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ФИЗИКА ЭЛЕМЕНТАРНЫХ ЧАСТИЦ И АТОМНОГО ЯДРА. ТЕОРИЯ

REPRESENTATIONS AND PARTICLES OF ORTHOSYMPLECTIC SUPERSYMMETRY GENERALIZATION [1]

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Orthosymplectic $osp(1|2n)$ supersymmetry (alternative names: Generalized conformal supersymmetry with tensorial central charges, conformal M-algebra, para-Bose algebra) has been considered as an alternative to d-dimensional conformal superalgebra. Due to mathematical difficulties, even classification of its unitary irreducible representations (UIRs) have not been entirely accomplished. We give this classification for $n = 4$ case (corresponding to four-dimensional space-time) and then show how the discrete subset of these UIRs can be constructed in a Clifford algebra variation of Green's ansatz.

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INTRODUCTION

Orthosymplectic type of space-time symmetry was first analyzed by C. Fronsdal [1], as early as 1985, and since then interest in this symmetry reappeared, sometimes independently, in many contexts: M-theory [2], BPS particles [3], higher spin fields [4] and others [5].

When considering a (super)group in the context of a space-time symmetry, one of the first and most natural steps to undertake is to find unitary irreducible representations (UIRs) of the group, as these give us basic information on the particle content of the free theory. And in the case of orthosymplectic supersymmetry, physically most important class of UIRs are so-called positive energy UIRs. The problem of finding these representations has been solved only for $n = 1$ and $n = 2$. We followed the approach of [6] and completed the task for $n = 4$ by using computer algorithms to analyze Verma module structure. In this way we managed to make a complete list of positive energy $osp(1|8)$ UIRs, together with explicit forms of the corresponding Verma module singular and subsingular vectors. In this short report we present the main features of the results, leaving the details to be published separately. In particular, we point out that there is a concrete number of discrete

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UIR families (precisely nine, or ten if the trivial representation is counted as a separate class) that physically should be related to elementary particles of $osp(1|8)$ models. In addition, we also point out a method to explicitly construct discrete representations, allowing one to easily perform concrete calculations in these spaces and, in that way, give physical interpretation to the states within. The method, directly generalizable to arbitrary n, is based on a Clifford algebra variation of Green's ansatz and is mathematically related to Howe duality.

1. POSITIVE ENERGY UIRs OF $osp(1|8)$

Structural relations of $osp(1|2n)$ superalgebra can be compactly written in the form of trilinear relations of odd algebra operators a_α and $a_\alpha^\dagger$:

$$[\{a_\alpha, a_\beta^\dagger\}, a_\gamma] = -2\delta_{\beta\gamma} a_\alpha, \quad [\{a_\alpha^\dagger, a_\beta\}, a_\gamma^\dagger] = 2\delta_{\beta\gamma} a_\alpha^\dagger, \tag{1}$$

$$[\{a_\alpha, a_\beta\}, a_\gamma], \quad [\{a_\alpha^\dagger, a_\beta^\dagger\}, a_\gamma^\dagger] = 0, \tag{2}$$

where operators $\{a_\alpha, a_\beta^\dagger\}$, $\{a_\alpha, a_\beta\}$ and $\{a_\alpha^\dagger, a_\beta^\dagger\}$ span the even part of the superalgebra and Greek indices take values $1, 2, \ldots, n$ (relations obtained from these by use of Jacobi identity are also implied). If we additionally require that the dagger symbol $\dagger$ above denotes Hermitian conjugation in the algebra representation Hilbert space (of positive definite metrics), then we have effectively constrained ourselves to the so-called positive energy UIRs of $osp(1|2n)$ [1]. Namely, in such a space, "conformal energy" operator $E \equiv (1/2)\sum_\alpha \{a_\alpha, a_\alpha^\dagger\}$ must be a positive operator. Operators a_α reduce the eigenvalue ϵ of E, so the Hilbert space must contain a subspace that these operators annihilate. This subspace is called vacuum subspace: $V_0 = \{|v\rangle, a_\alpha|v\rangle = 0\}$. From the algebra relations follows: $|v\rangle \in V_0 \Rightarrow \{a_\alpha, a_\beta^\dagger\}|v\rangle \in V_0$, with α, β arbitrary. Therefore, vacuum subspace carries a representation of $U(1) \times SU(N)$ group generated by operators $\{a_\alpha, a_\beta^\dagger\}$ (with $U(1)$ part generated by E). The positive energy UIRs of $osp(1|2n)$ are entirely labelled by UIR μ of $SU(N)$ (that can be given by a Young diagram) and a positive real number ϵ_0 (energy of the vacuum subspace) that labels $U(1)$ representation. However, for a given representation μ only certain values of ϵ_0 are allowed, and this connection is highly important and nontrivial.

In this paper we are interested in the $n = 4$ case. $SU(4)$ representation μ will be explicitly parameterized by three non-negative integers s_1, s_2, s_3 in a way that μ is determined by a Young diagram with $s_1 + s_2 + s_3$ boxes in the first row, $s_1 + s_2$ boxes in the second and s_1 boxes in the third row. In addition to these three numbers, we will use real parameter d given by $d = (1/4)(\epsilon_0 - s_1 + s_3)$ to label $osp(1|8)$ representations.

Classification of all positive energy UIRs was done by computer analysis of the Verma module structure, carried out in the following general manner (that we just briefly describe).

[1] Omitting a short proof, we note that in such a Hilbert space all superalgebra relations actually follow from one single relation — the first or the second of (1).

First, Kac determinant of a sufficiently high level was considered as a function of parameter d (for each given class of $SU(4)$ representation μ). In this way it was possible to locate the highest value of d for which the determinant vanishes and the Verma module becomes reducible. The singular or subsingular vector responsible for the singularity of the Kac matrix was then calculated effectively by solving an (optimized) system of linear equations. Next we would find the norm of this vector and look for possible additional discrete reduction points at (lower) values of d for which the norm also vanishes. If new reduction points with new (sub)singular vectors were found, it was also necessary to check that, upon removal of the corresponding submodules, no vectors with zero or negative norm remained. For this, it was enough to check that previously found (sub)singular vectors (i.e., those occurring for higher d values) belonged to the factored-out submodules. Optimized Wolfram Mathematica code was written to perform all these calculations. We now summarize the main results.

Parameter d can take the following values, depending on the labels s_1, s_2, s_3:

1. $s_1 = s_2 = s_3 = 0$: $d > 3/2$ and singular points $d = 0, 1/2, 1, 3/2$;
2. $s_1 = s_2 = 0, s_3 > 0$: $d > s_3/2+2$ and singular points $d = s_3/2+1, s_3/2+3/2, s_3/2+2$;
3. $s_1 = 0, s_2 > 0$: $d > (s_2 + s_3)/2 + 5/2$ and singular points $d = (s_2 + s_3)/2 + 2$, $(s_2 + s_3)/2 + 5/2$;
4. $s_1 > 0$: $d > (s_1 + s_2 + s_3)/2 + 3$ and a singular point $d = (s_1 + s_2 + s_3)/2 + 3$.

Case 1 corresponds to "unique vacuum" representations, i.e., when the vacuum subspace V_0 is one-dimensional and carries trivial representation of the $SU(4)$ group. Since spatial rotations are a subgroup of this $SU(4)$ group, in representations 1 the lowest conformal energy state is invariant to rotations. In this sense, representations 1 correspond to "fundamentally scalar" particles or, more precisely, multiplets (states of other spin values also belong to the multiplet of the full supersymmetry). Of particular physical interest are representations at singular points, and there are exactly three of such scalar representations. Namely, at singular points additional equations of motion appear directly related to the corresponding singular or subsingular vectors. In the case of the simplest nontrivial representation $d = 1/2, s_1 = s_2 = s_3 = 0$, singular vector yields the massless condition $p^2 = 0$ (this is the well known and studied UIR-containing tower of massless particles with increasing helicities). However, relating (sub)singular vectors to physical constraints (i.e., equations of motion) is in general complicated, and this problem is effectively solved by the explicit construction of representations that is discussed in the following section.

Case 2 corresponds to representations where V_0 subspace transforms w.r.t. $SU(4)$ subgroup as a Young diagram with s_3 boxes in a single row. The simplest representative of the kind is a single box representation — lowest energy state in these representations behaves as a Lorentz 1/2 spinor. There are again three singular points in case 2, corresponding to three physically interesting classes of representations, including three "fundamentally" spinor multiplets.

Cases 3 and 4 correspond to more complex classes of representations. However, it turns out that states from these classes can be naturally seen as composite states built from states belonging to representations of classes 2 or of classes 1.

Overall, there turns out to be 10 singular points corresponding to 9 different classes of nontrivial multiplets ($d = 0$ point corresponds to the trivial representation).

2. CONSTRUCTION OF REPRESENTATIONS

It turns out that all representations with (half-)integer d values (including all representations at (sub)singular points) can be obtained by representing the odd superalgebra operators a and $a^\dagger$ as the following sums:

$$a_\alpha = \sum_{a=1}^{p} b_\alpha^a \, e^a, \quad a_\alpha^\dagger = \sum_{a=1}^{p} b_\alpha^{a\dagger} \, e^a. \tag{3}$$

In these expressions, p is integer, e^a are elements of a real Clifford algebra:

$$\{e^a, e^b\} = 2\delta^{ab}, \quad a, b = 1, 2, \ldots, p, \tag{4}$$

and operators b_α^a together with adjoint $b_\alpha^{a\dagger}$ satisfy ordinary bosonic algebra relations: $[b_\alpha^a, b_\beta^{b\dagger}] = \delta_{\beta\alpha}\delta^{ab}, [b_\alpha^a, b_\beta^b] = 0$. The (reducible) representation space is spanned by the vectors:

$$\mathcal{H} = \text{l.s.}\,\{\mathcal{P}(b^\dagger)|0\rangle \otimes \omega\}, \tag{5}$$

where $\mathcal{P}(b^\dagger)$ are monomials in mutually commutative operators $b_\alpha^{a\dagger}$; $|0\rangle$ is a bosonic vacuum and $w \in \mathcal{H}_{\text{Cl}}$, where $\mathcal{H}_{\text{Cl}}$ is the representation space of real Clifford algebra (4).

Representation ansatz in the form (3) possesses certain intrinsic symmetries. Operators

$$G^{ab} \equiv \sum_{\alpha=1}^{n} i(b_\alpha^{a\dagger} b_\alpha^b - b_\alpha^{b\dagger} b_\alpha^a) + \frac{i}{4}[e^a, e^b] \tag{6}$$

commute with entire $osp(1|8)$ superalgebra. Operators G^{ab} themselves satisfy commutation relations of $so(p)$ algebra (the full symmetry is actually slightly larger, given by the orthogonal group). We will call this symmetry the gauge symmetry.

The gauge symmetry actually removes all degeneracy in decomposition of (5) to $osp(1|8)$ UIRs, i.e., the multiplicity of $osp(1|8)$ UIRs is fully taken into account by labeling transformation properties of the vector w.r.t. the gauge symmetry group. Furthermore, there is one-to-one correspondence between UIRs of $osp(1|8)$ and of the gauge group that appear in the decomposition, meaning that transformation properties under the gauge group action automatically fix the $osp(1|8)$ representation.

The vector $|v^0_{\{d,s_1,s_2,s_3\}}\rangle$ that is the lowest weight vector of $osp(1|8)$ positive energy UIR $\{d, s_1, s_2, s_3\}$ and the highest weight vector of the gauge group UIR (in a standardly defined root system) takes the following explicit form (up to multiplicative constant):

$$\begin{aligned} |v^0_{\{d,s_1,s_2,s_3\}}\rangle &= \left(B_{4+}^{(1)\dagger}\right)^{s_3} \times \\ &\times \left(B_{4+}^{(1)\dagger} B_{3+}^{(2)\dagger} - B_{4+}^{(2)\dagger} B_{3+}^{(1)\dagger}\right)^{s_2} \left(\sum_{k_1,k_2,k_3=1}^{3} \varepsilon_{k_1k_2k_3} B_{4+}^{(k_1)\dagger} B_{3+}^{(k_2)\dagger} B_{2+}^{(k_3)\dagger}\right)^{s_1} \times \\ &\times \left(\sum_{k_1,k_2,k_3,k_4=1}^{4} \varepsilon_{k_1k_2k_3k_4} B_{4+}^{(k_1)\dagger} B_{3+}^{(k_2)\dagger} B_{2+}^{(k_3)\dagger} B_{1+}^{(k_4)\dagger}\right)^{s_0} |0\rangle \otimes \omega_{\text{h.w.}}, \end{aligned} \tag{7}$$

with $d = s_0 + \dfrac{s_1 + s_2 + s_3 + p}{2}$, where $B^{(k)}_{\alpha\pm} = \dfrac{1}{\sqrt{2}}(b^{2k-1}_{\alpha} \mp i b^{2k}_{\alpha})$ and $(\mathrm{e}^{2k-1} + i\,\mathrm{e}^{2k})\omega_{\mathrm{h.w.}} = 0$, $k = 0, 1, \ldots, [p/2]$. The form above assumes that p is large enough that all $B^{(k)}_{\alpha\pm}$ can be defined, i.e., $p \geqslant 8$: s_0 must be 0 when $p < 8$, s_1 must be 0 when $p < 6$, s_2 must be 0 when $p < 4$, and all s_0, s_1, s_2, s_3 must be 0 when $p = 1$ ($p = 0$ is trivial UIR of $osp(1|8)$).

In this way all positive energy UIRs of $osp(1|8)$ classified in the previous section with integer or half-integer values of d can be constructed using ansatz (3) with $p \leqslant 9$. Physically, corresponding states have natural interpretation as particles composed from p of the simplest $osp(1|8)$ massless particles (belonging to $d = 1/2, s_1 = s_2 = s_3 = 0$ UIR).

REFERENCES

1. *Fronsdal C.* Massless Particles, Orthosymplectic Symmetry and Another Type of Kaluza–Klein Theory. Preprint UCLA/85/TEP/10 // Essays on Supersymmetry. Math. Phys. Studies. V. 8. Reidel, 1986.
2. *Lukierski J., Toppan F.* Generalized Spacetime Supersymmetries, Division Algebras and Octonionic M-Theory // Phys. Lett. B. 2002. V. 539. P. 266–276.
3. *Fedoruk S., Zima V. G.* Massive Superparticle with Tensorial Central Charges // Mod. Phys. Lett. A. 2000. V. 15. P. 2281–2296.
4. *Vasiliev M. A.* Conformal Higher Spin Symmetries of 4D Massless Supermultiplets and $osp(L, 2M)$ Invariant Equations in Generalized (Super)Space // Phys. Rev. D. 2002. V. 66. P. 066006.
5. *Salom I.* Single Particle Representation of Para-Bose Extension of Conformal Supersymmetry // Fortschr. Phys. 2008. V. 56. P. 505–509.
6. *Dobrev V. K., Zhang R. B.* Positive Energy Unitary Irreducible Representations of the Superalgebras $osp(1/2n, R)$ // Phys. Atom. Nucl. 2005. V. 68. P. 1660–1669.