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ON THE FOCK SPACE REALIZATIONS OF NONLINEAR ALGEBRAS DESCRIBING THE HIGH SPIN FIELDS IN AdS SPACES

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1 Introduction

It is well known that many physical systems are invariant with respect to nonlinear symmetries. As the most known examples one can mention the Kepler motion of the planets or the hydrogen atom. Though the general classification of nonlinear algebras does not exists, the examples of such algebras, as well as the description of physical systems in which they act, one can find in the literature. Some references and rather detailed investigation of the so called finite W-algebras see in [1].

The very interesting class of nonlinear symmetries arises in the description of the higher spin fields in the AdS space-time. They depend on the type of the corresponding Young tableaux[2] and become more complicated when the number N of raws is growing. The simplest of these symmetries is connected with the Young tableaux with $N = 1$ and describes the totally symmetric fields of higher ranks which correspond to unitary representations in the AdS space-time of arbitrary dimension.

Starting from a given physical system with known nonlinear symmetry one can construct the explicit expressions for the generators of the symmetry transformations, usually in terms of the harmonic oscillators connected with the physical observables. In the case at hand the generators are constructed in terms of a space-time derivatives and N additional harmonic oscillators, which describe spin degrees of freedom and can be viewed as some internal distances in the classical models of composite particles like a discrete string[3],[4]. No doubt that this representation of the algebra generators is not unique. The question arises, how one can construct other representations of nonlinear algebras? Moreover, the knowledge of such new realizations of the algebras can help in the search of physical systems having these algebras as a symmetries. It can also help in the investigations of mathematical properties of these algebras.

In the present paper we propose the regular method of construction of realizations for nonlinear algebras. We begin with the description how the method works in the case of linear algebras, [5],[6], and after that generalize it for the simplest case ($N = 1$) of totally symmetric fields describing the higher spins in the AdS space-time. For this goal we construct the Verma module for this nonlinear algebra. The representation of the algebra generators is constructed in terms of some auxiliary harmonic oscillators as a result of establishing the one-to-one correspondence of the Verma module vectors with the vectors of the Fock space generated by these oscillators. In conclusions we mention some interesting mathematical problems connected with the construction and some possible developments.

2 Construction of the Fock space realizations in the Lie algebra case

In this section we describe the method, based on the construction given in [5],[6]. Let $\hat{H}^i$, $(i = 1, ..., k)$ and $\hat{E}^\alpha$ be the Cartan generators and root vectors of the algebra with the following commutation relations

$$\left[\hat{H}^i, \hat{E}^\alpha\right] = \alpha(i)\hat{E}^\alpha, \tag{2.1}$$

$$\left[\hat{E}^\alpha, \hat{E}^{-\alpha}\right] = \sum \alpha^i \hat{H}^i, \tag{2.2}$$

$$\left[\hat{E}^\alpha, \hat{E}^\beta\right] = N^{\alpha\beta}\hat{E}^{\alpha+\beta}. \tag{2.3}$$

Roots $\alpha(i)$ and parameters α^i, $N^{\alpha\beta}$ are structure constants of the algebra in the Cartan – Weyl basis.

Consider the highest weight representation of the algebra under consideration with the highest weight vector $|0\rangle_V$, annihilated by the positive roots

$$E^{\alpha}|0\rangle_V = 0 \tag{2.4}$$

and being the proper vector of the Cartan generators

$$H^i|0\rangle_V = h^i|0\rangle_V. \tag{2.5}$$

The representation which is given by (2.4) and (2.5) in the mathematical literature is called the Verma module [7]. Following the Poincare – Birkhoff – Witt theorem, the basis space of this representation is given by vectors

$$|n_1, n_2, \ldots, n_r\rangle_V = (E^{-\alpha_1})^{n_1}(E^{-\alpha_2})^{n_2}\ldots(E^{-\alpha_r})^{n_r}|0\rangle_V \tag{2.6}$$

where $\alpha_1, \alpha_2, \ldots \alpha_r$ is some ordering of positive roots and $n_i \in N$.

Using the commutation relations of the algebra and the formula

$$AB^n = \sum_{k=0}^{n}\binom{n}{k}B^{n-k}[[\ldots[A,B],B]\ldots]$$

one can calculate the action of all generators on arbitrary vector of the Verma module. In [5] it was shown that, making use of the map

$$|n_1, n_2, \ldots, n_r\rangle_V \longleftrightarrow |n_1, n_2, \ldots, n_r\rangle \tag{2.7}$$

where $|n_1, n_2, \ldots, n_r\rangle$ are base vectors of the Fock space

$$|n_1, n_2, \ldots, n_r\rangle = (b_1^+)^{n_1}(b_2^+)^{n_2}\ldots(b_r^+)^{n_r}|0\rangle. \tag{2.8}$$

generated by creation and annihilation operators b_i^+, b_i $i = 1, 2, \ldots, r$ with the standard commutation relations

$$\left[b_i, b_j^+\right] = \delta_{ij}, \tag{2.9}$$

the generators of the algebra in the Fock space can be written as polynomials in creation operators.

As an explicit example of the construction given above let us consider the representations of the algebra, which can be used for the description of the higher spin fields in the flat space-time. This algebra is linear and its generators $L_1^+, L_2^+, L_1, L_2, G_0, L_0$, which single out the irreducible representation of the Poincare group have the following commutation relations

$$\begin{aligned}
&[L_1^+, L_1] = -L_0, \quad [L_1^+, L_2] = -L_1, \quad [L_2^+, L_1] = -L_1^+,\\
&[L_0, L_1] = 0, \quad [L_0, L_2] = 0, \quad [L_1^+, L_2^+] = 0,\\
&[L_1^+, L_0] = 0, \quad [L_1, L_0] = 0, \quad [L_1, L_2] = 0,\\
&[G_0, L_1] = -L_1, \quad [L_1^+, G_0] = -L_1^+, \quad [L_0, G_0] = 0,
\end{aligned} \tag{2.10}$$

with the $so(2,1)$ subalgebra

$$[G_0, L_2] = -2L_2, \quad [L_2^+, G_0] = -2L_2^+, \quad [L_2^+, L_2] = -G_0. \tag{2.11}$$

The defining relations for the highest vector of the Verma module are:

$$L_1|0\rangle_V = 0, L_2|0\rangle_V = 0, G_0|0\rangle_V = h|0\rangle_V, L_0|0\rangle_V = e|0\rangle_V.$$

As the base of the representation space one takes the following vectors

$$|n_1, n_2\rangle_V = (L_1^+)^{n_1}(L_2^+)^{n_2}|0\rangle_V \qquad (2.12)$$

and after the simple calculations one obtains

$$\begin{aligned}
L_1^+|n_1, n_2\rangle_V &= |n_1+1, n_2\rangle_V \\
L_2^+|n_1, n_2\rangle_V &= |n_1, n_2+1\rangle_V \\
G_0|n_1, n_2\rangle_V &= (n_1+2n_2+h)|n_1, n_2\rangle_V \\
L_0|n_1, n_2\rangle_V &= e|n_1, n_2\rangle_V \\
L_1|n_1, n_2\rangle_V &= n_2|n_1+1, n_2-1\rangle_V + en_1|n_1-1, n_2\rangle_V \\
L_2|n_1, n_2\rangle_V &= n_2(h+n_1+n_2-1)|n_1, n_2-1\rangle_V + e\frac{n_1(n_1-1)}{2}|n_1-2, n_2\rangle_V.
\end{aligned} \qquad (2.13)$$

Consider the auxiliary Fock space generated by the annihilation and creation operators b_1, b_2, b_1^+, b_2^+. If one takes the vectors in the Fock space

$$|n_1, n_2\rangle = (b_1^+)^{n_1}(b_2^+)^{n_2}|0\rangle \qquad (2.14)$$

in one-to-one correspondence with the vectors (2.12) in the Verma module, it can be seen that the action of the following operators in the Fock space

$$\begin{aligned}
L_1^+ &= b_1^+ \\
L_2^+ &= b_2^+ \\
G_0 &= b_1^+b_1 + 2b_2^+b_2 + h \\
L_0 &= e, \\
L_1 &= b_1^+b_2 + eb_1, \\
L_2 &= (b_1^+b_1 + b_2^+b_2 + h)b_2
\end{aligned} \qquad (2.15)$$

is identical to the expressions (2.13) for the Verma module. So, (2.15) gives realization of the algebra in the Fock space.

3 Construction of the Fock space realization in nonlinear algebra case

In the AdS space-time the algebra (2.10) is modified by terms proportional to the parameter r which is the square of the inverse radius of this space. When $r \neq 0$ the algebra is nonlinear and have the following nonzero commutation relations [2]

$$\begin{aligned}
&[L_1^+, L_1] = -L_0, \quad [L_1^+, L_2] = -L_1, \quad [L_2^+, L_1] = -L_1^+, \\
&[L_0, L_1] = -2rL_1 + 4rG_0L_1 - 8rL_1^+L_2, \\
&[L_1^+, L_0] = -2rL_1^+ + 4rL_1^+G_0 - 8rL_2^+L_1, \\
&[G_0, L_1] = -L_1, \quad [L_1^+, G_0] = -L_1^+,
\end{aligned} \qquad (3.1)$$

with the $so(2,1)$ subalgebra (2.11). The structure of this nonlinear algebra is simplified if one introduces new generator C instead of the generator L_0: $L_0 = C + 2K$, where K is the Casimir operator of the subalgebra $so(2,1)$

$$K = 4L_2^+ L_2 - G_0 G_0 + 2G_0. \tag{3.2}$$

The generator C commute with all other generators and is the central charge of the algebra.

It means that we are still able to define the Verma modules as the highest weight representation of this algebra under consideration with the highest weight vector $|0\rangle_V$, annihilated by L_1, L_2

$$L_1|0\rangle_V = 0, \quad L_2|0\rangle_V = 0, \tag{3.3}$$

and being the proper vector of the generators G_0, C and, correspondingly, of G_0, L_0

$$G_0|0\rangle_V = h|0\rangle_V, \quad L_0|0\rangle_V = e|0\rangle_V. \tag{3.4}$$

The basis of the representation space is given by vectors

$$(L_1^+)^{n_1}(L_2^+)^{n_2}|0\rangle_V = |n_1, n_2\rangle_V .$$

For further calculations we will define the operators K_1, K_2 with the help of the following relations

$$\begin{aligned}
&[K, L_1^+] = L_1^+ - 2L_1^+ G_0 + 4L_2^+ L_1 = K_1 \\
&[L_0, L_1^+] = 2rK_1 \\
&[K_1, L_1^+] = 4L_2^+ L_0 - 2(L_1^+)^2 = K_2 \\
&[K_2, L_1^+] = 8rL_2^+ K_1 \\
&[K_1, L_2^+] = 0, \quad [K_2, L_2^+] = 0 .
\end{aligned} \tag{3.5}$$

It is easy to see that

$$\begin{aligned}
K_1|0\rangle &= (1-2h)|1,0\rangle \\
K_2|0\rangle &= 4e|0,1\rangle - 2|2,0\rangle .
\end{aligned}$$

To obtain the explicit form of the representation we need the commutation rules with $(L_1^+)^n, L_2^+)^n$

$$\begin{aligned}
G_0(L_1^+)^n &= (L_1^+)^n G_0 + n(L_1^+)^n \\
G_0(L_2^+)^n &= (L_2^+)^n G_0 + 2n(L_2^+)^n \\
L_0(L_1^+)^n &= (L_1^+)^n L_0 + \sum_{k=1} \frac{(8r)^k}{4}\binom{n}{2k-1}(L_1^+)^{n-2k+1}(L_2^+)^{k-1}K_1 + \\
&+ \sum_{k=1} \frac{(8r)^k}{4}\binom{n}{2k}(L_1^+)^{n-2k}(L_2^+)^{k-1}K_2 \\
L_1(L_1^+)^n &= (L_1^+)^n L_1 + n(L_1^+)^{n-1}L_0 + \sum_{k=1} \frac{(8r)^k}{4}\binom{n}{2k}(L_1^+)^{n-2k}(L_2^+)^{k-1}K_1 + \\
&+ \sum_{k=1} \frac{(8r)^k}{4}\binom{n}{2k+1}(L_1^+)^{n-2k-1}(L_2^+)^{k-1}K_2
\end{aligned}$$

$$
\begin{aligned}
L_1(L_2^+)^n &= (L_2^+)^n L_1 + nL_1^+(L_2^+)^{n-1} \\
L_2(L_1^+)^n &= (L_1^+)^n L_2 + n(L_1^+)^{n-1}L_1 + \binom{n}{2}(L_1^+)^{n-2}L_0 + \\
&+ \sum_{k=1} \frac{(8r)^k}{4}\binom{n}{2k+1}(L_1^+)^{n-2k-1}(L_2^+)^{k-1}K_1 + \\
&+ \sum_{k=1} \frac{(8r)^k}{4}\binom{n}{2k+2}(L_1^+)^{n-2k-2}(L_2^+)^{k-1}K_2 \\
L_2(L_2^+)^n &= (L_2^+)^n L_2 + n(L_2^+)^{n-1}G_0 + n(n-1)(L_2^+)^{n-1}.
\end{aligned} \qquad (3.6)
$$

With the help of these formulas we obtain explicitly the Verma module representation

$$
\begin{aligned}
L_1^+|n_1,n_2\rangle_V &= |n_1+1,n_2\rangle_V \\
L_2^+|n_1,n_2\rangle_V &= |n_1,n_2+1\rangle_V \\
G_0|n_1,n_2\rangle_V &= (n_1+2n_2+h)|n_1,n_2\rangle_V \\
L_0|n_1,n_2\rangle_V &= e|n_1,n_2\rangle_V + (1-2h)\sum_{k=1}\frac{(8r)^k}{4}\binom{n_1}{2k-1}|n_1-2k+2,n_2+k-1\rangle_V + \\
&+ \sum_{k=1}\frac{(8r)^k}{4}\binom{n_1}{2k}\Big(4e|n_1-2k,n_2+k\rangle_V - 2|n_1-2k+2,n_2+k-1\rangle_V\Big) \\
L_1|n_1,n_2\rangle_V &= n_2|n_1+1,n_2-1\rangle_V + en_1|n_1-1,n_2\rangle_V + \\
&+ \frac{(1-2h)}{4}\sum_{k=1}(8r)^k\binom{n_1}{2k}|n_1-2k+1,n_2+k-1\rangle_V + \\
&+ e\sum_{k=1}(8r)^k\binom{n_1}{2k+1}|n_1-2k-1,n_2+k\rangle_V - \\
&- \frac{1}{2}\sum_{k=1}(8r)^k\binom{n_1}{2k+1}|n_1-2k+1,n_2+k-1\rangle_V \\
L_2|n_1,n_2\rangle_V &= n_2(h+n_1+n_2-1)|n_1,n_2-1\rangle_V + e\frac{n_1(n_1-1)}{2}|n_1-2,n_2\rangle_V + \\
&+ \frac{(1-2h)}{4}\sum_{k=1}(8r)^k\binom{n_1}{2k+1}|n_1-2k,n_2+k-1\rangle_V + \\
&+ e\sum_{k=1}(8r)^k\binom{n_1}{2k+2}|n_1-2k-2,n_2+k\rangle_V - \\
&- \frac{1}{2}\sum_{k=1}(8r)^k\binom{n_1}{2k+2}|n_1-2k,n_2+k-1\rangle_V .
\end{aligned} \qquad (3.7)
$$

Now again if we use the operators b_1, b_2, b_1^+, b_2^+ we obtain the Fock space realization of the algebra (3.1)

$$
\begin{aligned}
L_1^+ &= b_1^+ \\
L_2^+ &= b_2^+ \\
G_0 &= b_1^+b_1 + 2b_2^+b_2 + h \\
L_0 &= \sum_{k=0}(8r)^k\left(\frac{e}{(2k)!} - \frac{2r(2h-1)}{(2k+1)!}b_1^+b_1 - \frac{4r}{(2k+2)!}(b_1^+)^2b_1^2\right)b_1^{2k}(b_2^+)^k
\end{aligned} \qquad (3.8)
$$

$$
\begin{aligned}
L_1 &= b_1^+ b_2 + \sum_{k=0} (8r)^k \left(\frac{e}{(2k+1)!} - \frac{2r(2h-1)}{(2k+2)!} b_1^+ b_1 - \frac{4r}{(2k+3)!} (b_1^+)^2 b_1^2 \right) b_1^{2k+1} (b_2^+)^k \\
L_2 &= (b_1^+ b_1 + b_2^+ b_2 + h) b_2 + \\
&+ \sum_{k=0} (8r)^k \left(\frac{e}{(2k+2)!} - \frac{2r(2h-1)}{(2k+3)!} b_1^+ b_1 - \frac{4r}{(2k+4)!} (b_1^+)^2 b_1^2 \right) b_1^{2k+2} (b_2^+)^k .
\end{aligned}
$$

The infinite sums in this expressions are rather simple. For example, first of them can be written in terms of the formal variable $x = \sqrt{8rb_1^2 b_2}$ as

$$
e \sum_{k=0} \frac{(8r)^k}{(2k)!} b_1^{2k} (b_2^+)^k = e \cosh x .
$$

All the other sums can be also written in terms of the $\cosh x$ and $\sinh x$.

4 Conclusions

In this paper we have generalized the method of constructions of Fock space representations for nonlinear algebras, developed for linear algebras in [5],[6]. The method uses the notion of the Verma module for an algebra under consideration. Taking as an example the very important physically case of higher spin fields in the AdS space-time we constructed the Verma module, as well as the Fock space representation for the generators of the corresponding nonlinear algebra.

From the mathematical point of view the construction of the Verma module gives the possibility to analyze its structure and search for the singular vectors in it. The general procedure will then help to construct the finite dimensional representations of considered nonlinear algebra. It would be interesting to carry out investigations along this line, as well as make the generalization on the case of more complicated nonlinear algebras connected with irreducible unitary representations in the AdS space-time of arbitrary dimensions, described by the Young tableaux with more then one raw.

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О реализации в пространстве Фока нелинейных алгебр, описывающих поля высших спинов в пространстве анти-де Ситтера

Метод построения реализаций алгебр Ли в пространстве Фока обобщен на нелинейные алгебры. Рассмотрен пример нелинейной алгебры связей, описывающих полностью симметричные поля высших спинов в пространстве анти-де Ситтера.

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On the Fock Space Realizations of Nonlinear Algebras Describing the High Spin Fields in AdS Spaces

The method of construction of Fock space realizations of Lie algebras is generalized for nonlinear algebras. We consider as an example the nonlinear algebra of constraints which describe the totally symmetric fields with higher spins in the AdS space-time.

The investigation has been performed at the Bogoliubov Laboratory of Theoretical Physics, JINR.

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