УДК 539.123

AN INTEGRAL EQUATION FOR THE SPINOR AMPLITUDE OF A MASSIVE NEUTRAL DIRAC PARTICLE IN A CURVED SPACE-TIME WITH ARBITRARY GEOMETRY

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A set of integral equations for the spinor amplitude of the wave packet, which describes a massive neutral Dirac particle in a curved space-time with arbitrary geometry, are obtained. The equations enable one to describe, adequately to the quantum nature of spin, spin dynamics of fermions in gravitational fields. The question of the oscillations of massive neutrinos in external gravitational fields is discussed.

Получена система интегральных уравнений для спинорной амплитуды волнового пакета, описывающей массивную нейтральную дираковскую частицу в искривленном пространстве-времени произвольной геометрии. Эти уравнения позволяют адекватно квантовой природе спина описать спиновую динамику фермионов в гравитационных полях. Обсуждается вопрос об осцилляции массивных нейтрино во внешнем гравитационном поле.

In connection with the existence of compact astrophysical objects with strong gravitational fields near them, it is interesting to study a massive neutral Dirac particle in a curved space-time with arbitrary geometry. First of all, it is the studying of neutrino processes which play an important role in evolution of the neutron stars, black holes, etc.

Observing deficit of solar neutrinos gives rise to the question about revision of our knowledge of solar thermonuclear and neutrino processes.

The accelerators and astrophysical reactions generate only left neutrino ν_L and right antineutrino $\bar{\nu}_R$ with spin along and opposite to the direction of the particle moment, respectively. They interact with matter and so can be observed. On the contrary, ν_R and $\bar{\nu}_L$ do not practically interact with matter which is transparent for them. If the neutrino has no mass, its spirality does not change in external gravitational fields. However, if the neutrino has nonzero mass, $m_\nu \neq 0$ (maybe, very small), its oscillations $\nu_L \leftrightarrow \nu_R$, $\bar{\nu}_R \leftrightarrow \bar{\nu}_L$ are possible in gravitational fields. The urgency of this theme has strongly increased after a number of communications of the SuperK and KK experimental groups [1, 2] about nonzero rest mass $2 \cdot 10^{-2} \ll m_\nu \ll 7 \cdot 10^{-2}$ eV of neutrino.

Spin dynamics of a massive neutral Dirac particle in a curved space-time are considered in [3–6].

A direct continuation of the author's works [3–5], the present paper formulates a radically new mathematical approach to the study of the spinor amplitude of a Dirac particle in a curved space-time, based on the solving a set of integral equations with the Volterra kernels.

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The motion of a massive neutral Dirac particle of spin 1/2 in a curved space-time obeys the generally covariant Dirac equation:

$$i\left[e^{\mu}_{(a)}\gamma^a\partial_\mu - \frac{1}{4}\gamma_{abc}\gamma^c\gamma^b\gamma^a\right]\Psi = m\Psi, \tag{1}$$

where $\gamma_{abc} = e_{(a)\nu;\,\mu}e^{\nu}_{(b)};\, e^{\mu}_{(c)}$ are Ricci rotation coefficients; γ^a are Dirac matrices; m is the particle mass, and $c = \hbar = 1$.

The tetrad's vectors are chosen orthonormally:

$$g_{\mu\nu}e^{\mu}_{(a)}e^{\nu}_{(b)} = \eta_{ab},$$

where $g_{\mu\nu}$ is a metric and η_{ab} the Minkowsky matrix.

As in [5], we shall find the solution of the generally covariant Dirac equation in the form of the wave packet localized in a space region of width d^{-1}:

$$\Psi = \frac{d^{3/2}}{\pi^{1/4}}\int d^3q\, \exp\left\{-\left[i\int\limits_t^\infty E(\mathbf{q},\tau)d\tau + i(\mathbf{q},\mathbf{r}) + \frac{(\mathbf{q}-\mathbf{q}_0)}{2d^2}\right]\right\}U(\mathbf{q},t). \tag{2}$$

We suppose the reasonable condition $\kappa = d^{-1}/l_g \ll 1$ is fulfilled, where l_g is the characteristic scale of the changing of the metric. It is obvious that l_g has order of the gravitational field source size.

The wave function Ψ is normalized with accordance to the condition

$$\int \bar{\Psi}e^{0}_{(a)}\gamma^a\Psi\sqrt{-g}d^3x = 1. \tag{3}$$

From (2) and (3) the asymptotic normalizing condition for $U(\mathbf{q},t)$ follows:

$$U^{+}(\mathbf{q}, t\to\infty)U(\mathbf{q}, t\to\infty) = 1. \tag{4}$$

In order to describe spin dynamics we write Ψ in the form of the expansion in the spirality states

$$\Psi = \Psi_L + \Psi_R, \tag{5}$$

where

$$\Psi_{L,R} = \frac{1}{2}(1\pm\gamma_5)\Psi, \quad \gamma^5 = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}.$$

From the expressions for the probabilities W_L and W_R of the left and, respectively, the right states of a massive Dirac particle, it follows that

$$W_{L,R} = \iiint \bar{\Psi}_{L,R}\, e^{0}_{(a)}\gamma^a\Psi_{L,R}\sqrt{-g}d^3x. \tag{6}$$

The form of the function $E(\mathbf{q},\tau)$ will be defined below more precisely, although the asymptotic condition $E(\mathbf{q},\tau\to\infty) \to \sqrt{q^2+m^2}$ is obvious.

Expanding the metric components and the coefficients of the Dirac equation powers of the small parameter κ in a neighbourhood of the wave packet centre, one can reduce the equation to a set of ordinary differential equations which describe the evolution of the Dirac particle spinor amplitude $U(\mathbf{q}, t)$ ($\mathbf{q}$ is the particle moment).

The main contribution to the wave packet is made by values of $\mathbf{q}$ from the ball of radius d^{-1} with centre $\mathbf{q}_0$. To study spin dynamics, we write U as the sum $U = U_L + U_R$ of fields U_R and U_L with the left and, respectively, right spiralities, where $U_{LRs} = 1/2(1 \pm \gamma_5)U$.

Four-component spinors U_L and U_R can be expressed from two-component spinors F and H as follows:

$$U_L = \begin{pmatrix} F \\ -F \end{pmatrix}, \quad U_R = \begin{pmatrix} H \\ H \end{pmatrix}. \tag{7}$$

Using F and H, one can make up four-component spinor R:

$$R = \begin{pmatrix} F \\ H \end{pmatrix}, \tag{8}$$

which obeys the equation that can be written in the form of Schrödinger equation:

$$\dot{R} = -i\hat{H}R. \tag{9}$$

The explicit form $\hat{H}$ is presented in [5] for Kerr–Schild metric, and in [6] for a slightly curved space-time with arbitrary geometry in the first post-Newtonian approximation.

Let $\hat{H}(t \to \infty) = \hat{H}_0$, $R(t \to \infty) = R_0$, and $\mathbf{n} = \mathbf{q}/q$ ($q = |\mathbf{q}|$) be the unit vector in the direction of the Dirac particle moment $\mathbf{q}$. Let $\boldsymbol{\sigma}$ be the Pauli matrices, $\sigma_\pm = (1 \pm (\boldsymbol{\sigma}\mathbf{n}))$ and

$$\hat{S}^\pm = \frac{1}{2(\kappa_{01} + 1)} \begin{pmatrix} \sigma_\pm + \kappa_{01}, & \pm\kappa_0 \\ \pm\kappa_0, & \sigma_\mp + \kappa_{01} \end{pmatrix}, \tag{10}$$

where $\kappa_0 = m/q$; $\kappa_{01} = \sqrt{1 + \kappa_{01}^2} - 1$.

We have

$$\hat{H}_0 = q\hat{S}^+, \quad R_0 = \hat{S}^- \begin{pmatrix} r_{01} \\ r_{02} \end{pmatrix},$$

where r_{01} and r_{02} are the eigenvectors of the matrix $(\boldsymbol{\sigma}\mathbf{n})$:

$$(\boldsymbol{\sigma}\mathbf{n})r_{01,\,2} = \mp r_{01,\,2}. \tag{11}$$

Note the orthogonality condition $\hat{S}^+$ and $\hat{S}^-$, $\hat{S}^+ \cdot \hat{S}^- = 0$, which can be verified directly and is important in what follows.

Denote $\hat{V}$ as the difference $\hat{H} - \hat{H}_0$.

By means of simple transformations one can write equation (9) in the form of the set of integral equations with Volterra kernels:

$$R = i\int_t^\infty \left({}^-\hat{V}(\tau) + {}^+\hat{V}(\tau)\,\mathrm{e}^{2i\alpha(\tau - t)}\right) R(\tau)d\tau + R_0, \tag{12}$$

where $\alpha = \sqrt{q_2 + m^2}$; ${}^\mp\hat{V} = \hat{S}^\mp\hat{V}$.

One can find the solution of (12) by the iterative method:

$$R^{(n)} = i \int\limits_t^{\infty} \left({}^{-}\hat{V}(\tau) + {}^{+}\hat{V}(\tau)\, \mathrm{e}^{2i\alpha(\tau - t)} \right) R^{(n-1)}(\tau) d\tau + R_0. \tag{13}$$

We choose $R^{(0)} = R_0$. The limit $R = \lim\limits_{n \to \infty} R^{(n)}$ gives us an exact solution of (12).

The kernels of the integral equations contain the terms with the rapidly oscillating factor $\mathrm{e}^{2i\alpha(\tau - t)}$.

Note that the condition ${}^{+}\hat{V}(\tau \to \infty) \to 0$ is sufficient for convergence of the integrals. However, for convergence of the integrals with terms ${}^{-}\hat{V}(\tau)$ a more strong condition is needed:

$$\left| {}^{-}\hat{V}_{ik} \right|_{i,k=0,1,2,3} < \frac{A}{(r_0 + \tau)^{1+\delta}}, \tag{14}$$

where $\delta > 0$ and r_0 is characteristic radius of the gravitating object.

The main contribution to the integrals with the factor $\mathrm{e}^{2i\alpha(\tau - t)}$ is made by the integration region $t - \Delta\tau \leqslant \tau \leqslant t + \Delta\tau$, $\Delta\tau \sim 1/\alpha$. The condition $1/\alpha l_g \ll 1$ holds, so that the integrals with $\mathrm{e}^{i\alpha(\tau - t)}$ can be evaluated with the help of the following asymptotic expansion:

$$J(t) = \int\limits_t^{\infty} f(\tau)\, \mathrm{e}^{i\alpha(\tau - t)} d\tau =$$

$$= \frac{i}{\alpha} \sum_{k=0}^{N-1} \left(\frac{i}{\alpha} \right)^k \frac{d^k f(t)}{dt^k} + i \left(\frac{i}{\alpha} \right)^N \int\limits_t^{\infty} \left(\mathrm{e}^{i\alpha(\tau - t)} \frac{d^N f(\tau)}{dt^N} \right) d\tau. \tag{15}$$

If $f(t)$ obeys the condition $(N > 1)$

$$\left| \frac{d^N f(t)}{dt^N} \right| < \frac{\alpha A (N-1)!}{(t_0 + t)^N}, \tag{16}$$

we have

$$J(t) = \frac{i}{\alpha} \sum_{k=0}^{N-1} \left(\frac{i}{\alpha} \right)^k \frac{d^k f(t)}{dt^k} + O\left(\frac{(N-2)!}{\alpha (t_0 + t)^{N-1}} \right). \tag{17}$$

In our case $\alpha t_0 = \alpha l_g \gg 1$, so the omitted terms are small right up to $N \sim \alpha t_0$. The choice of N should be defined by the order of approximation in (17). $N = 1$ is sufficient for $R^{(1)}$, $N = 2$ for $R^{(2)}$, etc., so that $R^{(n)} N = n$. The procedure allows one to present R in the form of asymptotic series in terms of powers of the small parameter $1/\alpha l_g$.

Let us dwell on the choice of the function $E(\mathbf{q}, \tau)$. For each step of the approximation the function is chosen by the condition of absence in $R^{(n)}$ of large terms of order $(q l_g)$. This is in accordance with obvious physical condition of absence of rapidly changing time-dependent terms in the spinor amplitude. The convergence of the sequences $R^{(n)}$ can be proved analogously to the Volterra equations of the second kind. The transitions which change the massive Dirac particle spirality (that is neutrino in astrophysics) are most interesting in this investigation. It follows from (12) and (13) that such transitions are possible, if ${}^{\mp}\hat{V}_{12}$ and

$^{\mp}\hat{V}_{21}$ do not vanish. Here we write the four-row matrices $\hat{V}$ in the block form by using the two-row ones $^{\mp}\hat{V}_{a,b}(a, b = 1, 2)$:

$$^{\mp}\hat{V} = \begin{pmatrix} ^{\mp}\hat{V}_{11} & ^{\mp}\hat{V}_{12} \\ ^{\mp}\hat{V}_{21} & ^{\mp}\hat{V}_{22} \end{pmatrix}. \tag{18}$$

For arbitrary space-time geometry, $^{\mp}\hat{V}_{12}$ and $^{\mp}\hat{V}_{22}$ do not vanish only for $m \neq 0$. Therefore, the oscillations $\nu_L \leftrightarrow \nu_R$, $\bar{\nu}_L \leftrightarrow \bar{\nu}_R$ are possible only for massive neutrinos, and the value of the effect is determined by specific space-time geometry.

We suggest using the developed method for a detailed study of this phenomenon in the cases of some space-time geometries. The first post-Newtonian approximation is most interesting. In this case, it is sufficient to calculate a few first terms $R^{(n)}$ for nonvanishing effects to appear.

The present work was supported by the Ministry of Education of the Russian Federation, Grant E 00-3.3-101.

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Received on May 14, 2002.