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ON GLOBAL $L_1 \cap L_\infty$ SOLUTIONS OF THE VLASOV EQUATION WITH THE POTENTIAL r^{-2}

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1 Introduction. Notation. Results

Vlasov equations appear in the mean-field approximations of a dynamics of a large number of interacting classical particles (molecules). Currently, there is a numerous literature devoted to its mathematical treatments. In particular, in [1-4] a well-posedness for this equation supplied with initial data and its derivation from a molecular dynamics is considered in the case when the potential of interaction between particles is smooth and bounded. In [5-10], this equation is studied for the singular Coulomb potential $U(r) = r^{-1}$ (in [8], the Vlasov-Maxwell system is considered). In [11], an extension of these results for the cases of higher singularities is presented. We also mention paper [12] where a well-posedness of this equation supplied with a joint distribution of particles at two moments of time is proved.

In the present article, we consider the problem

$$\frac{\partial}{\partial t}f+v\cdot\nabla_x f+\nabla_v f\cdot w(x,t)=0, \quad f=f(t,x,v),\ t\in\mathbb{R},\ (x,v)\in\mathbb{R}^3\times\mathbb{R}^3, \tag{1}$$

$$w(x,t)=-\int_{\mathbb{R}^3\times\mathbb{R}^3}\nabla_x U(x-y)f(t,y,v)dydv, \tag{2}$$

$$f(0,x,v)=f_0(x,v), \tag{3}$$

where all quantities are real, $x, v \in \mathbb{R}^3$, $U(r) = r^{-2}$, ∇_x and ∇_v are the gradients in x and v, respectively, $v \cdot \nabla_x f$ and $\nabla_v f \cdot w(x,t)$ are the scalar products in $\mathbb{R}^3$, and f is an unknown function. For any fixed t, $f(t,x,v)$ regarded as a function of x and v has the sense of the distribution function of particles in $(x,v) \in \mathbb{R}^3 \times \mathbb{R}^3$. Therefore, the following requirements are natural:

$$f(t,\cdot,\cdot)\geq 0, \quad \int_{\mathbb{R}^3\times\mathbb{R}^3} f(t,x,v)dxdv=1, \quad \forall t\in\mathbb{R}. \tag{4}$$

Generally speaking, it is known that proving the existence of a solution for problem (1)-(4) is more difficult for the singular potential U than for a more regular one. Also, though the Vlasov equation appeared for the first time with the Coulomb potential $U(r) = r^{-1}$ for a description of plasma, it is well known that in statistical physics potentials with higher singularities occur: for example, the following one, the so-called Lennard-Jones potential, is known: $U(r) = Ar^{-12} - Br^{-6}$. So, the author of the present article believes that considerations of Vlasov equations for potentials with singularities of the degree higher than r^{-1} make a sense. Here we consider the case $U(r) = r^{-2}$ proving the existence of a weak solution of the problem (1)-(4). This case is critical in a sense because for singularities of the kind r^{-2-a} with $a > 0$ it is not clear so far how to determine the expression in the right-hand side of (2). So, a treatment of the problem in the latter case is still left open. As for the case $U(r) = r^{-a}$ with $a \in (1,2)$, here the problem becomes simpler, and we do not study this case.

Now, we introduce some notation and definitions. Let d be a positive integer. By M^+ we denote the space of nonnegative Borel measures in $\mathbb{R}^d$ satisfying $\mu(\mathbb{R}^d) = 1$. This space M^+ is equipped with the topology of the weak convergence of measures: a sequence $\{\mu_n\}_{n=1,2,3,\dots} \subset M^+$ is called weakly converging to a $\mu \in M^+$ if for any continuous bounded real-valued function $\varphi : \mathbb{R}^d \to \mathbb{R}$ one has

$$\lim_{n\to\infty} \int\limits_{\mathbb{R}^d} \varphi(z) d\mu_n(z) = \int\limits_{\mathbb{R}^d} \varphi(z) d\mu(z).$$

Let $C_b^1(\mathbb{R}^d)$ be the space of all real-valued continuously differentiable functions in $\mathbb{R}^d$ each of which is bounded with its gradient in $\mathbb{R}^d$; we set

$$||\varphi||_{C_b^1(\mathbb{R}^d)} = \sup_{z\in\mathbb{R}^d} \{|\varphi(z)| + |\nabla\varphi(z)|\}.$$

By $C_c^1(\mathbb{R}^d)$ we denote the subspace of $C_b^1(\mathbb{R}^d)$ consisting of finite functions. Also, for a $B \subset \mathbb{R}^d$ and a real-valued function φ in $\mathbb{R}^d$ denote

$$||\varphi||_{Lip} = \sup_{z\in B} |\varphi(z)| + \sup_{\substack{z_1,z_2\in B\\ z_1\neq z_2}} \frac{|\varphi(z_1) - \varphi(z_2)|}{|z_1 - z_2|}$$

and by $L(B)$ the space of all functions $\varphi : B \to \mathbb{R}$ satisfying $||\varphi||_{Lip} < \infty$. Now, it is well known (see, for example, [3] and also, Theorem 2.3 below) that the topological space M^+ is metrizable by the distance

$$\rho(\mu_1,\mu_2) = \sup_{\substack{\varphi\in L(\mathbb{R}^d)\\ ||\varphi||_{Lip}\leq 1}} \left| \int\limits_{\mathbb{R}^d} \varphi(z)(d\mu_1(z) - d\mu_2(z)) \right|$$

(one can easily verify that ρ satisfies all the axioms of a distance in a metric space), so that a sequence $\{\mu_n\} \subset M^+$ converges weakly to a $\mu \in M^+$ if and only if $\rho(\mu_n, \mu) \to 0$ as $n \to \infty$. In fact, (M^+, ρ) is a complete metric space.

Now, let M be an arbitrary compact metric space with a distance $d(\cdot, \cdot)$ and $I \subset \mathbb{R}$ be an interval. By $C(I; M)$ we denote the metric space consisting of all continuous bounded functions from I into M; the distance ν in $C(I; M)$ is defined by $\nu(g(\cdot); h(\cdot)) = \sup_{t \in I} d(g(t); h(t))$. The space $(C(I; M); \nu)$ is also compact.

For a set $\Omega \subset \mathbb{R}^d$ measurable in the Lebesgue sense by $L_p(\Omega)$ $(1 \leq p < \infty)$ and $L_\infty(\Omega)$ we denote the standard Lebesgue spaces with the norms

$$||g||_{L_p(\Omega)} = \left(\int_\Omega dz |g(z)|^p \right)^{1/p} \quad \text{and} \quad ||g||_{L_\infty(\Omega)} = \operatorname{vrai} \sup_{z \in \Omega} |g(z)|.$$

Finally for a Banach space B and an interval $I \subset \mathbb{R}$ by $C_w(I; B)$ we denote the space of bounded functions from I into B continuous in the weak topology of B.

Now, we define $L_1 \cap L_\infty$-(weak) solutions of (1)-(4).

Definition 1.1 *Let $T > 0$ be arbitrary and $f(\cdot) \in C_w([-T, T]; L_p(\mathbb{R}^3 \times \mathbb{R}^3))$ for all $p \in [1, \infty)$. Then, we call this function f a weak solution of problem (1)-(4) if f satisfies (2)-(4) and if for any $\varphi \in C_c^1(\mathbb{R}^3 \times \mathbb{R}^3)$ and $t \in [-T, T]$ one has*

$$\int_{\mathbb{R}^3 \times \mathbb{R}^3} (f(t, x, v)\varphi(x, v) - f_0(x, v)\varphi(x, v))dxdv -$$

$$- \int_0^t ds \int_{\mathbb{R}^3 \times \mathbb{R}^3} dxdv f(s, x, v) \{v \cdot \varphi_x(x, v) + \varphi_v(x, v) \cdot w(x, s)\} = 0. \quad (5)$$

Remark 1.2 As it is shown below (see Lemma 2.1), the operator in the right-hand side of (2) is continuous from $L_p(\mathbb{R}^3 \times \mathbb{R}^3)$ into $L_p(\mathbb{R}^3)$, where $1 < p < \infty$ is arbitrary, so that the expression in the left-hand side of (5) is well-defined.

Remark 1.3 Formally, (5) occurs from (1)-(3) by the multiplication of (1) by φ with further integration over (x, v) and t with an application of the integration by parts.

Our main result is the following.

Theorem 1.4 *Let $U(r) = r^{-2}$ and $f_0 \in L_1(\mathbb{R}^3 \times \mathbb{R}^3) \cap L_\infty(\mathbb{R}^3 \times \mathbb{R}^3)$ and satisfy (4). Then, there exists a function $f(\cdot)$ that belongs to $C_w(\mathbb{R}; L_p(\mathbb{R}^3 \times \mathbb{R}^3))$ for all $p \in [1, \infty)$ and is such that for any $T > 0$ it is a weak solution of (1)-(4) in the interval of time $[-T, T]$. In addition,*

$$\int\limits_{\mathbb{R}^3 \times \mathbb{R}^3} f(t, x, v) dx dv = 1 \text{ and } ||f(t, \cdot, \cdot)||_{L_p(\mathbb{R}^3 \times \mathbb{R}^3)} \leq ||f_0||_{L_p(\mathbb{R}^3 \times \mathbb{R}^3)}$$

$$\forall\, t \in \mathbb{R} \quad \forall\, 1 < p < \infty.$$

Remark 1.5 As the reader sees, the important question about the uniqueness of the weak solution given by Theorem 1.4 above is left open in the article as in [5] and [7] where this question for such solutions is left open, too, for the Caulomb potential.

We prove Theorem 1.4 in the next section. In addition, in this section we construct invariant measures for problem (1)-(4) in the case when $U(\cdot)$ is a smooth bounded function; we believe that this result may be of a separate interest.

2 Proof of Theorem 1.4

We begin with studying the properties of the integral operator T in the right-hand side of (2). It is well known that the integral operator

$$(Pg)(x) = P.V. \int\limits_{\mathbb{R}^3} \nabla_x \frac{1}{|x-y|^2} g(y) dy$$

maps any $g \in C_c^1(\mathbb{R}^3)$ into $Pg \in L_p(\mathbb{R}^3)$ with arbitrary $1 < p < \infty$ and $||Pg||_{L_p(\mathbb{R}^3)} \leq c_p ||g||_{L_p(\mathbb{R}^3)}$ with a constant $c_p > 0$ independent of g.

Therefore, for any $1 < p < \infty$ the operator P can be uniquely extended onto the whole space $g \in L_p(\mathbb{R}^3)$ by continuity.

Lemma 2.1 *The operator T can be uniquely extended from the space of all functions $g \in C_c^1(\mathbb{R}^3 \times \mathbb{R}^3)$ onto $L_p(\mathbb{R}^3 \times \mathbb{R}^3)$ becoming a bounded linear operator acting from $L_p(\mathbb{R}^3 \times \mathbb{R}^3)$ into $L_p(\mathbb{R}^3)$.*

Remark 2.2 The best constants c_p in the inequalities $||Tg||_{L_p(\mathbb{R}^3)} \leq c_p||g||_{L_p(\mathbb{R}^3\times\mathbb{R}^3)}$ become unbounded as $p \to 1+0$ and as $p \to +\infty$.

Proof of Lemma 2.1 is very simple. Indeed, in view of the above facts, we have for any $g \in C_c^1(\mathbb{R}^3 \times \mathbb{R}^3)$ and for $1 < p < \infty$:

$$||Tg||^p_{L_p(\mathbb{R}^3)} \leq c_p \left|\left| \int_{\mathbb{R}^3} |g(x,v)|^p dv \right|\right|_{L_p(\mathbb{R}^3)} = c_p||g||^p_{L_p(\mathbb{R}^3\times\mathbb{R}^3)}.$$

Thus, T can be uniquely extended onto the whole space $L_p(\mathbb{R}^3 \times \mathbb{R}^3)$ by continuity.□

Theorem 2.3 *Let d be a positive integer, $B \in \mathbb{R}^d$ be a bounded set measurable in the Lebesgue sense, $B_1 = \{z \in \mathbb{R}^d : \ \mathrm{dist}(z, B) \leq 1\}$, $1 \leq p < \infty$ and, for $g, h \in L_p(\mathbb{R}^d)$, denote*

$$\rho_B(g,h) = \sup_{\varphi\in L(B),\ ||\varphi||_{Lip}\leq 1} \left| \int_B \varphi(z)(g(z)-h(z))dz \right|$$

and

$$\rho_{1,B}(g,h) = \sup_{\substack{\varphi\in C_b^1(\mathbb{R}^d),\ ||\varphi||_{C_b^1(\mathbb{R}^d)}\leq 1 \\ \varphi\equiv 0 \text{ in } \mathbb{R}^d\setminus B_1}} \left| \int_B \varphi(z)(g(z)-h(z))dz \right|.$$

Let $\{f_n\}_{n=1,2,3,\ldots} \subset L_p(\mathbb{R}^d)$ be bounded and $f \in L_p(\mathbb{R}^d)$. Then, for $f_n \to f$ weakly in $L_p(B)$ each of the following two conditions is sufficient and necessary: 1). $\rho_B(f_n, f) \to 0$ as $n \to \infty$ and 2). $\rho_{1,B}(f_n, f) \to 0$ as $n \to \infty$. In addition, for any bounded set $A \subset L_p(\mathbb{R}^d)$ closed in the weak topology (A, ρ_B) and $(A, \rho_{1,B})$ are compact metric spaces.

Proof. Let us prove the first claim. We deal only with the function ρ_B because for $\rho_{1,B}$ the proof can be made by the complete analogy. First, let $f_n \to f$ weakly in $L_p(\mathbb{R}^d)$ and suppose $\rho_B(f_n, f) \not\to 0$ as $n \to \infty$. Then, there exist a subsequence of $\{f_n\}$, still denoted $\{f_n\}$, and a sequence $\{\varphi_n\} \subset L(B)$ satisfying $||\varphi_n||_{Lip} \leq 1$ such that

$$\int_B \varphi_n(z)(f_n(z) - f(z))dz \geq c > 0.$$

Then, the sequence $\{\varphi_n\}$ contains a subsequence, still denoted $\{\varphi_n\}$, that converges uniformly in $z \in B$ to some φ. But then

$$\int_B \varphi_n(z)(f_n(z) - f(z))dz =$$

$$= \int_B (\varphi_n(z) - \varphi(z))(f_n(z) - f(z))dz + \int_B \varphi(z)(f_n(z) - f(z))dz \to 0,$$

and we get a contradiction.

Now, let $\rho_B(f_n, f) \to 0$ and let us prove that $f_n \to f$ weakly in $L_p(\mathbb{R}^d)$. Suppose this is not so and there exist two subsequences $\{f_n'\}$ and $\{f_n''\}$ of $\{f_n\}$ converging weakly respectively to f' and f'' where $f' \neq f''$. Observe that the function ρ_B satisfies all the axioms of a distance in a metric space. Hence, we have

$$\rho_B(f_n', f') + \rho(f_n'', f'') \to 0 \quad \text{and} \quad \rho(f_n, f) \to 0$$

i. e. we get a contradiction. Thus, the first claim of Theorem 2.3 is proved. The second one is also obvious.□

Assume now the following.

(U) *Let $U(\cdot)$ be a twice continuously differentiable function in $\mathbb{R}^3$ all whose first and second partial derivatives are bounded in $\mathbb{R}^3$.*

Consider problem (1)-(4) with a function $U(\cdot)$ satisfying this assumption (U). In accordance with [4], we accept the following definition.

Definition 2.4 *Let $T > 0$ be arbitrary and U satisfy (U). We call a function $\mu(t) \in C([-T,T]; M^+)$ a generalized solution of problem (1)-(2) with initial data $\mu(0) = \mu_0 \in M^+$, if for any real-valued $\varphi \in C_b(\mathbb{R}^3 \times \mathbb{R}^3)$ and $t \in [-T,T]$ one has*

$$\int\limits_{\mathbb{R}^3\times\mathbb{R}^3} \varphi(x,v)(\mu(t)(dx\ dv) - \mu_0(dx\ dv)) -$$

$$-\int\limits_0^t ds \int\limits_{\mathbb{R}^3\times\mathbb{R}^3} \mu(s)(dx\ dv)\{v \cdot \varphi_x(x,v) + \varphi_v(x,v)w(x,s)\} = 0$$

where

$$w(x,t) = -\int\limits_{\mathbb{R}^3\times\mathbb{R}^3} \nabla_x U(x-y)d\mu(t)(dy\ dv). \tag{6}$$

The main well-known result under hypothesis (U) is the following.

Theorem 2.5 *Under assumption (U) for any $\mu_0 \in M^+$ and $T > 0$ system (1),(6) has a unique solution $\mu(\cdot) \in C([-T,T]; M^+)$ that satisfies $\mu(0) = \mu_0$ and the map $\mu(\cdot): M^+ \to C([-T,T]; M^+)$ is continuous.*

For the Proof, see [1-4].□

Proposition 2.6 *Under assumption (U) for any $\mu_0(dx\ dv) = f_0(x,v)dx\ dv \in M^+$ (so that $f_0(x,v) \geq 0$ and $\int\limits_{\mathbb{R}^3\times\mathbb{R}^3} f_0(x,v)dx\ dv = 1$) satisfying $f_0 \in C_b^1(\mathbb{R}^3 \times \mathbb{R}^3)$ and $\lim\limits_{(x,v)\to\infty} (1+|v|)f_0(x,v) = 0$ the solution of problem (1),(6) with the initial data $\mu(0) = \mu_0$ given by Theorem 2.5 for any t has the form $\mu(t)(dx\ dv) = f(t,x,v)dx\ dv$ where for any fixed t $f(t,\cdot,\cdot) \in C_b^1(\mathbb{R}^3 \times \mathbb{R}^3)$ (this means that for any Borel set $B \in \mathbb{R}^3 \times \mathbb{R}^3$ one has $\mu(t)(B) = \int\limits_B f(t,x,v)dx\ dv$) and $\lim\limits_{(x,v)\to\infty} (1+|v|)f(t,x,v) = 0$ uniformly in $t \in [-T,T]$.*

Proof. Let $\mu(t)$ be the measure-valued solution of (1),(6) given by Theorem 2.5 and $\mu(0)(dx\ dv) = f_0(x,v)dx\ dv$ where f_0 satisfies the conditions of Proposition. Consider the following linear problem:

$$\frac{\partial}{\partial t}g + v \cdot g_x + g_v \cdot E(x,t) = 0, \ \ g = g(t,x,v),\ t \in \mathbb{R},\ (x,v) \in \mathbb{R}^3 \times \mathbb{R}^3, \tag{7}$$

$$E(x,t) = -\int\limits_{\mathbb{R}^3\times\mathbb{R}^3} \nabla_x U(x-y)\mu(t)(dy\ dv), \tag{8}$$

$$g|_{t=0} = f_0. \tag{9}$$

Clearly, in view of our assumptions the function $E(x,t)$ is continuous and Lipschitz continuous in x. Consider also the following system:

$$\dot{x}(t,x_0,v_0) = v(t,x_0,v_0), \quad t \in \mathbb{R},\ (x_0,v_0) \in \mathbb{R}^3 \times \mathbb{R}^3, \tag{10}$$

$$\dot{v}(t,x_0,v_0) = E(x(t,x_0,v_0),t), \tag{11}$$

$$x(0,x_0,v_0) = x_0,\ v(0,x_0,v_0) = v_0. \tag{12}$$

For any (x_0,v_0) this system obviously has a unique solution $(x(t,x_0,v_0),v(t,x_0,v_0))$ defined for all $t \in \mathbb{R}$, and clearly, by the uniqueness theorem the trajectories do not intersect. Also, by the standard result, its solution $(x(t),v(t)) = (x(t,x_0,v_0),v(t,x_0,v_0))$ is continuously differentiable in (x_0,v_0). Hence, denoting by S_t the operator in $\mathbb{R}^3 \times \mathbb{R}^3$ that maps any (x_0,v_0) into $(x(t,x_0,v_0),v(t,x_0,v_0))$, we obtain that for any fixed t S_t is a diffeomorphism (i. e. it is a one-to-one operator continuously differentiable with the inverse).

The direct verification shows that the function defined by the rule: $g(t,x(t,x_0,v_0),v(t,x_0,v_0)) = g_0(x_0,v_0)$, where the point (x_0,v_0) runs over the whole space $\mathbb{R}^3 \times \mathbb{R}^3$, is a solution of the linear problem (10)-(12). Indeed,

$$0 \equiv \frac{d}{dt} g(t,x(t),v(t)) = \frac{\partial g}{\partial t} + v \cdot \nabla_x g + \nabla_v g \cdot E(x(t),t) \tag{13}$$

and this identity holds for all $(x,v) = (x(t),v(t)) \in \mathbb{R}^3 \times \mathbb{R}^3$ and $t \in \mathbb{R}$ because $(x(t),v(t))$ runs over the whole $\mathbb{R}^3 \times \mathbb{R}^3$ when (x_0,v_0) runs over the whole $\mathbb{R}^3 \times \mathbb{R}^3$.

Let us show that

$$||g(t,\cdot,\cdot)||_{L_1(\mathbb{R}^3\times\mathbb{R}^3)} = 1\ \ \forall\ t. \tag{14}$$

It is easy to see that in (8) the function $E(x,t)$ is bounded uniformly in x and t. Therefore, given $T > 0$, there exists a $D > 0$ such that

$$|x(t,x_0,v_0) - v_0 t| \le D \ \text{ and } \ |v(t,x_0,v_0) - v_0| \le D \ \ \forall\ t \in [-T,T].$$

This easily implies that $(1+|v|)g(t,x,v) \to 0$ as $(x,v) \to \infty$ uniformly in t from any finite interval. Now, integrating (13) over (x,v), we obtain relation (14).

The definition of M^+-solutions of linear problem (7)-(9) can be given completely as the nonlinear case, and the proof of its uniqueness repeats the proof in the nonlinear case presented in [1-4]. So, $g(t,x,v)dx\ dv$ is a M^+-solution of linear problem (7)-(9) and by the uniqueness of this solution and since $\mu(t)$ is obviously a M^+-solution of (7)-(9), we have $\mu(t) = g(t,x,v)dx\ dv$.□

Proposition 2.7 (conservation laws) *Under assumption (U) for any C_b^1-solution $f(t,x,v)$ of (1)-(2) such that $(1+|v|)f_0(x,v) \to 0$ as $(x,v) \to \infty$ given by Proposition 2.6 one has*

$$\frac{d}{dt}||f(t,\cdot,\cdot)||_{L_p(\mathbb{R}^3\times\mathbb{R}^3)} = 0 \quad \forall\, t\ \forall\, 1 \le p \le \infty.$$

Proof. For $1 \le p < \infty$, the proof repeats those presented when proving Proposition 2.6. To do this for $p = \infty$, it suffices to observe that for $h \in \bigcap\limits_{p\ge 1} L_p(\mathbb{R}^3 \times \mathbb{R}^3)$ one has

$$||h||_{L_\infty(\mathbb{R}^3\times\mathbb{R}^3)} = \lim_{p\to\infty} ||h||_{L_p(\mathbb{R}^3\times\mathbb{R}^3)}.\square$$

Theorem 2.8 (invariant measures) *Under assumption (U) for an arbitrary C_b^1-solution $f(t,x,v)$ of (1)-(2) satisfying $(1+|v|)f(0,x,v) \to 0$ as $(x,v) \to \infty$ and for any Borel set $\Omega_0 \subset \mathbb{R}^3 \times \mathbb{R}^3$ and $t \in \mathbb{R}$ one has:*

$$\int\limits_{\Omega_0} f(0,x,v)dx\ dv = \int\limits_{\Omega_t} f(t,x,v)dx\ dv$$

where $\Omega_t = \{(x(t,x_0,v_0), v(t,x_0,v_0)) :\ (x_0,v_0) \in \Omega_0\}$.

Proof. Approximate the measure $\mu_0 = f(0,x,v)dx\ dv$ by expressions $\mu_0^N = N^{-1}\sum\limits_{n=1}^{N} \delta(x - x_{0,n}^N) \times \delta(v - v_{0,n}^N)$, where $x_{0,n}^N$ and $v_{0,n}^N$ are constants and N runs over positive integers. It is known that there exist $x_{0,n}^N$ and $v_{0,n}^N$ such that $\mu_0^N \to \mu_0$ in M^+. Set

$$w_N(x,t) = -\int\limits_{\mathbb{R}^3\times\mathbb{R}^3} \nabla_x U(x-y)\mu_N(t)(dy\ dv)$$

where $\mu_N(t)$ are the corresponding M^+-solutions of (1)-(2) with the initial conditions $\mu_N(0) = \mu_0^N$. It is known (see [1-4] and also [12]) that $\mu_N(t) = = N^{-1} \sum_{n=1}^{N} \delta(x - x_n^N(t)) \times \delta(v - v_n^N(t))$ where the functions $x_n^N(t)$ and $v_n^N(t)$ obey the following system of equations:

$$\dot{x}_n^N(t) = v_n^N(t), \tag{15}$$

$$\dot{v}_n^N(t) = w_N(x^N(t), t), \quad n = 1, 2, ..., N \tag{16}$$

$$x_n^N(0) = x_{0,n}^N, \quad v_n^N(0) = v_{0,n}^N$$

(we denote $x^N = (x_1^N, ..., x_N^N)$ and $v^N = (v_1^N, ..., v_N^N))$. This is a system of ODEs and it is easy to see that, under condition (U), the right-hand sides in (15) and (16) are continuous in x_n^N, v_n^N and t and are Lipschitz continuous in (x^N, v^N). So, this system has a unique solution and hence, it is $(x^N(t), v^N(t))$.

Let us show that $w_N(x,t) \to w(x,t)$ uniformly with respect to (x,t) from an arbitrary compact set $B \times I \subset \mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{R}$. Take a $T > 0$ and set $\mathrm{osc}_D U = \sup_{x \in D} U(x) - \inf_{x \in D} U(x)$ and take an arbitrary $\epsilon > 0$. Let $B_R = \{(x,v) \in \mathbb{R}^3 \times \mathbb{R}^3 : \ |(x,v)| \leq R\}$. As it is proved in the proof of Proposition 2.6, there exists $R > 0$ such that

$$\int_{(\mathbb{R}^3 \times \mathbb{R}^3) \setminus B_{R-1}} f(t,x,v) dx\ dv < \epsilon/8 \quad \forall\, t \in [-T, T].$$

Take a Lipschitz continuous function $0 \leq \varphi(x,v) \leq 1$, equal to 1 in $(\mathbb{R}^3 \times \mathbb{R}^3) \setminus B_R$ and to 0 in B_{R-1}. Then, we have

$$\left| \int_{\mathbb{R}^3 \times \mathbb{R}^3} \varphi(x,v) \mu_N(t)(dx\ dv) \right| < \epsilon/8 \quad \forall\, t \in [-T, T]$$

for all sufficiently large N. Hence

$$\int_{(\mathbb{R}^3 \times \mathbb{R}^3) \setminus B_R} f(t,x,v) dx\ dv + \mu_N(t)((\mathbb{R}^3 \times \mathbb{R}^3) \setminus B_R) < \epsilon/3 \quad \forall t \in [-T, T]$$

for all sufficiently large N.

Take now a partition of $\mathbb{R}^3 \times \mathbb{R}^3$ into cubes $a_i < x_i \le b_i$, $c_j < v_j \le d_j$, $i, j = 1, 2, 3$, such that $\sup_{x \in \mathbb{R}^3} \operatorname{osc}_y \nabla_x U(x-y) < \epsilon/3$ in each cube and let $K_1, ..., K_P$ be a reindexing of all cubes the intersection of each of which with B_R is nonempty, and $K = \bigcup_{i=1}^{P} K_i$. Obviously, $\mu(t)(\partial K_i) = = 0$, $i = 1, ..., P$, and hence by the well-known property of weakly converging sequences of measures for any t $\mu_N(t)(K_i) \to \mu(t)(K_i)$ as $N \to \infty$, $i = 1, 2, ..., P$. Also, as it can be easily verified, this convergence is uniform in $t \in [-T, T]$ because of the uniform convergence of $\mu_N(t)$ to $\mu(t)$ in $t \in [-T, T]$. Thus, we have for any $x \in \mathbb{R}^3$:

$$|w_N(x,t) - w(x,t)| < C\epsilon/3 + \left| \int_K \nabla_x U(x-y)(\mu_N(t) - \mu(t))(dy\ dv) \right| \le$$

$$\le C\epsilon/3 + 2\sum_{i=1}^{P} [\sup_{x \in \mathbb{R}^3} \operatorname{osc}_{y \in K_i} \nabla_x U(x-y) + \epsilon] \times |\mu_N(t)(K_i) - \mu(t)(K_i)| < C_1 \epsilon$$

for all $t \in [-T, T]$ and all sufficiently large N.

Now, take an arbitrary t, a bounded set $\Omega_0 \subset \mathbb{R}^3 \times \mathbb{R}^3$ with a smooth, of the class C^1 boundary, and the corresponding set Ω_t. Since as noted earlier, S_t is a diffeomorphism, the boundary $\partial\Omega_t$ is also compact and of the class C^1. In particular, $\mu_0(\partial\Omega_0) = \mu(t)(\partial\Omega_t) = 0$. Take an arbitrary $\alpha > 0$ and set

$$\Omega_{t,\alpha} = \{(x, v) \in \mathbb{R}^3 \times \mathbb{R}^3 : \ \operatorname{dist}((x, v); \Omega_t) < \alpha\}$$

and

$$\Omega'_{t,\alpha} = \{(x, v) \in \mathbb{R}^3 \times \mathbb{R}^3 : \ \operatorname{dist}((x, v); \partial\Omega_t) < \alpha\}.$$

Then, $\mu(t)(\Omega_{t,\alpha} \setminus \Omega_t) \to 0$ as $\alpha \to +0$. Denote by S_t^N the operator defining as S_t and corresponding to the solution $\mu_N(t)$ and let $\Omega_t^N = S_t^N \Omega_0$. Then, by the arguments above,

$$\Omega_t \setminus \Omega'_{t,\alpha} \subset \Omega_t^N \subset \Omega_{t,\alpha}$$

for all sufficiently large N and clearly, by construction,

$$\mu_N(t)(\Omega_t^N) = \mu_N(0)(\Omega_0).$$

Hence,

$$\mu_N(t)(\Omega_t^N) - \gamma(\alpha) - \delta_N^1 \le \mu(0)(\Omega_0) \le \mu_N(t)(\Omega_t^N) + \gamma(\alpha) + \delta_N^2$$

where $\gamma(\alpha) \to +0$ as $\alpha \to +0$ and $\delta_N^i \to 0$ as $N \to \infty$. Thus, taking the limit $N \to \infty$, we get $\mu(t)(\Omega_t) - \gamma_1(\alpha) \le \mu(0)(\Omega_0) \le \mu(T)(\Omega_t) + \gamma_1(\alpha)$, where $\gamma_1(\alpha) \to +0$ as $\alpha \to +0$, and due to the arbitrariness of α, we obtain $\mu(t)(\Omega_t) = \mu(0)(\Omega_0)$.

If Ω_0 is an unbounded open set with a boundary of the class C^1, then for each $K = 1, 2, 3, \ldots$ we set $\Omega_{0,K} = \Omega \cap B_K$ and $\Omega_{t,K} = S_t(\Omega_{0,K})$. We have $\mu(0)(\Omega_{0,K}) = \mu(t)(\Omega_{t,K})$ for each K and therefore, since $\Omega_0 = \bigcup_{K=1}^{\infty} \Omega_{0,K}$ and $\Omega_t = \bigcup_{K=1}^{\infty} \Omega_{t,K}$, we deduce that $\mu(0)(\Omega_0) = \mu(t)(\Omega_t)$. The latter equality yields that $\mu(0)(\Omega_0) = \mu(t)(\Omega_t)$ for any closed set Ω_0 with a piecewise smooth boundary. Now, for an arbitrary set Ω_0 this equality can be obtained by approximations of Ω_0 by closed sets with piecewise smooth boundaries from inside.□

Corollary 2.9 *Let assumption (U) be valid and let $f(t,x,v)$ be a C_b^1-solution of (1)-(2) satisfying $(1+|v|)f(0,x,v) \to 0$ as $|(x,v)| \to \infty$. Then, for solutions $(x(t,x_0,v_0), v(t,x_0,v_0))$ of system (8),(10)-(12) the determinant of the Jacobi matrix*

$$J(t,x_0,v_0) = \frac{\partial(x(t,x_0,v_0), v(t,x_0,v_0))}{\partial(x_0,v_0)}$$

is identically equal to 1 for all (x_0,v_0) and t.

Proof. First, we consider the case $f_0(x,v) > 0$ for all (x,v). Then, $f(t,x',v') = f_0(x(-t,x',v'), v(-t,x',v')) > 0$ for all (t,x',v'). Take an arbitrary point (x_0,v_0), a $t \in \mathbb{R}$ and a ball $B_r(x_0,v_0) = \{(x,v) \in \mathbb{R}^3 \times \mathbb{R}^3 : |(x,v) - (x_0,v_0)| < r\}$. Then, we have by Theorem 2.8:

$$\int_{B_r(x_0,v_0)} f_0(x_0,v_0) dx_0 dv_0 = \int_{S_t(B_r(x_0,v_0))} f(t,x,v) dx\, dv =$$

$$= \int_{B_r(x_0,v_0)} f(x_0,v_0)\, |\det\, J(t,x_0,v_0)|\, dx_0\, dv_0.$$

Letting here $r \to +0$, we deduce: $|\det J(t, x_0, v_0)| \equiv 1$. Hence, by continuity and since $\det J(0, x_0, v_0) \equiv 1$, we obtain $\det J(t, x_0, v_0) \equiv 1$ for all (x_0, v_0) and t.

Now, consider the general case when $f_0(x_0, v_0)$ may be equal to 0 at some (x_0, v_0). Consider an arbitrary sequence $\{f_0^n(x_0, v_0)\}_{n=1,2,3,\dots} \subset C_b^1(\mathbb{R}^3 \times \mathbb{R}^3)$ converging to $f_0(x_0, v_0)$ in $C_b^1(\mathbb{R}^3 \times \mathbb{R}^3)$, satisfying $(1 + |v|)f_0^n(x, v) \to 0$ as $(x_0, v_0) \to \infty$ uniformly in n and such that the corresponding measures μ_0^n converge to $\mu_0 = f_0(x_0, v_0)dx_0\, dv_0$ in M^+. Let $\mu^n(t) = f_n(t, x, v)dx\, dv$ be the corresponding sequence of solutions of (1)-(2). Then,as when proving Theorem 2.8, it is easy to show that the corresponding sequence $w_n(x, t) = -\int\limits_{\mathbb{R}^3 \times \mathbb{R}^3} \nabla_x U(x - y) f_n(t, y, v) dy\, dv$ converges to $w(x, t)$ as $n \to \infty$ uniformly with respect to (x, t) from any compact set. By analogy, for $i = 1, 2, 3$

$$\int\limits_{\mathbb{R}^3 \times \mathbb{R}^3} \frac{\partial}{\partial x_i} \nabla_x U(x-y) f_n(t, y, v) dy\, dv \to \int\limits_{\mathbb{R}^3 \times \mathbb{R}^3} \frac{\partial}{\partial x_i} \nabla_x U(x-y) f(t, y, v) dy\, dv$$

uniformly with respect to (x, t) from any compact set. Hence, by the standard theorem on the differentiability of solutions of systems of ODEs over a parameter, we have $1 \equiv \det J_n(t, x_0, v_0) \to \det J(t, x_0, v_0)$.□

Now, we return to our original problem (1)-(4) with the potential $U(r) = r^{-2}$ and consider its approximations. Let ω be an infinitely differentiable nonnegative even function in $\mathbb{R}^3$ with a compact support satisfying

$$\int\limits_{\mathbb{R}^3} \omega(x) dx = 1.$$

We set $U_n(\cdot) = U(\cdot) * n^3 \omega(n \cdot)$, where the star means the convolution, and consider the following sequence of problems:

$$\frac{\partial f_n}{\partial t} + v \cdot \nabla_x f_n + \nabla_v f_n \cdot w_n, \quad f_n = f_n(t, x, v), \tag{17}$$

$$w_n(x, t) = -\int\limits_{\mathbb{R}^3 \times \mathbb{R}^3} \nabla_x U_n(x - y) f_n(t, y, v) dy\, dv, \tag{18}$$

$$f_n(0, x, v) = f_0^n(x, v), \tag{19}$$

where f_0^n is a sequence of C_b^1-functions for an arbitrary $1 \le p < \infty$ converging to f_0 in $L_p(\mathbb{R}^3 \times \mathbb{R}^3)$, satisfying $(1+|v|)f_0^n(x,v) \to 0$ as $|(x,v)| \to \infty$ and such that

$$f_0^n \ge 0, \quad ||f_0^n||_{L_\infty(\mathbb{R}^3\times\mathbb{R}^3)} \le const \quad \text{and} \quad ||f_0^n||_{L_1(\mathbb{R}^3\times\mathbb{R}^3)} = 1.$$

Denote, for each n, by $f_n(t,x,v)$ the corresponding C_b^1-solution of (17)-(19); we also have $\mu_n(t) = f_n(t,x,v)dx\, dv \in C([-T,T];M^+)$ for any $T > 0$.

Proposition 2.10 (compactness) *For any $\epsilon > 0$ and $T > 0$ there exists compact $Q_\epsilon \subset \mathbb{R}^3 \times \mathbb{R}^3$ such that*

$$\int\limits_{(\mathbb{R}^3\times\mathbb{R}^3)\setminus Q_\epsilon} f_n(t,x,v)dx\, dv < \epsilon \quad \forall\, t \in [-T,T],\ \forall\, n = 1,2,3,...$$

Proof. First of all, we shall show that for any $1 \le p < \infty$ there exists $D_p > 0$ such that

$$||w_n(\cdot,t)||_{L_p(\mathbb{R}^3\times\mathbb{R}^3)} \le D_p \quad \forall t \in [-T,T] \quad \forall n = 1,2,3,... \tag{20}$$

For this aim, we approximate $f_n(t,x,v)$ in $L_p(\mathbb{R}^3 \times \mathbb{R}^3)$ by infinitely differentiable functions $\tilde{f}_n$ with compact supports. Consider the expression

$$\int\limits_{\mathbb{R}^3} dx \left\{ \int\limits_{\mathbb{R}^3\times\mathbb{R}^3} \nabla_x (U(x-y-\cdot) * n^3\omega(n\cdot))\tilde{f}_n(t,y,v)dy\, dv \right\}^p . \tag{21}$$

Since $U(\cdot) * n^3\omega(n(\cdot))$ is infinitely differentiable and bounded with all its derivations and since $\tilde{f}_n$ are infinitely differentiable, the expression $\int\limits_{\mathbb{R}^3\times\mathbb{R}^3} \nabla_x(U * n^3\omega(n\cdot))(x-y)\tilde{f}_n(t,y,v)dy\, dv$ is determined. So, we have that expression (21) is equal to

$$\int\limits_{\mathbb{R}^3} dx \left\{ \int\limits_{\mathbb{R}^3} dz \nabla_x U(x-z) \left[\int\limits_{\mathbb{R}^3\times\mathbb{R}^3} n^3\omega(n(z-y))\tilde{f}_n(t,y,v)dy\, dv \right] \right\}^p .$$

Letting here $\tilde{f}_n \to f_n$ in $L_p(\mathbb{R}^3 \times \mathbb{R}^3)$, we easily get, because as well known

$$\int\limits_{\mathbb{R}^3\times\mathbb{R}^3} n^3\omega(n(\cdot - y))\tilde{f}_n(t,y,v)dy\, dv \to \int\limits_{\mathbb{R}^3\times\mathbb{R}^3} n^3\omega(n(\cdot - y))f_n(t,y,v)dy\, dv$$

in $L_p(\mathbb{R}^3)$ as $\tilde{f}_n \to f_n$ in $L_p(\mathbb{R}^3 \times \mathbb{R}^3)$, that

$$||w_n(\cdot,t)||^p_{L_p(\mathbb{R}^3)} = \int_{\mathbb{R}^3} dx \Big\{ \int_{\mathbb{R}^3} dz \nabla_x U(x-z) \times$$

$$\times \left[\int_{\mathbb{R}^3\times\mathbb{R}^3} n^3 \omega(n(z-y)) f_n(t,y,v) dy\, dv \right] \Big\}^p \leq$$

$$\leq C_p ||f_n(t,\cdot,\cdot)||^p_{L_p(\mathbb{R}^3\times\mathbb{R}^3)}$$

(here we applied the known estimate

$$\left|\left| \int_{\mathbb{R}^3\times\mathbb{R}^3} n^3\omega(n(x-y))g(y,v)dy \right|\right|_{L_p(\mathbb{R}^3\times\mathbb{R}^3)} \leq ||g(\cdot,\cdot)||_{L_p(\mathbb{R}^3\times\mathbb{R}^3)}).$$

Now, (20) follows by Proposition 2.7.

Take now arbitrary $\epsilon > 0$ and $T > 0$. There exists a closed ball $B_\epsilon = B_R(0)$ such that

$$\int_{(\mathbb{R}^3\times\mathbb{R}^3)\setminus B_\epsilon} f_0^n(x,v)dx\, dv < \epsilon/2 \quad \forall\, n = 1,2,3,... \tag{22}$$

For each n, denote by S^n_t, $(x_n(t,x_0,v_0), v_n(t,x_0,v_0))$ and $J_n(t,x_0,v_0)$ the operator, functions and Jacobi matrix analogous to S_t, $(x(t,x_0,v_0), v(t,x_0,v_0))$ and $J(t,x_0,v_0)$ and corresponding to $f = f_n$. Then, in view of (20) and Corollary 2.9, we obtain:

$$\int_0^T dt \int_{B_\epsilon} |\dot{v}_n(t,x_0,v_0)| dx_0 dv_0 \leq$$

$$\leq C_{\epsilon,T} \int_0^T dt \left\{ \int_{B_\epsilon} |w_n(x(t,x_0,v_0),t)|^2 dx_0 dv_0 \right\}^{1/2} \leq$$

$$\leq C_{\epsilon,T} \int_0^T dt \left\{ 1 + \int_{S^n_t(B_\epsilon)} |w_n(x,t)|^2 \cdot |\det J_n(-t,x,v)| dx\, dv \right\} \leq$$

$$\leq C_{\epsilon,T}(D_2^2+1)T$$

where $C_{\epsilon,T}>0$ depends only on T and on the Lebesgue measure meas(B_ϵ) of B_ϵ. For the interval of time $[-T,0)$ all the estimates can be made by analogy. So, finally we obtain:

$$\int_{-T}^{T} dt \int_{B_\epsilon} |\dot{v}_n(t,x_0,v_0)|dx_0dv_0 \leq C_{3,\epsilon,T}. \tag{23}$$

Set $g_n(x_0,v_0)=\int_{-T}^{T} dt|\dot{v}_n(t,x_0,v_0)|$. Then, it follows from (23) that there exists $A>0$ for which

$$(1+||f_0^n||_{L_\infty(\mathbb{R}^3\times\mathbb{R}^3)})\cdot \text{meas }(R_A^n)<\epsilon/2 \ \forall\ n \text{ where}$$

$$R_A^n=\{(x_0,v_0)\in B_\epsilon:\ g_n(x_0,v_0)>A\}. \tag{24}$$

Let

$$\overline{x}=\sup\{|x|:\ \exists\ v:\ (x,v)\in B_\epsilon\} \text{ and } \overline{v}=\sup\{|v|:\ \exists\ x:\ (x,v)\in B_\epsilon\},$$

$$x_1=\overline{x}+\overline{v}T+(\overline{v}+A)T^2,\ v_1=\overline{v}+(\overline{v}+A)T. \tag{25}$$

Take now $Q_\epsilon=\{(x,v):\ |x|\leq x_1,\ |v\leq v_1\}$. Clearly, $B_\epsilon\subset Q_\epsilon$. Let us show that

$$||f_n(t,\cdot,\cdot)||_{L_1(Q_\epsilon)}>1-\epsilon \quad \forall\ t\in[-T,T],\ \forall\ n=1,2,3,... \tag{26}$$

Indeed, if $t=0$, then (26) holds. Let $t>0$ (the case $t<0$ can be treated by analogy). We have by (24) and (22):

$$||f_0^n||_{L_1(R_A^n)}<\epsilon/2 \quad \text{and} \quad ||f_0^n||_{L_1(B_\epsilon\backslash R_A^n)}>1-\epsilon.$$

Let $R_A^n(t)=\{(x(t),v(t)):\ (x_0,v_0)\in R_A^n\}$. Then, by Theorem 2.8 on invariant measures

$$||f_n(t,\cdot,\cdot)||_{L_1(R_A^n(t))}=||f_n(0,\cdot,\cdot)||_{L_1(R_A^n)}<\epsilon/2,$$

and for any $(x_0,v_0)\in B_\epsilon\backslash R_A^n$ we have by (23)-(25) that $(x_n(t),v_n(t))\in Q_\epsilon$. Hence, by Theorem 2.8 on invariant measures

$$||f_n(t,\cdot,\cdot)||_{L_1(Q_\epsilon)}\geq ||f_n(t,\cdot,\cdot)||_{L_1(Q_\epsilon\backslash R_A^n(t))}=$$

$$= ||f_n(0,\cdot,\cdot)||_{L_1(S_{-t}(Q_\epsilon\setminus R_A^n(t)))} \geq ||f_0||_{L_1(B_\epsilon\setminus R_A^n)} > 1-\epsilon.\square$$

Proposition 2.11 (compactness) *Under the assumptions of the previous Proposition 2.10 for any $\epsilon > 0$ there exists $\delta > 0$ such that for any n, $1 \leq p < \infty$ and $t, s \in [-T, T]$ satisfying $|t-s| < \delta$ one has*

$$\rho_{1,Q_\epsilon}(f_n(t,\cdot,\cdot), f_n(s,\cdot,\cdot)) < \epsilon$$

where Q_ϵ is the compact set from Proposition 2.10 corresponding to a given $\epsilon > 0$ and $\rho_{1,Q_\epsilon}(\cdot,\cdot)$ is the metric function from Theorem 2.3 corresponding to the compact set Q_ϵ.

Proof. Take an arbitrary $\epsilon > 0$, the corresponding set Q_ϵ and let $M = \{z = (x,v) \in \mathbb{R}^3 \times \mathbb{R}^3 : \ \mathrm{dist}(z; Q_\epsilon) \leq 1\}$. Take an arbitrary $\varphi \in C_b^1(\mathbb{R}^3 \times \mathbb{R}^3)$ satisfying $||\varphi||_{C_b^1(\mathbb{R}^3\times\mathbb{R}^3)} \leq 1$ and $\varphi \equiv 0$ in $(\mathbb{R}^3\times\mathbb{R}^3)\setminus M$, a number n and $t, s \in [-T, T]$ such that $t > s$. We obtain from Eq. (1) using the fact that $\sup\{|v| : \ \exists x : \ (x,v) \in M\} < \infty$ and applying the Hölder's inequality, Lemma 2.1 and Proposition 2.7:

$$\left| \int_M \varphi(x,v)[f_n(t,x,v) - f_n(s,x,v)]dx\ dv \right| =$$

$$= \left| \int_s^t dr \int_M dx\ dv f_n(r,x,v)\{v\cdot\varphi_x + \varphi_v \cdot w_n(x,r)\} \right| \leq$$

$$\leq \int_s^t dr \int_M dx\ dv f_n(r,x,v)(|v| + |w_n(x,r)|) \leq C(t-s),$$

where $C > 0$ does not depend on t and s.$\square$

Now, we turn to proving Theorem 1.4. According to Propositions 2.10 and 2.11 and Theorem 2.3 the sequence $\{f_n(t,\cdot,\cdot)\}_{n=1,2,3,...}$ contains a subsequence still denoted $\{f_n(t,\cdot,\cdot)\}_{n=1,2,3,...}$ and there exists a function $f(t,\cdot,\cdot) \in C_w(\mathbb{R}; L_p(\mathbb{R}^3\times\mathbb{R}^3))$, where $p \in [1,\infty)$ is arbitrary, so that for any positive integer k and $1 \leq p < \infty$ this subsequence converges to $f(t,\cdot,\cdot)$ in $C_w([-k,k]; L_p(\mathbb{R}^3\times\mathbb{R}^3))$ and that in addition

$$||f(t,\cdot,\cdot)||_{L_p(\mathbb{R}^3\times\mathbb{R}^3)} \leq ||f_0(\cdot,\cdot)||_{L_p(\mathbb{R}^3\times\mathbb{R}^3)} \quad \forall\, t \neq 0.$$

Let $K_\delta(a,p) = \{(x,v) \in \mathbb{R}^3\times\mathbb{R}^3 : \ -\delta \leq x_i - a_i \leq \delta, \ -\delta \leq v_i - p_i \leq \delta, \ i = 1,2,3\}$ and $U_\delta(x) = U(x)$ if $|x| < \delta$, $U_\delta(x) = 0$ for all other values of x.

Lemma 2.12 *Let* $\{g_n\}_{n=1,2,3,...} \subset L_1(\mathbb{R}^3 \times \mathbb{R}^3) \cap L_\infty(\mathbb{R}^3 \times \mathbb{R}^3)$,

$$||g_n||_{L_1(\mathbb{R}^3\times\mathbb{R}^3)} + ||g_n||_{L_\infty(\mathbb{R}^3\times\mathbb{R}^3)} \le C \tag{27}$$

and for any $\epsilon > 0$ *there exist compact* $B_\epsilon \subset \mathbb{R}^3 \times \mathbb{R}^3$ *such that*

$$||g_n||_{L_1((\mathbb{R}^3\times\mathbb{R}^3)\setminus B_\epsilon)} < \epsilon, \quad n = 1, 2, 3, ...$$

Let also $g_n \to 0$ *weakly in* $L_p(\mathbb{R}^3 \times \mathbb{R}^3)$ *for any* $1 < p < \infty$. *Then, for any* $\epsilon > 0$ *and* $1 < p < \infty$ *there exists* $\delta_0 > 0$ *such that*

$$\left|\left|\int_{\mathbb{R}^3\times\mathbb{R}^3} (U_\delta(\cdot) * n^3\omega(n\cdot))(\cdot - y)g_n(y, v)dy\, dv\right|\right|_{L_p(\mathbb{R}^3)} < \epsilon \ \ \forall\, n\ \forall\, 0 < \delta < \delta_0.$$

<u>Proof.</u> Take an arbitrary $\epsilon > 0$ and let $B_\epsilon = K_R(0, 0)$ be such that

$$||g_n||_{L_p((\mathbb{R}^3\times\mathbb{R}^3)\setminus B_\epsilon)} < \epsilon \ \ \forall\, n = 1, 2, 3, ..., \ \forall\, 1 < p < \infty.$$

Take an arbitrary $\delta > 0$ and consider the partition of $\mathbb{R}^3 \times \mathbb{R}^3$ into cubes $K_\delta(2\delta i, 2\delta j)$ where $i = (i_1, i_2, i_3)$, $j = (j_1, j_2, j_3)$ and i_k, j_m are integer. Let $K_{\delta,k}$, $k = 1, 2, ..., N$, be a reindexing of those cubes $K_\delta(2\delta i, 2\delta j)$ the intersection of each of which with B_ϵ is nonempty and let $K_\delta = \bigcup_{k=1}^{N} K_{\delta,k}$. Then also

$$||g_n||_{L_p((\mathbb{R}^3\times\mathbb{R}^3)\setminus K_\delta)} < \epsilon \ \ \forall\, n = 1, 2, 3, ..., \ \forall\, 1 < p < \infty. \tag{28}$$

Obviously, there exists $C_1 > 0$ such that

$$N \le C_1\delta^{-6} \tag{29}$$

for all sufficiently small $\delta > 0$. Also, by (27), there exist $\delta_0 > 0$ and $C_1 > 0$ such that

$$||g_n||_{L_p(K_{\delta,k})} < C_1\delta^{6/p} \ \ \forall\, n\ \forall\, 0 < \delta < \delta_0\ \forall\, k = 1, 2, ..., N.$$

Introduce the functions $h_{\delta,k,n}$, $k = 1, 2, ..., N$, where for each k $h_{\delta,k,n}(x, v) = g_n(x, v)$ in $K_{\delta,k}$ and $h_{\delta,k,n}(x, v) = 0$ from the outside of $K_{\delta,k}$. Let also $h_{\delta,n}(x, v) = g_n(x, v)$ in K_δ, $h_{\delta,n}(x, v) = 0$ otherwise.

By Theorem 2 from Section 2.2.3 of [13] and from the proof of this result (see estimate (13)), we have

$$\left\|\int_{\mathbb{R}^3\times\mathbb{R}^3} U_\delta(\cdot - y)g(y,v)dy\,dv\right\|_{L_q(\mathbb{R}^3)} \le c'_q\|g\|_{L_q(\mathbb{R}^3\times\mathbb{R}^3)}$$

for any $1 < q < \infty$ and for any $g \in L_q(\mathbb{R}^3 \times \mathbb{R}^3)$, for all $0 < \delta \le 1$. In addition, we obviously have:

$$\int_{\mathbb{R}^3\times\mathbb{R}^3} (U_\delta(\cdot) * n^3\omega(n\cdot))(x-y)g(y,v)dy\,dv =$$

$$= \int_{\mathbb{R}^3\times\mathbb{R}^3} U_\delta(x-y)(n^3\omega(n\cdot) *_x g(\cdot,\cdot))(y,v)dy\,dv$$

and as well known

$$\|n^3\omega(n\cdot) *_x g(\cdot,\cdot)\|_{L_q(\mathbb{R}^3\times\mathbb{R}^3)} \le \|g\|_{L_q(\mathbb{R}^3\times\mathbb{R}^3)} \quad \forall\, g\ \forall\, 1 < q < \infty.$$

Take an arbitrary $\delta_1 \in (0,\delta)$ and set $U_{\delta_1,\delta}(x) = U_\delta(x) - U_{\delta_1}(x)$. Let us show that

$$\int_{K_\delta} (U_{\delta_1,\delta}(\cdot) * n^3\omega(n\cdot))(x-y)g_n(y,v)dy\,dv \to 0 \quad \text{as} \quad n \to \infty \tag{30}$$

uniformly in $x \in \mathbb{R}^3$. First, we observe that the expression in the left-hand side of (30) is equal to zero if dist $(x; K_\delta) > \delta$. Suppose that (30) does not hold. Then, there exist $c > 0$ and a bounded sequence $\{x_n\}$ such that the absolute value of the left-hand side of (30) with $x = x_n$ is not less than c for all n. Without the loss of generality we can accept that $x_n \to x$ as $n \to \infty$. But then, since obviously the sequence $(n^3\omega(n\cdot) *_x h_{\delta,n}(\cdot,\cdot))(y,v) \to 0$ weakly in $L_p(\mathbb{R}^3 \times \mathbb{R}^3)$ with an arbitrary $1 < p < \infty$, we obtain that the absolute value of the left-hand side in (30) with $x = x_n$ is not larger than

$$\int_{K_\delta} |U_{\delta_1,\delta}(x_n - y) - U_{\delta_1,\delta}(x-y)| \cdot |(n^3\omega(n\cdot) *_x h_{\delta,n}(\cdot,\cdot))(y,v)|dy\,dv +$$

$$+\left|\int\limits_{K_\delta} U_{\delta_1,\delta}(x-y)(n^3\omega(n\cdot) *_x h_{\delta,n}(\cdot,\cdot))(y,v)dy\, dv\right| \to 0 \quad \text{as} \quad n\to\infty,$$

i. e. we get a contradiction. So, relations (30) are proved. In particular,

$$\left\|\int\limits_{K_\delta} (U_{\delta_1,\delta}(\cdot) * n^3\omega(n\cdot))(\cdot - y)g_n(y,v)dy\, dv\right\|_{L_p(\mathbb{R}^3\times\mathbb{R}^3)} \to 0 \text{ as } n\to\infty,$$

for all $1<p<\infty$.

Now, applying the Hölder's inequality, we obtain the following:

$$\left\{\int\limits_{\mathbb{R}^3} dx \left[\int\limits_{K_{\delta},k} (U_{\delta_1}(\cdot) * n^3\omega(\cdot))(x-y)h_{\delta,k,n}(y,v)dy\, dv\right]^p\right\}^{1/p} =$$

$$= \delta_1^3 \left\{\int\limits_{\mathbb{R}^3} dx \left[\int\limits_{\mathbb{R}^3\times\mathbb{R}^3} (U_{\delta_1}(\cdot) * n^3\omega(\cdot))(x-y)h_{\delta,k,n}(y,\delta_1 v)dy\, dv\right]^p\right\}^{1/p} \leq$$

$$\leq C_2\delta_1^3\delta^{2/p}\times$$

$$\times \left\{\int\limits_{\mathbb{R}^3} dx \left[\int\limits_{\mathbb{R}^3\times\mathbb{R}^3} (U_{\delta_1}(\cdot) * n^3\omega(\cdot))(x-y)h_{\delta,k,n}(y,\delta_1 v)dy\, dv\right]^{3p}\right\}^{1/3p} \leq$$

$$\leq C_3c'_{3p}\delta_1^3\delta^{2/p}\delta^{1/p}(\delta/\delta_1)^{1/p} = C_3c'_{3p}\delta^{4/p}\delta_1^{3-1/p}.$$

Hence, by (29)

$$\left\|\int\limits_{K_\delta} (U_{\delta_1} * n^3\omega(n\cdot))(\cdot - y)g_n(y,v)dy\, dv\right\|_{L_p(\mathbb{R}^3)} \leq C_4c'_{3p}\delta_1^{3-1/p}\delta^{4/p-6}. \quad (31)$$

Note also that in (31) the constants C_4 and c'_{3p} are independent of n, δ and δ_1. Take $\delta_1 = \delta^4$. Then, the right-hand side of (31) becomes equal to

$$C_4c'_{3p}\delta^6 \to 0 \quad \text{as} \quad \delta\to +0. \quad (32)$$

In view of (28) and (30)-(32), Lemma 2.12 is proved.□

Proposition 2.13 *Let* $\{g_n\}_{n=1,2,3,...} \subset L_1(\mathbb{R}^3 \times \mathbb{R}^3) \cap L_\infty(\mathbb{R}^3 \times \mathbb{R}^3)$ *be bounded in* $L_1(\mathbb{R}^3 \times \mathbb{R}^3)$ *and in* $L_\infty(\mathbb{R}^3 \times \mathbb{R}^3)$ *and let for any* $1 < p < \infty$ *this sequence converge weakly in* $L_p(\mathbb{R}^3 \times \mathbb{R}^3)$ *to zero. Let for any* $\epsilon > 0$ *there exist compact* $B_\epsilon \subset \mathbb{R}^3 \times \mathbb{R}^3$ *such that for any* $1 < p < \infty$

$$||g_n||_{L_p((\mathbb{R}^3 \times \mathbb{R}^3)\setminus B_\epsilon)} < \epsilon, \quad n = 1, 2, 3, ... \tag{33}$$

Then, for any $1 < p < \infty$

$$\int\limits_{\mathbb{R}^3 \times \mathbb{R}^3} (U(\cdot) * n^3\omega(n\cdot))(\cdot - y) g_n(y, v) dy\, dv \to 0$$

in $L_p(\mathbb{R}^3)$ *strongly.*

Proof. Take arbitrary $\epsilon > 0$ and $1 < p < \infty$. Due to (33) there exists compact $B_\epsilon \subset \mathbb{R}^3 \times \mathbb{R}^3$ such that

$$\left|\left| \int\limits_{(\mathbb{R}^3 \times \mathbb{R}^3)\setminus B_\epsilon} (U(\cdot) * n^3\omega(n\cdot))(\cdot - y) g_n(y, v) dy\, dv \right|\right|_{L_p(\mathbb{R}^3)} < \frac{\epsilon}{3} \quad \forall\, n = 1, 2, 3, ...$$

By Lemma 2.12 there exists $\delta > 0$ such that

$$\left|\left| \int\limits_{B_\epsilon} (U_\delta(\cdot) * n^3\omega(n\cdot))(\cdot - y) g_n(y, v) dy\, dv \right|\right|_{L_p(\mathbb{R}^3)} < \frac{\epsilon}{3} \quad \forall\, n = 1, 2, 3, ...$$

Let $h_n = g_n$ in B_ϵ, $h_n = 0$ from outside of B_ϵ. Obviously, $\varphi_n = n^3\omega(n\cdot) *_x h_n \to 0$ weakly in $L_p(\mathbb{R}^3 \times \mathbb{R}^3)$ and $\{\varphi_n\}$ is finite and bounded in $L_p(\mathbb{R}^3 \times \mathbb{R}^3)$. Clearly,

$$\left| \int\limits_{\mathbb{R}^3 \times \mathbb{R}^3} U_{\delta,\infty}(x - y) \varphi_n(y, v) dy\, dv \right| \le C|x|^{-3}, \quad |x| \ge 1, \ n = 1, 2, 3, ...$$

so that there exists $R > 0$ for which

$$\left|\left| \int\limits_{\mathbb{R}^3 \times \mathbb{R}^3} U_{\delta,\infty}(\cdot - y) \varphi_n(y, v) dy\, dv \right|\right|_{L_p(\{x \in \mathbb{R}^3 :\ |x| > R\})} < \frac{\epsilon}{6}$$

for all $n = 1, 2, 3, \ldots$. So, to prove Proposition, it suffices to show that

$$\left| \int_{\mathbb{R}^3 \times \mathbb{R}^3} U_{\delta,\infty}(x-y)\varphi_n(y,v)dy\, dv \right| \to 0$$

uniformly in $x \in \{x \in \mathbb{R}^3 : \ |x| \leq R\}$. But this can be made as when proving Lemma 2.12.□

Since due to Proposition 2.7 L_p-norms of f_n are bounded uniformly in t and n and in view of Proposition 2.13, we can pass to the limit $n \to \infty$ in identity (5) obtaining it for the function f and for arbitrary large $T > 0$. Now, taking an arbitrary $\epsilon > 0$, we have by Proposition 2.10 $||f(t,\cdot,\cdot)||_{L_1(\mathbb{R}^3 \times \mathbb{R}^3)} \geq 1 - \epsilon$. Hence, due to the arbitrariness of $\epsilon > 0$, we have $||f(t,\cdot,\cdot)||_{L_1(\mathbb{R}^3 \times \mathbb{R}^3)} \equiv 1$, which completes our proof of Theorem 1.4.□

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Жидков П. Е. E5-2002-269
О глобальных $L_1 \cap L_\infty$-решениях уравнения Власова
с потенциалом r^{-2}

Рассматривается задача Коши для уравнения Власова с потенциалом взаимодействия r^{-2}. Доказано существование в целом слабого решения этой задачи. Кроме того, построены инвариантные меры для уравнений Власова с регулярными потенциалами.

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On Global $L_1 \cap L_\infty$ Solutions of the Vlasov Equation
with the Potential r^{-2}

We consider the initial value problem for the Vlasov equation with the potential of interaction r^{-2} and prove the existence of a global weak solution for this problem. In addition, we construct invariant measures for Vlasov equations with regular potentials.

The investigation has been performed at the Bogoliubov Laboratory of Theoretical Physics, JINR.

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