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ФИЗИКА ЭЛЕМЕНТАРНЫХ ЧАСТИЦ И АТОМНОГО ЯДРА. ТЕОРИЯ

ON THE LORENTZ GROUP $SO(3,1)$, GEOMETRICAL SUPERSYMMETRIC ACTION FOR PARTICLES AND SQUARE ROOT OPERATORS II. SQUEEZED STATES AND RELATIVISTIC WAVE EQUATIONS

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The geometical relativistic superparticle action is analyzed from theoretical group point of view. To this end an alternative technique of quantization, outlined by the authors in a previous work and based on the correct interpretation of the square root Hamiltonian, is used. We show that the obtained spectrum of physical states and the Fock construction in this previous work consist of squeezed states with the even and odd representations with the lowest weights $\lambda = 1/4$ and $\lambda = 3/4$ corresponding to four possible (nontrivial) fractional representations for the group decomposition of the spin structure. The conserved currents are computed, and a new relativistic wave equation is proposed and explicitly solved for the time-dependent case. The relation between the relativistic Schrödinger equation and the time-dependent harmonic oscillator is analyzed and discussed.

Проводится анализ геометрического релятивистского действия для суперчастиц с теоретической точки зрения. С этой целью используется альтернативная техника квантования, изложенная авторами в предыдущей работе и основанная на корректной интерпретации гамильтониана в форме квадратного корня. В упомянутой работе показано, что полученный спектр физических состояний и фоковское построение состоят из сжатых состояний, для которых соблюдается соответствие четных и нечетных представлений с низшими весами $\lambda = 1/4$ и $\lambda = 3/4$ четырем возможным (нетривиальным) дробным представлениям для группового разложения спиновой структуры. Вычисляются сохраняющиеся токи, и предлагается новое релятивистское волновое уравнение, которое явно решается для случая с зависимостью от времени. Анализируется и обсуждается связь релятивистского уравнения Шредингера и гармонического осциллятора с зависимостью от времени.

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1. INTRODUCTION AND SUMMARY

The quantum behaviour of a relativistic particle in the superspace, as well as being a useful tool for certain studies and applications of Quantum Field Theory (QFT), is of notable importance in many physical contexts. Time-dependent Landau systems and the electron-monopole system are described naturally by the Super-Heisenberg–Weyl and $OSP(1/2)$

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algebras [14–16]. If several more or less well known physical systems are intrinsically supersymmetric in nature, an obvious question is: Can any supersymmetric toy model give us a good picture of not so well known physical systems? Part of the purpose of this paper is to demonstrate the positive answer to this question showing that a relativistic particle in the superspace can describe particles with fractionary spin for which no concrete action is known.

On the other hand, the Time-Dependent Harmonic Oscillator (TDHO) was demonstrated to be powerful to describe systems with more complicated dynamics in closed form. From the famous reports of Ermakov [20] and Husimi [19] we can see that if any physical problem with a complicated or involved dynamics can be represented faithfully or «mapped» to a TDHO system, then this complicated dynamics admits a Coherent State (CS) or Squeezed States (SS) realization. It is clearly important that the model proposed here admits a CS and SS realization. Most notably, squeezed states have been used in the context of quantum optics [21] and in the context of gravitational wave detection [22]. The correct choice for the realization of the physical states, however, will depend on the symmetry group that defines in some meaning the particular physical system under study. The other part of this work will be devoted to discussion of this point and what happens when different algebras can characterize the same physical problem.

In a previous work [7], to which we will refer in the number of all equations here as Part I, we considered the simple model of superparticle of Volkov and Pashnev [1] in order to quantize it and to obtain the spectrum of physical states with the Hamiltonian remaining in the natural square root form. In the present paper we will complete this previous work, giving in more explicit form how the states can be faithfully represented and realized from the geometrical and from the dynamics of the group manifold point of view. We also find the link with the TDHO problem, and for instance, with the CS and SS representations of the physical states obtained in [7]. The plan of this paper is as follows: Section 2 is devoted to description of the obtained spectrum of the superparticle model under consideration in Part I, emphasizing the relation between the group representation of the physical states and their CS or SS realizations. In Secs. 3 and 4 the relation of the model with the relativistic Schrödinger equation is discussed and a new relativistic wave equation is proposed. Finally some conclusions and remarks are given in Sec. 5.

2. SQUEEZED STATE REALIZATION AND NON-COMPACT GROUPS

In Part I we showed that the wave functions which transform as linear irreducible representation of $ISO(2,1)$, subgroup of $ISO(3,1)$ generated by operators (I.33) are

$$\Psi_{1/4}(x,\theta,q) = \sum_{k=0}^{+\infty} f_{2k}(x,\theta)\varphi_{2k}(q), \tag{1}$$

$$\Psi_{3/4}(x,\theta,q) = \sum_{k=0}^{+\infty} f_{2k+1}(x,\theta)\varphi_{2k+1}(q) \tag{2}$$

(analogously for the $\overline{\Psi}_{1/4}$ and $\overline{\Psi}_{3/4}$ states with contrary helicity). We can easily see that the Hamiltonian $\mathcal{H}_{\rm cm} = \sqrt{m^2 - M^2 + \frac{2^{3/2}M}{|a|}\left[1 - (\sigma_0)_{\alpha\dot{\beta}}\bar{s}^{\dot{\beta}}s^{\alpha}\right]}$ Eq. (I.28) operates over the states $|\widehat{\Psi}\rangle$, which enter into $\mathcal{H}$ as its square Φ_α and $\Phi_{\dot{\alpha}}$. It is natural to associate, up to a proportional factor, the spinors d_α and $\overline{d}_{\dot{\alpha}}$ with

$$d_\alpha \to (\Phi_{1/4})_\alpha \equiv \left\langle \Psi_{1/4} \right| \mathbb{L}_\alpha \left| \Psi_{1/4} \right\rangle, \qquad \overline{d}_{\dot{\alpha}} \to \left(\overline{\Phi}_{1/4}\right)_{\dot{\alpha}} \equiv \left\langle \overline{\Psi}_{1/4} \right| \mathbb{L}_{\dot{\alpha}} \left| \overline{\Psi}_{1/4} \right\rangle \tag{3}$$

and in an analogous manner the spinors s_α and $\overline{s}_{\dot{\alpha}}$ with

$$s_\alpha \to (\Phi_{3/4})_\alpha \equiv \left\langle \Psi_{3/4} \right| \mathbb{L}_\alpha \left| \Psi_{3/4} \right\rangle, \qquad \overline{s}_{\dot{\alpha}} \to \left(\overline{\Phi}_{3/4}\right)_{\dot{\alpha}} \equiv \left\langle \overline{\Psi}_{3/4} \right| \mathbb{L}_{\dot{\alpha}} \left| \overline{\Psi}_{3/4} \right\rangle, \tag{4}$$

where the new spinors $\mathbb{L}_\alpha(\mathbb{L}_{\dot{\alpha}})$ are defined as

$$\mathbb{L}_\alpha = \begin{pmatrix} a_1 a_1 \\ a_1^+ a_1^+ \end{pmatrix}, \qquad \mathbb{L}_{\dot{\alpha}} = \begin{pmatrix} a_2 a_2 \\ a_2^+ a_2^+ \end{pmatrix}. \tag{4a}$$

The reason for this choice is the following: as was shown in Ref. [10], the Hilbert space for each subgroup $ISO(2,1) \approx SU(1,1)$ [23, 24] can be decomposed as direct sum of two independent subspaces characterized for the states of helicity $\lambda = 1/4$ and $\lambda = 3/4$, respectively. Each subspace is composed of the even ($\lambda = 1/4$) and odd states ($\lambda = 3/4$) given by the expressions (1), (2). These «cat» states admit (after a convenient choice for the functions $f_{2k}(x,\theta)$ and $f_{2k+1}(x,\theta)$) a coherent state realization being eigenvectors not of the ladder operator a of the Heisenberg–Weyl algebra with the operators (I.31) $L_\alpha = \begin{pmatrix} a_1 \\ a_1^+ \end{pmatrix}$ and $L_{\dot{\alpha}} = \begin{pmatrix} a_2 \\ a_2^+ \end{pmatrix}$, but of the quadratic ladder operator aa of the $SU(1,1)$ algebra defined in general by

$$K_+ = \frac{1}{2} a^+ a^+; \quad K_- = \frac{1}{2} aa; \quad K_0 = \frac{1}{4}(a^+ a + a a^+). \tag{4b}$$

This means that when we are in the full Hilbert space the algebra is Heisenberg–Weyl one and the states $|\Psi\rangle = \frac{1}{\sqrt{2}}\left(\left|\Psi_{1/4}\right\rangle + \left|\Psi_{3/4}\right\rangle\right)$ are eigenvectors of the operator a, and when we pass to the decomposed space (by means of a suitable unitary transformation), the algebra becomes the $SU(1,1)$ algebra with the quadratic ladder operators given by expression (4b).

3. RELATION WITH THE RELATIVISTIC SCHRÖDINGER EQUATION: COMPATIBILITY CONDITIONS AND PROBABILITY CURRENTS

According to formula (I.25) the new Hamiltonian operates as (the metric tensor signature is given here by $g_{\mu\nu} = (+ - - -)$)

$$\sqrt{m^2 - \mathcal{P}_0 \mathcal{P}^0 - \left(\mathcal{P}_i \mathcal{P}^i + \frac{1}{a} \Pi^\alpha \Pi_\alpha - \frac{1}{a^*} \Pi^{\dot{\alpha}} \Pi_{\dot{\alpha}}\right)} \, |\Psi\rangle = 0;$$

for instance, the action of the radical operator is

$$\left\{\left[m^2-\mathcal{P}_0\mathcal{P}^0-\left(\mathcal{P}_i\mathcal{P}^i+\frac{1}{a}\Pi^\alpha\Pi_\alpha-\frac{1}{a^*}\Pi^{\dot\alpha}\Pi_{\dot\alpha}\right)\right]_\beta^\alpha(\Psi L_\alpha)\right\}^{1/2}\Psi=0, \tag{5}$$

which seems as a parabosonic supersymmetric version of the relativistic Schrödinger–De Broglie equation. In the next paragraph we will see that this equation corresponds to the family of equations given first by E. Majorana [11] and P. A. M. Dirac [3], and in its parabosonic version by Sudarshan, N. Mukunda and C. C. Chiang in 1981 [9].

We can see from the above expression that if we put the (super)momenta together in the $\Box$ operator, we obtain a more suitable equation in order to compute the currents as in the Fock–Klein–Gordon case

$$\begin{aligned}&\left[\left(\Box+m^2\right)_\beta^\alpha(\Psi L_\alpha)\right]^{1/2}\Psi=0,\\&\left[\left(\Box+m^2\right)_\beta^\alpha(\Psi L_\alpha\Psi)\right]^{1/2}=\left[\left(\Box+m^2\right)_\beta^\alpha\Phi_\alpha\right]^{1/2}=0,\end{aligned} \tag{6}$$

now eliminating the exponent $1/2$ and taking the Hermitian conjugation to equation we have

$$\left[\left(\Box+m^2\right)_\beta^\alpha\left(\Psi^\dagger L_\alpha^\dagger\Psi^\dagger\right)\right]^{1/2}=\left(\Box+m^2\right)_\beta^\alpha\Phi_\alpha^\dagger=0. \tag{7}$$

Following the same procedure as Dirac in Ref. [3], we multiply the expression (6) from the left side by $\Phi_\alpha^\dagger$ and multiply the expression (7) from the left side by Φ_α, integrating and subtracting the final expressions we obtain

$$\Phi_\alpha^\dagger\Box\Phi_\beta-\Phi_\alpha\Box\Phi_\beta^\dagger=0. \tag{8}$$

Using the relations $\Phi^\dagger\Box\Phi=\partial_\mu\left(\Phi^\dagger\partial^\mu\Phi\right)-\partial_\mu\Phi^\dagger\partial^\mu\Phi$; $\Phi\Box\Phi^\dagger=\partial_\mu\left(\Phi\partial^\mu\Phi^\dagger\right)-\partial_\mu\Phi\partial^\mu\Phi^\dagger$ in expression (56), the current for the square states Φ_α is

$$\partial_\mu\left(\Phi^\alpha\partial^\mu\Phi_\alpha^\dagger-\Phi^{\alpha\dagger}\partial^\mu\Phi_\alpha\right)=0=-\partial_\mu j^\mu \tag{9}$$

with $j^\mu(x)\equiv-(i)\left[\Phi^\alpha\partial^\mu\Phi_\alpha^\dagger-\Phi^{\alpha\dagger}\partial^\mu\Phi_\alpha\right]$.

If we suppose that a link between the relativistic Schrödinger equation (5) and our new Hamiltonian $\mathcal{H}$ holds, the relation with the quartionic states is the following:

$$i\dot\Psi=E\Psi.$$

Squaring the above expression and having account as $\mathcal{H}$ operates over Ψ and $\Psi^\dagger$, we can easily obtain

$$-\dot\Phi_\beta=E^2\Phi_\beta,\quad \dot\Phi_\beta^\dagger=E^2\Phi_\beta^\dagger \tag{10}$$

which after substitution into the explicit expression for $j_0(x)$ permits us to analyze the positivity of this component of the current for the square states Φ_α

$$j_0(x)=2E^2\Phi^{\alpha\dagger}\Phi_\alpha.$$

As we have obtain of from expression (10) $j_0(x)$ for the square states Φ_α is positive definite because the energy E appears squared.

Now in order to find the current vector for the para-Bose states Ψ we proceed analogously as above for the states Φ_α but also using the consistency condition (13) arriving at

$$\Psi^\dagger \Box \Psi - \Psi \Box \Psi^\dagger = 0$$

as we expected because these square root states obey the square root operator equation and, for instance, also obey the equation for the squared operator (the inverse is not true in general). In fact in some references in the literature the authors do not take care in that we can pass to the equation with the square root Klein–Gordon operator to its squared traditional version operating on the same state but not the inverse (see e.g. Ref. [12]). The correct form to do this is as follows: if we start with

$$\sqrt{(-\Delta + m^2)}\Psi = i\partial_t \Psi, \tag{11}$$

the relation with any pseudo-differential operator A is

$$A\Psi = \sqrt{(-\Delta + m^2)}\Psi = i\partial_t \Psi \Rightarrow AA\Psi = iA\partial_t \Psi = A\sqrt{(-\Delta + m^2)}\Psi = \left(-\Delta + m^2\right)\Psi.$$

This happens clearly because Ψ obeys (11). Finally the current for the quartionic states that we were looking for is

$$\partial_\mu \left[\left(\Psi \partial^\mu \Psi^\dagger\right) - \left(\Psi^\dagger \partial^\mu \Psi\right)\right] = 0 = -\partial_\mu j^\mu \tag{12}$$

with $j^\mu(x) \equiv -i\left[\Psi \partial^\mu \Psi^\dagger - \Psi^\dagger \partial^\mu \Psi\right]$. It is not difficult to see that in this case from expression (12) the zero component of the current is not positive definite one given explicitly by

$$j_0(x) = 2E\Psi^\dagger \Psi.$$

The compatibility condition, as usual, is given by the following expression:

$$\left[\tau_\alpha, \tau_\beta\right]\Psi = 0, \tag{13}$$

where we defined $\tau_\beta \equiv \left[\left(\Box + m^2\right)_\beta^\alpha \left(\Psi L_\alpha\right)\right]^{1/2}$. After a little algebra and using expression (13) we arrive at

$$\left[\left(\Box + m^2\right)_\alpha^\delta \left(\Box + m^2\right)_\beta^\gamma \epsilon_{\delta\gamma}\right]^{1/2} \Psi = 0. \tag{14}$$

It is good to remember here that Eq. (5) describes a *free* particle in an $N = 1$ superspace and the term of interaction appears from the supersymmetry between the bosonic and fermionic fields. The last expression shows that our Eq. (5) is absolutely compatible and consistent because its fermionic character comes from the supersymmetric part, and, for instance, it is not necessary to introduce any extra term in order to include spin. As is well known, these terms (putted «by hand» in equations containing second-order derivatives) destroy the compatibility condition, leading to the impossibility of including interactions [13].

4. RELATIVISTIC WAVE EQUATION

Following the arguments given in the precedent paragraphs, it is natural to propose the following form for a square root of the second-order supersymmetric wave equation:

$$\left\{\left[m^2 - \mathcal{P}_0\mathcal{P}^0 - \left(\mathcal{P}_i\mathcal{P}^i + \frac{1}{a}\Pi^\alpha\Pi_\alpha - \frac{1}{a^*}\Pi^{\dot\alpha}\Pi_{\dot\alpha}\right)\right]_\beta^\alpha (\Psi L_\alpha)\right\}^{1/2}\Psi = 0. \tag{15}$$

Passing to the center of mass and rescaling the variables, we obtain the following expression:

$$\left\{\left[|a|^2\,\partial_0^2 + \frac{1}{4}\left(\partial_\eta - \partial_\xi + i\;\partial_0(\sigma^0)_{\alpha\dot\beta}\left(\overline{\theta}^{\dot\beta} - \theta^\alpha\right)\right)^2 - \right.\right.$$
$$\left.\left.-\frac{1}{4}\left(\partial_\eta + \partial_\xi + i\;\partial_0(\sigma^0)_{\alpha\dot\beta}\left(\overline{\theta}^{\dot\beta} - \theta^\alpha\right)\right)^2 + m^2\right]_\beta^\alpha\Phi_\alpha\right\}^{1/2} = 0, \tag{16}$$

where (e.g., Ref. [2] for the nonsupersymmetric case)

$$\eta \equiv (\overline{\theta} + \theta) \quad \xi \equiv (\overline{\theta} - \theta). \tag{17}$$

Imposing the condition $\partial_\eta\Phi_\alpha = 0 \Rightarrow \Phi_\alpha(\xi)$, the «square» of the solution eigenfunction of Eq. (15) takes the form

$$\Phi_\gamma(t) = \mathrm{e}^{A(t)+\xi\varrho(t)}\,\Phi_\gamma(0) \tag{18}$$

with $\varrho(t) = \phi_\alpha + \overline{\chi}_{\dot\alpha}$ (i.e., chiral plus antichiral parts). The system of equations for $A(t)$ and $\varrho(t)$ that we are looking for is easily obtained inserting the expression (18) in Eq. (16)

$$|a|^2\,\ddot{A} + m^2 = 0,$$
$$\ddot{\overline{\chi}}_{\dot\alpha} - i\frac{\omega}{2}(\sigma^0)_{\dot\alpha}^{\alpha}\phi_\alpha = 0,$$
$$-\ddot{\phi}_\alpha + i\frac{\omega}{2}(\sigma^0)_\alpha^{\dot\beta}\,\overline{\chi}_{\dot\beta} = 0.$$

The above system can be solved giving us the following result:

$$A = -\left(\frac{m}{|a|}\right)^2 t^2 + c_1 t + c_2; \quad c_1, c_2 \in \mathbb{C} \tag{19}$$

and

$$\phi_\alpha = \overset{\circ}{\phi}_\alpha\left(\alpha\,\mathrm{e}^{i\omega t/2} + \beta\,\mathrm{e}^{-i\omega t/2}\right) + \frac{2i}{\omega}(\sigma^0)_\alpha^{\dot\beta}\,\overline{Z}_{\dot\beta}, \tag{20}$$

$$\overline{\chi}_{\dot\alpha} = (\sigma^0)_{\dot\alpha}^{\alpha}\,\overset{\circ}{\phi}_\alpha\left(\alpha\,\mathrm{e}^{i\omega t/2} - \beta\,\mathrm{e}^{-i\omega t/2}\right) + \frac{2i}{\omega}(\sigma^0)_{\dot\alpha}^{\alpha}\,Z_\alpha, \tag{21}$$

where $\overset{\circ}{\phi}_\alpha Z_\alpha$ and $\overline{Z}_{\dot\beta}$ are constant spinors. The superfield solution for the square states that we are looking for has the following form:

$$\Phi_\gamma(t) = \exp\left(-\left(\frac{m}{|a|}\right)^2 t^2 + c_1 t + c_2\right)\,\mathrm{e}^{\xi\varrho(t)}\,\Phi_\gamma(0) \tag{22}$$

with

$$\varrho(t) = \overset{\circ}{\phi}_\alpha \left[\left(\alpha\, e^{i\omega t/2} + \beta\, e^{-i\omega t/2}\right) - (\sigma^0)^\alpha_{\dot\alpha}\left(\alpha\, e^{i\omega t/2} - \beta\, e^{-i\omega t/2}\right)\right] + \\ + \frac{2i}{\omega}\left[(\sigma^0)^{\dot\beta}_\alpha \overline{Z}_{\dot\beta} + (\sigma^0)^\alpha_{\dot\alpha} Z_\alpha\right] \quad (23)$$

and

$$\Phi_\gamma(0) = \langle\Psi(0)|\, L_\gamma\, |\Psi(0)\rangle \quad (24)$$

which is nothing else than the mean value of L_γ between the states $|\Psi\rangle$ in the initial time, where the subalgebra is the Heisenberg–Weyl algebra (with generators a, a^+ and $(n+1/2)$). As we have pointed out in Sec. 2, the states $|\Psi\rangle$ span all the Hilbert space and, for instance, we cannot obtain useful information from the point of view of the topology of the group manifold, for instance, also about the spin.

The dynamics of the square root fields, in the representation that we are interested in, can be simplified considering these fields as coherent states in the sense that they are eigenstates of a^2:

$$\begin{aligned} \left|\Psi_{1/4}(0,\xi,q)\right\rangle &= \sum_{k=0}^{+\infty} f_{2k}(0,\xi)\,|2k\rangle = \sum_{k=0}^{+\infty} f_{2k}(0,\xi)\frac{(a^\dagger)^{2k}}{\sqrt{(2k)!}}|0\rangle, \\ \left|\Psi_{3/4}(0,\xi,q)\right\rangle &= \sum_{k=0}^{+\infty} f_{2k+1}(0,\xi)\,|2k+1\rangle = \sum_{k=0}^{+\infty} f_{2k+1}(0,\xi)\frac{(a^\dagger)^{2k+1}}{\sqrt{(2k+1)!}}|0\rangle. \end{aligned} \quad (25)$$

From a technical point of view these states are one-mode squeezed states constructed by the action of the generators of the $SU(1,1)$ group over the vacuum. For simplicity, we will take all normalization and fermionic dependence or possible CS fermionic realization, into the functions $f(\xi)$. Explicitly at $t = 0$

$$\begin{aligned} \left|\Psi_{1/4}(0,\xi,q)\right\rangle &= f(\xi)|\alpha_+\rangle, \\ \left|\Psi_{3/4}(0,\xi,q)\right\rangle &= f(\xi)|\alpha_-\rangle, \end{aligned} \quad (26)$$

where $|\alpha_\pm\rangle$ are the CS basic states in the subspaces $\lambda = 1/4$ and λ =3/4 of the full Hilbert space. From expression (22) and expressions (4) we obtain

$$\Phi_\alpha(t,\lambda) = \langle\Psi_\lambda(t)|\,\mathbb{L}_\alpha|\Psi_\lambda(t)\rangle = \\ = \exp\left(-\left(\frac{m}{|a|}\right)^2 t^2 + c_1 t + c_2\right) e^{\xi\varrho(t)}\,\langle\Psi_\lambda(0)|\begin{pmatrix} a^2 \\ (a^+)^2 \end{pmatrix}_\alpha |\Psi_\lambda(0)\rangle, \quad (27)$$

$$\Phi_\alpha(t,\lambda) = \exp\left(-\left(\frac{m}{|a|}\right)^2 t^2 + c_1 t + c_2\right) e^{\xi\varrho(t)}\,|f(\xi)|^2 \begin{pmatrix} \alpha_\lambda^2 \\ \alpha_\lambda^{*2} \end{pmatrix}_\alpha, \quad (28)$$

where λ label the helicity or the spanned subspace (e.g., $\pm$). The «square root» states eigenfunctions of the square root wave operator (15) are

$$\Psi_\lambda = \exp\left(-\frac{1}{2}\left[\left(\frac{m}{|a|}\right)^2 t^2 + c_1 t + c_2\right]\right) \exp\left(\frac{\xi\varrho(t)}{2}\right) |f(\xi)| \begin{pmatrix} \alpha \\ \alpha^* \end{pmatrix}_\lambda, \quad (29)$$

where $\lambda = 1/4, 3/4$. Notice the difference from the case in which we used the HW realization for the states Ψ

$$|\Psi\rangle = \frac{f(\xi)}{2}\left(|\alpha_+\rangle + |\alpha_-\rangle\right) = f(\xi)\,|\alpha\rangle\,, \tag{30}$$

where, however, the linear combination of the states $|\alpha_+\rangle$ and $|\alpha_-\rangle$ spans now the full Hilbert space with λ in this CS basis $\lambda = 1/2$. The «square» state at $t = 0$ is

$$\Phi_\alpha(0) = \langle\Psi(0)|\,L_\alpha\,|\Psi(0)\rangle = \langle\Psi(0)|\begin{pmatrix} a \\ a^+ \end{pmatrix}_\alpha |\Psi(0)\rangle = f^*(\xi)f(\xi)\begin{pmatrix} \alpha \\ \alpha^* \end{pmatrix}_\alpha. \tag{31}$$

The square state at time t is

$$\Phi_\gamma(t) = \exp\left(-\left(\frac{m}{|a|}\right)^2 t^2 + c_1' t + c_2'\right)\mathrm{e}^{\xi\varrho(t)}\,|f(\xi)|^2\begin{pmatrix} \alpha \\ \alpha^* \end{pmatrix}_\alpha. \tag{32}$$

And the «square root» solution becomes

$$\Psi(t) = \exp\left(-\frac{1}{2}\left[\left(\frac{m}{|a|}\right)^2 t^2 + c_1' t + c_2'\right]\right)\exp\left(\frac{\xi\varrho(t)}{2}\right)|f(\xi)|\begin{pmatrix} \alpha^{1/2} \\ \alpha^{*1/2} \end{pmatrix}. \tag{33}$$

We can see the change in the solutions from the choice in the representation of the Hilbert space. The algebra (topological information of the group manifold) is «mapped» over the spinors solutions through the eigenvalues α and α^*. Notice that the constants c_1', c_2' in the exponential functions in expressions (32) and (33) differ from the c_1 and c_2 in (28) and (29), because these exponential functions of the Gaussian type come from the action of a unitary operator over the respective CS basic states in each representation (h_3 or HW). These constants can be easily determined as functions of the frequency ω as in Ref. [19] for the Schrödinger equation. A detailed analysis of this point and the other type of solutions will be given elsewhere [8].

The possible algebras that contain an $SU(1,1)$ as subgroup that can lead or explain the fermionic factors of type $\exp\left(\frac{\xi\varrho(t)}{2}\right)|f(\xi)|$ in the solutions are 2 that are strong candidates [22]: the supergroup $OSP(2,2)$ [15] and the supergroup $OSP(1/2,\mathbb{R})$ [17]. In the case of the $OSP(2,2)$ we have bosonic and fermionic realizations and the CS and SS can be constructed from the general procedure given by M. Nieto et al. in Refs. [14–16]. On the other hand, the $OSP(1/2,\mathbb{R})$ realization is more «economic», the number of generators is less than in the $OSP(2,2)$ case and the realization is bosonic: the $K_\pm$ and K_0 generators operate over the Bose states and the HW algebra given by a and a^+ operates over the fermionic part. In this case the CS and the SS that can be constructed are eigenstates of the displacement and squeezed operators, respectively, but they cannot minimize the dispersion of the quadratic Casimir operator, so that they are not minimum uncertainty states.

The important point to remark here is that when we describe from the mostly geometrical grounds any physical system through $SU(1,1)$ CS or SS, the orbits will appear as the intersections of curves that represent constant-energy surfaces, with one sheet of a two sheeted hyperboloid (the curved phase space of $SU(1,1)$ or Lobachevsky plane) in the space of averaged algebra generators. In the specific case treated in this paper, the group containing the $SU(1,1)$ as subgroup linear and bilinear functions of the algebra generators can factorize operators as the Hamiltonian or the Casimir operator (when averaged with respect to group CS or SS), defining corresponding curves in the averaged algebra space. If we notice that the validity of Ehrenfest's theorem for CS (SS) implies that, if the exact dynamics is confined to the $SU(1,1)$ hyperboloid, it necessarily coincides with the variational motion, the variational motion coming from the Euler–Lagrange equations for the Lagrangian

$$\mathcal{L} = \left\langle z \left| i\frac{\widehat{\partial}}{\partial t} - \widehat{H} \right| z \right\rangle$$

will be different if $|z\rangle = |\alpha\rangle$ or $z = |\alpha_{\pm}\rangle$, as is evident. It is interesting to note also that a similar picture is in the context of the pseudospin $SU(1,1)$ dynamics in the frame of the mean field approximation induced by the variational principle on nonlinear Hamiltonians [18].

Important Considerations. We can be tempted, instead of the choices given by Eqs. (17) and (18), e.g., superfield solution with chiral and antichiral parts, to use directly a chiral or antichiral superfield as for the three-dimensional case (represented by the Wigner little group $SO(2,1)$) in Refs. [9–11] of Part I. But this choice is absolutely inconsistent when we treat the four-dimensional problem described for the Lorentz group $SO(3,1)$: this enforces only bosonic field as solution instead of superfield with its corresponding anticommuting part.

The particular choices (17) and (18) are based on the observation that the term $i\partial_0(\sigma^0)_{\alpha\dot{\beta}}(\overline{\theta}^{\dot{\beta}} - \theta^{\alpha})$ in Eq. (16) is the equivalent in the superspace to the term for the ordinary Dirac or Klein–Gordon equation for a particle in a constant electromagnetic field [2] where in our case $\mathbb{A} \equiv \left(0, -i\mathcal{P}_{\mu}(\sigma^{\mu})_{\alpha\dot{\beta}}\overline{\theta}^{\dot{\beta}}, i\mathcal{P}_{\mu}(\sigma^{\mu})_{\alpha\dot{\beta}}\overline{\theta}^{\dot{\beta}}\right)$ plays the role of the electromagnetic potential 1-form. As we said in the previous paragraph, still the action is for a free particle in an $N = 1$ superspace, the supersymmetry of the model enforces to mix even and odd superspace coordinates giving a similar picture of the ordinary electromagnetic interaction in QFT. The claims in Ref. [10] Part I about the impossibility of parastates in $D = 4$ are based on a not so good choice in the appropiate coordinates well adapted to the physical problem under consideration. A more detailed analysis will be given elsewhere.

5. CONCLUDING REMARKS

In this work the problem of the physical interpretation of the square root quantum operators and possible relation with the TDHO and coherent and squeezed states was analyzed considering the simple model of superparticle of Volkov and Pashnev [1]. Besides the extension and clarification of the results of our previous work [7], now we make complete this research with the following new results:

i) The relation between the structure of the Hilbert space of the states, the spin content of the sub-Hilbert spaces and the CS and SS realization of the physical states was established for the particular model presented here.

ii) The relation between the relativistic Schrödinger equation and other type of equations that involve variables with fractional spin and the model analyzed here was established and discussed.

iii) As for the Klein–Gordon equation, the conserved currents for the «square-root» states (parafields) and for the square states were explicitly computed and analyzed. The component zero of the current is linearly dependent on the energy E in the parafield case and for the «square» state the dependence on the energy is quadratic.

iv) The compatibility conditions were analyzed and the consistency of the proposed equation was established. The explanation of this consistency and the relation with the free dynamics and the supersymmetry of the model was given.

v) New wave equation is proposed and explicitly solved for the time-dependent case. As for the TDHO the physical states are realized in the CS and SS basis, and the link between the topology of the (super)group manifold and the obtained solution from the algebraic and group theoretical point of view was discussed and analyzed.

It is interesting to see that the results presented here for the superparticle are in complete agreement with the results, symmetry group and discussions for nonsupersymmetric examples given in Refs. [4–6], where group and geometrical quantization was used. This fact gives a high degree of reliability of our method of quantization and the correct interpretation of the radical Hamiltonian operator. It is clear that the ordinary canonical method of quantization fails when the reparametrization procedure affects the power of the starting Hamiltonian modifying inexorably the obtained spectrum of the physical states (see, e.g., [5, 6]).

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