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THE CASIMIR EFFECT AND CRITICAL PHENOMENA

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Эффект Казимира и критические явления

Дается обзор теоретических и экспериментальных результатов, подтверждающих существование эффекта Казимира в системах с критическими флуктуациями. Показано, что эффект проявляется в системах с ограниченной геометрией вблизи фазового перехода. В качестве примера рассмотрено наблюдение эффекта в пленках жидкого ^{4}He в окрестности точки λ.

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The Casimir Effect and Critical Phenomena

In the present review we focus our attention on the theory and the experimental confirmations of the Casimir effect in critical phenomena. Since the effect is related to the boundary conditions imposed on a system undergoing a phase transition and its consequences, the theory of critical phenomena in finite-size systems is an indispensable part of the theoretical description. Experiments with liquid films near a critical point are of particular experimental relevance to the studied phenomenon.

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INTRODUCTION

In 1948 the Dutch physicist H. G. B. Casimir showed that a physical force can be generated by the change of zero point vacuum fluctuations in quantum electrodynamics due to the presence of two parallel conducting flat plates separated by a distance L. This force $F(L)$ (per unit area A) has the magnitude

$$\frac{F(L)}{A} = -\frac{\pi^2 \hbar c}{240 L^4}. \tag{1}$$

Only the fundamental Planck's constant $\hbar$ and the speed of light c enter Eq. (1). One can consider the Casimir force as a striking macroscopic observation of the effects of quantum vacuum energy. In the 1950s and 1960s the initial experiments to detect the effect were of very limited accuracy due to the presence of various interfering phenomena. Only five decades later new improved experimental measurements, based on microelectromechanical systems (MEMS), atomic force microscope, etc, allowed for a reliable confirmation of the effect. At the moment, a spectacular agreement with theory at the level smaller than 1 per cent is attained. For details one can see [1, 2] and references therein.

Further it becomes clear that an analogous effect [3] takes place in wetting phenomena. The confinement of a liquid near the critical point gives rise to an effective force between the substrate-liquid and liquid-vapour interfaces. This effect is called «statistical-mechanical» or «thermodynamic» Casimir effect. In terms of the behaviour of the two-point correlation function of a system one can stay on a more general point of view. Confinement conditions, imposed on a system whose correlation function decays as *power* law in space, induce a long-range force between the surfaces limiting the system. One can generally call this phenomenon the Casimir effect. In other words, the Casimir effect is a phenomenon common to all systems characterized by fluctuating quantities on which external boundary conditions are imposed [3–6].

In the present review we will focus our attention on Casimir effect in critical phenomena. The effect is related to the geometrical constraints on a system undergoing a phase transition. That is why the theory of critical phenomena in finite-size systems is an indispensable part of the theory of «statistical-mechanical» Casimir effect.

1. FINITE-SIZE SCALING

Experimental samples have a certain shape and are characterized by the presence of surfaces. They are always of finite size. The partition function of such a finite system is a polynomial of a finite degree, and thus never shows singularities. From the theoretical point of view, critical points of the system in a literal sense are the result of the thermodynamic limit at which the volume has become infinite at constant particle density in the bulk. Exactly in the thermodynamic limit, critical points are characterized by singularities in the thermodynamic functions and by an infinite correlation length ξ. To which extent an experimental sample can be described by the bulk or finite-size theory depends on the value of the ratio $y = \xi/L$, where L is the effective value of the linear extensions of the system. If $y \ll 1$ (physically this means that we are away from the critical point) any finite-size effects will be invisible. If, however, $y = \mathrm{O}(1)$, strong deviation from the bulk critical behaviour will be observed. Such a behaviour has been called finite-size critical behaviour. In a finite system its correlation length ξ_L will not become infinite and the singularities in the free energy are replaced by rounded extrema, located at somewhat shifted position as compared to the position of the bulk singularities. The description of this rounding near to critical temperature and crossover from finite-size to bulk critical behaviour first formulated by M. Fisher (1972) and subsequently elaborated by a number of authors is called Finite-Size Scaling (FSS) (see, e.g., [5]).

Scaling hypotheses are made in the form of statement about properties of thermodynamic quantities in terms of homogeneous functions. For our purposes it is convenient to distinguish «regular» and «singular» parts for every thermodynamic function of a finite sample in the vicinity of the bulk critical point. If a thermodynamic function depends on several variables the singular part of this function only depends on a certain combination of these variables. In the case of two variables, e.g., L and ξ, one is scaled by a certain power of the other. If the singular part of a thermodynamic function depends on $y = \xi/L$, such a dependence is known as standard FSS. In experiments, if the properly scaled thermodynamic function is plotted vs a properly scaled variable, FSS manifests itself as data collapse. Note that standard FSS reflects the *ordinary homogeneity* of the corresponding thermodynamic function of the finite system as a function of the two natural macroscopic lengths L and ξ. It is known that standard FSS fails above the upper critical dimension. Renormalization group analysis reveals that the violation of the standard FSS, as well as the breakdown of hyperscaling, is a consequence of the appearance in the theory of the so-called «dangerous irrelevant variables». In this case, the finite-size effects are controlled by another ratio, namely, l_∞/L, where l_∞ is called thermodynamic length [7]. Such a dependence is known as modified FSS. It reflects the *generalized homogeneity* of the corresponding thermodynamic function of the finite system.

A critical d-dimensional system confined in a finite geometry can be found in four qualitatively different situations depending on the value of the extended up to infinity dimensions d'.

1. If $d' < d_<$, the system is below its lower critical dimension and a (d-dimensional) critical behaviour appears only in the thermodynamic limit.

2. In the borderline case of $d' = d_<$, the system is at its lower critical dimension and may have only a zero-temperature critical point.

3. If d' of the system is above its lower critical dimension $d_<$, it exhibits a true critical behaviour. A crossover from d'-dimensional to d-dimensional critical behaviour takes place when $L \to \infty$.

4. If $d' \geqslant d_>$, the system is above its upper critical dimension and it exhibits a mean-field type critical behaviour.

The role of overall dimension d needs some comments. Normally one may expect modified FSS above the upper critical dimension $d_>$. However, for some special geometries it is not the case. Considering the exactly solvable $5d$ spherical model with one finite dimension, Barber and Fisher, as early as 1973 [8], stated that the scaling variable should be the standard one, i.e., L/ξ in our notation. In Ref. [9] it was shown that for the Ising model in block geometry $(d > 4, d' = 0)$ and under Dirichlet boundary conditions $\xi \sim L$. The finite-size geometry determines the level of the fluctuations at $T \approx T_c$, which is different in comparison with that of the bulk system. The applicability of the mean-field theory to the study of finite-size critical behaviour, for $d > d_>$, is strictly valid only as $d \to \infty$ [10]. A list, focusing on various aspects of standard and modified FSS, can be found in [4, 5] and references therein. While the FSS description of isotropic systems is relatively better understood, some specific problems arise in the case of anisotropic systems, systems with disorder, etc. It is possible to demonstrate that FSS in its standard form takes place also for a certain class of systems regardless of the nature of their anisotropic properties: both anisotropy of shape (e.g., rectangular systems with linear dimensions different in two lattice direction; $L_{||} \gg L_\perp$) and anisotropic critical behaviour (correlation lengths $\xi_{||}$ and $\xi_\perp$diverging with different critical exponents $\nu_{||}$ and $\nu_\perp$in different directions) are considered in [11, 12]. In such a system the FSS behaviour is governed by the «perpendicular» scaling combination $y = \xi_\perp/L$ only. In reality the description of effects of disorder on the finite-size critical behaviour is also of important relevance. A formulation of general FSS concepts in this case is strongly complicated due to the additional averaging over the different samples. Here, an important theoretical problem of interest is related to the property of self-averaging (SA) [13]. The lack of SA in disordered systems implies that the standard FSS breaks down. Due to the presence of randomness it is necessary to deal with a two-variable problem: the standard scaling variable y and one additional λ. In the mean-field regime $d > 4$, the second variable λ depends on the distribution of the random variable in the problem. In the case $d = 4 - \varepsilon, \varepsilon \ll 1$, the ε expansion to first order in ε shows that close to the critical temperature we are really dealing with a one-variable problem, since the second variable λ is a fixed universal number. For more details one can see [14] and references therein.

Since the Casimir force is a consequence of confinement and boundary conditions upon the critical point approach its theoretical understanding is based on the FSS theory.

2. THE CASIMIR FORCE

The films provide the simplest geometry for studying the Casimir force theoretically and experimentally as well. Let us consider a statistical-mechanical system with slab geometry $\mathrm{L}\times\infty^{d-1}$ under given boundary conditions τ imposed in the finite direction. If it is at a critical point T_C (i.e., undergoing a second order phase transition) or in a phase with broken continuous symmetry, the system in its «bulk phase» (i.e., in the limit $L \to \infty$) exhibits long-range correlations. These correlations decaying algebraically rather than exponentially fast give rise to fluctuation-induced long-range force — the Casimir force in the critical phenomena.

The Casimir force between the slab faces is defined as [4, 5]

$$F^{(a,b)}(T,\, L) = -\frac{\partial f_{\mathrm{ex}}^{(a,b)}(T, L)}{\partial L}, \tag{2}$$

where $f_{\mathrm{ex}}^{(a,b)}\,(T,\, L)$ is the so-called *excess free* energy defined as

$$f_{\mathrm{ex}}^{(a,b)}(T, L) = f_L^{(a,b)}(T) - L f_{\mathrm{bulk}}(T).$$

Here, $f_L^{(a,b)}(T)$ is the total free energy density (per unit area and per $k_B T$) of a d-dimensional critical system in the form of a slab with thickness L, area A, and boundary conditions (a) and (b) at the opposite surfaces. At the bulk critical point T_C, it has the asymptotic form

$$f_L^{(a,b)}(T_C) \cong L f_{\mathrm{bulk}}(T_C) + f_{\mathrm{surf}}^{(a)}(T_C) + f_{\mathrm{surf}}^{(b)}(T_C) + L^{-(d-1)}\Delta^{(a,b)}(T_C) + \dots \tag{3}$$

as $A \to \infty$, $L \gg 1$. Here, $f_{\mathrm{surf}}^{(a)}$ and $f_{\mathrm{surf}}^{(b)}$ are the surface energy contributions and $\Delta^{(a,b)}$ is the so-called Casimir amplitude. The L dependence of the last term in Eq. (3) follows from the scale invariance of the free energy as it was pointed out by Fisher and de Gennes [3].

Let us consider the *finite-size* part of the excess free energy

$$\delta f_L^{(a,b)}(T) \equiv f_L^{(a,b)}(T) - L f_{\mathrm{bulk}}(T) - f_{\mathrm{surf}}^{(a)}(T) - f_{\mathrm{surf}}^{(b)}(T). \tag{4}$$

The finite-size scaling analysis for the «singular part» of $\delta f_L^{(a,b)}(T)$ shows

$$\delta f_{\mathrm{ex,sin}\,g}^{(a,b)}(T,\, L) = L^{-(d-1)} X_{\mathrm{ex}}^{(a,b)}(a_t t L^{1/\nu}), \tag{5}$$

where $t = (T - T_C)/T_C$ is the reduced temperature, a_t is a nonuniversal scaling factor, $X_{\mathrm{ex}}^{(a,b)}$ is a *universal scaling function,* $X_{\mathrm{ex}}^{(a,b)}(0) \equiv \Delta^{(a,b)}(T_C)$, and ν is the critical exponent of the correlation length. The non-singular at T_C part of $f_{\mathrm{ex}}^{(a,b)}(T,\, L)$ is usually proportional to $\mathrm{O}(L^{-d})$ and can be omitted. The relation (5) is valid in the finite-size scaling region, when $tL^{1/\nu}$= O(1). When the system leaves this region away from the critical temperature (i.e., towards high temperatures) one usually expects small excess free energy

$$f_{\mathrm{ex}}^{(a,b)}(T,\, L) = \mathrm{O}\left(\mathrm{e}^{-tL}\right), \tag{6}$$

thus the Casimir force is equal to zero.

The *universal* amplitude $\Delta^{(a,b)}$ depends on the bulk universality class and the universality classes of the boundary conditions [4, 5]. It is just the subject of the theory in the framework of different model calculations.

3. MODEL CALCULATIONS

First, let us consider the model of Perfect Bose-Gas (PBG). It is known that the PBG undergoes a phase transition (the *Bose–Einstein condensation*) in the grand canonical ensemble if the chemical potential $\mu \leqslant 0$ equals its critical value $\mu_c = 0$. The PBG is the simplest system showing spontaneously broken continuous symmetry with a long-distance power decrease of particle–particle correlations in the condensed phase. As so, the PBG may be a nice illustration of a critical system where the Casimir phenomenon can take place. For example, in the case of slab with thickness L and with Dirichlet–Dirichlet b.c. (D), for the Casimir force the following result

$$F_{\mathrm{PBG}} = -\frac{\varsigma(3)}{4\pi}\frac{1}{L^3}, \tag{7}$$

was obtained [15].

Simple dimension analysis can explain the difference in power of L in Eqs. (1) and (7). The thermodynamic Casimir force (it is per unit area and per $k_B T$), as one can see, is classical in origin. This reflects the fact that the phase transition in the condensed state is governed by classical (or thermodynamic) fluctuations. This is in contrast to the electromagnetic Casimir force (per unit area), Eq. (1), which is quantum in origin and hence proportional to $\hbar c$ ($\hbar c$ has dimension of (energy times length)).

The above result is relevant in the critical point, $\mu = \mu_c = 0$. It can be extended outside of the critical point as it was pointed out in [16], (see also [6]). In terms of the polylogarithms $Li_p(p)$ defined by the series

$$Li_p(z) = \sum_{k=1}^{\infty} \frac{z^k}{k^p}$$

with $|z| \leqslant 1$ for $p \geqslant 2$ and $-1 \leqslant z < 1$ for $p = 1$, the Casimir force has the form

$$F_{\mathrm{PBG}}\,(u) = -\left[Li_3\left(\mathrm{e}^{-2u}\right) + 2uLi_2\left(\mathrm{e}^{-2u}\right) + 2u^2 \ln\left(1 - \mathrm{e}^{-2u}\right)\right] \frac{1}{4\pi L^3}. \quad (8)$$

Here, $u = (-2\beta\mu)^{\frac{1}{2}} L/\lambda \sim L/\xi_+$ is the intrinsic scaling combination ($\xi_+ \sim (\mu - \mu_c)^{-\frac{1}{2}}$). Equation (8) is obtained under the condition $L/\lambda \gg 1$ ($\lambda = \hbar\sqrt{\beta/m}$ denotes the thermal wavelength). If we are in the bulk regime $L/\xi_+ \gg 1$, the Casimir force becomes exponentially small

$$F_{\mathrm{PBG}}\,(u) = -\left[\left(\mathrm{e}^{-2u}\right) + 2u^2\left(\mathrm{e}^{-2u}\right) + ...\right] \frac{1}{4\pi L^3} \quad (9)$$

and so invisible. The above study can be also performed for the other types of boundary conditions. Being of a pedagogical interest, the PBG underlies a number of its ultimate features [6, 15, 16]. For example, the Casimir effect is an order of magnitude larger for Periodic and Antiperiodic b.c. In the case of equal boundary conditions the Casimir force is attractive. Unequal boundary conditions (Antiperiodic and Dirichlet–Naumann) lead to positive Casimir amplitude, which means a repulsive Casimir force between the confining plates, so that the PBG may be a nice illustration of a critical system in which the Casimir phenomenon takes place.

More elaborate finite-size theory is possible in the framework of the O(N) Ginzburg–Landau model. In the Gaussian approximation the universal scaling functions for the Casimir force (for different boundary conditions) are calculated by Krech and Dietrich [17, 18]. The results, if $N = 2$, coincide with the corresponding results for the PBG. The two-component Gaussian model and the PBG share the same universality class, see also [14]. It is important that this statement is valid under the condition $L \gg \lambda$. Strictly speaking, the Gaussian model, as an approximation, becomes relevant to one-loop order above the upper critical dimension $d = 4$. A further more refined consideration of the problem in the framework of the Ginzburg–Landau model is due to the field-theoretical analysis in $d = 4 - \varepsilon$ dimensions ($\varepsilon \ll 1$). It allows universal quantities, like the scaling function at $d = 3$, to be computed by a perturbation theory around the upper critical dimension $d_c = 4$, so far up to the first order in $\varepsilon = 4 - d$ [17, 18]. The question of how trustworthy such results are for a three dimensional system (by setting $\varepsilon = 1$) is quite difficult. The problem manifests itself in the case of Periodic b.c. Having in mind that we know the exact result in the limit $N \to \infty$ [19], contrary to the common expectation, the $\mathrm{O}(\varepsilon)$ results for Δ^P do not approach, but deviate from the exact one as N grows (see Fig. 2 of [20]). Note that Monte-Carlo result for the Ising model $\Delta^P = -0.1526 \pm 0.0010$ [21] is surprisingly close to the exact value $\Delta^P (N \to \infty) = -2\varsigma(3) / (5\pi) \approx -0.1530$ [19]. Thus, one may conclude that the first order in ε results has an incorrect N-dependent behaviour, being too crude approximation in order to capture the way in which the exact $N \to \infty$ result is approached [5]. Especially for the case of Periodic b.c. as it was pointed out in [22], the ε expansion is ill-defined beyond two-loop order, because of infrared singularities. The presented revised theory yields well-defined small ε expansion involving fractional powers and logarithms of ε. The estimated values of Δ^P are: -0.1967 (for $N = 1$), -0.2147 (for $N = 2$) and -0.2311 (for $N = 3$) [22], i.e., again we have a deviation from the exact $N \to \infty$ result -0.1530 as N grows. Although, in the case of Dirichlet–Dirichlet b.c., the ε expansion has no infrared complications, further work is necessary to answer the question of «how reliable the ε expansion is in the case of slab geometry in order to obtain $d = 3$ results?».

4. EXPERIMENTAL CONFIRMATIONS: ^{4}He

While micro- and nanoscale experiments precisely verify the original electromagnetic Casimir force, the experimental characterization of the thermodynamic Casimir force has been scarce and, often, ambiguous. The most reliable experimental testing of the thermodynamic Casimir effect is related to measurements on thin ^{4}He films at and near the superfluid/normal transition $T_c = T_\lambda = 2.1768$ K. This is due to the nearly-ideal impurity-free nature of liquid ^{4}He and its low-sensitivity to gravitational rounding errors [23, 24]. As the superfluid order parameter vanishes at both film interfaces the Dirichlet–Dirichlet boundary conditions seem to be relevant [25]. This causes an attractive Casimir force, as follows from the theoretical considerations. If this force appears, it must produce near T_λ a temperature-dependent change in the equilibrium film thickness.

Let us consider a surface of a substrate placed at a height h of a reservoir of liquid ^{4}He that is in coexistence with its vapor. A thin liquid film with thickness d is formed on the substrate. The film thickness d can be determined by the force balance equation among the gravitational, van der Waals and

Casimir forces (see, e.g., [24, 26])

$$mgh = \frac{\gamma_0}{d^3}\left(1+\frac{d}{d_{1/2}}\right)^{-1} + \frac{k_B T_\lambda V}{d^3} X_{\rm Cas}^{(D,D)}\left(d/\xi\right).$$

Here, g is the gravitational acceleration, m is the atomic weight of helium, γ_0 and $d_{1/2}$ are specific interpolation parameters that characterize the van der Waals interaction of the liquid with the substrate, $V = 45.81$ $\text{Å}^3/\mathrm{atom}$ is the specific volume of liquid ^{4}He, and $X_{\rm Cas}^{(D,D)}$ is the dimensionless Casimir force scaling function. Since the Casimir force is attractive, it favors a thinner film.

Thinning of the superfluid films was experimentally observed and studied in [23, 24]. The ^{4}He films were formed on the surfaces of polished Cu capacitor plates, set in a cell containing liquid ^{4}He at the bottom [23]. The force balance equation and the experimental data were utilized in order to obtain the Casimir force scaling function. A number of obstacles arise in such experiments related to minimization of surface roughness, precise control of the cleanness of the surfaces, precise control of the temperature, etc. For example, the roughness of the Cu surface changes the effective areas of the Cu plates and makes the accurate determination of the film thickness impossible. Recently there have been more precise capacity measurements [24], where films of 238, 285, and 340 Å thickness adsorbed on N-doped silicon substrates with roughness ≈ 8 Å, were studied. In the region below T_λ, where the effect is the greatest, the scaling function $X_{\rm Cas}^{(D,D)}(x)$, deduced from the thinning of these three films, collapses onto a single universal curve, attaining a min $X_{\rm Cas}^{(D,D)}(x) = -1.30 \pm 0.03$ at the value of the scaling variable $x = tL^{1/\nu}, x_m = -9.7 \pm 0.8$ $\text{Å}^{1/\nu}$. The collapse confirms the finite-size scaling origin in the film thickness. Also, the presence down to 2.13 K of the Goldstone/surface fluctuation force makes the superfluid film ~ 2 Å thinner than the normal film [24].

One might ask about the correlation of the theoretical estimations with the experimental results. From a theoretical point of view, the situation is again quite complicated. The existing renormalization group calculations [4, 17, 18] are valid only for $T \geqslant T_\lambda$. They are in good agreement with experiment only at the transition temperature [24, 26]. However, the regime $T < T_\lambda$ is more interesting, since the effect is the greatest there. The amplitude of the Casimir force at x_m is approximately an order of magnitude larger than its value at T_λ [23, 24]. This regime is apparently far more difficult for a theoretical description. The existing theory is still under development [25–31] exhibiting some controversies.

Deep in the superfluid state, coming from the Goldstone modes in the bulk of the film, the following result for the Casimir force (per unit area and per $k_B T$)

$$F_{\rm bulk} = -\frac{\varsigma(3)}{8\pi}\frac{1}{L^3},$$

has been obtained [32]. If only these modes caused the thinning of the film, the film would be roughly the same at the critical point and beneath the transition, which is not the experimental situation.

To resolve the discrepancy with the experimental observations, Zandi et al. [26] added the effect of surface fluctuations, which also act as a source of the Casimir force. It was shown that surface fluctuations lead to an additional force

$$F_{\text{surf}} = -\frac{7}{4}\frac{\varsigma(3)}{8\pi}\frac{1}{L^3}$$

nearly twice as large as the bulk one. As pointed out in [23], the experimentally obtained thinning of the film is consistent with the following asymptotic, *low-temperature* value of the Casimir amplitude $\Delta_{\text{exp}}^{^4\text{He}} \approx -0.30 \pm 0.10$, which is marginally larger than the theoretical result $\Delta_{\text{theor}}^{^4\text{He}} \approx -0.15$ [25], due to both the Goldstone modes and the surface fluctuations.

The most challenging for the theory is the region immediately below T_λ, where the experimental scaling function has a deep minimum. In [29], due to the fact that Helium film is in equilibrium with the bulk Helium liquid, the authors postulated an additional atom transfer contribution to the film thinning below point λ. Utilizing a simple mean-field calculation with appropriately renormalized critical fluctuations some qualitative consistence with the experimental data is obtained in [29]. However, the mean-field analysis of the classical XY model (which one believes to describe pure ^{4}He near the critical point) also exhibits the behaviour of the scaling function with a characteristic deep minimum, which can be interpreted in encouraging comparison with the experimental data [30]. At least, the Monte-Carlo simulations [31] of the $3d - XY$ model with Dirichlet–Dirichlet b.c. give results in excellent agreement with the experimental results [23, 24] below the transition, as well as with theoretical calculations for $T \geqslant T_\lambda$ [17, 18].

As a result, one may conclude that the finite-size scaling theory and the thermodynamic Casimir effect are unambiguously confirmed by the experiment on liquid ^{4}He films. However, an accurate comparison between theory and experiment requires further precise measurements and more reliable theoretical estimations.

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