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STABLE REGIMES OF *d*-DIMENSIONAL **MHD** TURBULENCE

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1 Introduction

The using of renormalization group (RG) approach to the analysis of fully developed magnetohydrodynamic (MHD) turbulence belongs to often discussed topics of classical stochastic processes [1-5]. In these investigations the randomly forced HD and MHD equations have been used to obtain a regular expansion of scaling exponents in the small parameter $\epsilon = 2 - \lambda$. It is the deviation of the power of wave-number λ in the correlation function of the random force from the critical value $\lambda_c = 2$, at which the corresponding field theory is logarithmic [6].

Recently the RG approach to the fully developed turbulence most has been carried out for the analysis of 2-dimensional turbulence [7], or in general, d-dimensional ($d \geq 2$) turbulence [8]. The authors notice that at two dimensions an additional class of divergences appears because the long-range correlation function of the random force is a power-like function of the wave-number proportional to $k^{4-d-2\epsilon}$, which is a singular function of k^2 at the origin supposing $d = 2$. At two dimensions this correlation function is renormalized by counter-terms proportional to k^2, which are added to the force correlation function at the outset. Therefore, in d-dimensional case one must use an additional expansion, the parameter of which is $2\delta = d - 2$ besides 2ϵ [3].

Here we apply a modified minimal substraction scheme [9] based on the fact that the tensor structure of counter-terms is left generally d-dependent in the calculation of divergent part of Green's functions, and it allows us to investigate behaviour of the system under continual transition to $d = 3$ beginning from $d = 2$. We attempt to restore the limit Prandtl number for $d \to 3$ and also to establish the stability region supposing an arbitrary dimension d, $2 \leq d \leq 3$. This paper revises the analysis of recent paper [10] of the randomly forced MHD equations with the proper account of the additional UV - divergences (appeared in $d = 2$) in the developed MHD turbulence in the frame of double expansion approach [4], and, it is alternative to performed analysis of [5]. Renormalization of the corresponding field-theoretic model is performed in one loop approximation, and, it is logical continuation our previous paper [4].

2 Formulation of problem

For convenience there are remained some fundamental treatments of the problem formulation. The present paper deals with study of the general model of stochastic MHD. Therefore, unlike the previous paper [4], where the Lorentzian term was omitted in the Navier-Stokes equation, here the model is described by system of equations for the fluctuating local incompressible velocity field, $\mathbf{v}(x)$, $x \equiv (\mathbf{x}, \mathbf{t})$, $\nabla \cdot \mathbf{v} = 0$, and magnetic field, $\mathbf{b}(x)$ in the general form

$$\partial_t \mathbf{v} + (\mathbf{v} \cdot \nabla)\mathbf{v} - (\mathbf{b} \cdot \nabla)\mathbf{b} - \nu \nabla^2 \mathbf{v} = \mathbf{f}^{\mathbf{v}} , \quad (1)$$

$$\partial_t \mathbf{b} + (\mathbf{v} \cdot \nabla)\mathbf{b} - (\mathbf{b} \cdot \nabla)\mathbf{v} - \nu u \nabla^2 \mathbf{b} = \mathbf{f}^{\mathbf{b}} , \quad (2)$$

with $\nabla \cdot \mathbf{f^v} = 0$ and $\nabla \cdot \mathbf{f^b} = 0$ because the fields $\mathbf{v}, \mathbf{b}$ are solenoidal, $\nabla.\mathbf{v} = \nabla.\mathbf{b} = 0$. The statistics of $\mathbf{v}, \mathbf{b}$ is completely determined by both the non-linear equations (1,2) and the statistics of the external large-scale random forces $\mathbf{f^v}$, $\mathbf{f^b}$. The dissipation is controled by the parameter of kinematic viscosity ν; u denotes inverse Prandtl number.

As usually, statistical properties of the Gaussian forcing with zero mean values of $\langle \mathbf{f^v} \rangle = 0$, $\langle \mathbf{f^b} \rangle = 0$ are determined by relations:

$$\begin{aligned} \langle f_j^v(x_1) f_s^b(x_2) \rangle &= 0 \,, \\ \langle f_j^v(x_1) f_s^v(x_2) \rangle &= u\, \nu^3\, D_{js}\,(x_1 - x_2; [1, g_{v1}, g_{v2}]) \\ \langle f_j^b(x_1) f_s^b(x_2) \rangle &= u^2\, \nu^3\, D_{js}\,(x_1 - x_2; [a, g_{b10}, g_{b20}]) \,, \end{aligned} \tag{3}$$

where the correlation matrix

$$\begin{aligned} D_{js}\,(x; [a, g_1, g_2]) &= \delta(t_1 - t_2) \int \frac{\mathrm{d}^d \mathbf{k}}{(2\pi)^d}\, P_{js}(\mathbf{k}) \exp[\,\mathbf{ik.x}\,] \\ &\quad \times \left[g_1\, k^{2-2\delta-2\,a\epsilon} + g_2\, k^2 \right] \end{aligned} \tag{4}$$

with transverse second-rank projector $P_{js}(\mathbf{k}) = \delta_{js} - k_j k_s / k^2$, is determined by constants g_1, g_2, and, the relation $d = 2 + 2\delta$ was used in exponent of k. The free parameter a controls the power form of magnetic forcing. The necessity to introduce a combined forcing and also to include the additional couplings (g_{v2}, g_{b2}) for obtaining of multiplicatively renormalizable two dimensional stochastic MHD, is absent in traditional formulation of stochastic hydrodymamics. The definition (4) includes two principal – low- and high-wave number – scale kinetic forcing separated by a transition region at the vicinity of the characteristic wave-number of order $O([g_{v10}/g_{v20}]^{\frac{1}{\delta}})$. In language of classical hydrodynamics the forcing contribution $\propto k^2$ corresponds to the appearence of large eddies convected by small and active ones and it is represented by the local term of $\mathbf{v}'\nabla^2\mathbf{v}'$. In its analogy the term $\mathbf{b}'\nabla^2\mathbf{b}'$ is added to the magnetic forcing. So, our stochastic MHD system can be described by the field-theoretical action

$$\begin{aligned} S &= \frac{1}{2} \int \mathrm{dx}_1 \int \mathrm{dx}_2 \\ &\quad \Big\{ u_0\, \nu_0^3\, v_j'(x_1) D_{js}\,(x_1 - x_2; [1, g_{v10}, g_{v20}])\, v_s'(x_2) + \\ &+ \; u_0^2\, \nu_0{}^3\, b_j'(x_1) D_{js}\,(x_1 - x_2; [a, g_{b10}, g_{b20}])\, b_s'(x_2) \Big\} + \\ &+ \; \int \mathrm{dx}\, \mathbf{v}' \cdot \Big(-\partial_t \mathbf{v} + \nu_0\, \nabla^2 \mathbf{v} - (\mathbf{v} \cdot \nabla)\mathbf{v} - (\mathbf{b} \cdot \nabla)\mathbf{b} \Big) \\ &+ \; \mathbf{b}' \cdot \Big(-\partial_t \mathbf{b} + u_0\, \nu_0 \nabla^2 \mathbf{b} + (\mathbf{b} \cdot \nabla)\mathbf{v} - (\mathbf{v} \cdot \nabla)\mathbf{b} \Big) \,. \end{aligned} \tag{5}$$

All dimensional constants $g_{v10}, g_{b10}, g_{v20}$ and g_{b20}, which control the amount of randomly injected energy given by (3), (4), play the role of coupling constants of the perturbative expansion. Their universal values have been determined after the parameters ϵ, δ have been choosen to give the desired power form of forcing and desired dimension.

For the convenience of further calculations the factors $\nu_0^3 u_0$ and $\nu_0^3 u_0^2$ including the "bare" (molecular) viscosity ν_0 and the "bare" (molecular or microscopic) magnetic inverse Prandtl number u_0 have been extracted. The bare (non-renormalized) quantities are denoted by subscript "0".

3 One loop order renormalization

We apply usual RG procedure and corresponding pertubative techniques described elsewhere in details [11]. The model (5) is renormalizable by the standard power-counting rules, and for limits $\epsilon \to 0$, $\delta \to 0$ possesses the ultraviolet (UV) divergences which are present in one-particle irreducible two-point Green functions $\Gamma^{vv'}, \Gamma^{v'v}, \Gamma^{bb'}, \Gamma^{b'b}, \Gamma^{b'b'}, \Gamma^{v'v'}$ and vertex function $\Gamma^{v'bb}$. Due to the last the field $\mathbf{b}$ (together with $\mathbf{b}'$) also must be renormalized. Free propagators $\hat{\Delta}$ have been calculated in [4].

The UV divergences proportional to $1/\epsilon, 1/\delta, 1/(2\epsilon+\delta), 1/(2a\epsilon+\delta), 1/(\epsilon(1+a)+\delta)$, have been removed by adding suitable counter terms to the basic action obtained from (5). Namely, the original form of the action S implies the counter terms

$$
\begin{aligned}
S_{count} &= \int \mathrm{dx}\,[\nu\,(1-\mathrm{Z}_1)\,\mathbf{v}'\nabla^2\,\mathbf{v} + \mathrm{u}\nu\,(1-\mathrm{Z}_2)\,\mathbf{b}'\nabla^2\,\mathbf{b} \\
&+ \frac{1}{2}(\,Z_4-1)u\nu^3 g_{v2}\,\mu^{-2\delta}\,\mathbf{v}'\nabla^2\mathbf{v}' + \frac{1}{2}\,(Z_5-1)\,u^2\nu^3 g_{b2}\,\mu^{-2\delta}\,\mathbf{b}'\nabla^2\mathbf{b}' \\
&+ (1-Z_3)\,\mathbf{v}'(\mathbf{b}\cdot\nabla)\mathbf{b}]\,. \qquad (6)
\end{aligned}
$$

Within UV renormalization the divergences appearing in form of Laurent series in the poles are contained in the constants Z_1, Z_2, Z_4, Z_5 renormalizing the "bare" parameters $e_0 \equiv \{g_{i0}, \nu_0, u_0\}$ and also the constant Z_3 renormalizing fields $\mathbf{b}, \mathbf{b}'$. The remaining fields $\vec{v}', \vec{v}$ are not renormalized due to the Galilean invariance of the model (5).

Renormalized Green functions are expressed in terms of the renormalized parameters

$$
\begin{aligned}
g_{v1} &= g_{v10}\,\mu^{-2\epsilon}\,Z_1^2 Z_2, & g_{v2} &= g_{v20}\,\mu^{2\delta}\,Z_1^2 Z_2 Z_4^{-1}, \\
g_{b1} &= g_{b10}\,\mu^{-2a\epsilon}\,Z_1 Z_2^2 Z_3^{-1}, & g_{b2} &= g_{b20}\,\mu^{2\delta}\,Z_1 Z_2^2 Z_3^{-1} Z_5^{-1}, \\
\nu &= \nu_0\,Z_1^{-1}, & u &= u_0\,Z_2^{-1} Z_1 \qquad (7)
\end{aligned}
$$

appearing in the renormalized action S^R connected with the action (5) by the relation of multiplicative renormalization: $S^R\{\mathbf{e}\} = S\{\mathbf{e_0}\}$. The renormalized action S^R, which depends on the renormalized parameters $e(\mu)$, yields renormalized Green functions without UV divergences. The expressions (7) yield the β-functions analogous to obtained in [3, 4]:

$$\begin{aligned}
\beta_{gv1} &= g_{v1}\left(-2\epsilon + 2\gamma_1 + \gamma_2\right), & \beta_{gv2} &= g_{v2}\left(2\delta + 2\gamma_1 + \gamma_2 - \gamma_4\right), \\
\beta_{gb1} &= g_{b1}\left(-2\,a\epsilon + \gamma_1 + 2\gamma_2 - \gamma_3\right), & \beta_{gb2} &= g_{b2}\left(2\delta + \gamma_1 + 2\gamma_2 - \gamma_3 - \gamma_5\right) \\
\beta_u &= u\left(\gamma_1 - \gamma_2\right). & & \qquad (8)
\end{aligned}$$

The calculation of UV divergences gives Z-constants in the form

$$\begin{aligned}
Z_1 &= 1 + \frac{S_d}{(2\pi)^d}\left[u\,\lambda_5\left(\frac{g_{v2}}{2\delta} - \frac{g_{v1}}{2\epsilon}\right) + \lambda_6\left(\frac{g_{b2}}{2\delta} - \frac{g_{b1}}{2a\epsilon}\right)\right], \\
Z_2 &= 1 + \frac{S_d}{(2\pi)^d(u+1)}\left[\lambda_1\left(\frac{g_{v2}}{2\delta} - \frac{g_{v1}}{2\epsilon}\right) + \lambda_3\left(\frac{g_{b2}}{2\delta} - \frac{g_{b1}}{2a\epsilon}\right)\right], \qquad (9) \\
Z_3 &= 1 + \frac{S_d}{(2\pi)^d}\lambda_7\left(\frac{g_{v1}}{2\epsilon} - \frac{g_{v2}}{2\delta} - \frac{g_{b1}}{2a\epsilon} + \frac{g_{b2}}{2\delta}\right), \\
Z_4 &= 1 + \frac{S_d}{(2\pi)^d}\frac{\lambda_4}{g_{v2}}\left(\frac{ug_{v1}^2}{2\delta + 4\epsilon} + \frac{2ug_{v1}g_{v2}}{2\epsilon} - \frac{ug_{v2}^2}{2\delta} + \frac{g_{b1}^2}{2\delta + 4a\epsilon} + \frac{2g_{b1}g_{b2}}{2a\epsilon} - \frac{g_{b2}^2}{2\delta}\right), \\
Z_5 &= 1 + \frac{S_d}{(2\pi)^d}\frac{\lambda_2}{(u+1)g_{b2}}\left(\frac{g_{v1}\,g_{b1}}{2\delta + 2\epsilon(1+a)} + \frac{g_{v1}\,g_{b2}}{2\epsilon} + \frac{g_{v2}\,g_{b1}}{2a\epsilon} - \frac{g_{v2}\,g_{b2}}{2\delta}\right),
\end{aligned}$$

and in consequence one obtains γ-functions :

$$\begin{aligned}
\gamma_1 &= \frac{S_d}{(2\pi)^d}\left(u\,\lambda_5\,g_v + \lambda_6\,g_b\right), & \gamma_2 &= \frac{S_d}{(2\pi)^d}\frac{\left(\lambda_1\,g_v + \lambda_3\,g_b\right)}{u+1}, \\
\gamma_3 &= \frac{S_d}{(2\pi)^d}\lambda_7(-g_v + g_b), & \gamma_4 &= \frac{S_d}{(2\pi)^d}\frac{\lambda_4}{g_{v2}}(u\,g_v^2 + g_b^2), \qquad (10) \\
\gamma_5 &= \frac{S_d}{(2\pi)^d}\frac{\lambda_2}{(1+u)}\frac{g_v\,g_b}{g_{b2}},
\end{aligned}$$

where S_d denote d-dimensional sphere, $S_d = 2\pi^{d/2}/\Gamma(d/2)$, and $g_v = g_{v1} + g_{v2}$, $g_b = g_{b1} + g_{b2}$, and d-dependent λ-coefficients are

$$\begin{aligned}
\lambda_1 &= \frac{d-1}{2d}, & \lambda_2 &= \frac{d-2}{2d}, & \lambda_3 &= \frac{d-3}{2d}, \\
\lambda_4 &= \frac{d^2-2}{4d(d+2)}, & \lambda_5 &= \frac{d-1}{4(d+2)}, & \lambda_7 &= \frac{1}{d(d+2)}, \\
& & \lambda_6 &= \frac{d^2+d-4}{4d(d+2)}.
\end{aligned}$$

4 Fixed points

4.1 "Kinetic" fixed point

Within the approach discussed in Ref.[4] the nontrivial stable "kinetic" fixed point of RG equations has been found:

$$
\begin{aligned}
u^* &= \left(\sqrt{(16+9d)/d} - 1\right)/2\,, \\
g^*_{v1} &= \frac{(2\pi)^d}{S_d}\frac{8\epsilon(u^*+1)[3d^3-(9-4\epsilon)d^2-6d(\epsilon-1)+4\epsilon]}{9(d-1)^2(d+2\epsilon-2)}\,, \\
g^*_{v2} &= \frac{(2\pi)^d}{S_d}\frac{8\epsilon^2(u^*+1)(d^2-2)}{9(d-1)^2(d+2\epsilon-2)}
\end{aligned}
\qquad (11)
$$

and $g^*_{b1} = g^*_{b2} = 0$. This fixed point is identical with found one in [4] for case when the magnetic field is considered as a passive admixture.

4.2 "Magnetic" fixed point

Let one examines a possibility of existence a nontrivial "magnetic fixed point". The magnetic fixed point is characterized by zeroth g^*_{v1} and u^*, so, the system of five β-equations reduces to the three equations which can be obtained substituting γ-functions (10) into the system of β-functions (8). Applying $g_{v1} = u = 0$ one obtains

$$
\begin{aligned}
a_1 g_{v2} + a_2 g^2_{v2} + a_3 g_{v2} g_b - a_4 g^2_b &= 0\,, \\
-A_0 + a_5 g_{v2} + a_6 g_b &= 0\,, \\
a_1 g_{b2} + a_5 g_{v2} g_{b2} + a_6 g_{b2} g_b - a_7 g_{v2} g_b &= 0\,,
\end{aligned}
\qquad (12)
$$

where

$$
\begin{aligned}
A_0 &= \frac{2a\epsilon}{S_d}\,, \quad a_1 = \frac{2(d-2)}{S_d}\,, \quad a_2 = \frac{(d-1)}{2d}\,, \\
a_3 &= \frac{(d^2-5)}{d(d+2)}\,, \quad a_4 = \frac{(d^2-2)}{4d(d+2)}\,, \quad a_5 = \frac{(d^2+d-3)}{d(d+2)}\,, \\
a_6 &= \frac{(5d^2-3d-24)}{4d(d+2)}\,, \quad a_7 = \frac{(d-2)}{2d}\,.
\end{aligned}
\qquad (13)
$$

This system can be analytically solved with respect to g_{v2}, g_{b1}, g_{b2} . Because all g_i must be positive, the system (12) with $g_{v1} = u = 0$ gives the only solution,

$$
g_{v2} = \frac{a_6 g_b - A_0}{a_5}\,,
$$

$$g_{b1} = g_b - \frac{g_b(a_6 g_b - A_0) - a_5 a_7}{a_5(a_1 + 2a_6 g_b - A_0)} ,$$
$$g_{b2} = \frac{g_b(a_6 g_b - A_0) - a_5 a_7}{a_5(a_1 + 2a_6 g_b - A_0)} , \qquad (14)$$

where

$$g_b = \frac{-a_1 a_5 a_6 + a_3 a_5 A_0 - 2a_2 a_6 A_0 + a_5\sqrt{D}}{2(a_4 a_5^2 + a_3 a_5 a_6 - a_2 a_6^2)} ,$$
$$D = a_1^2 a_6^2 + 4a_1 a_4 a_5 A_0 + 2a_1 a_3 a_6 A_0 + a_3^2 A_0^2 + 4a_2 a_4 A_0^2.$$

Note that the parameters a and ϵ appears in the solution only as the product $a\epsilon$ in A_0. Numerical analysis of the expressions (14) shows that all g_i have a discontinuity at $d_d = 2.02303$, and, a physical solution non exists for any a, ϵ if $d \le d_d$. The stability region of the magnetic fixed point is demonstrated in Fig. 1.

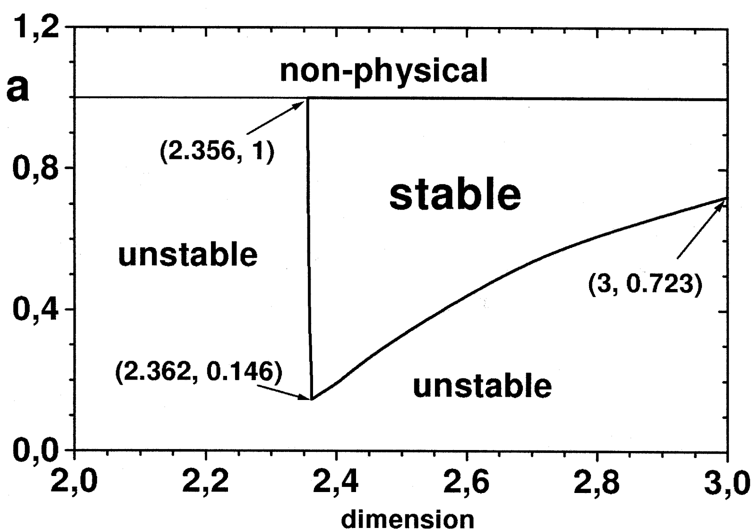

Figure 1: *The stability region of the magnetic fixed point in the plane of* $\{d, a\}$ *for the physical value of* $\epsilon = 2$.

5 Conclusions

In this paper we revised the calculations of stability ranges of developed magnetohydrodynamic turbulence [10] and it is logical continuation of previous paper [4], where the magnetic field has been considered as a passive admixture. The modified standard minimal substraction scheme [9] has been used in the dimension region of $d \geq 2$ up to $d = 3$. Two stable fixed point has been found. The first, the kinetic one, corresponds to the fixed point found in [4] with nonzero inverse Prandtl number u. A new nontrivial results of the present paper is connected with analytical calculation of the nontrivial stable fixed magnetic point with $u = g_{v1} = 0$ but nonzero g_{v2}, g_{b1} and g_{b2} . A physical region of the renormalization group fixed point lies below the $a\epsilon = 2$ line, see in Fig. 1, where a stability region of the Kolmogorov scaling regime is also demonstrated. This point losses stability below critical value of dimension $d_c = 2.36$ (independently on the a-parameter of a magnetic forcing) as well as below the value of $a_c = 0.146$ (independently on the dimension). This result slightly modifies the result of a numerical calculation in Ref.[5] performed beyond the frame of the double expansion method which was used here.

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Юрчишин М., Стеглик М. E17-2002-31

Устойчивый режим d-мерной МГД-турбулентности

Методом ренормализационной группы рассматривается развитая магнитогидродинамическая турбулентность с двойным разложением в окрестности двумерного пространства в интервале $d=\langle 2,3 \rangle$. Работа является логическим продолжением сообщения ОИЯИ Е17-2001-20 (Дубна, 2001). Для анализа устойчивости режима колмогоровского скейлинга использована некоторая модификация стандартной минимальной схемы вычитаний. Кроме известной кинетической фиксированной точки рассчитана стабильная магнитная точка и проверена область ее стабильности. Она теряет стабильность ниже критического значения $d_c = 2{,}36$ (независимо от параметра a магнитной накачки), а также ниже критического значения $a_c = 0{,}146$ (независимо от размерности d).

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Jurcisin M., Stehlik M. E17-2002-31

Stable Regimes of d-Dimensional MHD Turbulence

Developed magnetohydrodynamic turbulence near two dimensions d up to three dimensions has been investigated by means of renormalization group approach and double expansion regularization, and it is logical continuation of the previous Communication of JINR E17-2001-20 (Dubna, 2001). Some modification of standard minimal substraction scheme has been used to analyze the stability of the Kolmogorov scaling regime which is governed by renormalization group fixed point. Besides the known kinetic fixed point the magnetic stable fixed point has been calculated and its stability region has been examined. The point loses stability below the critical value of dimension $d_c = 2.36$ (independently of the a-parameter of a magnetic forcing) as well as below the value of $a_c = 0.146$ (independently of the d-dimension).

The investigation has been performed at the Bogoliubov Laboratory of Theoretical Physics, JINR.

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