ФИЗИКА ЭЛЕМЕНТАРНЫХ ЧАСТИЦ И АТОМНОГО ЯДРА. ТЕОРИЯ

A MATHEMATICAL STRUCTURE FOR NUCLEI

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A method is proposed to obtain a translationally invariant solution to the Schrödinger equation including two-body correlation. The conditions to obtain translationally invariant excited state are discussed. It is explained why ground states are surrounded by coupled satellite states cooperating to the binding energy.

В работе предлагается метод получения трансляционно-инвариантного решения уравнения Шредингера, включая корреляцию двух частиц. Обсуждаются условия получения трансляционно-инвариантного возбужденного состояния. Объясняется, почему основные состояния окружены связанными состояниями-спутниками, дающими вклад в энергию связи.

This paper is the result of a long procedure leading to the proposal of a mathematical structure translationally invariant for nuclei.

We assume that the wave functions of nuclei are eigenstates of a Schrödinger equation for bound systems of identical fermions. The starting point of a mathematically exact method originates from papers published in "Yadernaya Fizika" on the K-harmonic during the 1960s, see references in [1]. The method was plagued by the large degeneracy of the K-harmonics, i.e., by the large number of polynomial of the same degree, a number increasing exponentially with the number of particles, i.e., with the dimension of the space spanned by the particle coordinates of the interacting system leading to an intractable number of coupled differential equations needed to obtain a converged solution of the Schrödinger equation.

This problem was solved in 1969 by the introduction of the potential harmonics reducing to only one polynomial for each degree, the set of polynomials needed for a complete expansion of a two-body function where pairs are in s-state [2], for instance, a pairwise central potential.

When Mayer and Jensen discovered the similarity between the quantum numbers of nuclei in ground state and those of eigenstates of the Harmonic Oscillator Model (HOM) in ground state, two interpretations became available.

Writing the Schrödinger equation as

$$\sum_1^A (-\Delta_i + x_i^2 - E_i)\Psi_i(\mathbf{x}_i) = 0, \quad \Psi_i(\mathbf{x}_i) = \prod_1^A \Psi(\mathbf{x}_i), \quad E = \sum_1^A E_i \qquad (1)$$

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for a system of A identical particles of coordinates $(\mathbf{x}_1, \ldots, \mathbf{x}_A)$ leads to an Independent Particle Model (IPM) where each particle moves in a common central potential well $V(x_i)$, $x_i = |\mathbf{x}_i|$ and Δ_i is the Laplace operator of the ith particle and E_i the eigen value for an eigenstate $\Psi_i(\mathbf{x}_i) \xrightarrow[\mathbf{x}_i \to \infty]{} 0$.

But when the Schrödinger equation is written as

$$(-\Delta + r^2 - E)\Psi(\mathbf{x}) = 0, \tag{2}$$

where $\Delta = \sum_1^A \Delta_i$ is the Laplace operator in the $D = 3A$ dimensional space spanned by the particles coordinates, for $r^2 = \sum_1^A x_i^2$ and $\mathbf{x} = (\mathbf{x}_1, \ldots, \mathbf{x}_A)$, the wave function $\Psi(\mathbf{x}) \xrightarrow[|\mathbf{x}| \to \infty]{} 0$ is invariant by rotation in the whole space spanned by the particles leading to an eigenstate

$$\Psi(\mathbf{x}) = Y_{[L]}(\Omega)\Phi(r), \quad r = |\mathbf{x}|. \tag{3}$$

It is the product of a Hyperspherical Harmonic (HH) $Y_{[L]}(\Omega)$, where $[L]$ is a set of $3A - 1$ quantum numbers including the degree L of the associated Harmonic Polynomial (HP) $r^L Y_{[L]}(\Omega)$ in polar coordinates $\mathbf{x}(r, \Omega)$ in the D-dimensional space.

This time the discovery of Mayer and Jensen is interpreted as a collective model, the "Hyperspherical Model" (HM).

By writing that $r^L Y_{[L]}(\Omega)$ is an HP, i.e., $\Delta r^L Y_{[L]}(\Omega) = 0$, one finds the eigenvalue $(\Delta + L(L + D - 2))Y_{[L]}(\Omega) = 0$ and with $\Phi(r) = u(r)/r^{(D-1)/2}$, the radial equation for a Hypercentral Potential $V(r)$ becomes

$$\left\{ -\frac{d^2}{dr^2} + \frac{\mathcal{L}(\mathcal{L}+1)}{r^2} + V(r) - E \right\} u(r) = 0, \tag{4}$$

where $\mathcal{L} = L + (D - 3)/2$ in a center-of-mass coordinate system $(X, \boldsymbol{\xi}_i)$ and $\boldsymbol{\xi}_i$ are Jacobi coordinates with $r^2 = \sum_1^{A-1} \xi_i^2$.

It results from Eq. (4) that ground states are obtained when the degree L of the HP describing the state is L_m the lowest available.

For bosons where all polynomials can be in $1s$ state $L_m = 0$ and the polynomial is a constant $Y_{[0]}$. In this case the set of HH in the D-dimensional space can be written as a product of Jacobi polynomials [2].

They are polynomials associated with the angular coordinates Ω which fulfill the equation $\int P_{[K]}(\Omega)P_{[K']}(\Omega)\, d\Omega = \delta_{[K],[K']}$, where $[K]$ and $[K']$ are the $D - 1$ quantum numbers defining the HP of degree K and K', respectively.

For fermions the HP must be antisymmetric by exchange of the particles. The antisymmetry property is provided by a Slater determinant. Since an HP is a homogeneous polynomial, the building blocks are also homogeneous polynomials in the space where fermions move.

One requires that the state must be translationally invariant. From this needed property the IPM must be rejected.

If the ground state is described by a homogeneous polynomial Slater determinant of lowest degree, any symmetrical scalar operator seeking to lower the degree of such a polynomial gives zero when it is applied to the determinant.

Thus, the determinant is a translationally invariant harmonic polynomial.

Let us assume that the fermions are moving in a one-dimensional space. The position of the ith particle is given by one coordinate x_i and the Slater determinant becomes a Vandermond determinant. $D(\mathbf{x}) = \|1xx^2 \ldots x^n\| = \underbrace{\Pi}_{i>j}(x_i - x_j)$ for a fermions system.

It is obviously translationally invariant.

If one raises the degree of the determinant, by, for instance, raising n to $n + 1$, it loses the translational invariance.

Therefore, there is only one determinant for defining a ground state for fermions in a one-dimensional space.

But by introducing the spin of the fermions into the determinant, it becomes $D(x, \mathrm{spin}) = \|\alpha \, \alpha x_1 \ldots \alpha x_1^n \, \beta\beta x_1 \ldots \beta x_1^n\|$ for a system of $2(n + 1)$ fermions where the spins are filled with $\alpha = $ spin up and $\beta = $ spin down.

Let us call D_α the determinant where n is raised by one for the α spin and D_β the same operation with the spin β. Then $D_\alpha - D_\beta$ becomes translationally invariant and describes an excited state.

The introduction of spin enables one to generate new states.

The degree of the Vandermond determinant is $n(n+1)/2$ in an $(n+1)$-dimensional space.

Taking the two spins into account, $\mathcal{L} = n(n + 2) - 1/2$ in (4) is half integer.

Let us turn now to fermions systems in physical d-dimensional space. While there is only one homogeneous polynomial (x_1^K) for each degree K in a one-dimensional space, there are many independent homogeneous polynomials for $d > 1$. The number of independent polynomials of degree K is equal to the number of terms in the expansion of $(x_1 + x_2 + \ldots + x_d)^K$ in a d-dimensional space.

The Slater determinant is constructed like the Vandermond determinant except that the term x_1^K becomes $P_{[K]}(\mathbf{x}_1)$ the set of homogeneous polynomials of degree K. Then one fills the determinant from left to right by using successively all polynomials of degree K for growing K. The building blocks can be written as the product $|\mathbf{x}_d|^{2n_j} P_{[\ell_j]}(\mathbf{x}_d)$, $|\mathbf{x}_d|^2 = \sum_{i=1}^{d} x_i^2$, $2n_j + \ell_j = \ell_m$, where $P_{[\ell_j]}(\mathbf{x}_d)$, $\mathbf{x}_d = (x_1, x_2, \ldots, x_d)$ is a harmonic polynomial of degree ℓ_j in the d-space and $[\ell_j]$ the set of $d - 1$ quantum numbers defining the HP. The ℓ_j extend from 0 or 1 modulo 2 according to the K parity up to $\ell_m = K$ the degree of the polynomial. The set of homogeneous polynomials of degree ℓ_m are the elements of the ℓ_m-shell.

The concept of Shells is therefore associated with the degree of the homogeneous polynomials used in the construction of the antisymmetric harmonic polynomial defining a state.

In the physical $d = 3$-dimensional space the number of independent polynomials of degree ℓ_m is $(\ell_m + 1)(\ell_m + 2)/2$, i.e., 1, 3, 6, 10 for $\ell_m = 0, 1, 2, 3$, respectively. By introducing the spin, the number of terms in a closed shell are 2, 6, 12, 20 for $\ell_m = 0, 1, 2, 3$. The closed shell self-conjugate $N = Z$ nuclei appear for $A = 4, 16, 40$ and 80, respectively.

The building blocks in the Slater determinant are the homogeneous polynomials of degree $(2n_j + \ell_j)$ which can be written as the product of $x_i^{2n_j + \ell_j} Y_{\ell_j}^{m_j}(\omega_i)$ for the ith particle and jth column in polar coordinates $\mathbf{x}_i(x_i, \omega_i)$, in terms of the spherical harmonics $Y_{\ell_j}^{m_j}(\omega_i)$.

On the average, the binding energy of nuclei increases linearly with the number of nucleons. But after the closure of a Shell the increase of binding energy of the next nucleons undergo a drop.

It is customarily believed, in the IPM, that these shell effects proceed from the potential well.

In the Hyperspherical Model it is generated by a sudden increase in the strength of the kinetic energy central barrier.

Indeed, when a new nucleon is introduced in the last open ℓ_m-shell the $\mathcal{L}$ in (4) increases by $\ell_m + 3/2$ all over the shell, but when the ℓ_m-shell is closed, the next nucleon appears in the next $(\ell_m + 1)$-shell and $\mathcal{L}$ increases by $\ell_m + 5/2$, i.e., generating a supplementary increase of the strength of the kinetic energy central barrier and a smaller increase of binding energy.

In order to generate excited states, the nucleons might either play with the free states in the last shell or to be raised beyond the last shell.

Let ℓ_m be the degree of the building blocks in the last shell. If one raises a nucleon to the $(\ell_m + 2)$-shell, it cannot generate a translationally invariant Slater determinant and then cannot describe a state.

Indeed, if $\mathbf{x}_i - \mathbf{X}$, where $\mathbf{X}$ is the center-of-mass coordinate, is substituted for $\mathbf{x}_i$ in $x_i^{2n_j+\ell_j} Y_{\ell_j}^{m_j}(\omega_i)$, it can be expanded in terms of the harmonic polynomials $X^L Y_L^M(\omega_X)$ $\sum_{L,M} X^L Y_L^M(\omega_X) P_{2n_j+\ell_j-L}^M(\mathbf{x}_i)$, where the homogeneous polynomial of degree $2n_j + \ell_j - L$ can be expanded in terms of the building blocks of degree $2n_j + \ell_j - L$.

If at least one of these building blocks is not occupied in the determinant, it is not translationally invariant.

Since the ℓ_m-shell is the last shell, the $(\ell_m + 1)$-shell is free and any excitation to the higher shell generates a not translationally invariant determinant and cannot describe a state.

Let us turn now to the $(\ell_m + 1)$-shell.

If one raises a nucleon from the ℓ_m-shell to the $(\ell_m + 1)$-shell in a state $x_i^\ell Y_\ell^m(\omega_i)$, where ℓ is larger than all filled $(\ell - 1)$-states, then according to the expansion formula it can be an excited state.

For instance, if in the $\ell_m = 3$-shell all the $1f$-states are filled, a $1g$-state can be introduced before the complete filling of the 3-shell states. It could even be a ground state if it gives more binding energy than a state constructed only with building blocks of the ℓ_m-shell.

Let us introduce the spin and isospin $\zeta = 3/2, 1/2, -1/2, -3/2$ for proton spin up and down and neutron spin up and down, respectively. The Slater determinant is filled independently for each ζ with growing degree of homogeneous polynomials. Let $D_{[1,\zeta]}$ be the determinant describing an alpha particle where the nucleon with spin–isospin state ζ is in p-state. It is not translationally invariant but the mixture $\sum_\zeta a_\zeta D_{[1,\zeta]}$, where $\sum_\zeta a_\zeta = 0$, is translationally invariant and describes an excited p-state.

Let $P_{ij}^{(\varepsilon)}$, $\varepsilon = \sigma, \tau, \sigma\tau$, be the operators for the exchange of spin, isospin, spin–isospin between the columns i and j of the determinant, respectively. When P_{ij}^σ or P_{ij}^τ operates between states where $\zeta_i + \zeta_j = 0$, it exchanges the couple of states $(3/2, -3/2)$ and $(1/2, -1/2)$. It creates a structure where two pairs of spin–isospin states are identical, then canceling the determinant.

When the scalar operator $\mathbf{x}_i \cdot \mathbf{x}_j P_{ij}^{(\varepsilon)}(\varepsilon = \sigma, \tau)$ is applied to an alpha particle, it creates an excited state where two nucleons are in p-state.

This state is coupled to the α particle in ground state by the spin and isospin exchange operators in the potential, the same mechanism holds for nuclei where the ground state is described by a determinant of degree minimum.

It creates coupled satellite states where two nucleons are excited to the next shell.

The same situation occurs when $(\mathbf{x}_i \otimes \mathbf{x}_j)$ is coupled to an $\ell = 2$ orbital state. Then the coupling is generated either by the deformation of the nucleus or by the tensor force.

Then by describing a closed shell nucleus by a single Slater determinant and computing the correlations with a Jastrov function, one neglects the contribution of all satellite states coupled by the exchange operators in the potential to the determinant of degree minimum defining the state.

One can wonder whether the existence of satellite states created by operators in s- or d-state is not related to the interacting boson model.

Our purpose is to give a method for solving the Schrödinger equation

$$\{\Delta + V(\mathbf{x}) - E\}\,\Psi(\mathbf{x}) = 0, \quad \mathbf{x} = (\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_A) \tag{5}$$

for a system of A identical interacting fermions in bound state. One assumes that the interaction $V(\mathbf{r}_{ij}, \zeta)$ is a pairwise potential. It is calculated theoretically by mesons exchanges and practically by fitting the $N - N$ phase shift with a suitable one- and two-bosons exchange shape and then it is spin–isospin (ζ) dependent. The kinetic energy operator and the wave function are defined in polar coordinates (r, Ω) in the $D = 3A - 3$ dimensional space in the center-of-mass frame where the radial coordinate r is defined by
$r^2 = 2\sum_{1}^{A}(\mathbf{x}_i - \mathbf{x})^2 = 2/A \sum_{i,j>i}(\mathbf{x}_i - \mathbf{x}_j)^2.$

The state is defined either by a single (for closed shell) or by a sum of determinants $D_{[L_m]}(\mathbf{x})$ homogeneous in the particles coordinates $\mathbf{x}_i$ and of degree minimum L_m. Since the state is translationally invariant, it can be expressed in Jacobi coordinates. The product

$$V(\mathbf{x})\,D_{L_m}(\mathbf{x}) = \sum_{i,j>i,\varepsilon} V(\mathbf{r}_{ij},\varepsilon)P_{ij}^{(\varepsilon)}D_{L_m}(\mathbf{x}), \quad \mathbf{r}_{ij} = \mathbf{x}_i - \mathbf{x}_j, \tag{6}$$

where ε is for all two-body operators including exchange operators in the potential, generates a wave function

$$\Psi(\mathbf{x}) = D_{[L_m]}(\Omega)\sum_{i,j>i} F(\mathbf{r}_{ij}, r), \tag{7}$$

leading to a Schrödinger equation

$$\left\{ -\frac{d^2}{dr^2} + \frac{\mathcal{L}(\mathcal{L}+1)}{r^2} - \frac{L^2(\Omega)}{r^2} + \sum_{i,j>i,\varepsilon} V(\mathbf{r}_{ij},\varepsilon)P_{ij}^{(\varepsilon)} - E \right\} D_{[L_m]}(\Omega)\sum_{k,\ell>k} F(\mathbf{r}_{k\ell}, r) = 0, \tag{8}$$

where $L^2(\Omega)$ is a grand orbital operator [2], generalization of the $\ell^2(\omega)$ orbital operator in the 3-dimensional space, $\mathcal{L} = L_m + (D-3)/2$ and $F(\mathbf{r}_{ij}, r)$ a two-body amplitude for the pair i, j.

Since $D_{[L_m]}(\mathbf{x})$ can be written in Jacobi coordinates $\boldsymbol{\xi}$, one has to choose a reference two-body coordinate, for instance, $\boldsymbol{\xi}_2 = \mathbf{x}_2 - \mathbf{x}_1$ corresponding for easiness to the two first rows of $D_{[L_m]}$.

Any potential can be separated into two pieces:

1) A part invariant under rotation in the whole D-dimensional space, the Hypercentral part of the potential, function of the radial coordinate r only.

2) A residual interaction, deformed in the D-space generating the correlations.

A harmonic oscillator is purely Hypercentral.

The effective Hypercentral potential given by a sum over all ε of

$$V_{[L_m]}^{(\varepsilon)}(r) = \langle D_{[L_m]}(\Omega) | V(\mathbf{x}) | D_{[L_m]}(\Omega) \rangle =$$
$$= \frac{A(A-1)}{2} \left\langle D_{[L_m]}(\Omega) \left| V(\mathbf{r}_{ij}, \varepsilon) P_{ij}^{(\varepsilon)} \right| D_{[L_m]}(\Omega) \right\rangle \quad (9)$$

is calculated in the same way for any pair i, j. It is computed by choosing this pair as reference pair.

For each spin–isospin operator $P_{ij}^{(\varepsilon)}$ with $P_{ij}^0 = 1$, one calculates a pseudo-weight function

$$W_{[L_m]}^{(\varepsilon)}(z, \omega) = \int D_{[L_m]}^*(\Omega) P_{ij}^{(\varepsilon)} D_{[L_m]}(\Omega) \, d\Omega_1 \quad \text{with} \quad \int \left| D_{[L_m]}(\Omega) \right|^2 d\Omega = 1. \quad (10)$$

One chooses $i, j = 1, 2$ where $P_{1,2}^{(\varepsilon)}(\Omega)$ operates on the two first rows of $D_{[L_m]}(\mathbf{x})$, while Ω_1 are for all angular coordinates but $z = 2r_{12}^2/r^2 - 1$ and ω which are for the reference pair 1, 2. The calculation of the weight functions is made by substitution of the normalized 3-dimensional harmonic oscillator eigenfunctions (n, ℓ, m) for $r_i^{2n+\ell} Y_\ell^m(\omega_i)$ in $D_{[L_m]}(\mathbf{x})/A!$ The angular coordinate $z = \cos 2\Phi$ for $r_{12}/r = \cos \Phi$ has been chosen because for bosons $W_{[0]}(z) = (1-z)^{(D-5)/2}\sqrt{1+z}$ is associated with the Jacobi polynomials.

The weight function is obtained by the calculation of the Fourier transform over all pairs of columns $\left\langle D_{[L_m]}(\Omega) \left| e^{i\mathbf{k}(\mathbf{x}_2-\mathbf{x}_1)} P_{1,2}^{(\varepsilon)} \right| D_{[L_m]}(\Omega) \right\rangle = F^{(\varepsilon)}(\mathbf{k})$ followed by an inverse Fourier transform providing $W^{(\varepsilon)}(z, \omega)$ [1, Subsec. 4.4].

The effective Hypercentral potential for the pair 1, 2 becomes a sum over all ε of

$$\left\langle D_{[L_m]}(\Omega) \left| V(\mathbf{r}_{12}, \varepsilon) P_{1,2}^{(\varepsilon)} \right| D_{[L_m]}(\Omega) \right\rangle = \int W_{[L_m]}^{(\varepsilon)}(z, \omega) V(\mathbf{r}_{12}, \varepsilon) \, dz \, d\omega,$$
$$|\mathbf{r}_{12}| = r\sqrt{\frac{1+z}{2}}.$$

The residual potentials $\tilde{V}^{(\varepsilon)}(r_{1,2}, r) = V(\mathbf{r}_{1,2}, \varepsilon) P_{1,2}^{(\varepsilon)} - V_{[L_m]}^{(\varepsilon)}(r)$ generate the correlations.

In order to calculate the contribution of the two-body correlations to the binding energy and the wave function, one must first choose the reference pair among all Jacobi coordinates in $D_{[L_m]}(\mathbf{x})$ and then extract from Eq. (8) an equation for one pair only.

This equation is obtained by multiplying at left (8) by $D_{[L_m]}^*(\Omega)$ and by integrating over all coordinates but those of the reference pair, i.e., here z and ω.

It results from the structure of the kinetic energy operator, i.e., Eqs. (A.8) to (A.16) in [3], that the integration procedure cancels the contribution of the kinetic energy of all pairs, but the one of the reference pair.

It also appears from numerous numerical applications that the radial and orbital motions described by the amplitude $F(\mathbf{r}_{12}, r)$ can be written as the product $F(\mathbf{r}_{12}, r) = u(r) P(z, \omega, r)$,

where $u(r)$ is an eigenfunction of the radial Eq. (11), where $\hbar^2/m$ is taken into account:

$$\left\{ \frac{\hbar^2}{m}\left(-\frac{d^2}{dr^2} + \frac{\mathcal{L}(\mathcal{L}+1)}{r^2} \right) + \frac{A(A-1)}{2}V_{[L_m]}(r) + U(r) - E \right\} u(r) = 0, \qquad (11)$$

while the two-body amplitude where r is a parameter is a solution of

$$\left[\frac{4}{r^2}\frac{\hbar^2}{m}\left\{ \frac{\partial}{\partial z}(1-z^2)W_{[L_m]}(z,\omega)\frac{\partial}{\partial z} + W_{[L_m]}(z,\omega)\frac{\ell^2(\omega)}{2(1+z)} \right\} + W_{[L_m]}(z,\omega)U(r) \right] P(z,\omega,r) =$$

$$= \sum_\varepsilon W_{[L_m]}^{(\varepsilon)}(z,\omega)\tilde{V}^{(\varepsilon)}(\mathbf{r}_{12},r)\left(P(z,\omega,r) + \mathcal{P}^0 \sum_{k,\ell\neq 1,2} P(z_{k\ell},\omega_{k\ell},r) \right), \quad (12)$$

$$(\ell^2(\omega) + \ell(\ell+1))Y_m^\ell(\omega) = 0$$

for $-1 \leqslant z \leqslant 1$ vanishing at $z = 1$, where $\tilde{V}^{(\varepsilon)}$ is the residual potential and $\mathcal{P}^0$ a projection operator which projects functions in the space of $r_{k\ell}$ onto the space of the reference pair 1, 2.

The $z_{k\ell}$ and $\omega_{k\ell}$ refer to the variable z and ω for the pair $k, \ell \neq 1, 2$.

Another contribution can be introduced in the r.h.s. of (12), where the projection of the residual potential $\mathcal{P}^0 \sum_{k,\ell\neq 1,2} \tilde{V}(r_{k\ell},r)$ projected onto the pair (1, 2) is multiplied by $P(z,\omega,r)$, leading to an equation symmetrical in the exchange of the amplitude and the potential like the Schrödinger equation.

It is interesting to notice that by integrating Eq. (12), $\int (12)\,dz\,d\omega$ for $-1 \leqslant z \leqslant 1$ the kinetic energy term disappears and $U(r)$ is given by the integral of the last term. It enables one to check the quality of the amplitude obtained by solving Eq. (12), indeed the eigen energy $U(r)$ given by the integral where $P(z,\omega,r)$ is solution of (12) must be identical to the one given by solving Eq. (12).

When the nucleus is spherical like for the closed shell or closed subshell nuclei the weight function $W_{[L_m]}(z)$ is also spherical, and when the amplitude is in s-state one obtains the equation of the traditional Integro Differential Equation Approach (IDEA):

$$\left[\frac{4\hbar^2}{mr^2}\left\{ \frac{1}{W_{[L_m]}(z)}\frac{\partial}{\partial z}(1-z^2)W_{[L_m]}(z)\frac{\partial}{\partial z} \right\} + U(r) \right] P(z,r) =$$

$$= \tilde{V}\left(r\sqrt{\frac{1+z}{2}} \right)\left[P(z,r) + \mathcal{P}^0 P(z',r) \right], \quad (13)$$

where the projection operator $\mathcal{P}^0$ applied to the amplitudes for all pairs $k, \ell \neq 1, 2$ is an integral:

$$\mathcal{P}^0 P(z',r) = \mathcal{P}^0 \sum_{k,\ell\neq 1,2} P(z_{k\ell},r) = \int_{-1}^{1} f(z,z')P(z',r)\,dz, \qquad (14)$$

where the kernel called the projection function $f(z, z')$ is given by

$$
f(z, z') = W_{[L_m]}(z') \sum_{k=0}^{\infty} (f_k^2 - 1) P_k^{[L_m]}(z) P_k^{[L_m]}(z'),
$$

$$
f_k^2 - 1 = (A - 2) \frac{2 P_k^{[L_m]}(-1/2) + (A - 3)/2 P_k^{[L_m]}(-1)}{P_k^{[L_m]}(1)},
$$

(15)

where $P_k^{[L_m]}(z)$ are the normalized polynomials associated with $W_{[L_m]}(z)$ [4].

For three nucleons in s-state the sum of the series can be performed generating the Faddeev equation for s-state projected potentials when $V_{[0]}(r) = 0$ [5]. To this respect this Faddeev equation becomes a by-product of the IDEA when the Hypercentral potential is set equal to zero.

Asymptotic and adiabatic properties of the IDEA are studied in [6].

The solution given by the IDEA (13) is similar to the one obtained with a Hyperspherical harmonic expansion method but avoids the need to calculate a potential matrix. Equation (13) can be computed with a simple algorithm used for solving a radial differential equation [7]. The convergence in terms of the number of polynomials in (15) is reached already with no more than about five polynomials up to ^{40}Ca.

Out of closed shell or subshell the nuclei are not spherical. The structure of the weight function, i.e., the two-body density in (12), is

$$
W_{[L_m]}(z, \omega) = N \sum_{n, \ell' \text{even}} \langle [L_m] \,|\, n, \ell' \rangle Y_{\ell'}^0(\omega)(1 + z)^{n+(\ell'+1/2)}(1 - z)^{L_m+(D-5/2)-n-(\ell'/2)},
$$

(16)

where $\langle [L_m] \,|\, n, \ell \rangle$ are coefficients and N is a normalization constant fixed by

$$
\int_{-1}^{1} W_{[L_m]}(z, \omega) \, dz \, d\omega = 1.
$$

The amplitude $P(z, \omega, r)$ can also be expanded in a similar way:

$$
P(z, \omega, r) = \sum_{\ell \text{even}} (1 + z)^{\ell/2} Y_{\ell}^0(\omega) P^{\ell}(z, r).
$$

The product gives

$$
W_{[L_m]} P(z, \omega, r) = \sum_{n, \ell', \ell, \lambda} \langle [L_m] \,|\, n, \ell' \rangle \langle \ell', \ell; 0, 0 \,|\, \lambda, 0 \rangle Y_{\lambda}^0(\omega) \times
$$

$$
\times (1 + z)^{n+(\ell'+\ell+1)/2}(1 - z)^{L_m+(D-5)/2-n-\ell'/2} P^{\ell},
$$

where $\langle \ell', \ell, 0, 0 \,|\, \lambda, 0 \rangle$ is a Clebsh–Gordon coefficient.

Assuming a rotational invariance $\lambda = 0$ selects $\ell' = \ell$:

$$
(W_{[L_m]} P(z, \omega, r))_0 = \sum_{n, \ell \text{even}} \langle [L_m] \,|\, n, \ell \rangle (1 + z)^{n+\ell+1/2}(1 - z)^{L_m+(D-5)/2-n-\ell/2} P^{\ell}, \quad (17)
$$

where P^ℓ are coefficients which specify the deformation. They must be chosen in order to obtain the largest binding energy.

Then a set of orthogonal polynomials $P_K(z)$ associated with (17) can be generated to define an IDEA equation.

For more details and applications of the IDEA, refer to [3, 7].

REFERENCES

1. *Fabre de la Ripelle M., Navarro J.* // Ann. Phys. 1979. V. 123. P. 185.

2. *Fabre de la Ripelle M.* // Proc. of the Intern. School on Nucl. Theor. Phys. Predeal. Inst. Atom. Phys. Bucarest, 1969; http://m-fabredelaripelle.com/?q=content/hyperspherical-formalism-applied-three-and-four-body-problem; Contribution to the Second Problem Symp. in Nucl. Phys. Novosibirsk, 1970; Ann. Phys. 1983. V. 147. P. 281.

3. *Fabre de la Ripelle M., Sofianos S. A., Adam R. M.* // Ann. Phys. 2005. V. 316. P. 107.

4. *Fabre de la Ripelle M.* // Acad. Sci. Paris. 1984. V. 299, Ser. II, No. 13. P. 839.

5. *Fabre de la Ripelle M.* // Few-Body Syst. 1986. V. 181. P. 1.

6. *Fabre de la Ripelle M.* // Ann. Phys. (N. Y.). 2013. V. 331. P. 146–159.

7. *Fabre de la Ripelle M.* // Nucl. Phys. A. 2010. V. 839. P. 11.