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G. N. Afanasiev[1], Yu. P. Stepanovsky[2]

ON THE RADIATION OF ELECTRIC, MAGNETIC AND TOROIDAL DIPOLES

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[1]E-mail: afanasev@thsun1.jinr.ru
[2]Institute of Physics and Technology, Kharkov, Ukraine

1 Introduction

The radiation of Compton electrons moving in water was observed by Cherenkov in 1934 (see his Doctor of Science dissertation published in [1]). In 1934-1937, Tamm and Frank associated it with the radiation of electrons moving with the velocity v greater than the light velocity in medium c_n (see, e.g., the Frank monography [2]). The radiation of electric and magnetic dipoles moving uniformly in medium with $v > c_n$ was first considered by Frank in [3,4]. The procedure used by him is as follows. The Maxwell equations are rewritten in terms of electric and magnetic Hertz vector potentials. The electric and magnetic field strengths are expressed through them uniquely. In the right-hand sides of these equations enter electric and magnetic polarizabilities which are expressed through the laboratory frame (LF) electric π and magnetic μ moments of a moving particle. These moments are related to the electric π' and magnetic μ' moments of the dipole rest frame (RF) via the relations [5]

$$\vec{\pi} = \vec{\pi}' - (1-\gamma^{-1})(\pi'\vec{n}_v)\vec{n}_v + \beta(\vec{n}_v \times \mu'),$$

$$\vec{\mu} = \vec{\mu}' - (1-\gamma^{-1})(\mu'\vec{n}_v)\vec{n}_v - \beta(\vec{n}_v \times \pi'). \tag{1.1}$$

Here $\beta = v/c, \quad \gamma = 1/\sqrt{1-\beta^2}, \quad \vec{n}_v = \vec{v}/v$, v is the velocity of a dipole realative to the LF. Let in the RF be only the electric dipole ($\mu' = 0$). Then,

$$\vec{\pi} = \vec{\pi}' - (1-\gamma^{-1})(\pi'\vec{n}_v)\vec{n}_v, \quad \vec{\mu} = -\vec{\beta}(\vec{n}_v \times \pi'). \tag{1.2}$$

Excluding π', one gets in the LF

$$\vec{\mu} = -\beta(\vec{n}_v \times \vec{\pi}). \tag{1.3}$$

Similarly, if only the magnetic moment differs from zero in the RF, then, in the LF

$$\vec{\pi} = \beta(\vec{n}_v \times \vec{\mu}). \tag{1.4}$$

Using these relations, Frank evaluated the electromagnetic field (EMF) strengths and the energy flux per unit frequency and per unit length of the cylinder surface coaxial with the motion axis. These quantities depended on the dipole spacial orientation. For the electric dipole and for the magnetic dipole paralell to the velocity, Frank obtained expressions which satisfied him. For the magnetic dipole perpendicular to the velocity, the radiated energy did not disappear for $v = c_n$. Its disappearance is intuitively expected and is satisfied, e.g., for the electric charge and dipole and for for the magnetic dipole parallel to the velocity. On these grounds, Frank declared [6] the formula for the radiation intensity of the magnetic dipole perpendicular to the velocity as to be incorrect. He also admitted that the correct expression for the above intensity is obtained if (1.4) is changed by

$$\vec{\pi} = n^2\beta(\vec{n}_v \times \vec{\mu}), \tag{1.5}$$

while (1.3) remains to be the same. Here n is the medium refractive index.

Equation (1.5) was supported by Ginsburg in [7] who pointed out that the internal structure of a moving magnetic dipole and the polarization induced inside it are essential. This idea was further elaborated in [8]. After many years, Frank returned in [9,10] to the original transformation law (1.4). In particular, in [10], the rectilinear current frame moving uniformly in medium was considered. The evaluated electric moment of the moving current distribution was in agreement with (1.4).

Another transformation law for the magnetic moment, grounding on the proportionality between the magnetic and mechanical moments was suggested in [11]. This proportionality taking place, e.g., for an electron, was confirmed experimentally with a great accuracy in $g-2$ experiments. In them, the electron precession is described by the Bargmann-Michel- Telegdi equation of motion for the spin. In this theory, the spin is a three-vector $\vec{s}$ in its rest frame. In another inertial frame (and, in particular, in the laboratory frame relative to which a particle with spin moves with the velocity $\vec{v}$), the spin has four components $(\vec{S}, S_0)$ defined by

$$\vec{S} = \vec{s} + \frac{\gamma^2}{\gamma+1}(\vec{\beta}\cdot\vec{s})\vec{\beta}, \quad S_0 = \gamma(\vec{\beta}\cdot\vec{s}).$$

A nice exposition of these questions may be found in [12].

The goal of this consideration is to obtain EMF potentials and strengths for the point-like electric and magnetic dipoles and elementary toroidal dipole moving in medium with arbitrary velocity v greater or smaller than the light velocity in medium c_n. In the reference frame attached to a moving source we have finite static distribution of charge and current densities. We postulate that charge and current densities in the laboratory frame, relative to which the source moves with a constant velocity, can be obtained from the rest frame densities via the Lorentz transformations, the same as in vacuum. The further procedure is in decreasing the dimensions of the LF charge-current sources to zero, in a straghtforward solution of the Maxwell equations for the EMF potentials with the LF point-like charge-current densities in their r.h.s., and in a subsequent evaluation of the EMF strengths. Formerly, in the time representation, this was done in [13]. The present consideration is just the translation of [13] into the frequency language which is extensively used by experimentalists. Another reason for using the spectral representation is to make possible to compare our results with ones of [1-10] written in the frequency representation.

The plan of this exposition is as follows. In section 2, radiation intensities are evaluated for the electric, magnetic and toroidal dipoles moving uniformly in an unbounded medium. A lot of misprints in previous publications is recovered. It is not our aim to recover these misprints, but we need reliable working formulae which can be applied to concrete physical problems. In section 3, radiation intensities are evaluated for electric, magnetic and toroidal dipoles moving uniformly on the finite medium interval (the Tamm problem for dipoles). In section 4, the EMF of a precessing magnetic dipole is obtained. This can be applied to astrophysical problems. A brief discussion of the results obtained and their summary is given in section 5.

2 Unbounded motion of magnetic, toroidal and electric dipoles in medium

2.1 Pedagogical example: Uniform unbounded charge motion in medium

Consider at first the uniform unbounded charge motion in medium along the z axis. Charge and current densities are given by

$$\rho_{Ch} = e\delta(z - vt)\delta(x)\delta(y), \quad j_z = e\rho_{Ch}.$$

Their Fourier components are given by

$$\rho_\omega = \frac{1}{2\pi}\int \rho_{Ch}\exp(i\omega t)dt = \frac{e}{2\pi v}\delta(x)\delta(y)\exp(\frac{ikz}{\beta}), \quad j_\omega = v\rho_\omega, \quad k = \frac{\omega}{c}.$$

Electromagnetic potentials corresponding to these densities are

$$\Phi = \frac{1}{2\pi v\epsilon}\int \exp[ik(\frac{z'}{\beta} + nR)]\frac{dz'}{R}, \quad A_z = \mu\epsilon\beta\Phi. \tag{2.1}$$

Here $R = [x^2 + y^2 + (z - z')^2]^{1/2}$, ϵ and μ are the electric and magnetic constants of medium, $n = \sqrt{\epsilon\mu}$ is refractive index. Making the change of the integration variable $z' = z + \rho\sinh\chi$, we rewrite (2.1) in the form

$$\Phi = \frac{1}{2\pi v\epsilon}\alpha, \quad A_z = \mu\epsilon\beta\Phi, \quad \text{where}$$

$$\alpha = \exp(i\psi)I, \quad I = \int_{-\infty}^{\infty}\exp[ik\rho(\frac{\sinh\chi}{\beta} + n\cosh\chi)]d\chi \quad \text{and} \quad \psi = \frac{kz}{\beta}. \tag{2.2}$$

The integral I can be evaluated in a closed form [14]. It is given by

$$I = 2K_0 \quad \text{for} \quad v < c_n \quad \text{and} \quad I = i\pi H_0^{(1)} \quad \text{for} \quad v > c_n, \tag{2.3}$$

where the arguments of all Bessel functions are $k\rho/\beta\gamma_n$, $\gamma_n = |1 - \beta_n^2|^{-1/2}$, $\beta_n = \beta n$ and $c_n = c/n$ is the light velocity in medium. The scalar electric potential is given by

$$\Phi = \frac{e}{\pi v\epsilon}\exp(i\psi)K_0 \quad \text{for} \quad v < c_n \quad \text{and} \quad \Phi = \frac{ie}{2v\epsilon}\exp(i\psi)H_0^{(1)} \quad \text{for} \quad v > c_n.$$

The magnetic potential is $A_z = \beta\epsilon\mu\Phi$. Correspondingly, the electromagnetic field strengths are equal to

$$E_\rho = \frac{ek}{\pi v\epsilon\beta\gamma_n}\exp(i\psi)K_1, \quad E_z = -\frac{iek}{\pi v\epsilon\beta}(1 - \beta^2n^2)\exp(i\psi)K_0,$$

$$H_\phi = \frac{ek}{\pi v \gamma_n} \exp(i\psi) K_1 \quad \text{for} \quad \beta_n < 1 \quad \text{and}$$

$$E_\rho = i\frac{ek}{2v\epsilon\beta\gamma_n} \exp(i\psi) H_1^{(1)}, \quad E_z = \frac{ek}{2v\epsilon\beta}(1-\beta^2 n^2) \exp(i\psi) H_0^{(1)},$$

$$H_\phi = i\frac{ek}{2v\gamma_n} \exp(i\psi) H_1^{(1)} \quad \text{for} \quad \beta_n > 1.$$

The radial energy flux per unit length and per unit frequency through the cylinder surface of the radius ρ coaxial with the motion axis is given by

$$\sigma_\rho = \frac{d^2\mathcal{E}}{d\omega dz} = -\pi\rho c(E_z H_\phi^* + E_z^* H_\phi).$$

It is equal to zero for $\beta_n < 1$ and

$$\sigma_\rho = \frac{e^2 \omega \mu}{c^2}(1 - \frac{1}{\beta^2 n^2}) \quad \text{for} \quad \beta_n > 1, \tag{2.4}$$

which coincides with the frequency distribution of radiation given by Tamm and Frank.

2.2 Radiation of magnetic dipole uniformly moving in medium

2.2.1 Lorentz transformations of charge-current densities

In what follows, we need the Lorentz transformation formulae for the charge-current densities. They may be found in any textbook on electrodynamics (see, e.g., [12, 15]). Let ρ' and $\vec{j}'$ be charge and current densities in the rest frame S' which moves with a constant velocity $\vec{v}$ relative to the laboratory frame (LF) S. Then,

$$\rho = \gamma(\rho' + \vec{\beta}\vec{j}'/c), \quad \vec{j} = \vec{j}' + \frac{\gamma - 1}{\beta^2}\vec{\beta}(\vec{\beta}\vec{j}') + \gamma\vec{v}\rho'. \tag{2.5}$$

Here $\gamma = (1-\beta^2)^{-1/2}$, $\vec{\beta} = \vec{v}/c$. If there is no charge density in S', then

$$\rho = \gamma\vec{\beta}\vec{j}'/c, \quad \vec{j}_{||} = \gamma\vec{j}'_{||}, \quad j_\perp = j'_\perp, \tag{2.6}$$

where $\vec{j}_{||}$ and $j_\perp$ are the components of $\vec{j}$ parallel and perpendicular to $\vec{v}$. If there is no current density in S', then

$$\rho = \gamma\rho', \quad \vec{j} = \gamma\vec{v}\rho'. \tag{2.7}$$

In what follows, we assume that charge-current densities in two inertial reference frames placed in medium are connected by the Lorentz transformation, the same as in vacuum.

2.2.2 The magnetic dipole is parallel to the velocity

Consider a conducting loop $\mathcal{L}$ moving uniformly in a medium with the velocity v directed along the loop symmetry axis (coinciding with the z axis). Let in this loop a constant current I flows. In the reference frame S' attached to the moving loop, the current density is equal to

$$\vec{j} = I\vec{n}_\phi \delta(\rho' - d)\delta(z'), \quad \rho' = \sqrt{x'^2 + y'^2} \tag{2.8}$$

(x', y', z' are the coordinates in S'.) In accordance with (2.4), one gets in the LF

$$j = I\vec{n}_\phi \delta(\rho - d)\delta(\gamma(z - vt)) = \frac{I}{\gamma}\vec{n}_\phi \delta(\rho - d)\delta(z - vt). \tag{2.9}$$

Here $\vec{n}_\phi = \vec{n}_y \cos\phi - \vec{n}_x \sin\phi, \quad \gamma = 1/\sqrt{1-\beta^2}$. Since the current direction is perpendicular to the velocity, no charge density arises in the LF. The current density $\vec{j}$ can be expressed through the magnetization

$$\vec{j} = curl \vec{M}$$

which is perpendicular to the plane of a current loop:

$$M_z = \frac{I_0}{\gamma}\Theta(d - \rho)\delta(z - vt).$$

Now, let the loop radius d tends to zero. Then,

$$\Theta(d - \rho) \to \pi d^2 \delta(x)\delta(y), \quad M_z \to \frac{I\pi d^2}{\gamma}\delta(x)\delta(y)\delta(z - vt),$$

$$j_x = \partial M_z/\partial y, \quad j_y = -\partial M_z/\partial x, \quad j_z = 0.$$

The Fourier components of the current density are

$$j_x(\omega) = \partial M_z(\omega)/\partial y, \quad j_y(\omega) = -\partial M_z(\omega)/\partial x, \quad j_z(\omega) = 0,$$

where

$$M_z(\omega) = \frac{Id^2}{2\gamma v}\delta(x)\delta(y)\exp(i\psi)$$

and ψ is the same as in (2.2). The vector magnetic potential satisfies the equation

$$\Delta\vec{A}_\omega + k_n^2 \vec{A}_\omega = -\frac{4\pi\mu}{c}\vec{j}_\omega, \quad k_n = kn.$$

Its non-vanishing components are given by

$$A_x = \frac{\mu m_d}{2\pi\gamma v}\frac{\partial\alpha}{\partial y}, \quad A_y = -\frac{\mu m_d}{2\pi\gamma v}\frac{\partial\alpha}{\partial x},$$

where α is the same as in (2.2) and $m_d = I\pi d^2/c$ is the magnetic moment of the current loop in its rest frame. It is seen that only the ϕ component of $\vec{A}_\omega$ differs from zero:

$$A_\omega = -\frac{m_d\mu}{2\pi\gamma v}\frac{\partial\alpha}{\partial\rho}.$$

The electromagnetic field strengths are

$$E_\phi = -\frac{ikm_d\mu}{2\pi\gamma v}\frac{\partial\alpha}{\partial\rho}, \quad H_\rho = \frac{ikm_d}{2\pi\gamma\beta v}\frac{\partial\alpha}{\partial\rho}, \quad H_z = \frac{m_d}{2\pi\gamma v\beta^2}k^2(\beta_n^2 - 1)\alpha.$$

In a manifest form, they are equal to

$$E_\phi = \frac{ik^2m_d\mu}{\pi\beta\gamma_n\gamma v}\exp(i\psi)K_1, \quad H_\rho = -\frac{ik^2m_d}{\pi\gamma_n\gamma\beta^2 v}\exp(i\psi)K_1,$$

$$H_z = -\frac{m_dk^2}{\pi\gamma v\beta^2\gamma_n^2}\exp(i\psi)K_0$$

for $\beta_n < 1$ and

$$E_\phi = -\frac{k^2m_d\mu}{2\beta\gamma_n\gamma v}\exp(i\psi)H_1^{(1)}, \quad H_\rho = -\frac{k^2m_d}{2\gamma_n\gamma\beta^2 v}\exp(i\psi)H_1^{(1)},$$

$$H_z = i\frac{m_dk^2}{2\gamma v\beta^2\gamma_n^2}\exp(i\psi)H_0^{(1)}$$

for $\beta_n > 1$. The energy emitted in the radial direction per unit length and per unit frequency

$$\sigma_\rho = \frac{d^2\mathcal{E}}{d\omega dz} = -\pi\rho c(E_\phi H_z^* + H_z E_\phi^*)$$

equals zero for $v < c_n$ and

$$\sigma_\rho = \frac{\omega^3 m_d^2\mu}{v^4\gamma^2\gamma_n^2} \tag{2.10}$$

for $v > c_n$. Formerly, this equation was obtained by Frank in [6, 9], but without the factor γ^2 in the denominator. It is due to the factor γ in the denominator of (2.9). On the other hand, this factor presents in [3,14,16]. When obtaining (2.10), we suggested that the current density is equal to (2.8) in the reference frame attached to a moving current loop. The current density in the LF is obtained from (2.8) by the Lorentz transformation. It follows from (2.10) that the intensity of radiation produced by the magnetic dipole parallel to the velocity differs from zero in the velocity window $c_n < v < c$. Therefore, v should not be too close either to c_n or c. For this, n should appreciably differ from unity. Probably, the best candidate to observe this radiation is a neutron moving in medium with large n. By comparing (2.10) with the radiation intensity of a moving charge ($\sigma_e = e^2\omega\mu/c^2\gamma_n^2$), we see that there is a chance to observe the radiation from a neutron moving in medium only for very high frequencies.

2.2.3 The magnetic dipole is perpendicular to the velocity

Let the current loop lies in the $z = 0$ plane with its velocity alomg the x axis (magnetic dipole is along the z axis). Then, in the rest frame S',

$$j'_x = -\frac{I}{d}\frac{y'}{\rho'}\delta(z')\delta(\rho'-d), \quad j'_y = \frac{I}{d}\frac{x'}{\rho'}\delta(z')\delta(\rho'-d), \quad j'_z = 0, \quad \rho'_{Ch} = 0.$$

Here $\rho' = \sqrt{z'^2 + y'^2}$. According to (2.6), in the laboratory frame

$$j_x = -I\delta(z)\frac{y\gamma}{d}\delta(\rho-d), \quad j_y = I\delta(z)\frac{(x-vt)\gamma}{d}\delta(\rho-d), \quad \rho_{Ch} = -I\delta(z)\frac{yv\gamma}{c^2 d}\delta(\rho-d).$$

Here $\rho = [(x-vt)^2\gamma^2 + y^2]^{1/2}$. The charge density arises because on a part of the loop, the current has a non-zero projection on the direction of motion. It is easy to check that

$$j_x = I\gamma\delta(z)\frac{\partial}{\partial y}M_z, \quad j_y = -I\frac{1}{\gamma}\delta(z)\frac{\partial}{\partial x}M_z, \quad \rho_{Ch} = I\frac{v\gamma}{c^2}\delta(z)\frac{\partial}{\partial y}M_z,$$

where $M_z = \Theta(d-\rho_1)$. In the limit of an infinitesimal loop ($d \to 0$),

$$M_z = \Theta(d-\rho) \to \delta(x-vt)\delta(y)\pi d^2/\gamma \quad \text{and}$$

$$j_x = I\pi d^2\delta(z)\delta(x-vt)\frac{\partial}{\partial y}\delta(y), \quad j_y = -\frac{1}{\gamma^2}I\pi d^2\delta(z)\delta(y)\frac{\partial}{\partial x}\delta(x-vt),$$

$$\rho_{Ch} = I\pi d^2\frac{v}{c^2}\delta(z)\delta(x-vt)\frac{\partial}{\partial y}\delta(y).$$

The Fourier components of these densities are

$$j_x(\omega) = \frac{m_d}{2\pi\beta}\exp(i\psi_1)\delta(z)\frac{\partial}{\partial y}\delta(y), \quad j_y(\omega) = -\frac{m_d}{2\pi\beta\gamma^2}\frac{\partial}{\partial x}\delta(z)\delta(y)\exp(i\psi_1),$$

$$\rho_{Ch}(\omega) = \frac{m_d}{2\pi c}\delta(z)\exp(i\psi_1)\frac{\partial}{\partial y}\delta(y).$$

Here $\psi_1 = kx/\beta$. The electromagnetic potentials are equal to

$$\Phi = \frac{m_d}{2\pi c\epsilon}\frac{\partial\alpha_1}{\partial y}, \quad A_x = \frac{m_d\mu}{2\pi v}\frac{\partial\alpha_1}{\partial y}, \quad A_y = -\frac{m_d\mu}{2\pi\gamma^2 v}\frac{\partial}{\partial x}\alpha_1. \tag{2.11}$$

Here

$$\alpha_1 = \exp(i\psi_1)\int_{-\infty}^{\infty}\exp[ik\rho_1(\frac{\sinh\chi}{\beta} + n\cosh\chi)]d\chi, \quad \rho_1 = \sqrt{y^2+z^2}.$$

This integral is evaluated along the same lines as α in (2.2). It equals $2K_0$ for $v < c_n$ and $i\pi H_0^{(1)}$ for $v > c_n$. The arguments of these Bessel functions are $k\rho_1/\beta\gamma_n$. The electromagnetic field strengths are

$$E_x = \frac{im_d k\cos\phi}{2\pi v\epsilon}(n^2-1)\frac{\partial\alpha_1}{\partial\rho_1}, \quad E_y = \frac{m_d}{2\pi c\epsilon}\{\frac{\cos 2\phi}{\rho_1}\frac{\partial\alpha_1}{\partial\rho_1}+[\cos^2\phi\frac{k^2(\beta_n^2-1)}{\beta^2}+\frac{k^2n^2}{\gamma^2\beta^2}]\alpha_1\},$$

$$E_z = +\frac{m_d}{2\pi c\epsilon}\sin\phi\cos\phi[\frac{k^2(\beta_n^2-1)}{\beta^2}\alpha_1 + \frac{2}{\rho_1}\frac{\partial\alpha_1}{\partial\rho_1}],$$

$$H_x = \frac{ikm_d\sin\phi}{2\gamma^2 v\beta}\frac{\partial\alpha_1}{\partial\rho_1}, \quad H_y = -\frac{m_d\sin\phi\cos\phi}{2\pi v}[\frac{k^2(\beta_n^2-1)}{\beta^2}\alpha_1 + \frac{2}{\rho_1}\frac{\partial\alpha_1}{\partial\rho_1}],$$

$$H_z = \frac{m_d}{2\pi v}[\frac{k^2\alpha_1}{\gamma^2\beta^2} + \frac{\cos 2\phi}{\rho_1}\frac{\partial\alpha_1}{\partial\rho_1} + \cos^2\phi\frac{k^2(\beta_n^2-1)}{\beta^2}\alpha_1].$$

The angle ϕ ($\cos\phi = y/\rho_1$, $\sin\phi = z/\rho_1$) defines the asimuthal position of the observation point in the yz plane. It is counted from the y axis. In a manifest form, the field strengths are equal to

$$E_x = -\frac{im_d k^2\cos\phi}{\pi v\beta\gamma_n\epsilon}(n^2-1)\exp(i\psi_1)K_1,$$

$$E_y = -\frac{km_d c}{\pi\epsilon}[\frac{\cos 2\phi}{\rho_1\beta\gamma_n}K_1 - \frac{k}{\beta^2}(\frac{n^2}{\gamma^2} - \frac{\cos^2\phi)}{\gamma_n^2})K_0]\exp(i\psi_1),$$

$$E_z = -\frac{m_d k\sin\phi\cos\phi}{\pi\beta\gamma_n c\epsilon}(\frac{2}{\rho_1}K_1 + \frac{k}{\beta\gamma_n}K_0)]\exp(i\psi_1), \quad H_x = -\frac{im_d k^2\sin\phi}{\gamma^2\gamma_n v\beta^2}K_1\exp(i\psi_1),$$

$$H_y = \frac{m_d k\sin\phi\cos\phi}{\pi v\beta\gamma_n}(\frac{k}{\beta\gamma_n}K_0 + \frac{2}{\rho_1}K_1)\exp(i\psi_1),$$

$$H_z = \frac{m_d k}{\pi v\beta}[k(\frac{1}{\beta\gamma^2} - \frac{\cos^2\phi}{\gamma_n^2})K_0 - \frac{cos2\phi}{\gamma_n\rho_1}K_1]\exp(i\psi_1) \qquad (2.12)$$

for $v < c_n$ and

$$E_x = \frac{m_d k^2\cos\phi}{2v\beta\gamma_n}(n^2-1)H_1^{(1)}\exp(i\psi_1), \quad H_x = \frac{m_d k^2\sin\phi}{2\gamma^2\gamma_n\beta^2 v}H_1^{(1)}\exp(i\psi_1),$$

$$E_y = -\frac{im_d k}{2\epsilon v}[\frac{\cos 2\phi}{\rho_1\gamma_n}H_1^{(1)} - \frac{k}{\beta}(\frac{\cos^2\phi}{\gamma_n^2} + \frac{n^2}{\gamma^2})H_0^{(1)}]\exp(i\psi_1),$$

$$E_z = \frac{im_d k\sin\phi\cos\phi}{2v\epsilon\gamma_n}(\frac{k}{\beta\gamma_n}H_0^{(1)} - \frac{2}{\rho_1}H_1^{(1)})\exp(i\psi_1),$$

$$H_y = -\frac{im_d k\sin\phi\cos\phi}{2v\beta\gamma_n}[\frac{k}{\beta\gamma_n}H_0^{(1)} - \frac{2}{\rho_1}H_1^{(1)}]\exp(i\psi_1),$$

$$H_z = \frac{im_d k}{2v\beta}[\frac{k}{\beta}(\frac{\cos^2\phi}{\gamma_n^2} + \frac{1}{\gamma^2})H_0^{(1)} - \frac{\cos 2\phi}{\rho_1\gamma_n}H_1^{(1)}]\exp(i\psi_1) \qquad (2.13)$$

for $v > c_n$. To evaluate the energy flux in the radial direction (that is, perpendicular to the motion axis), one should find the components of field strengths tangential to the surface of a cylinder coaxial with the motion axis and perpendicular it. They are given by

$$E_\phi = E_z \cos\phi - E_y \sin\phi, \quad H_\phi = H_z \cos\phi - H_y \sin\phi.$$

We rewrite them in a manifest form. It is easy to check that

$$E_\phi = -\frac{m_d k \sin\phi}{\pi v \epsilon}\left(\frac{1}{\rho_1\gamma_n}K_1 + \frac{kn^2}{\beta\gamma^2}K_0\right)\exp(i\psi_1),$$

$$H_\phi = \frac{k m_d \cos\phi}{\pi v \beta}\left[k\beta(n^2-1)K_0 - \frac{1}{\rho_1\gamma_n}K_1\right]\exp(i\psi_1) \qquad (2.14)$$

for $v < c_n$ and

$$E_\phi = -\frac{i m_d k \sin\phi}{2v\epsilon}\left[\frac{kn^2}{\beta\gamma^2}H_0^{(1)} + \frac{1}{\rho_1\gamma_n}H_1^{(1)}\right]\exp(i\psi_1),$$

$$H_\phi = +\frac{i m_d k \cos\phi}{2v\beta}\left[k\beta(n^2-1)H_0^{(1)} - \frac{1}{\rho_1\gamma_n}H_1^{(1)}\right]\exp(i\psi_1) \qquad (2.15)$$

for $v > c_n$. The energy flux through the cylindrical surface of the radius ρ_1 per unit length and per unit frequency is equal to

$$\frac{d^2\mathcal{E}}{dx\,d\omega} = \int_0^{2\pi} \sigma(\omega,\phi)\,d\phi,$$

where

$$\sigma(\omega,\phi) = \frac{d^3\mathcal{E}}{dx\,d\omega\,d\phi} = \frac{c}{2}\rho_1(E_\phi^* H_x + E_\phi H_x^* - H_\phi^* E_x - H_\phi E_x^*). \qquad (2.16)$$

Substituting here field strengths, one obtains that the differential intensity is zero for $v < c_n$ and

$$\sigma(\omega,\phi) = \frac{m_d^2 k^3}{2\pi\epsilon\beta v}\left[\frac{n^2}{\gamma^4\beta^2}\sin^2\phi + (n^2-1)^2\cos^2\phi\right] \qquad (2.17)$$

for $v > c_n$. The integration over ϕ gives

$$\sigma(\omega) = \frac{m_d^2 k^3}{2\beta\epsilon v}\left[\frac{n^2}{\gamma^4\beta^2} + (n^2-1)^2\right]. \qquad (2.18)$$

Equations (2.17) and (2.18) coincide with ones obtained by Frank [3,4,16] who noted that in the limit $\beta \to 1/n$, these intensities do no vanish as it is intuitively expected. On these grounds, Frank declared them as to be incorrect [6]. 30 years later, Frank returned to the same problem [9]. He attributed the non-vanishing of intensities (2.17) and (2.18) to the specific polarization of medium.

We analyse this question in some detail. Intensity (2.16) is non-zero for $\beta = 1/n + \epsilon$ and zero for $\beta = 1/n - \epsilon$, where $\epsilon << 1$. Since it consists of EMF strengths (see (2.16), the latter should exhibit jump at $\beta = 1/n$ too. Turning to Eqs. (2.12) and (2.13) defining EMF strengths, we observe that E_x and H_x are continuous at $\beta = 1/n$, while E_ϕ and H_ϕ entering into (2.16) exhibit jump. Further examination shows that this jump is due to the fact that first terms in the definition of E_ϕ and H_ϕ in (2.14) and (2.15) are not transformed into each other when β changes from $1/n - \epsilon$ to $1/n + \epsilon$. Further reflection shows that this is due to Eqs. (2.3). Separating in them real and imaginary parts, one gets

$$I_1 = \int\limits_0^\infty \cos(\frac{k\rho}{\beta}\sinh\chi)\cos(k\rho n\cosh\chi) = K_0 \quad \text{for} \quad \beta < 1/n \quad \text{and}$$

$$I_1 = -\frac{\pi}{2}N_0 \quad \text{for} \quad \beta > 1/n, \tag{2.19}$$

$$I_2 = \int\limits_0^\infty \cos(\frac{k\rho}{\beta}\sinh\chi)\sin(k\rho n\cosh\chi) = 0 \quad \text{for} \quad \beta < 1/n \quad \text{and}$$

$$I_2 = \frac{\pi}{2}J_0 \quad \text{for} \quad \beta > 1/n, \tag{2.20}$$

where the arguments of all Bessel functions are $k\rho/\beta\gamma_n$. Now, I_1 is continuous at $\beta = 1/n$, while I_2 is zero for $\beta < 1/n$ and tends to $\pi/2$ as $\beta \to 1/n$.

For $\beta = 1/n$, I_2 looks like ($y = k\rho n$):

$$I_2 = \int\limits_0^\infty \cos(y\sinh\chi)\sin(y\cosh\chi)d\chi = \frac{1}{2}\int\limits_{-\infty}^\infty \cos(y\sinh\chi)\sin(y\cosh\chi)d\chi =$$

$$= \frac{1}{2}Im\int\limits_{-\infty}^\infty \exp[iy(\sinh\chi + \cosh\chi)]d\chi = \frac{1}{2}Im\int\limits_{-\infty}^\infty \exp[iy\exp\chi]d\chi.$$

Putting $t = \exp(\chi)$, one gets

$$\int\limits_{-\infty}^\infty \exp[iy\exp(\chi)]d\chi = \int\limits_0^\infty \exp(iyt)\frac{dt}{t}.$$

and

$$Im\int\limits_0^\infty \exp(iyt)\frac{dt}{t} = \int\limits_0^\infty \sin(yt)\frac{dt}{t} = \frac{\pi}{2}$$

Therefore, I_2 equals

$$I_2 = \frac{\pi}{2} \quad \text{for} \quad \beta = 1/n + \epsilon, \quad I_2 = \frac{\pi}{4} \quad \text{for} \quad \beta = 1/n \quad \text{and}$$

$$I_2 = 0 \quad \text{for} \quad \beta = 1/n - \epsilon, \quad \epsilon << 1.$$

As a result, radiation intensities equal one half of (2.17) or (2.18) for $\beta = 1/n$.

Again, neutron moving in dielectric medium with n appreciably different from unity, is the best candidate to observe this radiation. The absence of the overall $1/\gamma$ factor in (2.17) and (2.18) makes easier to observe radiation from the neutron with the spin perpendicular to the velocity than from the neutron with the spin directed along it.

2.3 Electromagnetic field of the point-like toroidal solenoid uniformly moving in unbounded medium

The exposition of this subsection and the subsection (3.3) is grounded on the formalism of elementary toroidal sources treated in [17]. Consider the poloidal current flowing on the surface of a torus equation of which in the rest frame is

$$(\rho' - d)^2 + z'^2 = R_0^2$$

(R_0 and d are the minor and large radii of torus). It is convenient to introduce coordinates $\rho' = d + R' \cos\psi$, $z' = R' \sin\psi$. In these coordinates, the poloidal current flowing on the torus surface is given by

$$\vec{j}' = j_0 \frac{\delta(R_0 - R')}{d + R_0 \cos\psi} \vec{n}_\psi.$$

Here $\vec{n}_\psi = \vec{n}_z \cos\psi - \vec{n}_\rho \sin\psi$ is the vector lying on the torus surface in a particular $\phi = const.$ plane and defining the current direction, $R' = \sqrt{(\rho' - d)^2 + z'^2}$. The cylindrical components of $\vec{j}$ are

$$j_z = j_0 \frac{\delta(R_0 - R')}{d + R_0 \cos\psi} \cos\psi = j_0 \delta(R_0 - R') \frac{\rho' - d}{R_0 \rho'},$$

$$j_\rho = -j_0 \frac{\delta(R_0 - R')}{d + R_0 \cos\psi} \sin\psi = -j_0 \delta(R_0 - R') \frac{z'}{R_0 \rho'}.$$

2.3.1 The velocity is along the torus symmetry axis

Let this current distribution move uniformly along the z axis (directed along the torus symmetry axis) with the velocity v. According to (2.6), in the laboratory frame, the nonvanishing charge and current components are

$$\rho_{Ch} = j_0 \gamma \beta \frac{\rho - d}{c \rho R_0} \delta(R_0 - R), \quad j_\rho = -j_0 \gamma \frac{z - vt}{\rho R_0} \delta(R_0 - R),$$

$$j_z = j_0 \gamma \frac{\rho - d}{\rho R_0} \delta(R_0 - R).$$

Here $R = \sqrt{(\rho - d)^2 + (z - vt)^2\gamma^2}$. These components can be represented in the form

$$j_z = \frac{1}{\rho}\frac{\partial}{\partial\rho}(\rho M_\phi), \quad j_\rho = -\frac{1}{\gamma^2}\frac{\partial M_\phi}{\partial z}, \quad \rho_{Ch} = \frac{\beta}{c\rho}\frac{\partial}{\partial\rho}(\rho M_\phi),$$

where

$$M_\phi = -j_0\gamma\frac{1}{\rho}\Theta(R_0 - R).$$

The Cartesian components of $\vec{M}$ are

$$M_x = j_0\gamma\frac{y}{\rho^2}\Theta(R_0 - R), \quad M_y = -j_0\gamma\frac{x}{\rho^2}\Theta(R_0 - R).$$

Then,

$$j_x = -\frac{1}{\gamma^2}\frac{\partial M_y}{\partial z}, \quad j_y = \frac{1}{\gamma^2}\frac{\partial M_x}{\partial z}.$$

Let the minor torus radius R_0 tend to zero. Then,

$$\Theta(R_0 - R) \to \frac{\pi R_0^2}{\gamma}\delta(\rho - d)\delta(z - vt)$$

and

$$M_x = -\frac{j_0}{d}\pi R_0^2\frac{\partial}{\partial y}\Theta(d - \rho)\delta(z - vt), \quad M_y = \frac{j_0}{d}\pi R_0^2\frac{\partial}{\partial x}\Theta(d - \rho)\delta(z - vt).$$

Therefore,

$$j_x = -\frac{1}{\gamma^2}\frac{\partial M_y}{\partial z} = -\frac{j_0\pi R_0^2}{\gamma^2 d}\frac{\partial^2}{\partial z\partial x}\Theta(d - \rho)\delta(z - vt),$$

$$j_y = \frac{1}{\gamma^2}\frac{\partial M_x}{\partial z} = -\frac{j_0\pi R_0^2}{\gamma^2 d}\frac{\partial^2}{\partial z\partial y}\Theta(d - \rho)\delta(z - vt),$$

$$j_z = \frac{\partial M_y}{\partial x} - \frac{\partial M_x}{\partial y} = \frac{j_0\pi R_0^2}{d}(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2})\Theta(d - \rho)\delta(z - vt),$$

$$\rho_{Ch} = \frac{\beta}{c}(\frac{\partial M_y}{\partial x} - \frac{\partial M_x}{\partial y}) = \frac{\beta j_0\pi R_0^2}{cd}(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2})\Theta(d - \rho)\delta(z - vt).$$

Let the major torus radius also tend to zero. Then,

$$\Theta(d - \rho) = \pi d^2\delta(x)\delta(y)$$

and

$$j_x = -\frac{cm_t}{\gamma^2}\frac{\partial^2}{\partial z\partial x}\delta(x)\delta(y)\delta(z - vt), \quad j_y = -\frac{cm_t}{\gamma^2 d}\frac{\partial^2}{\partial z\partial y}\delta(x)\delta(y)\delta(z - vt),$$

$$j_z = cm_t(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2})\delta(x)\delta(y)\delta(z - vt),$$

$$\rho_{Ch} = \beta m_t(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2})\delta(x)\delta(y)\delta(z - vt).$$

Here $m_t = \pi^2 j_0 d R_0^2/c$ is the toroidal moment. Fourier transforms of these densities are

$$\rho_{Ch}(\omega) = \frac{m_t}{2\pi c}(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2})D, \quad j_z(\omega) = \frac{m_t}{2\pi\beta}(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2})D,$$

$$j_x = -\frac{m_t}{2\pi\beta\gamma^2}\frac{\partial^2}{\partial z\partial x}D, \quad j_y = -\frac{m_t}{2\pi\beta\gamma^2}\frac{\partial^2}{\partial z\partial y}D,$$

where $D = \delta(x)\delta(y)\exp(i\psi)$ and $\psi = kz/\beta$. Electromagnetic potentials are given by

$$\Phi = \frac{m_t}{2\pi\epsilon c}(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2})\alpha, \quad A_z = \frac{\mu m_t}{2\pi v}\exp(i\psi)(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2})\alpha,$$

$$A_x = -\frac{\mu m_t}{2\pi\gamma^2 v}\frac{\partial^2\alpha}{\partial z\partial x}, \quad A_y = -\frac{\mu m_t}{2\pi\gamma^2 v}\frac{\partial^2\alpha}{\partial z\partial y},$$

where α is the same as in (2.2). Electromagnetic field strengths are

$$E_x = \frac{m_t k^2}{2\pi\epsilon v\beta}(n^2 - 1)\frac{\partial\alpha}{\partial x}, \quad E_y = \frac{m_t k^2}{2\pi\epsilon v\beta}(n^2 - 1)\frac{\partial\alpha}{\partial y},$$

$$E_z = \frac{ik^3 m_t}{2\pi\epsilon v\beta^2}(n^2 - 1)(1 - \beta_n^2)\alpha, \quad H_x = -\frac{m_t k^2}{2\pi v}(n^2 - 1)\frac{\partial\alpha}{\partial y},$$

$$H_y = \frac{m_t k^2}{2\pi v}(n^2 - 1)\frac{\partial\alpha}{\partial x}, \quad H_z = 0.$$

Or, explicitly,

$$E_\rho = -\frac{m_t k^3}{\pi\epsilon v\beta^2\gamma_n}(n^2 - 1)\exp(i\psi)K_1, \quad E_z = \frac{ik^3 m_t}{\pi\epsilon v\beta^2}(n^2 - 1)\exp(i\psi)(1 - \beta_n^2)K_0,$$

$$H_\phi = -\frac{m_t k^3}{\pi v\beta\gamma_n}(n^2 - 1)\exp(i\psi)K_1$$

for $\beta_n < 1$ and

$$E_\rho = -i\frac{m_t k^3}{2\epsilon v\beta^2\gamma_n}(n^2 - 1)\exp(i\psi)H_1^{(1)}, \quad E_z = \frac{k^3 m_t}{2\epsilon v\beta^2}(n^2 - 1)\exp(i\psi)(\beta_n^2 - 1)H_0^{(1)},$$

$$H_\phi = -i\frac{m_t k^3}{2v\beta\gamma_n}(n^2 - 1)\exp(i\psi)H_1^{(1)}$$

for $\beta_n > 1$. The energy loss through the cylinder surface of the radius ρ coaxial with the motion axis per unit frequency and per unit length is

$$\sigma_\rho(\omega) = \frac{d^2\mathcal{E}}{dzd\omega} = -\pi c\rho(E_z H_\phi^* + E_z^* H_\phi).$$

It equals zero for $v < c_n$ and

$$\sigma_\rho(\omega) = \frac{k^5 m_t^2}{\epsilon v \beta^3}(\beta_n^2 - 1)(n^2 - 1)^2 \tag{2.21}$$

for $v > c_n$. Formerly, this equation was obtained in [18]. The absence of overall $1/\gamma$ factor in (2.21) and its proportionality to ω^5 show that the radiation intensity for the toroidal dipole directed along the velocity is maximal for large frequencies and $v \sim c$.

2.3.2 The velocity is perpendicular to the torus axis

Let a toroidal solenoid move in medium with the velocity perpendicular to the torus symmetry axis coinciding with the z axis. For definiteness, let the TS move along the x axis. Then, in the LF

$$\rho_{Ch} = -\frac{j_0 v\gamma^2}{c^2 R_0}\frac{z(x-vt)}{\rho_1^2}\delta(R_1 - R_0), \quad j_x = -j_0\frac{\gamma^2}{R_0}\frac{z(x-vt)}{\rho_1^2}\delta(R_1 - R_0),$$

$$j_y = -j_0\frac{zy}{\rho_1^2}\frac{\delta(R_1 - R_0)}{R_0}, \quad j_z = j_0\frac{\rho_1 - d}{\rho_1}\frac{\delta(R_1 - R_0)}{R_0}.$$

Here

$$\rho_1 = \sqrt{(x-vt)^2\gamma^2 + y^2}, \quad R_1 = \sqrt{(\rho_1 - d)^2 + z^2}.$$

It is easy to check that

$$j_x = -\frac{\partial M_y}{\partial z}, \quad j_y = \frac{\partial M_x}{\partial z}, \quad j_z = \frac{1}{\gamma^2}\frac{\partial M_y}{\partial x} - \frac{\partial M_x}{\partial y}, \quad \rho_{Ch} = -\frac{\beta}{c}\frac{\partial M_y}{\partial z},$$

where

$$M_y = -j_0\gamma^2\frac{x - vt}{\rho_1^2}\Theta(R_0 - R_1), \quad M_x = j_0\frac{y}{\rho_1^2}\Theta(R_0 - R_1), \quad M_z = 0.$$

Let the minor radius R_0 of a torus tend to zero. Then,

$$\Theta(R_0 - R_1) = \pi R_0^2 \delta(\rho_1 - d)\delta(z)$$

and

$$M_x = -j_0\frac{\pi R_0^2}{d}\frac{\partial}{\partial y}\Theta(d - \rho_1)\delta(z), \quad M_y = j_0\frac{\pi R_0^2}{d}\frac{\partial}{\partial x}\Theta(d - \rho_1)\delta(z).$$

Therefore,

$$\rho_{Ch} = -\frac{\beta j_0 \pi R_0^2}{cd}\frac{\partial^2}{\partial x \partial z}\Theta(d - \rho_1)\delta(z), \quad j_x = -\frac{j_0 \pi R_0^2}{d}\frac{\partial^2}{\partial x \partial z}\Theta(d - \rho_1)\delta(z),$$

$$j_y = -\frac{j_0 \pi R_0^2}{d}\frac{\partial^2}{\partial y \partial z}\Theta(d - \rho_1)\delta(z), \quad j_z = \frac{j_0 \pi R_0^2}{d}[\frac{1}{\gamma^2}\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}]\Theta(d - \rho_1)\delta(z).$$

Now we let the major radius d also tend to zero. Then,

$$\Theta(d-\rho_1)=\frac{\pi d^2}{\gamma}\delta(x-vt)\delta(y), \quad \rho_{Ch}=-\frac{\beta m_t}{\gamma}\frac{\partial^2}{\partial x\partial z}\delta(x-vt)\delta(y)\delta(z),$$

$$j_x=-\frac{cm_t}{\gamma}\frac{\partial^2}{\partial x\partial z}\delta(x-vt)\delta(y)\delta(z), \quad j_y=-\frac{cm_t}{\gamma}\frac{\partial^2}{\partial y\partial z}\delta(x-vt)\delta(y)\delta(z),$$

$$j_z=\frac{cm_t}{\gamma}[\frac{1}{\gamma^2}\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}]\delta(x-vt)\delta(y)\delta(z).$$

The Fourier transforms of these densities are

$$\rho_{Ch}=-\frac{m_t}{2\pi c\gamma}\frac{\partial^2}{\partial x\partial z}\exp(i\psi_1)\delta(y)\delta(z),$$

$$j_x=-\frac{m_t}{2v\pi\gamma}\frac{\partial^2}{\partial x\partial z}\exp(i\psi_1)\delta(y)\delta(z), \quad j_y=-\frac{m_t}{2v\pi\gamma}\frac{\partial^2}{\partial y\partial z}\exp(i\psi_1)\delta(y)\delta(z),$$

$$j_z=\frac{m_t}{2v\pi\gamma}[\frac{1}{\gamma^2}\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}]\exp(i\psi_1)\delta(y)\delta(z).$$

Here $\psi_1=kx/\beta$. As a result, we arrive at the following electromagnetic potentials:

$$\Phi=-\frac{\beta m_t}{2c\pi\gamma\epsilon}\frac{\partial^2}{\partial x\partial z}\alpha_1, \quad A_x=-\frac{m_t\mu}{2v\pi\gamma}\frac{\partial^2}{\partial x\partial z}\alpha_1,$$

$$A_y=-\frac{m_t\mu}{2v\pi\gamma}\frac{\partial^2}{\partial y\partial z}\alpha_1, \quad A_z=\frac{m_t\mu}{2v\pi\gamma}[\frac{1}{\gamma^2}\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}]\alpha_1,$$

where α_1 is the same as in (2.11). We give without derivation EMF strengths

$$E_x=\frac{k^2m_t}{2\pi v\beta\gamma\epsilon}(n^2-1)\sin\phi\frac{\partial\alpha_1}{\partial\rho_1},$$

$$E_y=\frac{ikm_t}{2\pi\gamma\epsilon v}(n^2-1)\sin\phi\cos\phi[\frac{k^2(\beta_n^2-1)}{\beta^2}\alpha_1+\frac{2}{\rho_1}\frac{\partial\alpha_1}{\partial\rho_1}],$$

$$E_z=-\frac{ikm_t}{2\pi\gamma\epsilon v}(n^2-1)[\frac{k^2}{\beta^2}(1+(\beta_n^2-1)\cos^2\phi)\alpha_1+\frac{\cos 2\phi}{\rho_1}\frac{\partial\alpha_1}{\partial\rho_1}],$$

$$E_\phi=-\frac{ikm_t}{2\pi\gamma\epsilon v}(n^2-1)\cos\phi[k^2n^2\alpha_1+\frac{1}{\rho_1}\frac{\partial\alpha_1}{\partial\rho_1}],$$

$$H_x=-\frac{m_t}{2v\pi\gamma}k^2(n^2-1)\cos\phi\frac{\partial\alpha_1}{\partial\rho_1}, \quad H_y=\frac{im_tk^3}{2\pi v\gamma\beta}(n^2-1)\alpha_1, \quad H_z=0,$$

$$H_\phi=-\frac{im_tk^3\alpha_1}{2\pi v\gamma\beta}(n^2-1)\sin\phi,$$

where ϕ is the angle defining the observation point in the yz plane. It is counted from the y axis and is defined by (2.12) and (2.13). In a manifest form, EMF strengths are given by

$$E_x = -\frac{k^3 m_t}{\pi v \beta^2 \gamma \gamma_n \epsilon}(n^2-1)\sin\phi K_1 \exp(i\psi_1),$$

$$E_y = \frac{ikm_t}{\pi v \gamma \epsilon}(n^2-1)\sin\phi\cos\phi[\frac{k^2}{\beta^2}(\beta_n^2-1)K_0 - \frac{2k}{\rho\beta\gamma_n}K_1]\exp(i\psi_1),$$

$$E_z = -\frac{ikm_t}{\pi v \gamma \epsilon}(n^2-1)[\frac{k^2}{\beta^2}(1+\cos^2\phi(\beta_n^2-1))K_0 - \frac{k}{\rho\beta\gamma_n}\cos 2\phi K_1]\exp(i\psi_1),$$

$$E_\phi = -\frac{ikm_t}{\pi v \gamma \epsilon}(n^2-1)\cos\phi[k^2 n^2 K_0 - \frac{k}{\rho\beta\gamma_n}K_1]\exp(i\psi_1),$$

$$H_x = \frac{m_t k^3}{\pi v \beta \gamma \gamma_n}(n^2-1)\cos\phi K_1 \exp(i\psi_1), \quad H_y = \frac{i m_t k^3}{\pi v \gamma \beta}(n^2-1)K_0 \exp(i\psi_1),$$

$$H_z = 0, H_\phi = -\frac{i m_t k^3}{\pi v \gamma \beta}(n^2-1)\sin\phi K_0 \exp(i\psi_1)$$

for $v < c_n$ and

$$E_x = -\frac{i m_t k^3}{2 v \epsilon \beta^2 \gamma \gamma_n}\sin\phi(n^2-1)H_1^{(1)}\exp(i\psi_1),$$

$$E_y = -\frac{km_t}{2v\gamma\epsilon}(n^2-1)\sin\phi\cos\phi[\frac{k^2}{\beta^2}(\beta_n^2-1)H_0^{(1)} - \frac{2k}{\rho\beta\gamma_n}H_1^{(1)}]\exp(i\psi_1),$$

$$E_z = \frac{km_t}{2v\gamma\epsilon}(n^2-1)[\frac{k^2}{\beta^2}(1+\cos^2\phi(\beta_n^2-1))H_0^{(1)} - \frac{k}{\rho\beta\gamma_n}\cos 2\phi H_1^{(1)}]\exp(i\psi_1),$$

$$E_\phi = \frac{km_t}{2v\gamma\epsilon}(n^2-1)\cos\phi[k^2 n^2 H_0^{(1)} - \frac{k}{\rho\beta\gamma_n}H_1^{(1)}]\exp(i\psi_1),$$

$$H_x = \frac{i m_t k^3}{2 v \beta \gamma \gamma_n}(n^2-1)\cos\phi H_1^{(1)}\exp(i\psi_1), \quad H_y = -\frac{m_t k^3}{2 v \gamma \beta}(n^2-1)H_0^{(1)}\exp(i\psi_1),$$

$$H_z = 0, \quad H_\phi = \frac{m_t k^3}{2 v \gamma \beta}(n^2-1)\sin\phi H_0^{(1)}\exp(i\psi_1)$$

for $v > c_n$. Again, E_ϕ and H_ϕ are tangential to the torus surface and perpendicular to torus velocity directed along the x axis. The energy flux through the cylindrical surface of the radius ρ_1 per unit length and per unit frequency is equal to

$$\frac{d^2\mathcal{E}}{dx d\omega} = \int_0^{2\pi} \sigma(\omega,\phi)d\phi,$$

where

$$\sigma(\omega,\phi) = \frac{d^3\mathcal{E}}{dx d\omega d\phi} = \frac{c}{2}\rho_1(E_\phi^* H_x + E_\phi H_x^* - H_\phi^* E_x - H_\phi E_x^*).$$

Substituting here field strengths , one obtains that the differential intensity is zero for $v < c_n$ and

$$\sigma(\omega,\phi) = \frac{k^5 m_t^2}{2\epsilon v\beta\pi\gamma^2}(n^2-1)^2(n^2\cos^2\phi + \frac{1}{\beta^2}\sin^2\phi). \tag{2.22}$$

for $v > c_n$. The integration over ϕ gives

$$\sigma(\omega) = \frac{k^5 m_t^2}{2\epsilon v\beta\gamma^2}(n^2-1)^2(n^2 + \frac{1}{\beta^2}). \tag{2.23}$$

As far as we know, radiation intensities (2.22) and (2.23) are obtained here for the first time. They are discontinuous: in fact, they fall from (2.22) or (2.23) for $\beta_n > 1$ to their one-half for $\beta = 1/n$ and to zero for $\beta < 1/n$. Also, we observe the appearance of the velocity window $c_n < v < c$ where the radiation differs from zero. Following to the Frank terminology, we conclude that the magnetic dipole parallel (perpendicular) to the velocity polarizes the medium in the same way as the toroidal dipole perpendicular (parallel) to the velocity.

2.4 Unbounded motion of a point-like electric dipole

Consider an electric dipole consisting of point-like electric charges:

$$\rho_d = e[\delta^3(\vec{r}+a\vec{n}) - \delta^3(\vec{r}-a\vec{n})]. \tag{2.24}$$

Here $\vec{r}$ defines the dipole center-of-mass, $2a$ is the distance between charges and vector $\vec{n} = (\sin\theta_0\cos\phi_0, \sin\theta_0\sin\phi_0, \cos\theta_0)$ defines the dipole orientation. Let the dipole move uniformly along the z axis. Then, in the laboratory frame

$$\rho_d = e\gamma\{\delta(x+an_x)\delta(y+an_y)\delta[(z-vt)\gamma+an_z] - \delta(x-an_x)\delta(y-an_y)\delta[(z-vt)\gamma-an_z]\},$$

$$j_z = v\rho_d.$$

Let the distance between charges tend to zero. Then,

$$\rho_d = 2ea(\vec{n}\vec{\nabla})\delta(x)\delta(y)\delta(z-vt), \quad j_z = v\rho_d.$$

Here

$$(\vec{n}\vec{\nabla}) = \vec{n}_x\nabla_x + \vec{n}_y\nabla_y + \frac{1}{\gamma}\vec{n}_z\nabla_z, \quad \nabla_i = \frac{\partial}{\partial x_i}.$$

Fourier components of these densities are

$$\rho_d(\omega) = \frac{ea}{\pi v}(\vec{n}\vec{\nabla})\delta(x)\delta(y)\exp(\frac{ikz}{\beta}), \quad j_z(\omega) = v\rho_d(\omega).$$

The electromagnetic potentials are equal to

$$\Phi = \frac{ea}{\pi v\epsilon}(\vec{n}\vec{\nabla})\alpha, \quad A_z = \frac{ea\mu}{\pi c}(\vec{n}\vec{\nabla})\alpha,$$

where α is the same as in (2.2). The nonvanishing components of EMF strengths are

$$E_x = -\frac{ea}{\pi v\epsilon}\frac{\partial}{\partial x}(\vec{n}\vec{\nabla})\alpha, \quad E_y = -\frac{ea}{\pi v\epsilon}\frac{\partial}{\partial y}(\vec{n}\vec{\nabla})\alpha, \quad E_z = -\frac{ea}{\pi v\epsilon}(1-\beta_n^2)\frac{\partial}{\partial z}(\vec{n}\vec{\nabla})\alpha,$$

$$H_x = \frac{ea}{\pi c}\frac{\partial}{\partial y}(\vec{n}\vec{\nabla})\alpha, \quad H_y = -\frac{ea}{\pi c}\frac{\partial}{\partial x}(\vec{n}\vec{\nabla})\alpha.$$

In a manifest form, we write out only those components of field strengths which are needed for the evaluation of the radial cylindric energy flux. They are equal to

$$E_z = \frac{2ek^2a}{\pi\epsilon\beta^2 v}(1-\beta_n^2)(\frac{n_z}{\gamma}K_0 + i\frac{\tilde{n}_\rho}{\gamma_n}K_1)\exp(i\psi),$$

$$H_\phi = \frac{2eak}{\pi v}\{\tilde{n}_\rho[\frac{k}{\beta}(\beta_n^2-1)K_0 - \frac{1}{\gamma_n\rho}K_1] + \frac{ikn_z}{\beta\gamma\gamma_n}K_1\}\exp(i\psi)$$

for $v < c_n$ and

$$E_z = \frac{ek^2a}{\epsilon\beta^2 v}(\beta_n^2-1)(\frac{\tilde{n}_\rho}{\gamma_n}H_1^{(1)} - i\frac{n_z}{\gamma}H_0^{(1)})\exp(i\psi),$$

$$H_\phi = \frac{eak}{v}\{i\tilde{n}_\rho[\frac{k}{\beta}(\beta_n^2-1)H_0^{(1)} - \frac{1}{\rho\gamma_n}H_1^{(1)}] - n_z\frac{k}{\beta\gamma\gamma_n}H_1^{(1)}\}\exp(i\psi)$$

for $v > c_n$. Here $\psi = kz/\beta$, $\tilde{n}_\rho = \sin\theta_0\cos(\phi-\phi_0)$; θ_0 is the angle between the symmetry axis of the electric dipole and its velocity; ϕ is the asimuthal position of the observation point on the cylinder surface and ϕ_0 defines the orientation of the electric dipole in the plane perpendicular to the motion axis.

The radiation intensity per unit length of the cylindrical surface coaxial with the motion axis, per unit asimuthal angle and per unit frequency is

$$\sigma(\phi,\omega) = \frac{d^3\mathcal{E}}{dzd\phi\, d\omega} = -\frac{c\rho}{2}(E_zH_\phi^* + E_z^*H_\phi).$$

It equals

$$\sigma_\rho(\phi,\omega) = \frac{4e^2a^2k^3n_z\tilde{n}_\rho}{\pi^2\epsilon\beta^3 v\gamma}(1-\beta_n^2)[\frac{k\rho}{\beta}(1-\beta_n^2)(K_0^2+K_1^2) + \frac{1}{\gamma_n}K_0K_1] \qquad (2.25)$$

for $v < c_n$ and

$$\sigma_\rho(\phi,\omega) = \frac{2e^2a^2k^3}{\pi\epsilon\beta^3 v}(\beta_n^2-1)\{\tilde{n}_\rho^2(\beta_n^2-1) + n_z^2(1-\beta^2)+$$

$$+\tilde{n}_\rho n_z\frac{\pi}{2\gamma}[\frac{k\rho}{\beta}(\beta_n^2-1)(J_0^2+N_0^2+J_1^2+N_1^2) - \frac{1}{\gamma_n}(N_0N_1+J_0J_1)]\} \qquad (2.26)$$

for $v > c_n$. Integrating over the asimuthal angle ϕ one finds that $\sigma_\rho(\omega) = 0$ for $v < c_n$ and

$$\sigma_\rho(\omega) = \frac{2e^2a^2k^3}{\pi\epsilon\beta^3 v}(\beta_n^2 - 1)[(\beta_n^2 - 1)\sin^2\theta_0 + 2(1-\beta^2)\cos^2\theta_0] \qquad (2.27)$$

for $v > c_n$. For the symmetry axis along the velocity ($\theta_0 = 0$) and perpendicular to it ($\theta_0 = \pi/2$) one gets

$$\sigma_\rho(\omega, \theta_0 = 0) = \frac{4e^2a^2k^3}{\epsilon\beta^3 v}(\beta_n^2 - 1)(1-\beta^2) \qquad (2.28)$$

and

$$\sigma_\rho(\omega, \theta_0 = \theta/2) = \frac{2e^2a^2k^3}{\epsilon\beta^3 v}(\beta_n^2 - 1)^2, \qquad (2.29)$$

resp. Again, the same confusion with (2.28) and (2.29) takes place in the physical literature. In Refs. [6,9,19], the factor $(1-\beta^2)$ in (2.28) is absent. Yet, it presents in [3,4,16]. In Ref.[16], $(\beta_n^2 - 1)$, instead of $(\beta_n^2 - 1)^2$, enters into (2.29). The expression given in [19] is two times larger than (2.29). The correct expession for (2.29) is given in [3,4,6,9].

It is rather surprising that for $\beta_n < 1$, the non-averaged radiation intensities are equal to zero when the symmetry axis is either parallel or perpendicular to the velocity, but differs from zero for the intermediate inclination of the symmetry axis (see (2.25)). Integration over the asimuthal angle gives $\sigma_\rho(\omega, \theta) = 0$ for $\beta_n < 1$.

Again, it should be mentioned that we did not intend to recover misprints in the papers of other authors. What we need are the reliable formulae suitable for practical applications.

3 The Tamm problem for electric charge, magnetic, electric and toroidal dipoles

3.1 Pedagogical example: the Tamm problem for the electric charge

Tamm considered the following problem [20]. A point charge is at rest at the point $z = -z_0$ of the z axis up to a moment $t = -t_0$ and at the point $z = z_0$ after the moment $t = t_0$. In the time interval $-t_0 < t < t_0$, it uniformly moves along the z axis with the velocity v greater than the light velocity in medium $c_n = c/n$. The nonvanishing z Fourier component of the vector potential (VP) is given by

$$A_z(x, y, z) = \frac{e\mu}{2\pi c}\alpha_T, \qquad (3.1)$$

where

$$\alpha_T = \int_{-z_0}^{z_0} \frac{dz'}{R}\exp[ik(\frac{z'}{\beta} + nR)], \quad R = [\rho^2 + (z-z')^2]^{1/2}, \quad \rho^2 = x^2 + y^2.$$

Tamm presents R in the form $R = r - z' \cos\theta$, thus, disregarding the second order terms relative to z'. Imposing the conditions:
i) $R >> z_0$ (this means that the observation distance is much larger than the motion interval); ii) $k_n R_0 >> 1$, $k_n = \omega/c_n$ (this means that the observations are made in the wave zone); iii) $nz_0^2/2R_0\lambda << 1$, $\lambda = 2\pi c/\omega$ (this this means that the second-order terms in the expansion of R should be small compared with π since ψ is a phase in (3.1); λ is the observed wavelength), Tamm obtained the following expression for α_T

$$\alpha_T = \frac{2}{kr} \exp(ik_n r) q$$

and for the vector magnetic potential

$$A_z = \frac{e\mu}{\pi \omega r} \exp(iknr) q. \tag{3.2}$$

Here

$$q = \frac{1}{1/\beta - n\cos\theta} \sin[kz_0(\frac{1}{\beta} - n\cos\theta)].$$

In the limit $kz_0 \to \infty$,

$$q \to \pi\delta(1/\beta - n\cos\theta) \quad \text{and} \quad A_z \to \frac{e\mu}{\omega n r} \exp(iknr)\delta(\cos\theta - 1/\beta n).$$

Using (3.2), Tamm evaluated the EMF strengths and the energy flux through the sphere of the radius R_0 for the whole time of observation

$$\mathcal{E} = R_0^2 \int S_r d\Omega dt = \int \frac{d^2\mathcal{E}}{d\Omega d\omega} d\Omega d\omega, \quad d\Omega = \sin\theta d\theta d\phi, \quad S_r = \frac{c}{4\pi} E_\theta H_\phi$$

where

$$\frac{d^2\mathcal{E}}{d\Omega d\omega} = \frac{e^2 \mu n}{\pi^2 c} [\sin\theta \frac{\sin kz_0(1/\beta - n\cos\theta)}{n\cos\theta - 1/\beta}]^2, \quad \beta_n = \beta n \tag{3.3}$$

is the energy emitted into the solid angle $d\Omega$, in the frequency interval $d\omega$. This famous formula obtained by Tamm is frequently used by experimentalists for the identification of the Cherenkov radiation. When kz_0 is large,

$$\frac{d^2\mathcal{E}}{d\Omega d\omega} = \frac{e^2 \mu k z_0}{\pi c} (1 - 1/\beta_n^2) \delta(\cos\theta - 1/\beta n). \tag{3.4}$$

Integrating this equation over the solid angle, one gets

$$\frac{d\mathcal{E}}{d\omega} = \frac{2e^2 \mu k z_0}{c} (1 - 1/\beta_n^2). \tag{3.5}$$

Correspondingly, the energy radiated per unit frequency and per unit length (it is obtained by dividing (3.5) by the motion interval $L = 2z_0$) is

$$\frac{d\mathcal{E}}{d\omega dL} = \frac{e^2 \mu}{c} (1 - 1/\beta_n^2). \tag{3.6}$$

The typical experimental situations described by the Tamm formula are:
i) β decay of a nucleus at one space point accompanied by a subsequent absorption of the emitted electron at another point;
ii) A high energy electron consequently moves in vacuum, enters into the dielectric slab, leaves the slab and propagates again in vacuum. Since the electron moving uniformly in vacuum does not radiate (apart from the transition radiation arising at the boundaries of the dielectric slab), the experimentalists describe this situation via the Tamm formula, assuming that the electron is created at one side of the slab and is absorbed at the other.

3.2 The Tamm problem for the magnetic dipole

3.2.1 The magnetic dipole is parallel to the velocity

In this case the Fourier components of the current density differ from zero only on the motion interval $(-z_0, z_0)$. Correspondingly, magnetic potential and field strengths are given by

$$A_\phi = -\frac{\mu m_d}{2\pi v\gamma}\frac{\partial \alpha_T}{\partial \rho}, \quad \mu H_\theta = -\frac{\partial A_\phi}{\partial r} - \frac{\cot\theta}{r}A_\phi,$$

where α_T is the same as in (3.1). Using approximations i)-iii), one gets

$$H_\theta = -\frac{m_d k^2 n^2 \sin\theta}{2\pi\gamma v}\alpha_T.$$

The electric field strengths are obtained from the relation

$$curl\vec{H} = -ik\epsilon\vec{E}$$

valid outside the motion interval. This gives

$$E_\phi = \frac{k^2 n\mu m_d}{2\pi\gamma v}\alpha_T \sin\theta.$$

When evaluating field strengths, we dropped the terms which decrease at infinity faster than $1/r$ and which do not contribute to the radiation flux. The distribution of the radial energy flux on the sphere of the radius r is given by

$$\sigma_r(\theta,\phi) = \frac{d^2\mathcal{E}}{d\Omega d\omega} = -\frac{c}{2}r^2(E_\phi H_\theta^* + E_\phi^* H_\theta) = \frac{m_d^2 k^2 n^3 \mu \sin^2\theta}{\pi^2\gamma^2\beta v}q^2. \quad (3.7)$$

In the limit $kz_0 \to \infty$, one gets

$$\frac{d^2\mathcal{E}}{d\Omega d\omega} = \frac{m_d^2 k^2 n^2 \mu k z_0}{\pi\gamma^2\beta v}(1 - 1/\beta_n^2)\delta(\cos\theta - 1/\beta_n). \quad (3.8)$$

Integration over the solid angle gives the frequency distribution of the emitted radiation per unit frequency and per unit length

$$\frac{d\mathcal{E}}{dLd\omega} = \frac{m_d^2\omega^3\mu}{v^4\gamma^2\gamma_n^2}. \tag{3.9}$$

This coincides with (2.10).

3.2.2 The magnetic dipole is perpendicular to the velocity

Let the magnetic dipole directed along the z axis move on the interval $(-x_0, x_0)$ of the x axis with the constant velocity v. We write out without derivation electromagnetic field strengths contributing to the radial energy flux

$$E_\theta = \frac{m_d k^2 \mu n}{2\pi v}\alpha'_T(1-\beta^2\cos^2\theta)\cos\phi, \quad E_\phi = -\frac{m_d k^2 \mu n}{2\pi v\gamma^2}\alpha'_T\cos\theta\sin\phi,$$

$$H_\theta = \frac{m_d k^2 n^2}{2\pi v\gamma^2}\alpha'_T\cos\theta\sin\phi, \quad H_\phi = \frac{m_d k^2 n^2}{2\pi v}\alpha'_T(1-\beta^2\cos^2\theta)\cos\phi.$$

where

$$\alpha'_T = (2/kr)q\exp(ik_n r), \quad q = (1/\beta - n\cos\theta)^{-1}\sin[kx_0(1/\beta - n\cos\theta)].$$

The θ is the angle between the radius-vector of the observation point and the motion axis (which is the x axis). The ϕ is the observation asimuthal angle in the yz plane. The value $\phi = 0$ corresponds to the y axis, the magnetic moment is along the z axis. The distribution of the radial energy flux on the sphere of the radius r is given by

$$\sigma_r(\theta,\phi,\omega) = \frac{d^2\mathcal{E}}{d\Omega d\omega} = \frac{c}{2}r^2(E_\theta H_\phi^* + E_\theta^* H_\phi - E_\phi H_\theta^* - E_\phi^* H_\theta) =$$

$$= \frac{m_d^2 k^2 n^3\mu}{\pi^2\beta v}[\cos^2\phi(1-\beta^2\cos^2\theta)^2 + \gamma^{-4}\sin^2\phi\cos^2\theta]q^2. \tag{3.10}$$

In the limit $kz_0 \to \infty$ this gives

$$\frac{d^2\mathcal{E}}{d\Omega d\omega} = \frac{m_d^2 k^3 z_0 n^2\mu}{\pi\beta v}[\cos^2\phi(1-1/n^2)^2 + \frac{1}{\gamma^4\beta_n^2}\sin^2\phi]\delta(\cos\theta - 1/\beta_n). \tag{3.11}$$

Integration over the solid angle gives

$$\frac{d^2\mathcal{E}}{dLd\omega} = \frac{m_d^2 k^3 n^2\mu}{2\beta v}[(1-1/n^2)^2 + \frac{1}{\gamma^4\beta_n^2}]. \tag{3.12}$$

This coincides with (2.18).

3.3 The Tamm problem for the toroidal dipole

3.3.1 The toroidal dipole is parallel to the velocity

The direction of the toroidal dipole coincides with the direction of its symmetry axis. The electromagnetic vector potential and field strengths contributing to the radial energy flux are given by

$$E_\theta = \frac{i m_t k^3 n^2 \mu}{2\pi v} \sin\theta(1-\beta^2\cos^2\theta)\alpha_T, \quad H_\phi = \frac{i m_t k^3 n^3}{2\pi v} \sin\theta(1-\beta^2\cos^2\theta)\alpha_T,$$

where α_T is the same as above. The distribution of the radial energy flux on the sphere of the radius r is given by

$$\sigma_r = \frac{d^2\mathcal{E}}{d\Omega d\omega} = \frac{c}{2} r^2 (E_\theta H_\phi^* + E_\theta^* H_\phi) = \frac{m_t^2 k^4 n^5 \mu}{\pi^2 \beta v} \sin^2\theta(1-\beta^2\cos^2\theta)^2 q^2. \quad (3.13)$$

Here θ is the polar angle of the observation point. In the limit $kz_0 \to \infty$, (3.13) goes into

$$\frac{d^2\mathcal{E}}{d\Omega d\omega} = \frac{m_t^2 k^5 z_0 n^4 \mu}{\pi \beta v}(1-1/\beta_n^2)(1-1/n^2)^2 \delta(\cos\theta - 1/\beta_n). \quad (3.14)$$

Integration over the solid angle gives

$$\frac{d^2\mathcal{E}}{dL d\omega} = \frac{m_t^2 k^5 n^4 \mu}{\beta v}(1-1/\beta_n^2)(1-1/n^2)^2. \quad (3.15)$$

This coincides with (2.21).

3.3.2 The symmetry axis is perpendicular to the velocity

In this case, the electromagnetic field strengths contributing to the radial energy flux are given by

$$E_\theta = -\frac{i\mu m_t k^3 n^2 \alpha_T'}{2v\pi\epsilon\gamma}(1-\beta^2\cos^2\theta)\cos\theta\sin\phi, \quad E_\phi = -\frac{i\mu m_t k^3 n^2 \alpha_T'}{2v\pi\epsilon\gamma}(1-\beta^2\cos^2\theta)\cos\phi,$$

$$H_\theta = \frac{i m_t k^3 n^3 \alpha_T'}{2v\pi\gamma}(1-\beta^2\cos^2\theta)\cos\phi, \quad H_\phi = -\frac{i m_t k^3 n^3 \alpha_T'}{2v\pi\gamma}(1-\beta^2\cos^2\theta)\cos\theta\sin\phi.$$

Correspondingly, the radial energy flux is

$$\sigma_r(\theta,\phi,\omega) = \frac{d^2\mathcal{E}}{d\omega d\Omega} = \frac{1}{2} c r^2 (E_\theta H_\phi^* + E_\theta^* H_\phi - E_\phi H_\theta^* - E_\phi^* H_\theta) =$$

$$= \frac{m_t^2 k^4 n^5 \mu}{\gamma^2 \pi^2 v \beta}(1-\beta^2\cos^2\theta)^2(\cos^2\theta\sin^2\phi + \cos^2\phi)q^2. \quad (3.16)$$

Again, θ is the polar angle of the observation point; the toroidal dipole is along the z axis, the angle ϕ defining the position of the observation point in the (yz) plane

perpendicular to the velocity, is counted from the y axis.
In the limit $kz_0 \to \infty$, (3.16) goes into

$$\frac{d^2\mathcal{E}}{d\omega d\Omega} = \frac{m_t^2 k^5 z_0 n^4 \mu}{\gamma^2 \pi v \beta}(1-1/n^2)^2(\frac{1}{\beta_n^2}\sin^2\phi + \cos^2\phi)\delta(\cos\theta - 1/\beta_n). \tag{3.17}$$

The integration over the solid angle ϕ gives

$$\frac{d^2\mathcal{E}}{d\omega dL} = \frac{m_t^2 k^5 n^4 \mu}{2\gamma^2 v \beta}(1-1/n^2)^2(\frac{1}{\beta_n^2}+1). \tag{3.18}$$

This coincides with (2.23).

3.4 Tamm's problem for the electric dipole with arbitrary orientation of the symmetry axis

Let the electric dipole move along the z axis and let it be directed along the vector $\vec{n} = (n_x, n_y, n_z)$ defining the direction of its symmetry axis in the laboratory reference frame. In this case, the vector potential and electromagnetic field strengths contributing to the radial energy flux are given by

$$A_z = \frac{iea\mu}{\pi c}(\vec{n}\vec{\nabla})\alpha_T,$$

$$E_\theta = \frac{eak^2 n\mu}{\pi c}\sin\theta(\tilde{n}_\rho \sin\theta + \frac{1}{\gamma}n_z\cos\theta)\alpha_T, \quad H_\phi = \frac{eak^2 n^2}{\pi c}\sin\theta(\tilde{n}_\rho\sin\theta + \frac{1}{\gamma}n_z\cos\theta)\alpha_T,$$

where $\tilde{n}_\rho = \sin\theta_0\cos(\phi - \phi_0)$ and $n_z = \cos\theta_0$; θ and ϕ define the position of the observation point; θ_0 and ϕ_0 define the orientation of electric dipole. Correspondingly, the radial energy flux is

$$\sigma_r(\theta,\phi,\omega) = \frac{d^2\mathcal{E}}{d\omega d\Omega} = \frac{1}{2}cr^2(E_\theta H_\phi^* + E_\theta^* H_\phi) =$$

$$= \frac{4e^2a^2k^2n^3\mu}{\pi^2 c}(\tilde{n}_\rho\sin\theta + \frac{1}{\gamma}n_z\cos\theta)^2\sin^2\theta q^2. \tag{3.19}$$

For the electric dipole oriented along the velocity ($\tilde{n}_\rho = 0, n_z = 1$), (3.19) is reduced to

$$\sigma_r^{||}(\theta,\phi,\omega) = \frac{4e^2a^2k^2n^3\mu}{\gamma^2\pi^2 c}\cos^2\theta\sin^2\theta q^2. \tag{3.20}$$

Correspondingly, for the electric dipole orientation perpendicular to the motion axis ($\tilde{n}_\rho = \cos(\phi-\phi_0)$, $n_z = 0$), one gets

$$\sigma_r^{\perp}(\theta,\phi,\omega) = \frac{4e^2a^2k^2n^3\mu}{\pi^2 c}q^2\sin^4\theta\cos^2(\phi-\phi_0). \tag{3.21}$$

In the limit $kz_0 \to \infty$ one gets

$$\frac{d^2\mathcal{E}}{d\omega d\Omega} = \frac{4e^2a^2k^3z_0n^2\mu}{\pi c}(\tilde{n}_\rho\sqrt{1-1/\beta_n^2} + \frac{1}{\gamma\beta_n}n_z)^2(1-1/\beta_n^2)\delta(\cos\theta - 1/\beta_n). \quad (3.22)$$

$$\sigma_r^{\|}(\theta,\phi,\omega) = \frac{4e^2a^2k^3z_0n^2\mu}{\gamma^2\pi c}\frac{1}{\beta_n^4\gamma_n^2}\delta(\cos\theta - 1/\beta_n), \quad (3.23)$$

$$\sigma_r^{\perp}(\theta,\phi,\omega) = \frac{4e^2a^2k^3z_0n^2\mu}{\pi c\beta_n^4\gamma_n^4}\cos^2(\phi-\phi_0)\delta(\cos\theta - 1/\beta_n). \quad (3.24)$$

The integration over the solid angle gives

$$\frac{d^2\mathcal{E}}{d\omega dL} = \frac{2e^2a^2k^3n^2\mu}{c}(\sin^2\theta_0(1-1/\beta_n^2) + \frac{2}{\gamma^2\beta_n^2}\cos^2\theta_0)(1-1/\beta_n^2). \quad (3.25)$$

$$(\frac{d^2\mathcal{E}}{d\omega dL})_{\|} = \frac{4e^2a^2k^3n^2\mu}{\gamma^2 c}\frac{1}{\beta_n^2}(1-\frac{1}{\beta_n^2}), \quad (3.26)$$

$$(\frac{d^2\mathcal{E}}{d\omega dL})_{\perp} = \frac{2e^2a^2k^3n^2\mu}{c}(1-\frac{1}{\beta_n^2})^2. \quad (3.27)$$

These equations coincide with (2.27)-(2.29).

Concluding remarks to this section. As expected, the integral Tamm intensities (that is, integrated over the solid angle), in the limit $kz_0 \to \infty$ (large motion interval) coincide with the radiation intensities correponding to the unbounded motion treated in section 2. The radiation intensities obtained in subsections (3.2)-(3.4) differ considerably from those given by Frank in [3,4]. There is essential difference between our derivation and that of [3,4]. The method used by Frank is rather complicated. He writes Maxwell equations in terms of electric and magnetic vector Hertz potentials which are related to the electromagnetic field strengths. In the right-hand sides of Maxwell equations there are electric and magnetic polarizations proportional to the LF electric and magnetic moments, resp. Electric and magnetic moments in the LF are connected with ones in the the dipole RF through the well-known linear relations (see, e.g. [5]). When in the dipole RF there is only electric or magnetic dipole, one may exclude from these relations the non-zero magnetic moment of the RF, thus, obtaining the relation between the electric and magnetic moments of the LF. On the other hand, we define the charge-current densities in the RF. Using the Lorentz transformation, the same as in vacuum, we recalculate them to the LF. Then, we tend the dimensions of these distributions to zero, thus obtaining infinitesimal charge-current distributions corresponding to the electric, magnetic or toroidal dipoles. With these infinitesimal charge-current distributions we solve Maxwell equations finding electromagnetic potentials and field strengths. Using them, we evaluate the radiated energy flux.

4 Electromagnetic field of the precessing magnetic dipole

Consider an infinitely thin circular turn with the constant current flowing in it. Let the centre of this current loop coincdes with the origin, while its symmetry axis precesses around the z axis with a constant angular velocity ω_0. We chose the rest frame (RF) of this loop as follows. Let $\vec{n}_x, \vec{n}_y$ and $\vec{n}_z$ be the orthogonal basis vectors of the laboratory frame (LF). The $\vec{e}_z$ vector of RF we align along the loop symmetry axis $\vec{n}$. Being expressed in terms of the LF basis vectors, it is given by

$$\vec{n} = \vec{e}_z = \cos\theta_0 \vec{n}_z + \sin\theta_0 \vec{n}_\rho = \vec{n}_r,$$

where $\vec{n}_\rho = \cos\omega_0 t \vec{n}_x + \sin\omega_0 t \vec{n}_y$ and θ_0 is the inclination angle of the loop symmetry axis towards the laboratory z axis. Other two basis vectors of RF lying in the plane of loop, we choose in the following way

$$\vec{e}_x = \frac{1}{\sin\theta_0}(\vec{n} \times \vec{n}_z) = \cos\omega_0 t \vec{n}_y - \sin\omega_0 t \vec{n}_x = \vec{n}_\phi,$$

$$\vec{e}_y = \frac{1}{\sin\theta_0}(\vec{n} \times (\vec{n} \times \vec{n}_z)) = \cos\omega_0 t \vec{n}_\rho - \sin\omega_0 t \vec{n}_z = \vec{n}_\theta,$$

that is, $\vec{e}_x, \vec{e}_y$ and $\vec{e}_z$ coincide with the spherical basis vectors.
Let x, y, z and x', y', z' be the coordinates of the same point in the laboratory and proper reference frames, resp. They are related as follows

$$x' = x\sin\omega_0 t - y\cos\omega_0 t, \quad y' = \rho\cos\theta_0 - z\sin\theta_0, \quad z' = \rho\sin\theta_0 + z\cos\theta_0, \quad (4.1)$$

where $\rho = x\cos\omega_0 t + y\sin\omega_0 t$. The current density in the RF is given by

$$\vec{j}' = \vec{e}_\psi I_0 \delta(z')\delta(\rho' - d),$$

where $\rho' = \sqrt{x'^2 + y'^2}$, $e_\psi = \vec{e}_x\cos\psi - \vec{e}_y\sin\psi$ is the vector lying in the plane of loop and defining the direction of current and ψ is the asimuthal angle in the plane of loop defined by $\cos\psi = x'/d$, $\sin\psi = y'/d$. In the LF, the components of the current density are given by

$$j_x = (\cos\theta_0\frac{\partial}{\partial y} - \sin\omega_0 t\sin\theta_0\frac{\partial}{\partial z})M, \quad j_y = (-\cos\theta_0\frac{\partial}{\partial x} + \cos\omega_0 t\sin\theta_0\frac{\partial}{\partial z})M,$$

$$j_z = \sin\theta_0(\sin\omega_0 t\frac{\partial}{\partial x} - \cos\omega_0 t\frac{\partial}{\partial y})M, \quad (4.2)$$

where

$$M = I_0\delta(z')\Theta(d - \sqrt{x'^2 + y'^2}).$$

x', y' and z' should be expressed through the coordinates (x, y, z, t) of the LF via the relations (4.1). We are interested to study the point-like ($d \to 0$) current loop,which is equivalent to the magnetic dipole. In this limit,

$$M = \pi d^2 I_0 \delta(x)\delta(y)\delta(z).$$

The vector magnetic potential is given by

$$\vec{A} = \frac{1}{c}\int \frac{1}{R}\vec{j}(\vec{r'}, t')\delta(t' - t + R/c)dV'dt'.$$

After integration, one gets for the spherical components of $\vec{A}$:

$$A_r = 0, \quad A_\theta = -\frac{\pi d^2 I_0}{c}\sin\theta_0 \frac{\partial}{\partial r}\frac{\sin\psi}{r},$$

$$A_\phi = \frac{\pi d^2 I_0}{c}\left(\frac{1}{r^2}\cos\theta_0 \sin\theta + \sin\theta_0\cos\theta \frac{\partial}{\partial r}\frac{\sin\psi}{r}\right), \tag{4.3}$$

Here $\psi = \omega_0 t - k_0 r - \phi$. The non-vanishing components of the field strengths are

$$E_r = 0, \quad E_\phi = \frac{\pi d^2 I_0 k_0}{c}\sin\theta_0\cos\theta\frac{\partial}{\partial r}\frac{\sin\psi}{r}, \quad E_\theta = \frac{\pi d^2 I_0 k_0}{c}\sin\theta_0\frac{\partial}{\partial r}\frac{\cos\psi}{r},$$

$$H_r = \frac{2\pi d^2 I_0}{cr}\left(\frac{1}{r^2}\cos\theta_0\cos\theta - \sin\theta_0\sin\theta\frac{\partial}{\partial r}\frac{\cos\psi}{r}\right),$$

$$H_\phi = -\frac{\pi d^2 I_0}{c}\sin\theta_0\frac{1}{r}\frac{\partial}{\partial r}r\frac{\partial}{\partial r}\frac{\sin\psi}{r},$$

$$H_\theta = -\frac{\pi d^2 I_0}{cr}\frac{\partial}{\partial r}\left(\frac{1}{r}\cos\theta_0\sin\theta + r\sin\theta_0\cos\theta\frac{\partial}{\partial r}\frac{\cos\psi}{r}\right). \tag{4.4}$$

To evaluate the radiation field, one should leave in (4.4) the terms which decrease not faster than $1/r$ for $r \to \infty$:

$$E_r = 0, \quad E_\theta = -H_\phi \approx \frac{\pi d^2 k_0^2 I_0}{cr}\sin\theta_0\sin\psi,$$

$$H_r \approx 0, \quad E_\phi = H_\theta \approx \frac{\pi d^2 k_0^2 I_0}{cr}\sin\theta_0\cos\theta\cos\psi. \tag{4.5}$$

The radial energy flux per unit time through the surface element $r^2 d\Omega$ is

$$S_r = \frac{d\mathcal{E}}{dt d\Omega} = \frac{cr^2}{4\pi}(E_\theta H_\phi - H_\theta E_\phi) = \frac{\pi}{4c}(d^2 k_0^2 I_0 \sin\theta_0)^2(\sin^2\psi + \cos^2\theta\cos^2\psi). \tag{4.6}$$

However, experimentalists usually measure not the time distribution of the energy flux flowing through the observation sphere, but the photons with definite frequency. For this, we evaluate the Fourier transforms of the field strengths

$$\vec{E}(\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\exp(i\omega t)\vec{E}(t)dt, \quad \vec{H}(\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\exp(i\omega t)\vec{H}(t)dt.$$

In the wave zone where $kr >> 1$ one gets

$$E_\theta(\omega) = H_\phi(\omega) = -\frac{i\pi k_0^2 I_0 d^2}{2cr}\sin\theta_0[\exp(-i\Phi_0)\delta(\omega+\omega_0) - \exp(i\Phi_0)\delta(\omega-\omega_0)],$$

$$E_\phi(\omega) = -H_\theta(\omega) = -\frac{\pi k_0^2 I_0 d^2}{2cr}\sin\theta_0\cos\theta[\exp(-i\Phi_0)\delta(\omega+\omega_0) + \exp(i\Phi_0)\delta(\omega-\omega_0)], \tag{4.7}$$

where $\Phi_0 = k_0 r + \phi$. The energy radiated into the unit solid angle, per unit frequency is

$$\frac{d^2\mathcal{E}}{d\omega d\Omega} = \frac{cr^2}{4\pi}(E_\theta H_\phi^* - H_\theta^* E_\phi + c.c.) = \frac{\pi k_0^4 I_0^2 d^4}{8c}\sin^2\theta_0(1+\cos^2\theta)[\delta(\omega-\omega_0)]^2). \tag{4.8}$$

This means that only the photons with the energy ω_0 should be observed.

A question arises, why we did not use the instantaneous Lorentz transformation when transforming charge-current densities from the dipole non-inertial RF to the inertial LF. The reason for this may be illustrated using the circular loop with the current density $j = \vec{j}_0\delta(\rho - a)\delta(z)/2\pi a$ as an example. Let this loop rotate with a constant angular velocity ω around its symmetry axis. Then, in the LF the charge density $\sigma = a\omega j\gamma/c^2$ and the charge

$$q = \int \sigma dV = a\omega j_0\gamma/c^2$$

arise. Here a is the loop radius, $\gamma = 1/\sqrt{1-\beta^2}$, $\beta = a\omega/c$. This absurd result is due to the fact that it is not always possible to apply the instantaneous Lorentz transformation for the transformation between the inertial and non-inertial reference frames. The correct approach is as follows. In the inertial reference frame (that is, in the laboratory one) there is only the static current density. In the non-inertial reference frame (attached to a rotating current loop), both charge and current densities differ from zero. There is no charge in this reference frame since a charge is no longer space integral over the charge density, but includes integration over other hypersurfaces [21].
The content of this section may be applied to the explanation of radiation observed from neutron stars (magnetars) with superstrong magnetic fields (see e.g., [22]).

5 Discussion and Conclusion

In previous sections we evaluated the electromagnetic fields of electric, magnetic and toroidal dipoles moving im medium. We use the following procedure. At first, in the rest dipole reference frame we consider finite charge-current densities which in the infinitesimal limit are reduced to electric, magnetic and toroidal dipoles. Then, we transform these finite charge-current densities to the laboratory frame using the Lorentz transformation, the same as in vacuum. Then, we tend the dimensions of these densities to zero, thus obtaining ones describing moving electric, magnetic and toroidal dipoles. With these densities, we solve Maxwell equations, find electromagnetic potentials, field strengths and the radiated energy flux. This procedure is straightforward, without any ambiguities. On the other hand, complications arise when one formulates the same problem in terms of electric and magnetic polarizations (see Introduction). The ambiguity is due to the transformation laws between electric and magnetic moments in two inertial reference frames. Since these two approaches should be equivalent, the question arises, whether the same ambiguity takes place for the charge-current densities. Or, more exactly: Is it true that charge-current densities in two inertial reference frames placed in medium are related via the vacuum Lorentz transformation ? It should be noted that a standard treatment of a moving bodies electrodynamics (see, e.g., [23-25]) definitely supports the same transformation law for the charge-current densities both in medium and vacuum.

Another ambiguity is that there is another formulation of relativistic spin theory. We mean the so-called Bargmann-Michel-Telegdi theory. In it, there are three spin components in the spin rest frame, four components in any other reference frame and there is no electric moment in this reference frame.

We briefly enumerate the main results obtained:

1. Representing electric, magnetic and toroidal dipoles as an infinitesimal limit of corresponding charge-current densities, we study how they radiate when moving uniformly in an unbounded medium. The frequency and velocity domains where radiation intensities are maximal are defined. The behaviour of radiation intensities near the Cherenkov threshold is investigated in some detail.
2. Radiation intensities are obtained for electric, magnetic and toroidal dipoles moving uniformly in a medium finite space interval (Tamm problem).
3. It is investigated how radiates the precessing magnetic dipole.

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Афанасьев Г. Н., Степановский Ю. П. E2-2002-142
Об излучении электрических, магнитных и тороидальных диполей

Рассматривается излучение электрических, магнитных и тороидальных диполей, равномерно движущихся в неограниченной среде (это соответствует задаче Тамма–Франка). Плотности этих диполей получаются из соответствующих плотностей заряда и тока переходом к точечному пределу. Изучается поведение интенсивности излучения вблизи черенковского порога $\beta = 1/n$. Определены интервалы скоростей и частот, в которых интенсивность излучения максимальна. Дано сравнение с предыдущими работами. Рассматривается также излучение электрических, магнитных и тороидальных диполей, равномерно движущихся в среде на конечном интервале (это соответствует задаче Тамма). Изучаются свойства излучения, возникающего при прецессии магнитного диполя.

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Afanasiev G. N., Stepanovsky Yu. P. E2-2002-142
On the Radiation of Electric, Magnetic and Toroidal Dipoles

We consider the radiation of electric, magnetic and toroidal dipoles uniformly moving in unbounded medium (this corresponds to the Tamm–Frank problem). The densities of these dipoles are obtained from the corresponding charge-current densities in an infinitesimal limit. The behaviour of radiation intensities in the neighbourhood of the Cherenkov threshold $\beta = 1/n$ is investigated. The frequency and velocity regions are defined where radiation intensities are maximal. The comparison with previous attempts is given. We consider also the radiation of electric, magnetic and toroidal dipoles uniformly moving in medium, in a finite space interval (this corresponds to the Tamm problem). The properties of radiation arising from the precession of a magnetic dipole are studied.

The investigation has been performed at the Bogoliubov Laboratory of Theoretical Physics, JINR.

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