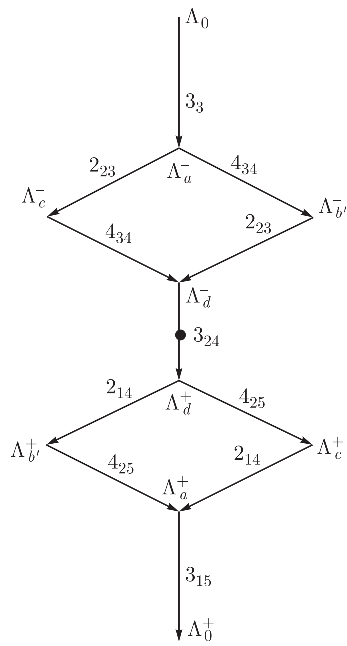

Fig. 7. $su(3,3)$ reduced multiplets of type R^3_{15}

We note a peculiarity in the last case, namely, the operator between $\chi_d^\pm$ is not a differential operator. It is a reduction of the Knapp–Stein operator, which does not change the conformal weight, but only conjugates the signature of $\mathcal{M}$.

The reduced multiplets of type R^3_{24} contain 10 ERs/GVMs, whose signatures can be given in the following pair-wise manner:

$$\begin{aligned}
\chi_0^\pm &= \left\{(m_1,0,0,m_5)^\pm ; \pm\frac{1}{2}(m_{1,3,5}+m_3)\right\}, \\
\chi_b^\pm &= \left\{(m_1,m_3,m_3,m_5)^\pm ; \pm\frac{1}{2}m_{1,5}\right\}, \\
\chi_c^\pm &= \left\{(0,m_3,m_{1,3},m_5)^\pm ; \pm\frac{1}{2}(m_5-m_1)\right\}, \\
\chi_{d'}^\pm &= \left\{(m_1,m_{3,5},m_3,0)^\pm ; \pm\frac{1}{2}(m_1-m_5)\right\}, \\
\chi_e^\pm &= \left\{(0,m_{3,5},m_{1,3})^\pm ; \mp\frac{1}{2}m_{15}\right\}.
\end{aligned} \tag{22}$$

The multiplets are given in Fig. 8.

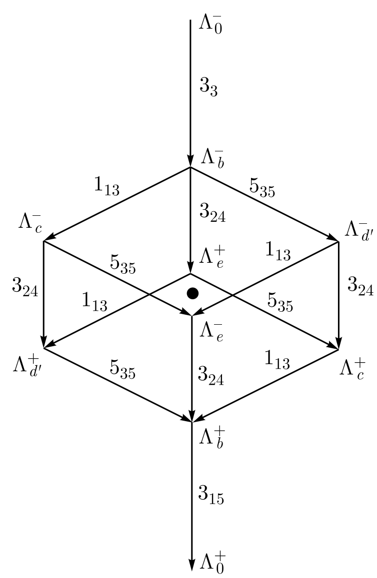

Fig. 8. $su(3,3)$ reduced multiplets of type R^3_{24}

3.4. Last Reduction of Multiplets. There are further reductions of the multiplets — triple and quadruple, but only one triple reduction contains representations of physical interest. Namely, this is the multiplet R^3_{135}, which may be obtained from the main multiplet by setting formally $m_1 = m_3 = m_5 = 0$. It contains 7 ERs/GVMs, whose signatures can be given in the following manner:

$$\begin{aligned}
\chi_a^{\pm} &= \left\{(0, m_2, m_4, 0)^{\pm};\ \pm\frac{1}{2}m_{2,4}\right\},\\
\chi_b^{\pm} &= \left\{(0, m_{2,4}, 0, m_4)^{\pm};\ \pm\frac{1}{2}m_2\right\},\\
\chi_{b'}^{\pm} &= \left\{(m_2, 0, m_{2,4}, 0)^{\pm};\ \pm\frac{1}{2}m_4\right\},\\
\chi_d &= \{(m_2, m_4, m_2, m_4);\ 0\}.
\end{aligned} \tag{23}$$

The multiplets are given in Fig. 9. The representation χ^d is a singlet, not in a pair, since it has zero weight c, and the $\mathcal{M}$ entries are self-conjugate under (7). It is placed in the middle of the figure as the bullet. That ER contains the *minimal irreps* of $su(3,3)$ characterized by two positive integers, which are denoted in this context as m_2, m_4. Each such an irrep is the kernel of the two invariant differential operators $\mathcal{D}^{m_2}_{14}$ and $\mathcal{D}^{m_4}_{25}$, which are of order m_2, m_4, respectively, and correspond to the noncompact roots α_{14}, α_{25}, respectively, cf. (5).

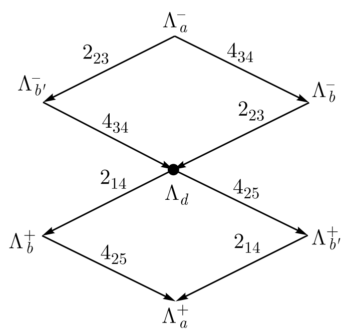

Fig. 9. $su(3,3)$ reduced multiplets of type R^3_{135}

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