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COHERENT STATES FOR A QUANTUM PARTICLE ON A MÖBIUS STRIP[1]

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The coherent states for a quantum particle on a Möbius strip are constructed and the relation with the natural phase space for fermionic fields is shown. The explicit comparison of the obtained states with those obtained in the previous works, where the cylindrical quantization was used and the spin 1/2 was introduced by hand, is given.

Конструируются когерентные состояния квантовой частицы на листе Мёбиуса, и приводится соотношение для естественного фазового пространства для фермионных полей. Представлено сравнение полученных состояний с состояниями, полученными в предыдущих работах, где используется цилиндрическое квантование, а спин 1/2 вводится вручную.

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INTRODUCTION

Coherent States (CS) have attracted much attention in many branches of physics [1]. In spite of their importance, the theory of CS when the configuration space has nontrivial topology is far from complete. The CS for a quantum particle on a circle [2] and a sphere have been introduced very recently, and also the case of the torus has been treated. Although in all these works the different CS constructions for the boson case are practically straighforward, the simple addition by hand of spin 1/2 to the angular momentum operator J for the femionic case into the corresponding CS remains obscure and non-natural. The question that naturally arises is: Is there exist any geometry for the phase space where the CS construction leads precisely to a fermionic quantization condition? The purpose of this paper is to demonstrate the positive answer to this question showing that the CS for a quantum particle on the Möbius strip geometry is the natural candidate to descibe fermions exactly as the cylinder geometry for bosons.

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1. ABSTRACT COHERENT STATES

The position of a point on the Möbius strip geometry can be parametrized as $P_0 = (X_0, Y_0, Z_0)\, P_0 = (X_0 + X_1, Y_0 + Y_1, Z_0 + Z_1)$. The coordinates of P_0 describe the central cylinder (generated by the invariant fiber of the middle of the strip): $Z_0 = l$, $X_0 = R\cos\varphi$, $Y_0 = R\sin\varphi$. We use the standard spherical coordinates: R, θ, φ with $d\Omega = R^2(d\theta^2 + \sin^2\theta d\varphi)$, and r is the secondary radius of the torus.

The coordinates of P_1 (the boundaries of the Möbius band) are the coordinates of P_0 plus $Z_1 = r\cos\theta$, $X_1 = r\sin\theta\cos\varphi$, $Y_1 = r\sin\theta\sin\varphi$. The weight of the band is obviously $2r$, then our space of phase is embedded into of the torus: $X = R\cos\varphi + r\sin\theta\cos\varphi$, $Y = R\sin\varphi + r\sin\theta\sin\varphi$, $Z = l + r\cos\theta$. The important point is that the angles are not independent in the case of the Möbius band and are related by the following constraint: $\theta = \dfrac{\varphi + \pi}{2}$.

In order to introduce the CS for a quantum particle on the Möbius strip geometry, we follow the Barut–Girardello construction and we seek the CS as the solution of the eigenvalue equation $X\,|\xi\rangle = \xi\,|\xi\rangle$ with complex ξ. Taking $R = 1$ and inserting the constraint into the parametrization of the torus, we obtain the parametrization of the band: $X = \cos\varphi + r\cos(\varphi/2)\cos\varphi$, $Y = \sin\varphi + r\cos(\varphi/2)\sin\varphi$, $Z = l + r\sin(\varphi/2)$.

Taking account on the initial condition, and the transformations: $X' = \mathrm{e}^{-Z}X$, $Y' = \mathrm{e}^{-Z}Y$, $Z' = Z$, finally

$$\xi = \exp[-\left(l + r\sin(\varphi/2)\right) + i\varphi]\left(1 + r\cos(\varphi/2)\right).$$

Inserting the above expression into the expansion of the CS in the j basis, the CS in explicit form is obtained:

$$|\xi\rangle = \sum_{j=-\infty}^{\infty} \xi^{-j}\mathrm{e}^{-\frac{j^2}{2}}\,|j\rangle = \sum_{j=-\infty}^{\infty} \mathrm{e}^{l'j - i\varphi j}\mathrm{e}^{-\frac{j^2}{2}}\,|j\rangle\,,$$

where $l' \equiv \left(l + r\sin(\varphi/2)\right) - \ln\left(1 + r\cos(\varphi/2)\right) - i\varphi$. From the above expression, the fiducial vector is $|1\rangle = \sum\limits_{j=-\infty}^{\infty} \mathrm{e}^{-\frac{j^2}{2}}\,|j\rangle$, then

$$|\xi\rangle = \sum_{j=-\infty}^{\infty} \mathrm{e}^{-(\ln\xi)\widehat{J}}\,|1\rangle\,. \tag{1}$$

As is easily seen, the vector $|1\rangle$ is $|0,0\rangle_{r=0}$ in the (l, φ) parametrization. This fact permits us to rewrite expression (1) as $|l,\varphi\rangle = \exp\{[(l + r\sin(\varphi/2)) - \ln(1 + r\cos(\varphi/2)) - i\varphi]j\}|0,0\rangle_{r=0}$. The non-ortogonality formulas (overlap) are explicitly derived[1]:

$$\langle\xi\,|\eta\rangle = \sum_{j=-\infty}^{\infty} (\xi^*\eta)^{-j}\,\mathrm{e}^{-j^2} = \Theta_3\left(\frac{i}{2\pi}\ln(\xi^*\eta)\mid\frac{i}{\pi}\right),$$

$$\langle l,\varphi\,|h,\psi\rangle = \Theta_3\left(\frac{i}{2\pi}(\varphi - \psi) - \frac{l' + h'}{2}\frac{i}{\pi}\mid\frac{i}{\pi}\right).$$

[1]The normalization as a function of Θ_3: $\langle\xi\,|\xi\rangle = \Theta_3\left(\dfrac{i}{\pi}\ln|\xi|\mid\dfrac{i}{\pi}\right)$ or $\langle l,\varphi\,|l,\varphi\rangle = \Theta_3\left(\dfrac{il'}{\pi}\mid\dfrac{i}{\pi}\right)$.

2. THE PHYSICAL PHASE SPACE AND THE NATURAL QUANTIZATION

From the expressions obtained in the previous section and $\widehat{J}\,|j\rangle = j\,|j\rangle$ we notice that the normalization, for the cylinder [2] (boson case) that is φ-dependent, now depends on φ through $l' \equiv (l + r\sin(\varphi/2)) - \ln(1 + r\cos(\varphi/2))$. As $\widehat{J}\,|l,\varphi\rangle = j\,|l,\varphi\rangle$, then

$$\frac{\langle\xi|\,\widehat{J}\,|\xi\rangle}{\langle\xi\,|\xi\rangle} = l' + 2\pi\sin(2l'\pi)\sum_{n=1}^{\infty}\frac{\mathrm{e}^{-\pi^2(2n-1)}}{\left(1+\mathrm{e}^{-\pi^2(2n-1)}\mathrm{e}^{2i\pi l'}\right)\left(1+\mathrm{e}^{-\pi^2(2n-1)}\mathrm{e}^{-2i\pi l'}\right)},$$

where the well-known identities for Θ functions were introduced. Notice the important result coming from the above expression: the fourth condition required for the CS [3], namely $\langle\widehat{J}\rangle = l$, demands not only l to be integer or semi-integer (as the case for the circle quantization), but also that $\varphi = (2k+1)\,\pi$ leading a natural quantization similar as the charge quantization in the Dirac monopole. Precisely, this condition over the angle fixes the position of the particle in the internal or external border of the Möbius band, that for $r = 1/2$ is $s = \pm 1/2$ how it is required to be.

In order to compare our case with the CS constructed in [2], we consider the existence of the unitary operator $U \equiv \mathrm{e}^{i\varphi}$ obeying $[J, U] = U$, then $U\,|j\rangle = |j+1\rangle$. The same average as before for the $\widehat{J}$ operator is

$$\frac{\langle\xi|\,U\,|\xi\rangle}{\langle\xi\,|\xi\rangle} = \mathrm{e}^{-\frac{1}{4}}\mathrm{e}^{i\varphi}\frac{\Theta_2\,(il'/\pi \mid i/\pi)}{\Theta_3\,(il'/\pi \mid i/\pi)} = \mathrm{e}^{-\frac{1}{4}}\mathrm{e}^{i\varphi}\frac{\Theta_3\,(l'+1/2 \mid i\pi)}{\Theta_3\,(l' \mid i\pi)}, \tag{2}$$

where in the last equality the relation $\Theta_2(\nu) = \exp\left[i\pi\left(\frac{1}{4}\tau + \nu\right)\right]\Theta_3\left(\nu + \frac{\tau}{2}\right)$ was introduced. As in [2] we also can perform the relative average for the operator U in order to eliminate the factor $\mathrm{e}^{-\frac{1}{4}}$, then at the first order expression (2) coincides with the unitary circle. It is clear that the denominator in quotient (2) (average with respect to the fiducial CS) serves to centralize the expression of the numerator. However, the claim that U is the best candidate for the position operator is still obscure and requires a special analysis that we will give elsewhere.

3. THE DYNAMICS

To study the dynamics in this nontrivial geometry, we construct the nonrelativistic Lagrangian and the corresponding Hamiltonian:

$$H = \frac{1}{2}\left\{\dot{\varphi}^2\left[(1 + r\cos(\varphi/2))^2 - \frac{r^2}{4}\cos\,\varphi\right] + L_0^2\right\}.$$

Now, $\widehat{H}\,|E\rangle = E\,|E\rangle$. If $|E\rangle = |j\rangle$, imposing the fourth CS requirement [3], we have $\varphi = (2k+1)\pi$ and the expression for the energy takes the form: $E = \dfrac{2j^2}{4+r^2} + \dfrac{L_0^2}{2}$.

From the dynamical expressions given above, it is not difficult to make the following remarks:

1) the Hamiltonian is not a priori T invariant. The H_{MS} is T invariant iff $TL_0 = -L_0$: the variable conjugate to the external momenta l changes under T as J is manifesting with

this symmetry the full inversion of the motion of the particle on a Möbius strip (evidently this is not the case of the motion of the particle on the circle);

2) the distribution of energies is Gaussian: from the Bargmann representation $\phi_j(\xi^*) \equiv \langle\xi|\, E\rangle = (\xi^*)^{-j}\,\mathrm{e}^{-\frac{j^2}{2}}$, and by using the approximate relation from the definition of the Θ function[1], the expression for the distribution of energies can be written as

$$\frac{|\langle j|\xi\rangle|^2}{\langle\xi|\xi\rangle} \approx \frac{1}{\sqrt{\pi}}\,\mathrm{e}^{-\left(j-l'\right)^2}.$$

It is useful to remark here that, when $\varphi = (2k+1)\pi$, $l = l'$, this expression coincides exactly in form with the boson case given in [2], but now l is semi-integer valuated.

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3. *Gazeau J. P., Klauder J. R.* // J. Phys. A. 1999. V. 32. P. 123.

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[1] $\Theta_3\left(\frac{il'}{\pi} \mid \frac{i}{\pi}\right) = \mathrm{e}^{\left(l'\right)^2}\sqrt{\pi}\left(1 + 2\sum_{n=1}^{\infty}\mathrm{e}^{-\pi^2 n^2}\cos\left(2l'\pi n\right)\right) \approx \mathrm{e}^{\left(l'\right)^2}\sqrt{\pi}.$