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A METHOD FOR SOLVING DIFFERENTIAL EQUATIONS VIA APPROXIMATION THEORY

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1 General remarks

In this paper, we propose a method for approximate solving well-posed boundary-value problems for differential equations. In fact, this method is a variant of the well-known method of quasi-reversibility considered in [1]. A principal difference between considerations in [1] and ours is that, unlike in [1], we principally apply it to well-posed problems. Roughly speaking, it is applicable to certain classes of such problems so that, for approximate solutions to converge to an exact solution of a differential equation, one does not need to look for standard questions as a stability of a finite-difference scheme. Also, though we consider only applications of the method to linear problems, it can also be exploited in nonlinear cases.

First, to describe our idea, we consider an abstract construction. So, let we deal with an abstract equation

$$Lu = f \tag{1}$$

where $L: \ X \to Y$ is an isomorphism (i. e. a one-to-one linear map continuous together with the inverse) of a Banach space X onto another Banach space Y equipped respectively with norms $||\cdot||_X$ and $||\cdot||_Y$, let $f \in Y$, and $u \in X$ be an unknown element. So, for any $f \in Y$ equation (1) has a unique solution $u = L^{-1}f \in X$ and

$$||u||_X \leq ||L^{-1}|| \ ||f||_Y \tag{2}$$

where, as usually, $||L|| = \sup\limits_{0 \neq x \in X} \frac{||Lx||_Y}{||x||_X}$ and $||L^{-1}|| = \sup\limits_{0 \neq y \in Y} \frac{||L^{-1}y||_X}{||y||_Y}$. We assume the existence of a sequence $\{X_n\}_{n=1,2,3,\ldots}$ of subsets of X such that for any $u \in X$ one has

$$\lim_{n\to\infty} \inf_{v\in X_n} ||u-v||_X = 0. \tag{3}$$

In a real situation when X is, for example, a Sobolev or Lebesgue space subsets X_n may be, for example, finite-dimensional subspaces of X spanned by all splines on a fixed network of the net width n^{-1} or by all algebraic or trigonometric polynomials of order $\leq n$. In a real situation, one often has an additional information about the unknown u, such as its additional smoothness, so that for this concrete element u the convergence in (3) is more quick than in the general case. So, for a given $f \in Y$ let $\{\alpha_n\}_{n=1,2,3,...} \subset \mathbb{R}$ be such that $\alpha_n \to 0$ as $n \to \infty$ and that we have for our unknown u:

$$\inf_{v\in X_n} ||v-u||_X \leq \alpha_n, \quad n=1,2,3,... \tag{4}$$

We call elements $u_n \in X_n$, $n = 1,2,3,...$, approximate solutions of problem (1). The method we propose looks as follows. For each n we seek $u_n \in X_n$ so that

$$||Lu_n - f||_Y \leq \epsilon_n + \inf_{v\in X_n} ||Lv-f||_Y \leq \epsilon_n + \inf_{v\in X_n} ||L(u-v)||_Y \leq \epsilon_n + ||L||\alpha_n \tag{5}$$

where $\epsilon_n \to +0$ as $n \to \infty$. Then, we obtain a sequence $\{u_n\}$ of approximate solutions. In view of (2),(4) and (5) we have

$$||u_n - u||_X \leq ||L^{-1}||\ ||Lu_n - f||_Y \leq ||L^{-1}||(\epsilon_n + ||L||\alpha_n) \to 0 \quad \text{as} \quad n \to \infty.$$

If one is able to find u_n so that $\epsilon_n \leq C\alpha_n$ for a constant $C > 0$ independent of n, then he finally gets:

$$||u - u_n||_X \le ||L^{-1}||(C + ||L||)\alpha_n \le C_1\alpha_n \to +0 \quad \text{as} \quad n \to \infty$$

where $C_1 = ||L^{-1}||(C + ||L||) > 0$ is independent of n.

What is considered above is a general description of the method we propose. One can see that its idea is quite elementary. In the next sections, we consider several applications of the method to concrete problems of mathematical physics. With these examples, we try to show that, in a number of cases, the method works not quite bad or, at least, that it may be considered as a candidate to be exploited.

Now, we introduce some notation used in what follows. Let d be a positive integer, $\mathbb{R}^d$ be the standard Euclidian space (we denote $\mathbb{R} = \mathbb{R}^1$) of vectors $x = (x_1, ..., x_d)$, with the scalar product $(x, y) = x_1y_1 + ... + x_dy_d$ and the corresponding norm $||x|| = \sqrt{x_1^2 + ... + x_d^2}$. Everywhere $\Omega \subset \mathbb{R}^d$ is an open bounded domain with a smooth boundary $\partial\Omega$, $\Delta = \frac{\partial^2}{\partial x_1^2} + ... + \frac{\partial^2}{\partial x_d^2}$ is the Laplace operator, $C_0^\infty(\Omega)$ is the linear space of infinitely differentiable real-valued functions in Ω continuous in the closure $\overline{\Omega}$ of Ω and equal to zero on $\partial\Omega$. For each nonnegative integer k $C^k(\Omega)$ denotes the linear space of real-valued k times continuously differentiable functions in Ω, all derivatives of the order $\le k$ of each of which are bounded in Ω, equipped with the norm

$$||g||_{C^k(\Omega)} = \sum_{k_1 + ... + k_d \le k} \sup_{x \in \Omega} \left| \frac{\partial^{k_1 + ... + k_d} g(x)}{\partial x_1^{k_1} ... \partial x_d^{k_d}} \right|,$$

and $C_0^k(\Omega)$ denotes the subspace of $C^k(\Omega)$ that consists of all $g \in C^k(\Omega)$ each of which has a continuous extension $\overline{g}$ onto $\overline{\Omega}$ such that $\overline{g} \equiv 0$ on $\partial\Omega$.

Let $L_2(\Omega)$ be the standard Lebesgue space consisting of real-valued square integrable functions $g, h, \ldots$ in Ω, with the inner product

$$(g,h) = \int_{\Omega} g(x)h(x)dx$$

and the corresponding norm $||g|| = (g,g)^{1/2}$. By D we denote the closure of the operator $-\Delta$, taken with the domain $C_0^\infty(\Omega)$, in $L_2(\Omega)$. In fact, as well known, D is a positive self-adjoint (unbounded) operator in $L_2(\Omega)$. For any $s \in \mathbb{R}$, let $H^s(\Omega)$ be the Sobolev space being the completion of the space $C_0^\infty(\Omega)$ taken with the inner product $(g,h)_{H^s(\Omega)} = (D^{s/2}g, D^{s/2}h)$; in fact, $H^s(\Omega)$ are Hilbert spaces. Clearly, $H^0(\Omega) = L_2(\Omega)$. Also, for a positive integer s, as well known, the norm $||\cdot||_{H^s(\Omega)}$ is equivalent to the norm

$$|||g|||_{H^s(\Omega)} = \left\{ \sum_{\substack{k_1+\ldots+k_d=s \\ k_i \geq 0}} \left(\frac{\partial^s g(x)}{\partial x_1^{k_1} \ldots \partial x_d^{k_d}} \right)^2 dx \right\}^{\frac{1}{2}}$$

(this is a norm because $g|_{\partial\Omega} = 0$). We recall the known fact that, for any s, D is an isomorphism of $H^s(\Omega)$ onto $H^{s-2}(\Omega)$. Also, for two Banach spaces X and Y, the norms in which are denoted $||\cdot||_X$ and $||\cdot||_Y$, respectively, $\mathcal{L}(X;Y)$ denotes the Banach space of linear bounded operators A acting from X into Y, equipped with the norm

$$||A||_{\mathcal{L}(X;Y)} = \sup_{0 \neq x \in X} \frac{||Ax||_Y}{||x||_X}.$$

Finally, $c, C, C_1, C_2, C', C'', \ldots$ denote positive constants.

2 Application of the method to an ODE

Here, we consider the standard problem

$$Lu = -\frac{d^2}{dx^2}u + q(x)u = f(x), \quad u = u(x), \quad x \in (0,1), \tag{6}$$

$$u(0) = u(1) = 0, \tag{7}$$

where $q, f \in C^2([0,1])$, $q(x) \geq q_0 > 0$ and u is an unknown real-valued function. It is well known that problem (6),(7) has a unique solution $u(\cdot) \in C_0^4([0,1])$. Let n be a positive integer and S_h be the network $\{0, h, 2h, ..., (n-1)h, 1\}$ where $h = n^{-1}$. We take $X = L_2(0,1)$ and, for X_n, the linear space of all cubic splines $v(x)$ on the network S_h satisfying $v(0) = v(1) = 0$. Then, as well known [2], we have

$$\inf_{v \in X_n} ||v - u||_{C([0,1])} \leq Ch^4$$

with a $C > 0$ independent of n, therefore

$$\alpha_n = \inf_{v \in X_n} ||v - u||_{L_2(0,1)} \leq C_1 h^4. \tag{8}$$

Further, it is well known that the operator L taken with the conditions (7) is an isomorphism of $L_2(0,1)$ onto $H^{-2}(0,1)$. The problem of seeking u_n minimizing the expression

$$||Lv - f||^2_{H^{-2}(0,1)} = ||v + D^{-1}(q(\cdot)v) - D^{-1}f||^2_{L_2(0,1)}$$

over $v \in X_n$ obviously reduces to a linear system of algebraic equations for coefficients $\{z\}$ of v (coefficients $\{z\}$ may be, for example, $\{z_k = v(kh),\ k = 1, 2, ..., n-1\}$); in addition, we have $\lim_{|z| \to \infty} ||Lv - f||^2_{H^{-2}(0,1)} =$

$+\infty$ because $D^{-1}L$ is an isomorphism of $L_2(0,1)$ onto itself and, also, because $||v||_{L_2(0,1)} \to +\infty$ as $|z| \to \infty$. Hence, due to (8), we have

$$||u_n - u||_{L_2(0,1)} = ||L^{-1}(Lu_n - f)||_{L_2(0,1)} \leq ||L^{-1}|| \; ||Lu_n - f||_{H^{-2}(0,1)} \leq C_2 h^4$$

for a constant $C_2 > 0$ independent of n.

So, the degree of convergence of our method is $O(h^4)$. In this connection, it should be noted that the standard three-point finite-difference approximations

$$-\frac{u_n((k-1)h) - 2u_n(kh) + u_n((k+1)h)}{h^2} + q(kh)u_n(kh) = f(kh),$$

$$k = 1, 2, ..., n-1,$$

$$u_n(0) = u_n(1) = 0$$

give only a $O(h^2)$-convergence.

3 Schrödinger-type eigenvalue problems in higher dimensions

In this section, we propose a method for seeking the minimal eigenvalue and corresponding eigenfunction of a Schrödinger-type eigenvalue problem in many spatial dimensions. In fact, for a standard iteration method involving inversions of the operator $-\Delta + V(\cdot)$ we introduce an approximate procedure for such inversions. The author hopes that with this, though estimates of the speed of convergence are not obtained, it may become possible in some particular cases to make an approximate computation of the eigenvalues and eigenfunctions of quantum-mechanical Schrödinger eigenvalue problems in higher dimensions $d \geq 3$ by the use of modern computers.

So, we accept that a d-dimensional Schrödinger equation is already approximated by the following problem considered in a cube:

$$-\Delta u + V(x)u = \lambda u, \quad u = u(x), \quad x \in K_R \subset \mathbb{R}, \tag{9}$$

$$u|_{\partial K_R} = 0, \tag{10}$$

where $\lambda \in \mathbb{R}$ is a spectral parameter, $K_R = \{x = (x_1, ..., x_d) \in \mathbb{R}^d : -R < x_i < R, \ i = 1, 2, ..., d\}$ is a cube in $\mathbb{R}^d$, and $V(\cdot)$ is a real-valued potential to be assumed to be in $C_0^\infty(K)$. By the trivial substitution $y_i = \frac{x_i + R}{2R}$ problem (9),(10) reduces to the following:

$$Lu = -\Delta_y u + V^1(y)u = \lambda u, \quad u = u(y), \quad y \in K, \tag{11}$$

$$u|_{\partial K} = 0, \tag{12}$$

where $K = \{y = (y_1, ..., y_d) \in \mathbb{R} : \ 0 < y_i < 1, \ i = 1, 2, ..., d\}$. In what follows, for a simplicity of the notation, we rename y by x and $V^1(y)$ by $V(x)$. We also suppose

$$V(x) \geq V_0 > 0, \quad x \in K.$$

As well known, problem (11),(12) has a sequence of eigenvalues

$$0 < \lambda_0 < \lambda_1 \leq ... \leq \lambda_n \leq ...,$$

where each λ_n is corresponded by an eigenfunction u_n satisfying $||u_n|| = 1$ and such that $\{u_n\}_{n=0,1,2,...}$ is an orthonormal basis in $L_2(K)$. A standard iteration process for seeking λ_0 and u_0 is the following (see, for example, [3]). We take an arbitrary w^1 positive and continuous in K and satisfying (12). For each $n = 1, 2, 3, \ldots$ we set

$$w^{n+1} = L^{-1} w^n$$

where L is the operator $-\Delta + V(\cdot)$ taken with the boundary condition (12). It is known [4] that for each continuous f there exists a unique weak solution u of the equation $Lu = f$ taken with boundary conditions (12) and that this weak solution is also continuous in $\overline{K}$. Thus our iteration process is well-defined. As well known, one has

$$\frac{(w^{n+1}, w^n)}{||w^n||^2} = \lambda_0^{-1} + O\left(\left(\frac{\lambda_0}{\lambda_1}\right)^n\right)$$

and

$$\frac{w^n}{||w^n||} = u_0 + O\left(\left(\frac{\lambda_0}{\lambda_1}\right)^n\right)$$

which allows to calculate λ_0 and u_0 approximately arbitrary closely to the exact ones. A simple modification of this method also allows to find higher eigenvalues and eigenfunctions of the operator L.

One of the main difficulties when applying this method in higher dimensions is the problem of inverting the operator L, that is the problem of solving the equation

$$Lu = f, \quad u = u(x), \quad x \in K, \tag{13}$$

$$u|_{\partial K} = 0, \tag{14}$$

where u is an unknown function, so that one should take $u = w^{n+1}$ and $f = w^n$ to obtain the above iterations. We assume $f|_{\partial K} = 0$. In this section, our main aim is to establish a procedure for approximate solving (13),(14) in higher dimensions (for instance, when $d \geq 3$). Here, we apply classical Korobov's approximations of smooth functions. To describe them, we introduce some new notions. Let

$$e_{n_1,...,n_d}(x) = 2^{d/2} \sin \pi(n_1+1)x_1 \sin \pi(n_2+1)x_2 \times ... \times \sin \pi(n_d+1)x_d,$$

so that $\{e_{n_1,...,n_d}(x)\}_{n_i=0,1,2,...}$ is an orthonormal basis in $L_2(K)$, and let

$$f = \sum_{n_1,...,n_d=0}^{\infty} f_{n_1,...,n_d} e_{n_1,...,n_d} \in L_2(k).$$

Then, for $C > 0$ and $\alpha > 0$ we say that $f \in E^\alpha(C)$ if

$$|f_{n_1,...,n_d}| \leq \frac{C}{(\overline{n}_1 ... \overline{n}_d)^\alpha} \tag{15}$$

for all values of indices (here $\overline{n}_i = \max\{1; n_i\}$). One can easily verify by integrating by parts in the representations for Fourier coefficients of f that (15) holds for a $C > 0$ if $f \in C^\infty_{\text{fin}}(K)$ where $C^\infty_{\text{fin}}(K)$ is the space of functions infinitely differentiable in K each of which is equal to zero in a neighborhood of the boundary ∂K.

Let also $B_r(t)$, where $t \in \mathbb{R}$, $B_1(t) = t - \frac{1}{2}$ and $B_r(1) = B_r(0)$, $B_r'(t) = rB_{r-1}(t)$ for $r \geq 2$, be the Bernoulli polynomials. The following result is due to Korobov [5].

<u>Theorem</u> *Let $r \geq 2$ be integer, $\alpha \geq 2r$ be integer, too, and $a_1, ..., a_d$ be the optimal coefficients by module p an effective computation procedure of finding which is presented in [5]. Then, for any $g \in E^\alpha(C)$ one has*

$$g(x_1,...,x_d) = p^{-1} \sum_{k=1}^{p} \sum_{\tau_1,...,\tau_d=0}^{1} g^{r\tau_1,...,r\tau_d}\left(\left\{\frac{a_1 k}{p}\right\},...,\left\{\frac{a_d k}{p}\right\}\right) \times$$

$$\times \prod_{\nu=1}^{d} \left[\frac{(-1)^{r-1}}{r!} B_r \left(\left\{ \frac{a_\nu k}{p} - x_\nu \right\} \right) \right]^{\tau_\nu} + O\left(\frac{\ln^\gamma p}{p} \right) \qquad (16)$$

where $g^{r\tau_1,\dots,r\tau_d} = \frac{\partial^{r(\tau_1+\dots+\tau_d)} g}{\partial x_1^{r\tau_1} \dots \partial x_d^{r\tau_d}}$*, the constant* $\gamma > 0$ *depends only on* r *and* s *and* $\{q\}$ *is the fractional part of a real* q *so that* $\{q\} = q - [q]$ *with* $[q]$ *the maximal integer that is not larger than* q*. Thus, in (16) the speed of convergence does not depend on the dimension* d *of the space.*

Applying this Theorem, we first approximate our functions in (13),(14) V and f by expressions similar to (16) with some real coefficients in place of the derivatives $g^{r\tau_1,\dots,r\tau_d}$. It is possible to choose these approximations $V_{\overline{p}}$ and $f_{\overline{p}}$ arbitrary closely to V and f in $C(\overline{K})$ because V and f are continuous and also, because we can approximate them by functions V_ϵ and f_ϵ from $C^\infty_{\text{fin}}(K)$ arbitrary closely in $C(\overline{K})$ and for these new functions V_ϵ and f_ϵ their approximations similar to (16) converge in $C(\overline{K})$ by Theorem above. So, we assume that problem (13),(14) is changed by the following:

$$\overline{L}u = -\Delta u + \overline{V}(x)u = \overline{f}, \quad u = u(x), \quad x \in K, \qquad (17)$$

$$u|_{\partial K} = 0, \qquad (18)$$

where $\overline{V} = V_{\overline{p}}$ and $\overline{f} = f_{\overline{p}}$. It is well known that the solution $\overline{u}$ of problem (17),(18) converges to the solution u of (13),(14) in $L_2(K)$ as $\overline{V} \to V$ and $\overline{f} \to f$ in $C(\overline{K})$, therefore we can accept that $||\overline{u} - u||_{L_2(K)}$ is sufficiently small. So, applying Theorem above, we set

$$u_p(\{z\}, x_1, \dots, x_d) = p^{-1} \sum_{k=1}^{p} \sum_{\tau_1,\dots,\tau_d=0}^{1} z_{\tau_1,\dots,\tau_d,k} \times$$

$$\times \prod_{\nu=1}^{d} \left[\frac{(-1)^{r-1}}{r!} B_r \left(\left\{ \frac{a_\nu k}{p} - x_\nu \right\} \right) \right]^{\tau_\nu}$$

where $\{z\}$ is the set of indefinite coefficients $z_{\tau_1,...,\tau_d,k}$. If it would be $\overline{u} \in E^\alpha(C)$, then we would have

$$\inf_z ||u_p(\{z\}, x_1, ..., x_d) - \overline{u}(x_1, ..., x_d)||_{L_2(K)} = O\left(\frac{\ln^\gamma p}{p^r}\right) \quad \text{as } p \to \infty.$$

To find an approximate solution of problem (17),(18), we, for each p, look for u_p minimizing the expression

$$||Lu_p - f||^2_{H^{-2}(K)} = ||u_p + D^{-1}[\overline{V}(\cdot)u_p] - D^{-1}\overline{f}||^2_{L_2(K)} = g_p(z)$$

over coefficients $\{z\}$. It is easy to see that $g_p(z)$ is a nonnegative quadratic function of the coefficients $\{z\}$. If it becomes zero at some $z = z_0^p$, then $u_p(\{z_0^p\}, x_1, ..., x_d)$ is an exact solution of problem (17),(18); otherwise $\lim_{|z|\to\infty} g_p(z) = +\infty$ so that $g_p(z)$ has a unique point of minimum $z = z_0^p$. As earlier, we have

$$\lim_{p\to\infty} ||u_p(z_0^p, x_1, ..., x_d) - \overline{u}(x_1, ..., x_d)||_{L_2(K)} = 0$$

and

$$\lim_{p\to\infty} ||Lu_p(z_0^p, x_1, ..., x_d) - \overline{f}(x_1, ..., x_d)||_{H^{-2}(K)} = 0,$$

therefore, we can construct approximate solutions $u_p(z_0^p, x_1, ..., x_d)$ arbitrary closely to the exact solution of (17),(18). Finally, if an approximate solution u_p is chosen, then it is continuous in $\overline{K}$ by construction so that the iteration process of seeking (λ_0, u_0) can be continued.

Now, we observe, first, that, to find the coefficients z_0^p from the condition

$$g_p(z_0^p) = \min_z g_p(z),$$

we may differentiate the function g_p over $z_{\tau_1,...,\tau_d,k}$ and set these expressions equal to zero obtaining a linear system of algebraic equations that has a solution. Second, as well known,

$$||h||^2_{H^{-2}(K)} = \sum_{i_1,...,i_d=0}^{\infty} \mu^{-2}_{i_1,...,i_d} \left\{ \int_K h(x_1,...,x_d) e_{i_1,...,i_d}(x_1,...,x_d) dx_1...dx_d \right\}^2$$

where $\mu_{i_1,...,i_d} = [\pi^d(i_1+1)...(i_d+1)]^2$ are the eigenvalues of D with the corresponding eigenfunctions $e_{n_1,...,n_d}$. So, we may calculate in this way coefficients of our linear system for $\{z\}$ approximately using similar expressions, and one can easily verify that the corresponding integrals can be taken by quadratures in elementary functions. By analogy, coefficients for $g_p(z)$ regarded as a quadratic function of $z_{\tau_1,...,\tau_d,k}$ can be approximately found analytically. These facts make it possible to calculate z_0^p effectively by a computer.

We should now remark the following. Korobov in [5] presents upper estimates for the differences between approximate and exact functions $\tilde{f}$ and f when the exact function is in $E^\alpha(C)$ with $\alpha \geq 2r$ that provides in particular that f is sufficiently smooth. For the iteration procedure we propose, the functions w^n are only continuous in general. This does not allow us to obtain upper estimates for the speed of convergence of our method of seeking w^{n+1} by w^n. However, the author hopes that this method could allow researchers to solve approximately Schrödinger eigenvalue problems in many spatial dimensions in certain particular cases.

4 Application of the method to a heat equation

In this section, we consider a method of numerical solving the problem

$$u_t = u_{xx} - c(x)u + f(x,t), \quad u = u(x,t), \quad (x,t) \in (0,1) \times (0,T), \quad (19)$$

$$u(0,t) = u(1,t) = 0, \quad (20)$$

$$u(x,0) = u_0(x) \in C_0^4([0,1]). \quad (21)$$

We assume that $c(\cdot) \in C^2([0,1])$, $c(\cdot) \geq c_0 > 0$, and that $f \in C^2([0,1] \times [0,T])$. Then, problem (19)-(21) has a unique solution $u(x,t)$ that belongs to $C_0^4([0,1])$ for any fixed t and is twice continuously differentiable in t.

Let M and N be positive integer, $h = N^{-1}$, $\tau = TM^{-1}$, $S_h = \{0, h, 2h, ..., (N-1)h, 1\}$ and $P_\tau = \{0, \tau, 2\tau, ..., (M-1)\tau, T\}$. First, we consider the following semidiscrete approximation of problem (19)-(21):

$$\frac{u_\tau(x,(r+1)\tau) - u_\tau(x,r\tau)}{\tau} = u_{\tau,xx}(x,(r+1)\tau) -$$

$$-c(x)u_\tau(x,(r+1)\tau) + f(x,(r+1)\tau), \quad r = 0,1,2,...,M-1, \quad (22)$$

$$u_\tau(0,r\tau) = u_\tau(1,r\tau) = 0, \quad r = 0,1,2,...,M, \quad (23)$$

$$u_\tau(x,0) = u_0(x). \quad (24)$$

Clearly, since $c_1(x) = c(x) + \tau^{-1} > c_0 > 0$, for each r equation (22) with boundary conditions (23) has a unique solution $u(x,(r+1)\tau)$; in addition, as in Section 2, one has $u(\cdot,(r+1)\tau) \in C_0^4([0,1])$ for each $r = 0,1,2,...,M-1$. The following result takes place.

Proposition. *There exists* $C_1 > 0$ *such that*

$$\max_{r=0,1,\dots,M-1} \int_0^1 dx |u_\tau(x,(r+1)\tau) - u(x,(r+1)\tau)|^2 \le C_1 \tau^2.$$

Proof. Due to our assumptions, the exact solution $u(x,t)$ is twice continuously differentiable over t and is four times continuously differentiable over x. Thus, we have:

$$\frac{u(x,(r+1)\tau) - u(x,r\tau)}{\tau} = O(\tau) + u_{xx}(x,(r+1)\tau) -$$

$$-c(x)u(x,(r+1)\tau) + f(x,(r+1)\tau), \quad r = 0,1,2,\dots,M-1. \tag{25}$$

It easily follows from (22) and (25) that

$$\int_0^1 dx (u_\tau(x,(r+1)\tau) - u(x,(r+1)\tau))^2 =$$

$$= \int_0^1 dx \Big\{ O(\tau^2)(u_\tau(x,(r+1)\tau) -$$

$$-u(x,(r+1)\tau)) - \tau(u_{\tau,x}(x,(r+1)\tau) - u_x(x,(r+1)\tau))^2 -$$

$$-\tau c(x)(u_\tau(x,(r+1)\tau) - u(x,(r+1)\tau))^2 + (u_\tau(x,r\tau) -$$

$$-u(x,r\tau))(u_\tau(x,(r+1)\tau) - u(x,(r+1)\tau)) \Big\}$$

which implies

$$\frac{1}{2} \int_0^1 dx (u_\tau(x,(r+1)\tau) - u(x,(r+1)\tau))^2 \le$$

$$\le C_1 \tau^2 \left\{ \int_0^1 dx (u_\tau(x,(r+1)\tau) - u(x,(r+1)\tau))^2 \right\}^{1/2} +$$

$$+\frac{1}{2}\int_0^1 dx(u_\tau(x,r\tau)-u(x,r\tau))^2.$$

The latter inequality immediately yields the statement of our Proposition. Indeed, setting $y(r\tau)=\int_0^1 dx(u_\tau(x,r\tau)-u(x,r\tau))^2$ so that $y(0)=0$, we obtain from it for some $C_2>0$ independent of τ and r:

$$y((r+1)\tau)\leq C_2\tau^3+(1+\tau)y(r\tau)\leq C_2\tau^3+(1+\tau)(C_2\tau^3+(1+\tau)y((r-1)\tau))\leq$$

$$\ldots\leq C_2\tau^3\sum_{q=0}^{r}(1+\tau)^q\leq C_2\tau^2\left[(1+\tau)^{1/\tau+1}-1\right]=C_3\tau^2$$

where $r=0,1,2,\ldots,M-1.\square$

As noted above, in fact $u(\cdot,r\tau)\in C_0^4([0,1])$ for each τ and each r. So, we can apply the method described in Section 2 to seek $u_\tau(x,(r+1)\tau)$ regarded as a function of x from equation (22). So, for each fixed τ, we obtain a $O(h^4)$-convergence of approximate solutions of problem (22)-(24) to the exact one.

5 Conclusion

With this paper, we did not aim to develop in detail the method of quasi-reversibility we consider. As noted earlier, we only wanted to show that this method may be considered as a candidate to be exploited for numerical solving certain differential equations. As for concrete algorithms realizing it, it seems to be that there are no principal difficulties to construct them. For example, to solve the typical problem for the ODE considered in Section 2, one could chose a h between 10 and 20 obtaining an accuracy of the method about 10^{-4}, and with this choice of h, the problem reduces to a not very large linear system of algebraic equations which seems to be not difficult so far for its numerical solving. The author hopes that this method could also be useful for researchers in order to solve many-body quantum-mechanical problems in certain particular cases.

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Жидков П. Е. E11-2002-61
Метод решения дифференциальных уравнений
на основе теории аппроксимации

Рассматривается метод квазиобращения для численного решения граничных задач для дифференциальных уравнений. Специфической особенностью предлагаемого подхода является корректность изучаемых задач. Главная идея метода проиллюстрирована на нескольких примерах типичных задач математической физики. В частности, предложена новая идея для решения квантово-механических шредингеровских задач на собственные значения в многомерном случае, основанная на классических аппроксимациях Коробова.

Работа выполнена в Лаборатории теоретической физики им. Н. Н. Боголюбова ОИЯИ.

Препринт Объединенного института ядерных исследований. Дубна, 2002

Zhidkov P. E. E11-2002-61
A Method for Solving Differential Equations
via Approximation Theory

We consider the method of quasi-reversibility for numerical solving boundary-value problems for differential equations. A specific feature of our approach is the well-posedness of the problems we study. We illustrate the main idea of the method with several examples of typical problems of mathematical physics. In particular, we propose a new idea for solving quantum-mechanical Shrödinger eigenvalue problems in many spatial dimensions based on the classical Korobov's approximations.

The investigation has been performed at the Bogoliubov Laboratory of Theoretical Physics, JINR.

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